

Recent research in Graph Theory

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During the last few years, my research has been in graph theory and has led to several publications [3, 12, 13, 11, 28, 14] and pre-publications [29, 30]. Firstly I will introduce the results of [12] and [13].

A well-known and fundamental property of graphs is *Hamiltonicity*. A connected graph is Hamiltonian if it contains a spanning cycle. Determining if a graph is Hamiltonian is known as a NP-complete problem and no satisfactory characterization exists. Nevertheless, many sufficient conditions for Hamiltonicity were found, usually expressed in terms of degree, connectivity, density, toughness, independent set, regularity and forbidden subgraphs.

In [20], Goodman and Hedetniemi gave two alternative sufficient conditions based on the existence of some type of clique-covering of the graph. In [13] one of these conditions is generalized using the notion of *eulerian clique-covering*. A polynomial algorithm to decide the existence of such a covering for every graph containing at least one simplicial vertex is also given in [12].

Now, several Hamiltonicity preserving closure concepts for claw-free graphs were defined recently in [25, 15]. The closure of the graph is Hamiltonian if and only if the graph is. Moreover, it is easier to look for a clique-covering of the closure than to look for a clique-covering of the graph itself. Starting from these facts, I defined recently three new Hamiltonicity preserving closure concepts for graphs (cf. [14, 29, 30]).

Another research topic concerns hypergraphs. A first theme of this research is about Cartesian products of hypergraphs. Besides giving some properties of these products, we designed in [11] an algorithm able to decompose any conformal hypergraph into its prime factors. We gave recently a generalization to every hypergraph of this algorithm. Notice that the decomposition in prime factors helps to reduce algorithm complexity and that hypergraphs have several applications, in particular in image analysis. A second theme in hypergraph theory is the generalization of flow on graphs using hypergraphs and hyperflows. We remind the reader that flow theory has numerous applications in particular in engineering, computer science and operational research.

Finally a last future research topic concerns a new way to associate groups and graphs, said G-graphs. The traditional way to associate a graph to a group G and a set S of generators of G is known as *Cayley graphs*. For every group,

there exists a corresponding Cayley Graph. These graphs have highly-regular properties, making them suitable for many applications in parallel computing, networks, cryptography and security (see for instance [16, 17, 18, 19] and the corresponding chapters in [1, 2, 21, 22]). However, Cayley graphs have also certain limitations:

- They are always regular and vertex-transitive, and so in particular they can never be semi-symmetric; many interesting graphs are not Cayley Graphs (cubic semi-symmetric graphs, Gray and Ljubljana graphs, the complete bipartite graphs $K_{m,n}$, used for instance in modelling matching problems or concurrent systems, and also widely used in Coding Theory, are not Cayley graphs whenever $m \neq n$).
- Cayley Graphs do not capture much information about the underlying groups. For instance, when the generator set is defined as the group itself, we have a limit case where none of the particular features of the group is captured by the graph. Indeed, in that case, all the possible groups of order n have the same Cayley graph which is the complete graph K_n , for every positive integer n .
- It is known that two isomorphic groups give rise to two isomorphic Cayley Graphs; yet the converse is not true, even for the rather restricted subcategory of Abelian groups.

In order to overcome these limitations, the concept of a *G-Graph* was recently introduced and developed in [4, 5, 7, 8, 6, 10]. G-graphs generalize Cayley graphs in the sense that every Cayley graph can be constructed from a G-graph (more precisely there exists a surjective functor from the category of G-graphs into the category of Cayley graphs). They share good properties of Cayley graphs: the automorphism group is non-trivial, many properties of groups can be seen on the graphs, they can naturally be constructed efficiently by a polynomial algorithm from the groups, they are connected. But they have also some additional interesting properties that Cayley graphs do not have. First, not every G-graph is regular which makes this kind of graph more suitable to model telecommunication and processor networks. Moreover, in the Cayley case, taking $S = G$, where G is the group and S the generating set, leads to a complete graph. It is not the case for G-graphs which means that the group structure is more adequately captured by the graph, and that the canonical association to the group of the G-graph induced by setting $S = G$ is meaningful. An other interesting application of G-graph is the generation of symmetric and semi-symmetric graphs in a very efficient way. In particular, it is possible to generate 95% of the Marston Conder's list of cubic symmetric and semi-symmetric graphs (which extends itself the Foster Census) in a much more efficient way. The first algorithm able to generate quartic and quintic symmetric and semi-symmetric graphs is based on G-graph. Finally notice that G-graphs are quite general. For instance Petersen, Heawood, Pappus, Moebius-Kantor, Gray, Ljubljana graphs are G-graphs as well as the cube, the hypercube, the octahedral and cub-octahedral graphs

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