Scritical Pairs and Sunification

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Abstract. A general critical pair theory is given for rewriting many sorted terms with overloaded operations modulo equations. A main notion is *sunification*, which yields a set of *scritical pairs*, such that a set of rules is locally confluent iff they all converge. We prove a sufficient condition for overlaps to work instead of sunification, show that complete sunifier sets always exist, and are finite in important special cases. We also sketch a generalization based on category theory, for rewriting in free objects, e.g., algebras with additional structure, such as many sorts, ordered sorts, equationally defined subsorts, or continuity.

1 Introduction

This paper generalizes the Knuth-Bendix [20] critical pair tradition, using "sunifier" and "scritical pair" notions[1] partly inspired by cognitive linguistics blending [8] and its mathematical formulation in [10]. We show that scritical pair sets always exist, give a separation condition that often reduces them, and treat important special cases, including C and AC. Section 1.1 briefly reviews some closely related work, Section 2 gives an algebraic review of basic term rewriting, Section 3 introduces sunification, and Section 4 gives a sufficient condition for sunification to be overlap. Section 5 generalizes to terms in free objects, which is needed to encompass rewriting in recent algebraic languages. For example, OBJ3 [17] and BOBJ [15] use order sorted rewriting modulo A/C/1[2] operations; CafeOBJ [7] and Maude [5] use rewriting over membership equational modulo A/C/1; and CASL [23] uses rewriting in many sorted partial algebras. Other applications, e.g., programming language semantics, use algebras with metric or topological structure, or subsorts defined by conditions. Section 6 gives some conclusions and discusses future work. Many proofs are omitted or abbreviated in this version.

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1.1 Some Related work

A basic limitation of the original Knuth-Bendix [20] approach is its failure to handle permutative equations like commutativity, since it orients equations for use as rewrite rules; this failure inspired much subsequent work. Results of Huet [18] assume rules in A are left linear and equations in B are balanced (i.e., have the same variables in both sides); also confluence here is not confluence of $\Rightarrow_{A/B}$, and critical pairs are obtained by overlapping rules with one another and with equations in B. Peterson and Stickel [24] restrict to balanced, linear equations; their "variable extension" notion resembles our sunifier notion, but their solution to the problem that every reduction has infinitely many variable extensions is restricting to reductions compatible with the equations. Jouannaud and Kirchner [19] generalize the compatibility of [24], and their critical pair algorithm assumes finite complete unification and finite equivalence classes. All these classic papers treat unsorted terms.

Prior categorical formulations of term rewriting have used 2-categories, sesqui-categories, and monads, but it seems fair to say, as in [21], that this has neither had great impact nor solved significant open

[1] The letter "s" abbreviates "super," from our use of a superterm to generalize the usual notions, but we use the contractions to avoid pretention.
[2] This indicates any subset of associative, commutative, and identity laws.

problems; it has been a search for the right formulation. While we do not claim success, we are encouraged that our formulation is a minimal weakening of Diaconescu's successful categorical generalization of equational logic [6, 11], and by our unified treatment of special cases.

2 Term Rewriting

To introduce algebraic concepts and notations needed later, we give an algebraic treatment of **MSO term rewriting**, which is many sorted term rewriting where operation symbols may be overloaded. Many sorted rewriting is needed for many computer science applications [13, 12]; overloading is less essential, but very helpful for readability in applicatons. Section 2.1 shows that the obvious generalization of the critical pair theorem does not hold for rewriting modulo equations.

Let S be a set whose elements are called **sorts**. The set of finite sequences of sorts is denoted S^*. An S-sorted **signature** Σ is an indexed family of sets $\Sigma_{w,s}$ for $w \in S^*$ and $s \in S$. The elements of $\Sigma_{w,s}$ are called **operation symbols** of **sort** s, **arity** w, and **rank** (w, s). An operation with empty arity is a constant. The S-sorted set of ground Σ-terms is denoted T_Σ and has a natural Σ-algebra structure with operation symbols interpreted as constructors. We assume a countable set Ω of variable symbols, viewed as a signature of constants disjoint from Σ. If X is a set of variables (i.e., a subsignature of Ω), then the set of Σ-terms with variables in X is denoted $T_\Sigma(X)$; it is the **free** Σ-algebra generated by X, characterized by the *universal property* that for any Σ-algebra M and S-sorted function $f\colon X \to M$ there is a unique Σ-homomorphic extension $\overline{f}\colon T_\Sigma(X) \to M$. For $t \in T_\Sigma(X)$, let $var(t)$ denote the least $Y \subseteq X$ such that $t \in T_\Sigma(Y)$.

A **substitution** is an S-sorted map $\theta\colon X \to T_\Sigma(Y)$ for some signatures of constants X, Y. The Σ-homomorphic extension of θ to $\overline{\theta}\colon T_\Sigma(X) \to T_\Sigma(Y)$ may also be denoted θ, and $\theta(t)$ may be written θt. If $\theta_1 : X \to T_\Sigma(Y)$ and $\theta_2 : Y \to T_\Sigma(Z)$ are two substitutions, then their **composition** $\theta_1;\overline{\theta_2}$ is denoted $\theta_1;\theta_2$. An n**-ary context** for $n \geq 1$ is a term in $T_\Sigma(X \cup Z)$ where $Z = \{z_1, ..., z_n\}$ with $z_1, ..., z_n$ "fresh" variables (i.e., not in X); this may be denoted $\gamma[z_1]...[z_n]$ or $\gamma[z_1, ..., z_n]$, or just γ when no confusion may arise. A unary context, or just context, is a term in $T_\Sigma(X \cup \{z\})$, denoted $\gamma[z]$; the notation $\cdot$ may also be used for fresh variables. A binary context may be denoted $\gamma[z_1, z_2]$ or $\gamma[z_1][z_2]$ or $\gamma[\cdot][\cdot]$. The result of substituting a term t for $\cdot$ in $\gamma[\cdot]$ is denoted $\gamma[t]$. A **subterm** of $t \in T_\Sigma(X)$ is $t' \in T_\Sigma(X)$ such that $t = \gamma[t']$ for some context γ; we write $t' \ll t$. If $\theta\colon X \to T_\Sigma(Y)$ is a substitution and $\gamma \in T_\Sigma(X \cup Z)$ is an n-ary context, then θ extends to $\theta\colon X \cup Z \to T_\Sigma(Y \cup Z)$ by mapping each $z \in Z$ to itself, so that $\theta(\gamma)$ is again an n-ary context.

A Σ**-equation** is a pair of terms t_1, t_2 in $T_\Sigma(X)$, written $(\forall X)\ t_1 = t_2$; the need for declaring variables in equations for many sorted deduction is shown in [13]. A Σ**-rewrite rule** is a Σ-equation $(\forall X)\ t_1 = t_2$ with $var(t_2) \subseteq var(t_1)$; we can omit $(\forall X)$ by assuming $X = var(t_1)$. A (**many sorted**) **term rewriting system** (or **TRS**) is a set of Σ-rewrite rules. The **rewriting relation** $\Rightarrow_A$ of a TRS A is as usual (the special case of Section 2.1 where the set B of equations is empty). **Termination**, **confluence**, and **local confluence** are as usual, from the abstract rewrite system (ARS) of the rewriting relation. Two terms **overlap** iff one unifies with a subterm of the other, and the subterm is not a variable; if the two terms are equal, the subterm must be strict. **More general overlap**, **most general overlap**, and **critical pair** are again special cases of Section 2.1 where $B = \emptyset$. A leisurely algebraic exposition of MSO rewriting is in [12], and an exposition of basic unsorted rewriting in [1]. Here is the (MSO version of the) classic Knuth-Bendix result:

Theorem 1. *A TRS is locally confluent iff all its critical pairs converge.*

2.1 Term Rewriting Modulo Equations

If B is a set of Σ-equations, then $t_1, t_2 \in T_\Sigma(X)$ are **equivalent modulo** B, denoted $t_1 \simeq_B t_2$, iff the corresponding Σ-equation follows from equations in B by equational deduction; $\simeq_B$ is a Σ-congruence relation. An equation is **balanced** iff its two sides have the same variables, and is **linear** iff each variable occurs at most once on each side. Two n-ary contexts γ, γ' are **equivalent modulo** B (written $\gamma \simeq_B \gamma'$) iff the terms $\gamma[z_1, ..., z_n]$ and $\gamma'[z_1, ..., z_n]$ are equivalent modulo B. A **many sorted term rewriting system modulo equations** (**MTRS**) is (Σ, A, B) for Σ a sorted signature, A a set of Σ-rewrite rules, and B a set of Σ-equations. If t_1, t_2 are Σ-terms, then t_1 **rewrites to** t_2 **with** A **modulo** B iff there are t'_1, t'_2 such that $t'_1 \Rightarrow_A t'_2$, $t_1 \simeq_B t'_1$ and $t_2 \simeq_B t'_2$. The **rewriting relation modulo** B is denoted $\Rightarrow_{A/B}$ and its reflexive transitive closure is denoted $\overset{*}{\Rightarrow}_{A/B}$. Terms t, t' **converge**, indicated $t \downarrow_{A/B} t'$, iff they can be rewritten with A modulo B to a common term.

Let $T_{\Sigma,B}(X)$ denote the quotient $T_{\Sigma \cup X}/_{\simeq_B}$; it is the **free** (Σ, B)-algebra generated by X, characterized by the property that for any (Σ, B)-algebra M and S-sorted function $f : X \to M$ there is a unique Σ-homomorphic extension $\overline{f} : T_{\Sigma,B}(X) \to M$. We will write $[t]_B$ (or just $[t]$ if confusion is unlikely) for the B-class of a Σ-term t. The **class rewriting relation** $\Rightarrow_{[A/B]}$ is defined for $c_1, c_2 \in T_{\Sigma,B}(X)$, by $c_1 \Rightarrow_{[A/B]} c_2$ iff $t_1 \Rightarrow_{A/B} t_2$ for some t_1 in c_1 and t_2 in c_2; we may omit subscripts A/B and $[A/B]$ if they are clear from context. Although it is in general inefficient or even impossible to implement directly, we consider class rewriting the "gold standard" semantics for concepts in term rewriting modulo equations.

A **class substitution** is $\theta : X \to T_{\Sigma,B}(Y)$; its homomorphic extension $\overline{\theta} : T_{\Sigma,B}(X) \to T_{\Sigma,B}(Y)$ may also be denoted θ. The **composition** of $\theta_1 : X \to T_{\Sigma,B}(Y)$ with $\theta_2 : Y \to T_{\Sigma,B}(Z)$ is $\theta_1;\overline{\theta_2}$, denoted $\theta_1;\theta_2$. A class term c in $T_{\Sigma,B}(X)$ can be viewed as a class substitution $c : \{w\} \to T_{\Sigma,B}(X)$ for some variable w. An n**-ary class context** for $n \geq 1$ is a class term in $T_{\Sigma,B}(X \cup Z)$ where $Z = \{z_1, ..., z_n\}$ contains fresh variables; terms representing class contexts may have multiple occurrences of some z_i. If $\theta : X \to T_{\Sigma,B}(Y)$ is a class substitution and $\gamma \in T_{\Sigma,B}(X \cup Z)$ is an n-ary class context, then θ extends to $\theta : T_{\Sigma,B}(X \cup Z) \to T_{\Sigma,B}(Y \cup Z)$ by mapping each $[z_i]$ to itself, so that $\theta(\gamma)$ is also an n-ary context.

If $\gamma \in T_{\Sigma,B}(X \cup Z)$ where $Z = \{z_1, ..., z_n\}$, and $c_i \in T_{\Sigma,B}(Y)$ for $i = 1, ..., n$, then $\gamma[c_1]...[c_n]$ or $\gamma[c_1, ..., c_n]$ denotes the class term $a_{c_1,...,c_n}(\gamma)$ where $a_{c_1,...,c_n} : X \cup Z \to T_{\Sigma,B}(X \cup Y)$ is the identity on X and $a_{c_1,...,c_n}(z_i) = c_i$. A **subterm modulo** B of $t \in T_\Sigma(X)$ is $t' \in T_\Sigma(X)$ that is a subterm of some term equivalent to t; we may write $t' \ll_B t$. Also, c' is a **class subterm** of c iff $c = \gamma[c']$ for some class context γ, written $c' \ll c$. Then:

Lemma 1. *The relation $\ll_B$ is reflexive and transitive. Also $t \ll_B t'$ iff $[t] \ll [t']$, for $t, t' \in T_\Sigma(X)$.*

Definition 1. *Two terms* **overlap modulo** B *iff one unifies modulo B with a subterm modulo B of the other, and the subterm is not just a variable. Thus rewrite rules with leftsides t, t'* **overlap modulo** B *iff for some substitution θ and term $\gamma[t_0]$ where t_0 is not just a variable, $t \simeq_B \gamma[t_0]$ and $\theta(t_0) \simeq_B \theta(t')$; the substitution θ is called the (***modulo** B*)* **overlap** *of t, t' at the subterm (modulo B) t_0 of t. The pair of terms obtained by rewriting $\theta(t)$ with A modulo B using the two rules is called the* **critical pair of the overlap**. *A critical pair is said to* **converge** *iff its two terms converge.*

Note that this definition of overlap is symmetric with respect to the two rules, and that any unifier is an overlap. The following shows that, unlike term rewriting modulo $B = \emptyset$, when the rules coincide, the subterm should not be required to be strict:

Example 1 Let Σ contain constants a, b and binary operations $+, *$, let B consist of the commutative law for $+$, and let A consist of just $x + y \to x * y$. Then $a + b$ rewrites with A modulo B to both $a * b$ and $b * a$. Hence this rule should be considered self-overlapping, with $(a * b, b * a)$ a critical pair. □

Lemma 2. *If θ is an overlap of t_1, t_2 at a subterm (modulo B) of t_2 then $\theta t_1 \ll_B \theta t_2$.*

The converse is not true. For example, let B contain associativity and commutativity and let a, b, c be constants. Consider the terms $t_1 = a + x, t_2 = b + y$ and the substitution $\theta x = b$, $\theta y = a + c$. Then $\theta t_1 \ll_B \theta t_2$ but θ is not an overlap.

Definition 2. *Given modulo B overlaps θ, θ' of t, t' at subterms modulo B t_0, t_0' of t such that $t \simeq_B \gamma[t_0] \simeq_B \gamma'[t_0']$, $\gamma \simeq_B \gamma'$ and $t_0 \simeq_B t_0'$, we say that θ is* **more general than** *(or* **subsumes***) θ' iff θ' is a substitution instance of θ, i.e., iff there is a substitution ρ such that $\theta' \simeq_B \theta;\rho$. A modulo B overlap θ of t, t' at a subterm modulo B of t is* **most general** *iff it is more general than any modulo B overlap of t, t' at the same subterm modulo B of t. An MTRS has* **most general overlaps modulo** B *iff whenever two terms t, t' overlap modulo B at a subterm of t, there is a most general overlap modulo B for t, t' at the same subterm (modulo B).*

Lemma 3. *Let θ, θ' be two overlaps modulo B of rules $\ell \to r, \ell' \to r'$ at subterms ℓ_0, ℓ_0' of ℓ, such that θ' is more general than θ. If the critical pair of θ' is convergent then the critical pair of θ is also convergent.*

If an MTRS has most general overlaps, then we need only consider critical pairs corresponding to most general overlaps. Most general overlaps modulo B do not always exist, but one can look instead for complete sets of overlaps and consider only the corresponding critical pairs. Complete sets of overlaps always exist, though they might be infinite (at most countable). However, minimal complete sets of overlaps do not always exist. If B consists of associative and idempotent laws for an operation, then minimal complete sets of overlaps need not exist, because minimal complete sets of unifiers modulo AI do not always exist. When B contains only associativity, finite complete sets of overlaps need not exist, since unification modulo associativity is infinitary, in that there exist terms with no finite minimal complete set of unifiers (e.g., [3]). Because unification modulo commutativity is finitary [3], the following implies that finite complete sets of overlaps modulo commutativity always exist:

Proposition 1. *If equivalence classes modulo B are finite then finite complete sets of unifiers modulo B exist iff finite complete sets of overlaps modulo B exist.*

An MTRS is **locally confluent** iff whenever $t \Rightarrow_{A/B} t_1$ and $t \Rightarrow_{A/B} t_2$, then t_1 and t_2 converge. Translating Theorem 1 to MTRS's gives the following conjecture:

An MTRS is locally confluent iff all its critical pairs converge.

The following shows the "if" part is false, which is why we generalize overlap and critical pair:

Example 2 Let Σ contain a binary operation $+$ and constants a, b, let B contain only the associativity law for $+$, and let $A = \{a + b \to b,\ b + a \to a\}$. Then there are no critical pairs because A is non-overlapping, but the MTRS (Σ, B, A) is not locally confluent because $a + b + a$ rewrites with A modulo B both to a and to $a + a$, which are not equivalent modulo B. □

3 Sunification

Definition 3. *A* **sunifier** *(or* **superunifier***) of $t_1, t_2 \in T_\Sigma(X)$ is (θ, u_1, u_2, t) where $\theta\colon X \to T_\Sigma(Y)$ is a substitution, $t \in T_\Sigma(Y)$, and $u_1, u_2 \in T_\Sigma(Y \cup \{z\})$ are contexts such that $u_1[\theta(t_1)] \simeq_B u_2[\theta(t_2)] \simeq_B t$; we call t the* **term** *of the sunifier. We may use notations $t_1 \Diamond t_2$, $\Diamond_{\theta,u_1,u_2,t}$ or just $\Diamond$ for (θ, u_1, u_2, t).*

Overlap is the special case where one of u_1, u_2 is the identity context ($u[z] = z$), and unification is the special case where both are the identity. Unlike overlap, sunification treats t_1, t_2 the same way. If θ is an overlap of t_1, t_2 at subterm t_0 of t_2, then $(\theta, \theta(u), z, \theta(t_2))$ is a sunifier of t_1, t_2, where u is the context of t_0 in the term equivalent to t_2, called the **sunifier corresponding to the overlap** θ**.**

Definition 4. *Let (Σ, A, B) be an MTRS with $\ell_1 \to r_1, \ell_2 \to r_2$ rules in A (having disjoint variables), and let (θ, u_1, u_2, t) be a sunifier of ℓ_1, ℓ_2. Then t rewrites with A modulo B to both $p_1 = u_1[\theta r_1]$ and $p_2 = u_2[\theta r_2]$, and $\langle p_1, p_2\rangle$ is called its associated* **scritical** *(or* **supercritical***)* **pair***. A scritical pair $\langle p_1, p_2\rangle$* **converges** *iff $p_1 \downarrow_{A/B} p_2$.*

A simple criterion for a scritical pair to converge is that p_1 and p_2 are equivalent modulo B; another is that $\ell_1 \to r_1$ applies to p_2 and $\ell_2 \to r_2$ applies to p_1 and some result of applying $\ell_1 \to r_1$ to p_2 is equivalent to some result of applying $\ell_2 \to r_2$ to p_1. In particular, if $\diamondsuit = (\theta, \gamma_1, \gamma_2, t)$ is a sunifier of ℓ_1, ℓ_2 and there exists some binary context γ such that $\gamma_1[z] \simeq_B \gamma[z][\theta\ell_2]$ and $\gamma_2[z] \simeq_B \gamma[\theta\ell_1][z]$ then the scritical pair of $\diamondsuit$ is convergent.

Proposition 2. *If $\diamondsuit = (\theta, \gamma_1, \gamma_2, t)$ is a sunifier of ℓ_1, ℓ_2 such that $\theta\ell_1$ is a subterm modulo B of θx for some variable x in ℓ_2 then the scritical pair of $\diamondsuit$ is convergent.*

Example 3 Let Σ have one sort, constants a, b, c and binary operation $+$, let B consist of associativity for $+$, and consider terms $t_1 = a + x$ and $t_2 = b + y$. There is a sunifer of t_1, t_2 having term $t = a + b + y$, substitution $\theta = (x \mapsto b,\ y \mapsto y)$, and contexts $u_1[z] = z + y,\ u_2[z] = a + z$, since $u_i[\theta(t_i)] = t$ for $i = 1, 2$. Another sunifer of t_1, t_2 has term $t' = a + b + c + a$, substitution $\theta' = (x \mapsto b,\ y \mapsto c)$, and contexts $u'_1[z] = z + c + a,\ u'_2[z] = a + z + a$, since $u'_i[\theta'(t_i)] = t'$ for $i = 1, 2$. □

We want to use as few sunifiers as possible in computing scritical pairs, and one way to reduce the set of sunifiers is to eliminate those that are less general than others, in the following sense:

Definition 5. *A sunifier (θ, u_1, u_2, t) of t_1, t_2* **subsumes** *another $(\theta', u'_1, u'_2, t')$ iff there exist a substitution ρ and context u such that $\theta' \simeq_B \theta; \rho$ and $u'_i[\cdot] \simeq_B u[\rho(u_i[\cdot])]$ for $i = 1, 2$. We also say $(\theta', u'_1, u'_2, t')$ is* **less general than** *(θ, u_1, u_2, t). If u is not the identity context then (θ, u_1, u_2, t)* **strictly** *subsumes $(\theta', u'_1, u'_2, t')$.*

This definition implies that $t' \simeq_B u[\rho t]$. Subsumption is reflexive and transitive but not total. Intuitively, the subsumed sunifier may be thought of as "larger," in both its substitution and its term; the substitution is larger in that some variables are replaced by terms, and the term is larger in being a superterm. If (θ, u_1, u_2, t) sunifies t_1, t_2 and t is a subterm of t', then (θ, u'_1, u'_2, t') also sunifies t_1, t_2, and is subsumed by (θ, u_1, u_2, t), where $u'_i[z] = u[u_i[z]]$ for u a context such that $t' \simeq_B u[t]$.

Proposition 3. *The convergence of the scritical pair obtained from a sunifier that is subsumed by a second sunifier is implied by the convergence of the scritical pair obtained from the subsuming sunifier.*

The following can eliminate some sunifiers that only yield convergent scritical pairs:

Definition 6. *Terms $t_1, t_2 \in T_\Sigma(X)$ are* **separated** *(modulo B) in a term $t \in T_\Sigma(X)$ iff t_1, t_2 are subterms modulo B of t and there is a binary context $\gamma \in T_\Sigma(X \cup \{z_1, z_2\})$ such that $t \simeq_B \gamma[t_1, t_2]$. A sunifier (θ, u_1, u_2, t) of t_1, t_2 is a* **separated sunifier** *iff $\theta(t_1), \theta(t_2)$ are separated in t.*

If B is such that no term is equivalent to a proper subterm, then the sunifier corresponding to an overlap is not separated.

Example 4 Continuing Example 3, note that both sunifiers (θ, u_1, u_2, t) and $(\theta', u'_1, u'_2, t')$ of t_1, t_2 are not separated, since there is no binary context u such that $t = u[\theta t_1, \theta t_2]$ or $t' = u[\theta' t_1, \theta' t_2]$. □

Example 5 Continuing Example 2, the terms $a + b$ and $b + a$ admit sunifiers $a + b + a$ and $b + a + b$, but neither is a subterm of the other, so neither subsumes the other. Thus most general sunifiers do not necessarily exist. □

This motivates the following:

Definition 7. *A set I of [unseparated] sunifiers of terms t_1, t_2 is* **complete** *iff every [unseparated] sunifier of t_1, t_2 is subsumed by one in I. A complete set of [unseparated] sunifiers of t_1, t_2 is* **minimal** *iff no proper subset is also complete.*

Proposition 4. *Complete [unseparated] sunifier sets always exist, and are at most countable (though the unseparated ones may be empty), and minimal complete [unseparated] sunifier sets exist whenever finite [unseparated] complete sunifier sets exist.*

We now give class versions of the concepts introduced above:

Definition 8. *A* **sunifier** *(or* **superunifier***) of $c_1, c_2 \in T_{\Sigma,B}(Y)$ is (θ, u_1, u_2, c) with $c \in T_{\Sigma,B}(X)$, $u_1, u_2 \in T_{\Sigma,B}(X \cup \{z\})$ and $\theta \colon Y \to T_{\Sigma,B}(X)$ such that $c = u_1[\theta(c_1)] = u_2[\theta(c_2)]$; c is the* **term** *of the sunifier. Class terms $c_1, c_2 \in T_{\Sigma,B}(Y)$ are* **separated in** *$c \in T_{\Sigma,B}(X)$ iff there exist a binary class context γ and class substitution $\theta \colon Y \to T_{\Sigma,B}(X)$ such that $c = \gamma[\theta(c_1), \theta(c_2)]$. A* **separated sunifier** *of c_1, c_2 is (θ, u_1, u_2, c) such that $\theta(c_1), \theta(c_2)$ are separated in c. The class version of subsumption replaces terms with class terms and equality with class equivalence in Definition 5.*

Proposition 5. *For $t_1, t_2 \in T_\Sigma(Y)$, $t \in T_\Sigma(X)$ and $\theta \colon Y \to T_\Sigma(X)$, then $(\theta, \gamma_1, \gamma_2, t)$ is a sunifier of t_1, t_2 iff $([\theta], [\gamma_1], [\gamma_2], [t])$ is a sunifier of $[t_1], [t_2]$, and $\theta t_1, \theta t_2$ are unseparated in t iff $[\theta][t_1], [\theta][t_2]$ are unseparated in $[t]$.*

Thus by Proposition 3, if a complete (preferably minimal) set of sunifiers exists, it suffices to use only the scritical pairs corresponding to these sunifiers. We want the same for unseparated sunifiers, but the following shows there are B such that separated sunifiers generate non-convergent scritical pairs:

Example 6 Let Σ be the one sorted signature with unary function symbols f, g, i, binary function symbol h, and constants a, b, c. Let B contain equations $f(i(a)) = g(i(b)) = h(i(a), i(b))$, and let A contain rules $i(a) \to a$ and $i(b) \to c$. Then there are no unseparated sunifiers of $i(a)$ and $i(b)$, and thus no unseparated scritical pairs. But this system is not confluent modulo B. □

In some cases all scritical pairs of separated sunifiers are convergent.

Definition 9. *A* **minimal sunifier** *is one that is minimal with respect to the strict subsumption relation (i.e., there is no other sunifier that strictly subsumes it.)*

Proposition 6. *If B consists of A/C laws and $\Diamond = (\theta, \gamma_1, \gamma_2, t)$ is a minimal separated sunifier, then its scritical pair is convergent.*

Proof Let the leftsides be l_1, l_2. Then by definition there is a binary context γ such that $\gamma_1[\theta l_1] \simeq_B \gamma_2[\theta l_2] \simeq_B \gamma[\theta l_1][\theta l_2]$. The AC laws require that the top operator is the same in each of $\gamma_1[\theta l_1], \gamma_2[\theta l_2], \gamma[\theta l_1][\theta l_2]$.

Assume first that the top operator is an m-ary operator f not in the signature of B, for some $m \geq 1$. Then $\gamma_i[\theta l_i] = f(t_1^i, \ldots, t_m^i)$ for some terms t_j^i, with $i = 1, 2$ and $1 \leq j \leq m$ such that $t_j^1 \simeq_B t_j^2$ for every j. Furthermore, there exist contexts γ_1', γ_2' and positions j, k such that $t_k^1 = \gamma_1'[\theta l_1]$ and $t_j^2 = \gamma_2'[\theta l_2]$. If j and k are distinct, then clearly the scritical pair of $\Diamond$ is convergent. Otherwise, $\gamma_1'[\theta l_1] \simeq_B \gamma_2'[\theta l_2]$. But this means $(\theta, \gamma_1', \gamma_2', \gamma_1'[\theta l_1])$ is a sunifier of l_1, l_2 that strictly subsumes $\Diamond$.

Assume now that the top operator $+$ is binary and associative but not commutative. Then $\gamma_i[\theta l_i] \simeq_B t_1^i + \ldots + t_m^i$ and $\gamma[\theta l_1][\theta l_2] \simeq_B t_1^3 + \ldots + t_m^3$ for some $m \geq 1$ and terms t_j^i, with $i = 1, 2, 3$ and $1 \leq j \leq m$ such that $t_j^1 \simeq_B t_j^2 \simeq_B t_j^3$ for every j and the top operator in each t_j^i is not $+$. Furthermore, in each of the 3 groups $\{t_j^i \mid 1 \leq j \leq m\}$ either there is a term with θl_1 as subterm or θl_1 is equivalent to the sum of a consecutive subset of the terms t_j^i, $j = 1, \ldots, m$. The same is true for θl_2 in place of θl_1. If the terms involved with the occurrence of θl_1 are disjoint from the terms involved with the occurrence of θl_2 then the scritical pair of $\Diamond$ is convergent. Suppose the opposite holds. Since in the third group the terms involved with the specified occurrence of θl_1 are disjoint from the terms involved with the specified occurrence of θl_2, it follows that in the first group there is an additional occurrence of θl_1 or θl_2 (or both), which is (are) disjoint from the specified occurrence of θl_1. Similarly, in the second group there is an additional occurrence of θl_1 or, respectively, of θl_2 (or both), which is (are) disjoint from the specified occurrence of θl_2. This implies the existence of a sunifier of l_1, l_2 that strictly subsumes $\diamondsuit$.

If the top operator $+$ is commutative but not associative then the proof is similar to the associative case, but the terms t^i_j may need to be reordered.

Assume now that the top operator is an AC binary operator $+$. Then $\gamma_i[\theta l_i] \simeq_B u_i + \theta l_i$ and $\gamma[\theta l_1][\theta l_2] \simeq_B u + \theta l_1 + \theta l_2$ for some terms u_1, u_2, u. From $u_1 + \theta l_1 \simeq_B u + \theta l_1 + \theta l_2$ follows $u_1 \simeq_B u + \theta l_2$ and from $u_2 + \theta l_2 \simeq_B u + \theta l_1 + \theta l_2$ follows $u_2 \simeq_B u + \theta l_1$. Then the scritical pair is convergent since both terms converge to $u + \theta r_1 + \theta r_2$. □

Example 7 Let Σ have one sort with constants a, b, c and a binary operation $+$, let B consist of associativity for $+$, and let A contain the rules $a + x \to r_1$ and $b + c \to r_2$. If a term has instances of $a + x$ and $b + c$ as subterms modulo associativity, then it must have the form $t_1 + a + t_2 + b + c + t_3$ or $t_1 + b + c + t_2 + a + t_3$ where each t_i is a term or else is absent. Any sunifier with the term of the second form is separated, therefore only the first form is considered below. Replacing each t_i by a variable in each of the 8 cases gives the following sunifier terms:

$a + b + c$	$y + a + b + c$	$a + y + b + c$
$a + b + c + y$	$y_1 + a + y_2 + b + c$	$y_1 + a + b + c + y_2$
$a + y_1 + b + c + y_2$	$y_1 + a + y_2 + b + c + y_3$	

If we eliminate sunifiers that are subsumed by others, we obtain a minimal complete set of sunifiers (see Proposition 3) for the terms $a + x$ and $b + c$. To obtain a minimal set of scritical pairs, we also eliminate separated sunifiers (by Proposition 6). The following lists these 6 unseparated sunifiers with their corresponding scritical pairs, in which θ denotes the substitution of its sunifier:

$((x \mapsto b + c), z, a + z, a + b + c)$	$\langle \theta(r_1),\, a + r_2 \rangle$
$((x \mapsto b), z + c, a + z, a + b + c)$	$\langle \theta(r_1) + c,\, a + r_2 \rangle$
$((x \mapsto y + b + c), z, a + y + z, a + y + b + c)$	$\langle \theta(r_1),\, a + y + r_2 \rangle$
$((x \mapsto y + b), z + c, a + y + z, a + y + b + c)$	$\langle \theta(r_1) + c,\, a + y + r_2 \rangle$
$((x \mapsto b + c + u), z, a + z + u, a + b + c + u)$	$\langle \theta(r_1),\, a + r_2 + u \rangle$
$((x \mapsto y + b + c + u), z, a + y + z + u, a + y + b + c + u)$	$\langle \theta(r_1),\, a + y + r_2 + u \rangle$

Note that only 4 of the original 8 terms appear, but that the same term can appear in more than one distinct sunifier, when its substitution and/or subterm inclusions are different. Note also that some terms may have a different substitution under which they are separated; one such example is $((x \mapsto y), z + b + c, a + y + z, a + y + b + c)$. Finally, one can check that by Proposition 2, the first, third, fifth and sixth sunifiers have convergent scritical pairs, because they satisfy the property $\theta l_2 \ll_B \theta x$. □

Theorem 2. *An MTRS is locally confluent iff all its scritical pairs converge.*

Proof For the only if, local confluence implies by definition that all scritical pairs converge.

For the converse, we may assume without loss of generality that rules have disjoint sets of variables. Now assuming that all scritical pairs converge, we prove local confluence. Let t, t_1, t_2 be Σ-terms such that $t \Rightarrow_{A/B} t_1$ and $t \Rightarrow_{A/B} t_2$. Then by definition, there exist rules $\ell_1 \to r_1$ and $\ell_2 \to r_2$ in A (not necessarily distinct), contexts γ_1, γ_2, and substitution θ such that $t \simeq_B \gamma_1[\theta\ell_1]$ and $t \simeq_B \gamma_2[\theta\ell_2]$ and $t_1 \simeq_B \gamma_1[\theta r_1]$ and $t_2 \simeq_B \gamma_2[\theta r_2]$. Then $\diamondsuit = (\theta, \gamma_1, \gamma_2, t)$ is a sunifier for ℓ_1, ℓ_2, and the scritical pair of $\diamondsuit$ is $\langle p_1, p_2\rangle = \langle \gamma_1[\theta r_1], \gamma_2[\theta r_2]\rangle$. Clearly, t_1, t_2 converge if p_1, p_2 converge. Let I be a complete set of sunifiers for ℓ_1, ℓ_2. By hypothesis, all scritical pairs of sunifiers in I converge. If $\diamondsuit$ is not in I, then there exists a sunifier $\diamondsuit' \in I$ for ℓ_1, ℓ_2 such that $\diamondsuit'$ subsumes $\diamondsuit$, and the scritical pair of $\diamondsuit'$ converges. So by Proposition 3, the scritical pair of $\diamondsuit$ also converges. □

Peterson and Stickel [24] give a similar result (their Theorem 8.12) involving unifiers of all "variable extensions of rules," but it always gives an infinite set of pairs to be checked, whereas the sets from

Theorem 2 can be finite, e.g., if B is C or AC. The following shows that finite complete sets of sunifiers modulo associativity do not always exist:

Example 8 Let B contain only associativity and let $x+x \to x$ and $a+y+y+a \to y$ be two rules in A. For every $n \geq 1$, the substitution θ_n is defined by $\theta_n(x) = a+na$ and $\theta_n(y) = na$. Then, for every $n \geq 1$, $(\theta_n, z, z, a+na+a+na)$ is a sunifier of $x+x$ and $a+y+y+a$. This infinite set of sunifiers cannot be subsumed by any finite set of sunifiers.

If the rules are $b+x+x \to x$ and $a+y+y+a+c \to y$ then $(\theta_n, b+z, z+c, b+a+na+a+na+c)$ is a sunifier of $b+x+x$ and $a+y+y+a+c$ for every $n \geq 1$. In this case, as opposed to the previous one, θ_n is not a unifier. □

Definition 10. *A set of [unseparated] sunifiers is* **sufficient** *iff every [unseparated] sunifier is either subsumed by one in the set, or else its scritical pair is convergent.*

Theorem 3. *Finite minimal sufficient sets of sunifiers modulo AC always exist.*

Proof First consider the "elementary" case where Σ contains only one AC operator $+$. Then any context γ other than z is equivalent to a context $z+t$ for some term $t \in T_\Sigma(X)$. Let $l_1, l_2 \in T_\Sigma(X)$ be terms (leftsides of rewrite rules); let u_1, u_2 be fresh variables (not occurring in either l_1 or l_2); let $l'_i = l_i + u_i$ for $i = 1, 2$; let I_1, I_2, I_3, I_4 be finite complete sets of unifiers respectively for l_1 and l_2, l_1 and l'_2, l'_1 and l_2, l'_1 and l'_2. Then every $\theta \in I_j$ for $j = 1,2,3,4$ determines a sunifier $\diamondsuit^\theta_j$ for l_1, l_2, namely $\diamondsuit^\theta_1 = (\theta', z, z, \theta l_1)$, $\diamondsuit^\theta_2 = (\theta', z, z+\theta u_2, \theta l_1)$, $\diamondsuit^\theta_3 = (\theta', z+\theta u_1, z, \theta l'_1)$, $\diamondsuit^\theta_4 = (\theta', z+\theta u_1, z+\theta u_2, \theta l'_1)$, where θ' is θ restricted to the variables in l_1, l_2. The set of all these sunifiers is finite. To show it is also complete, let $\diamondsuit = (\theta, \gamma_1, \gamma_2, t)$ be a sunifier for l_1, l_2. There are four possibilities.

1. If $\gamma_1 = \gamma_2 = z$ then θ is a unifier for l_1, l_2 therefore it is subsumed by a unifier $\theta^* \in I_1$. The sunifier corresponding to θ^* is $(\theta^*, z, z, \theta^* l_1)$ and it subsumes $\diamondsuit = (\theta, z, z, t)$.
2. If $\gamma_1 = z$ and $\gamma_2 \neq z$ then $\gamma_2 \simeq_B z + t_2$ for some term t_2. Let θ' be the substitution that maps u_2 to t_2 and is the same as θ for every variable in l_1, l_2. Then θ' is a unifier of l_1 and $l_2 + u_2$, therefore it is subsumed by some unifier $\theta^* \in I_2$. The sunifier corresponding to θ^* is $(\theta^{**}, z, z+\theta^* u_2, \theta^* l_1)$ where θ^{**} is the restriction of θ^* to the variables in l_1, l_2, and it subsumes $\diamondsuit = (\theta, z, z+t_2, t)$.
3. If $\gamma_2 = z$ and $\gamma_1 \neq z$ then $\gamma_1 \simeq_B z + t_1$ for some term t_1. The proof is symmetrical with 2., using I_3 instead of I_2.
4. If $\gamma_1 \neq z$ and $\gamma_2 \neq z$ then $\gamma_1 \simeq_B z + t_1$ and $\gamma_2 \simeq_B z + t_2$ for some terms t_1, t_2. Let θ' be the substitution that maps u_1 to t_1, u_2 to t_2 and is the same as θ for every variable in l_1, l_2. Then θ' is a unifier of $l_1 + u_1, l_2 + u_2$, and thus is subsumed by some unifier $\theta^* \in I_4$. The sunifier corresponding to θ^* is $(\theta^{**}, z+\theta^* u_1, z+\theta^* u_2, \theta^* l_1)$ where θ^{**} is the restriction of θ^* to the variables in l_1, l_2, and it subsumes $\diamondsuit = (\theta, z+t_1, z+t_2, t)$.

When Σ contains several AC operators, then the sets I_1, I_2, I_3, I_4 are built for each one. The proof is the same as above, but in the last case the top AC operator in γ_1 may be different from the top AC operator in γ_2.

If Σ contains other operators than the AC ones (including constants) then the problem is reduced to the "elementary" case. Let $\diamondsuit = (\theta, \gamma_1, \gamma_2, t)$ be a sunifier for l_1, l_2. Then $\gamma_1[\theta l_1] \simeq_B \gamma_2[\theta l_2]$ and thus the top operator in $\gamma_1[\theta l_1]$ is the same as in $\gamma_2[\theta l_2]$. Assume first that the top operator, f, is not an AC operator. The corresponding arguments of f must be equivalent. Moreover, one of the arguments of f in $\gamma_1[\theta l_1]$, say the ith, must be of the form $\gamma'_1[\theta l_1]$ for some context γ'_1. Similarly, one of the arguments of f in $\gamma_2[\theta l_2]$, say the jth, must be of the form $\gamma'_2[\theta l_2]$ for some context γ'_2. If i and j are distinct, then the scritical pair of $\diamondsuit$ is convergent. Otherwise, $i = j$ and $\gamma'_1[\theta l_1] \simeq_B \gamma'_2[\theta l_2]$ which means $(\theta, \gamma'_1, \gamma'_2, \gamma'_2[\theta l_2])$ is a sunifier of l_1, l_2 that subsumes $\diamondsuit$.

Assume that the top operator in $\gamma_1[\theta l_1], \gamma_2[\theta l_2]$ is an AC operator, say $+$, but there are other operators in γ_1, γ_2. Then each of $\gamma_1[\theta l_1], \gamma_2[\theta l_2]$ is a sum of terms whose top operators are not $+$. The number of such terms must be the same in $\gamma_1[\theta l_1]$ and $\gamma_2[\theta l_2]$, and the terms must be equivalent (one from $\gamma_1[\theta l_1]$ with one from $\gamma_2[\theta l_2]$). As in the previous case, there is a sunifier that subsumes $\diamondsuit$.

The only case left is when both γ_1 and γ_2 contain only + operators (there might be other operators in θl_1 or θl_2). This is the same as the case where the signature contains only the + operator.

In conclusion, the algorithm to obtain a complete set of sunifiers is the same. More precisely, there are three possibilities:

1. The top operators in l_1, l_2 are distinct. Then the set I_1 is empty and the sunifiers corresponding to unifiers in I_4 have convergent scritical pairs. Therefore a complete set of sunifiers is obtained from unifiers in I_2 and I_3.
2. The top operators in l_1, l_2 are the same but other than +. Then the unifiers in I_4 are either unifiers in I_1 or have corresponding sunifiers with convergent scritical pairs. Therefore a complete set of sunifiers is obtained from unifiers in I_1, I_2 and I_3.
3. The top operator is + in both l_1 and l_2. Then a complete set of sunifiers is obtained from unifiers in I_1, I_2 I_3 and I_4.

□

Note that the same algorithm works for AC1 because any number of occurrences of the identity element in terms (other than the identity element itself) can be ignored.

Proposition 7. *Finite minimal sufficient sets of sunifiers modulo AC1 always exist.*

4 A Sufficient Condition

Theorem 4 below shows that the following condition implies that Theorem 2 using overlapping and ordinary critical pairs, instead of sunification and scritical pairs, holds for MTRS's:

(C) *For all Σ-terms t, t', u such that $t \simeq_B t'$ and $t = \gamma[u]$ for some context $\gamma[\cdot]$, there exist a context $\gamma'[\cdot]$ and Σ-term u' such that $\gamma \simeq_B \gamma'$, $u \simeq_B u'$ and $t' = \gamma'[u']$.*

We view (C) as an abstract commutativity condition, and consider Theorem 4 below as explaining why equation sets like C are so well behaved. Neither A nor AC satisfy (C):

Example 9 Let $t = a + (b + c)$, let $t' = (a + b) + c$, let $\gamma[z] = a + z$, and let $u = b + c$. Then $t = \gamma[u]$ but there are no γ', u' such that $t' = \gamma'[u']$ with $\gamma \simeq_B \gamma'$ and $u \simeq_B u'$. □

Example 10 Condition (C) is satisfied by a commutative binary operation f, with $f(x, y) = f(y, x)$. More generally, (C) is satisfied by commutative n-ary operations; for example, for f ternary, B would contain all five of the combinations, $f(x, y, z) = f(y, x, z)$, $f(x, y, z) = f(y, z, x)$, etc. □

Below is the key technical result for proving Theorem 4, the main result of this section:

Proposition 8. *Let $l_1 \to r_1, l_2 \to r_2$ be rules in A. If condition (C) holds for B, then for every sunifier $(\theta, \gamma_1, \gamma_2, t)$ of l_1, l_2, either θ is an overlap of l_1, l_2 and the corresponding sunifier subsumes $(\theta, \gamma_1, \gamma_2, t)$ or else the scritical pair generated by $(\theta, \gamma_1, \gamma_2, t)$ is convergent. Moreover, if θ is an overlap then convergence of its critical pair implies the convergence of the scritical pair generated by $(\theta, \gamma_1, \gamma_2, t)$.*

Proof If $(\theta, \gamma_1, \gamma_2, t)$ is a sunifier of l_1, l_2 then $t \simeq_B \gamma_1[\theta(l_1)] \simeq_B \gamma_2[\theta(l_2)]$, and this occurrence of $\theta(l_2)$ is always meant whenever the term $\theta(l_2)$ appears in this proof. Condition (C) ensures that $\gamma_2[\theta(l_2)] = \gamma[u]$ some context $\gamma \simeq_B \gamma_1$ and for some term $u \simeq_B \theta(l_1)$. Since u is a subterm of $\gamma_2[\theta(l_2)]$, one of the following holds:

(i) $\theta(l_2)$ is a subterm of u in $\gamma_2[\theta(l_2)]$
(ii) u is a subterm of $\theta(l_2)$ in $\gamma_2[\theta(l_2)]$
(iii) u and $\theta(l_2)$ are disjoint subterms of $\gamma_2[\theta(l_2)]$.

If (i) holds then $\theta(l_2)$ is equivalent to a subterm of $\theta(l_1)$ because $u \simeq_B \theta(l_1)$ and condition (C) holds. This means $u = \gamma'[\theta(l_2)]$ and $l_1 = \gamma_0[l_0]$ such that $\gamma' \simeq_B \theta(\gamma_0)$ and $\theta(l_2) \simeq_B \theta(l_0)$. If l_0 is not just a variable, then θ is an overlap of l_1, l_2 at a subterm modulo B of l_1 and $(\theta, z, \theta(\gamma_0), \theta(l_1))$ is the corresponding sunifier of l_1 and l_2. (In this case the subterm is actually a subterm of l_1 and not just a subterm modulo B of l_1 as required by Definition 1.) The fact that $(\theta, z, \theta(\gamma_0), \theta(l_1))$ is more general than $(\theta, \gamma_1, \gamma_2, t)$ is ensured by the following: $\gamma_1[\theta(\gamma_0)] \simeq_B \gamma[\theta(\gamma_0)] \simeq_B \gamma[\gamma'] = \gamma_2$. If the critical pair $\langle \theta(r_1), \theta(\gamma_0)[\theta(r_2)] \rangle$ generated by the overlap is convergent then also the scritical pair $\langle \gamma_1[\theta(r_1)], \gamma_2[\theta(r_2)] \rangle$ is convergent because $\gamma_1[\theta(\gamma_0)[\theta(r_2)]] \simeq_B \gamma_2[\theta(r_2)]$.

Otherwise, $l_0 = x$ for some $x \in var(l_1)$. Let θ' be the substitution that coincides with θ everywhere except x, where $\theta'(x) = \theta(r_2)$. Then $\theta(r_1) \stackrel{*}{\Rightarrow}_{A/B} \theta'(r_1)$ by applying rule $l_2 \rightarrow r_2$ with redex $\theta(l_2)$ at every occurrence of x in r_1. Also $\theta(\gamma_0)[\theta(r_2)] \stackrel{*}{\Rightarrow}_{A/B} \theta'(l_1)$ by applying rule $l_2 \rightarrow r_2$ with redex $\theta(l_2)$ at every occurrence of x in γ_0, and finally $\theta'(l_1) \Rightarrow_{A/B} \theta'(r_1)$. It follows that the scritical pair $\langle \gamma_1[\theta(r_1)], \gamma_2[\theta(r_2)] \rangle$ is convergent because $\gamma_2[\theta(r_2)] \simeq_B \gamma_1[\theta(\gamma_0)[\theta(r_2)]]$.

If (ii) holds then $l_2 = \gamma_0[l_0]$ such that $u = \theta(l_0) \simeq_B \theta(l_1)$. If l_0 is not just a variable then θ is an overlap of l_1 and l_2 at a subterm modulo B of l_2 and $(\theta, \theta(\gamma_0), z, \theta(l_2))$ is the corresponding sunifier of l_1 and l_2. (Again, l_0 is actually a subterm of l_2 and not just a subterm modulo B of l_2.) Since $\gamma_2[\theta(\gamma_0)] = \gamma$ it follows that $\gamma_2[\theta(\gamma_0)] \simeq_B \gamma_1$, therefore $(\theta, \theta(\gamma_0), z, \theta(l_2))$ is more general than $(\theta, \gamma_1, \gamma_2, t)$. If the critical pair $\langle \theta(\gamma_0)[\theta(r_1)], \theta(r_2) \rangle$ generated by the overlap is convergent, then the scritical pair $\langle \gamma_1[\theta(r_1)], \gamma_2[\theta(r_2)] \rangle$ is also convergent, because $\gamma_1[\theta(r_1)] \simeq_B \gamma_2[\theta(\gamma_0)[\theta(r_1)]]$.

Otherwise, $l_0 = x$ for some $x \in var(l_2)$. Let θ' be the substitution that coincides with θ everywhere except x and $\theta'(x) = \theta(r_1)$. Then $\theta(r_2) \stackrel{*}{\Rightarrow}_{A/B} \theta'(r_2)$ by applying rule $l_1 \rightarrow r_1$ with redex $\theta(l_1)$ at every occurrence of x in r_2. Also $\theta(\gamma_0)[\theta(r_1)] \stackrel{*}{\Rightarrow}_{A/B} \theta'(l_2)$ by applying rule $l_1 \rightarrow r_1$ with redex $\theta(l_1)$ at every occurrence of x in γ_0, and finally $\theta'(l_2) \Rightarrow_{A/B} \theta'(r_2)$. It follows that the scritical pair $\langle \gamma_1[\theta(r_1)], \gamma_2[\theta(r_2)] \rangle$ is convergent because $\gamma_1[\theta(r_1)] \simeq_B \gamma_2[\theta(\gamma_0)[\theta(r_1)]]$.

If (iii) holds then $\gamma_2[\cdot] = \gamma'[u][\cdot]$ and $\gamma[\cdot] = \gamma'[\cdot][\theta(l_2)]$ for some binary context γ'. Then $\gamma_2[\theta(r_2)] = \gamma'[u][\theta(r_2)] \simeq_B \gamma'[\theta(l_1)][\theta(r_2)] \stackrel{*}{\Rightarrow}_{A/B} \gamma'[\theta(r_1)][\theta(r_2)]$ and $\gamma_1[\theta(r_1)] \simeq_B \gamma[\theta(r_1)] = \gamma'[\theta(r_1)][\theta(l_2)] \stackrel{*}{\Rightarrow}_{A/B} \gamma'[\theta(r_1)][\theta(r_2)]$. Therefore the scritical pair generated by $(\theta, \gamma_1, \gamma_2, t)$ is convergent. □

Theorem 4. *An MTRS that satisfies (C) is locally confluent iff all its critical pairs converge.*

Proof If the MTRS is locally confluent then all its critical pairs are convergent by definition. For the other direction, assume that $t \Rightarrow_{A/B} t_1$ and $t \Rightarrow_{A/B} t_2$ and t_1, t_2 are not equivalent modulo B. We will show that $t_1 \downarrow t_2$. By the definition of $\Rightarrow_{A/B}$, there exist terms u_i, v_i such that $t \simeq_B u_i \Rightarrow_A v_i \simeq_B t_i$ for $i = 1, 2$. Let $l_i \rightarrow r_i$ for $i = 1, 2$ be the rules in A used for these rewritings. There exist a substitution θ and contexts γ_1, γ_2 such that $u_i = \gamma_i[\theta(l_i)]$ and $v_i = \gamma_i[\theta(r_i)]$. Therefore $(\theta, \gamma_1, \gamma_2, t)$ is a sunifier of l_1, l_2 and the scritical pair it generates is $\langle v_1, v_2 \rangle$. By Proposition 8, either θ is an overlap or else $\langle v_1, v_2 \rangle$ is convergent. If θ is an overlap then it is subsumed by an overlap whose critical pair is convergent. Therefore the critical pair of θ is also convergent, by Lemma 3. Then $\langle v_1, v_2 \rangle$ is again convergent, by Proposition 8. Finally, $\langle t_1, t_2 \rangle$ is convergent iff $\langle v_1, v_2 \rangle$ is convergent. □

Proposition 9. *If B satisfies (C) and finite complete sets of overlaps modulo B exist, then finite sufficient sets of sunifiers also exist, and the resulting scritical pairs are the critical pairs of the overlaps.*

Proof Let I be a finite complete set of overlaps modulo B. Then the set U of sunifiers that correspond to overlaps in I is also finite and sufficient. To prove this, let $(\theta, \gamma_1, \gamma_2, t)$ be a sunifier for l_1, l_2. Proposition 8 ensures that either θ is an overlap of l_1, l_2 or else the scritical pair is convergent. Assume that θ is an overlap of l_1, l_2 at a subterm modulo B of, say, l_2 and that $l_2 = \gamma_0[l_0]$. (Equality holds here and not just equivalence because in the proof of Proposition 8 the overlap holds for a subterm and not for a subterm modulo B.) If θ is in I then the corresponding sunifier is in U and subsumes $(\theta, \gamma_1, \gamma_2, t)$ (by Proposition 8). Otherwise, let $\theta' \in I$ be a more general overlap of l_1, l_2 at the same subterm modulo B of l_2. Then $l_2 \simeq_B \gamma'_0[l'_0]$ such that $\gamma_0 \simeq_B \gamma'_0$ and $l_0 \simeq_B l'_0$. Also $\theta = \theta'; \rho$ for some substitution ρ. The sunifier in U corresponding to θ' is $(\theta', \theta'(\gamma'_0), z, \theta'(l_2))$, and it subsumes $(\theta, \gamma_1, \gamma_2, t)$ because $\gamma_2[\rho(\theta'(\gamma'_0))] = \gamma_2[\theta(\gamma'_0)] \simeq_B \gamma_2[\theta(\gamma_0)] \simeq_B \gamma_1$ (see proof of Proposition 8) and $\gamma_2 = \gamma_2[\rho(z)]$. If θ is an overlap of l_1, l_2 at a subterm modulo B of l_1 the proof is similar. □

When B contains just the commutative law for a binary operation $+$, then condition (C) is satisfied and finite complete sets of overlaps always exist (see Section 2.1). Theorem 4 thus yields an algorithm to decide local confluence modulo commutativity, provided an algorithm to compute complete sets of overlaps exists. (The same result can be obtained using sunifiers. By Proposition 9, finite sufficient sets of sunifiers always exist and Theorem 2 yields an algorithm to decide local confluence modulo commutativity.)

Proposition 10. *For any B, condition (C) is equivalent to the following:*

(C') *For every equation $(\forall X)\ t = t'$ in B, for every context $\gamma[\cdot]$ and Σ-term u, if $t = \gamma[u]$ then $t' = \gamma'[u']$ for some context $\gamma'[\cdot]$ and Σ-term u' such that $u \simeq_B u'$ and $\gamma \simeq_B \gamma'$.*

This implies if B and the equivalence classes are recursive, then it is decidable whether B satisfies (C). In particular, if B contains m equations and each equivalence class modulo B contains at most n elements, there is a straightforward algorithm running in time $O(mn^2p^2)$, where p is the maximum number of subterms in the left- or rightsides of equations in B. It would be interesting to find conditions like (C) that are both necessary and sufficient, but the next section generalizes the notions of unifier and critical pair, instead of constraining B.

5 Categorical Term Rewriting and Sunification

This section assumes familiarity with the basics of category theory, for which see e.g. [9, 25, 22]. Our setup was inspired by Diaconescu's category based equational logic [6, 11], the categorical approach to unification given in [9], the theory of conceptual blending from cognitive linguistics [8], and its mathematical formulation in [10]. These theories helped us in formulating the right definitions. By way of notation, we use the "bbold" font, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, etc., for categories; also we let $|\mathbb{C}|$ denote the class of objects of $\mathbb{C}$, we let ";" denote composition in any category, and we let 1_A denote the identity morphism at an object A. $\mathbb{S}et$ denotes the category of sets.

Our (**basic**) **setup** assumes an arbitrary faithful functor $\mathcal{U}\colon \mathbb{A} \to \mathbb{B}$ with left adjoint $\mathcal{F}\colon \mathbb{B} \to \mathbb{A}$, where $\mathbb{A}$ is the category of **models**, $\mathbb{B}$ the **base category**, $\mathcal{U}$ the **forgetful functor**, and $\mathcal{F}$ the **free functor**. In examples, $\mathbb{A}$ is a category of *algebras with additional structure*. We often further assume that $\mathbb{B}$ is a functor category of the form $\mathbb{S}et^S$ where S is a set (represented as a discrete category) whose elements are called **sorts**. For X an object of $\mathbb{B}$, we usually write X_s instead of $X(s)$, and call $x \in X_s$ an **element** of sort s, thinking of objects in $\mathbb{S}et^S$ as *S-indexed sets*, or *S-indexed tuples*, for which the notation $\langle X_s \mid s \in S\rangle$ may be more familiar. If X, Y are objects of $\mathbb{B}$, then $X \subseteq Y$ makes sense, meaning $X_s \subseteq Y_s$ for all $s \in S$, and similarly for $X \cup Y$, $X \cap Y$, $X - Y$, etc.[3]. Elements of free algebras, $t \in (\mathcal{U}(\mathcal{F}(X)))_s$ should be thought of *terms* with variables in X; these are class terms in many special cases. Maps $\theta\colon X \to \mathcal{U}(\mathcal{F}(Y))$ are indexed tuples of terms, sometimes used as *substitutions*. Let $\overline{\theta}$ denote the adjoint morphism $\mathcal{F}(X) \to \mathcal{F}(Y)$ to the morphism $\theta\colon X \to \mathcal{U}(\mathcal{F}(X))$, often abbreviated to θ. When

[3] Although this set theoretic notation runs counter to categorical ideology, it is very useful, and can be given a more politically correct basis, and much greater generality, by assuming $\mathbb{B}$ is an inclusive category in the sense of [4]. However, the special case is adequate for this paper.

X has a single element, say of sort s, then $X \to \mathcal{U}(\mathcal{F}(Y))$ represents a single term of that sort. Given $t \in \mathcal{U}(\mathcal{F}(X))_s$ let $var(t)$ denote the least $V \subseteq X$ such that $t \in \mathcal{U}(\mathcal{F}(V))_s$.

Let $\mathbb{K}$ denote the **Kleisli category** of the adjunction, the objects of which are those of $\mathbb{B}$, with morphisms from X to Y maps $X \to \mathcal{U}(\mathcal{F}(Y))$ in $\mathbb{B}$, and with the composition in $\mathbb{K}$ of $t\colon X \to Y$ with $t'\colon Y \to Z$, written $t;t'$, defined to be $t;\overline{t'}$ in $\mathbb{B}$; then the identity morphism on an object X is the unit $X \to \mathcal{U}(\mathcal{F}(X))$ of the adjunction. If $t = t';u$, then u serves as a **context** for t' in t, and we may write $u[t']$, especially if the target of u has just one element. If the source of u, which is $var(u)$, has n elements, then u is an n**-ary context**.

Definition 11. *A* **rewrite rule** *is a pair of terms $\ell, r \in \mathcal{U}(\mathcal{F}(X))_s$ such that $var(r) \subseteq var(\ell)$, usually written $\ell \to r$. A set A of rewrite rules is a* **categorical term rewriting system**, *abbreviated* **CTRS**. *A* **match** *of the leftside ℓ of a rule $\ell \to r$ to a term $t \in \mathcal{U}(\mathcal{F}(X))_{s'}$ is a pair (θ, u) such that $t = \theta;\ell;u$, where $\theta\colon X \to Y$ and u is a context of sort s'; then $t' = \theta;\ell$ is the* **redex** *of the match, and $\theta;r;u$ is the* **result** *of* **rewriting** *t with the rule $\ell \to r$, written $t \Rightarrow_A t'$. As usual, $\overset{*}{\Rightarrow}_A$ denotes the transitive reflexive closure of $\Rightarrow_A$.*

The notations $u[\theta(\ell)]$ and $u[\theta(r)]$, where the former indicates the match, may seem clearer than the compositions $\theta;\ell;u$ and $\theta;r;u$, respectively. Note that any CTRS A defines an ARS, as the set of all pairs (t, t') such that t rewrites to t' using a rule in A (though this should be viewed as an S-valued set). The notions of termination, confluence, local confluence, etc., and all ARS results, including the Newman Lemma are available for CTRS's.

Example 11 For the unsorted case, S is a one element set, so $\mathbb{B}$ is (isomorphic to) the category $\mathbb{S}et$ of sets. Fixing an unsorted signature Σ, the functor $\mathcal{F}$ maps a set X to $T_\Sigma(X)$, the free Σ-algebra on X, and the forgetful functor $\mathcal{U}$ returns the underlying sets of Σ-algebras. It is well known that $\mathcal{F}$ is left adjoint to $\mathcal{U}$, and it is not difficult to see that the resulting notions of term, context, rewrite rule, etc. agree with the usual ones. □

Example 12 The many sorted case discussed in Section 2 is nearly the same as the example above, except that an overloaded S-sorted signature Σ is used instead of an unsorted signature, and many sorted Σ-algebras are used instead of unsorted algebras. □

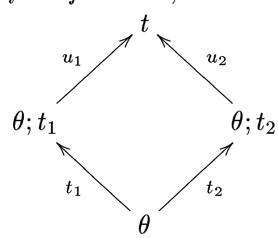
Example 13 To get overloaded many sorted term rewriting modulo equations (MTRS), as in Section 2.1, in addition to a fixed S-sorted signature Σ as in Example 12, assume a set B of Σ-equations, and let $\mathcal{F}(X)$ be $T_{\Sigma,B}(X)$, the free (Σ, B)-algebra generated by X, with $\mathcal{U}$ the usual forgetful functor. Again, it is known that these functors are adjoint, and it is not difficult to see that the resulting notions of term, rewrite rule, etc. agree with the usual class rewriting notions. It seems rather elegant to pull this out of our setup so easily. □

Example 14 Order sorted algebra (i.e., overloaded many sorted algebra with subsorts) goes much the same way, by fixing an S-sorted signature Σ that includes a partial order on the sorts, with a fixed set B of Σ-equations, and then letting $\mathcal{F}(X)$ be $T_{\Sigma,B}(X)$, the free (Σ, B)-algebra generated by X, with $\mathcal{U}$ the usual forgetful functor. These functors are known to be adjoint [14]. One also can go a little further and add sort constraints (which are equationally defined subsorts) to B. □

It is easy to impose various notions of order, continuity, etc. on algebras, by assuming the algebras in $\mathbb{A}$ have the desired structure; background is in [16] and other sources.

We will need the following construction: given a category $\mathbb{C}$, let $\mathbb{C}^\bullet$ denote the category whose objects are the morphisms of $\mathbb{C}$, and whose morphisms from f to g are all those morphisms t in $\mathbb{C}$ such that $g = f;t$ in $\mathbb{C}$, with composition in $\mathbb{C}^\bullet$ of $t\colon f \to g$ with $t'\colon g \to h$ defined to be $t;t'$ in $\mathbb{C}$, and with the identity at f in $\mathbb{C}^\bullet$ defined to be 1_Y, where Y is the target of f. We leave the reader to check that this is a category. It may be helpful to notice that if $t\colon f \to g$, then t and g have the same target in $\mathbb{C}$, and f and g have the same source, because $f;t = g$.

Definition 12. *Given morphisms* $t_1, t_2\colon X \to Y$ *in* $\mathbb{K}$*, a* (t_1, t_2)-**diamond** *is a diagram in* $\mathbb{K}$ *as shown below, where* $\theta\colon Z \to X$ *and* $t\colon Z \to W$ *are objects of* $\mathbb{K}^{\bullet}$ *(i.e., morphisms of* $\mathbb{K}$*), while* $u_i\colon \theta; t_i \to t$ *for* $i = 1, 2$ *are morphisms of* $\mathbb{K}^{\bullet}$*. Then* $t_i\colon \theta \to \theta; t_i$ *are also morphisms of* $\mathbb{K}^{\bullet}$*, and the diagram below commutes in* $\mathbb{K}^{\bullet}$*, since it satisfies* $\theta; t_i; u_i = t$ *for* $i = 1, 2$*. This diagram may be denoted* $\diamondsuit_{\theta,u_1,u_2,t}$.



A **sunifier** *(or* **superunifier***) of two morphisms* t_1, t_2 *in* $\mathbb{K}$ *is a diamond* $\diamondsuit_{\theta,u_1,u_2,t}$*. A sunifier* $\diamondsuit_{\theta,u_1,u_2,t}$ *is* **separated** *iff there exists a binary context* u *such that* $t = u[t_1, t_2]$.

In general, W, Y have just one element, so $t \in \mathcal{U}(\mathcal{F}(Z)), t_i \in \mathcal{U}(\mathcal{F}(X))$ are terms.

Example 15 The MTRS sunifier in Example 3 is also a CTRS sunifier, since MTRS is the special of CTRS with $\mathcal{F}(X) = T_{\Sigma,B}(X)$, the free (Σ, B)-algebra generated by X, according to Example 13. □

Definition 13. *Given rewrite rules* $\ell_1 \to r_1$ *and* $\ell_2 \to r_2$*, if* $\diamondsuit_{\theta,u_1,u_2,t}$ *is a sunifier for* ℓ_1, ℓ_2*, then both rules apply to* t*, giving the results* $p_1 = u_1[\theta(r_1)]$ *and* $p_2 = u_2[\theta(r_2)]$*; in this case, the pair* (p_1, p_2) *is called a* **scritical pair***, and is said to* **converge** *iff its two terms can be rewritten to a common term, indicated* $p_1 \downarrow_A p_2$*. If* $\diamondsuit_{\theta,u_1,u_2,t}$ *is a separated sunifier for* ℓ_1, ℓ_2*, call* (p_1, p_2) *a* **separated scritical pair**.

Definition 14. *Given morphisms* $t_1, t_2\colon X \to Y$ *in* $\mathbb{K}$ *and* (t_1, t_2)*-diamonds* $\diamondsuit_{\theta,u_1,u_2,t}$ *and* $\diamondsuit_{\theta',u'_1,u'_2,t'}$*, then a* **morphism** *from* $\diamondsuit_{\theta,u_1,u_2,t}$ *to* $\diamondsuit_{\theta',u'_1,u'_2,t'}$ *is a triple* (ρ, u) *of morphisms of* $\mathbb{K}$ *such that* $\theta' = \rho; \theta$ *and* $\rho; t = t'; u$ *and* $u_i; u = u'_i$ *in* $\mathbb{K}$ *for* $i = 1, 2$*. We may call this a* (t_1, t_2)-**diamond morphism**.

Example 16 The MTRS sunifiers and subsumption in Example 7 also define a CTRS subsumption, since MTRS is the special of CTRS with $\mathcal{F}(X) = T_{\Sigma,B}(X)$, the free (Σ, B)-algebra generated by X, according to Example 13. □

Definition 15. *Given morphisms* $t_1, t_2\colon X \to Y$ *in* $\mathbb{K}$*, and* (t_1, t_2)*-diamond morphisms* (ρ, u) *from* $\diamondsuit_{\theta,u_1,u_2,t}$ *to* $\diamondsuit_{\theta',u'_1,u'_2,t'}$ *and* (τ, v) *from* $\diamondsuit_{\theta',u'_1,u'_2,t'}$ *to* $\diamondsuit_{\theta'',u''_1,u''_2,t''}$*, then their* **composition** *is the diamond morphism* $(\tau; \rho, u; v)$ *from* $\diamondsuit_{\theta,u_1,u_2,t}$ *to* $\diamondsuit_{\theta'',u''_1,u''_2,t''}$*. The* **identity** (t_1, t_2)*-diamond morphism in* $\mathbb{D}_{t_1,t_2}$ *on* $\diamondsuit_{\theta,u_1,u_2,t}$ *is* $(1_Z, 1_Z, 1_W)$*.*

Proposition 11. *Given morphisms* $t_1, t_2\colon X \to Y$ *in* $\mathbb{K}$*, the* (t_1, t_2)*-diamonds with their morphisms form a category, denoted* $\mathbb{D}_{t_1,t_2}$*; in particular, the composition of* (t_1, t_2)*-diamond morphisms is a* (t_1, t_2)*-diamond morphism, this composition is associative, and has identities as in Definition 15.*

The proof is an easy diagram chase left to the reader. If we assume $\mathcal{U}(\mathcal{F}(X))$ is countable for any countable X, which holds for every interesting example we know, then $\mathbb{D}_{t_1,t_2}$ has at most countably many objects, although it could also be the empty category.

Proposition 12. *The convergence of the scritical pair obtained from a sunifier that is subsumed by a second sunifier is implied by the convergence of the scritical pair obtained from that second sunifier.*

Definition 16. *An object in a category* $\mathbb{C}$ *is* **weak initial** *iff there is a morphism from it to any other object in* $\mathbb{C}$*; this morphism need not be unique. A* **weak initial set** $\mathcal{I}$ *in* $\mathbb{C}$ *is a set* $\mathcal{I}$ *of objects in* $\mathbb{C}$ *such that for any object* C *in* $\mathbb{C}$*, there exists an object* I *in* $\mathcal{I}$ *and a morphism from* I *to* C*. A weak initial set is* **minimal** *iff no proper subset of it is weak initial.*

We would like to have a weak initial object in $\mathbb{D}_{t_1,t_2}$ but since these need not exist, we consider weak initial sets, which do always exist (e.g., all the objects in $\mathbb{D}_{t_1,t_2}$); however, minimal weak initial sets may not exist. As in Section 3, we use the following terminology:

Definition 17. *A weak initial set in $\mathbb{D}_{t_1,t_2}$ is called a* **complete** *set of sunifiers for t_1, t_2, and is* **minimal complete** *if no proper subset is also complete. Given a set $\mathcal{I}$ of sunifiers for t_1, t_2, call the resulting set of scritical pairs* **complete**, *and* **minimal complete**, *if $\mathcal{I}$ is. The same terminology is used for separated sunifiers. Given a CTRS A and a complete set of sunifiers for each pair of leftsides of rules in A, the set of* **scritical pairs** *of A is the union of the scritical pairs for each pair of rules. The* **unseparated scritical pairs** *are defined analogously. A set of [unseparated] sunifiers is* **sufficient** *iff every [unseparated] sunifier is either subsumed by one in the set, or else its critical pair is convergent.*

Lemma 4. *If a CTRS A has finite complete sets of sunifiers for all rule pairs, then it has a finite sufficient set of critical pairs.*

The following generalization of Theorem 2 follows from the construction of diamond morphisms:

Theorem 5. *A CTRS is locally confluent iff all its scritical pairs converge.*

6 Conclusions and Future Research

There are large literatures on term rewriting, unification, and completion modulo equations, with important applications, e.g., to theorem proving; we hope methods like those in this paper can lend greater unity and generality to this area, and provide a general basis for the modular construction of various algorithms. It is intriguing that our sunification construction resembles the blending of conceptual spaces developed in cognitive linguistics [8]; shared phenomena include the key role of "diamond" diagrams, and the non-uniqueness of results [12]. Our category theoretic approach is still at an early stage, but we believe it shows promise for unifying, generalizing, and simplifying, by better separating conceptual issues from algorithmic issues. Interesting future projects include generalizing Proposition 6, and extending methods for combining unification and constraint solving algorithms (e.g., [2]) to sunification. Toyama-style modularity results also seem suitable for our framework, using ideas like [21]. The scritical pair theorem (2) has a pleasing simplicity, and the sunifier and separation notions seem very natural. The sufficient condition for sunification to agree with overlapping is interesting, and technically non-trivial. It is also encouraging that we can treat the A/C cases, and we expect to treat more cases in future work.

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