

Hilbert and the concept of axiom

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First period

Axiomatic method in the first period

“The actual so-called *axioms* of geometry, arithmetic, statics, mechanics, radiation theory, or thermodynamics arose in this way. These axioms form a layer of axioms which lies deeper than the axiom-layer given by the recently-mentioned fundamental theorems of the individual field of knowledge. The procedure of the axiomatic method, as it is expressed here, amounts to a *deepening of the foundations* of the individual domains of knowledge – a deepening that is necessary to every edifice that one wishes to expand and to build higher while preserving its stability”. 1918 *Axiomatic Thought*.

Axiomatic method as a means to *deepening the foundations* of the individual domains of knowledge.

The axiomatic method requires an *historical* development of a science before formalization. This development can give *meaning* to the signs.

Against formalism

“The use of geometrical signs as a means of strict proof presupposes the exact knowledge and complete mastery of the axioms which underlie those figures; and in order that these geometrical figures may be incorporated in the general treasure of mathematical signs, there is necessary a rigorous axiomatic investigation of their conceptual content. [...] so the use of geometrical signs is determined by the axioms of geometrical concepts and their combinations.” 1900

Mathematical Problems

Where the meaning comes from?

“These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition.”
1899 *Grundlagen der Mathematic*.

Later on intuition.

Axioms in the first period

“When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. The axioms so set up are at the same time the definitions of those elementary ideas; and no statement within the realm of the science whose foundation we are testing is held to be correct unless it can be derived from those axioms by means of a finite number of logical steps.” 1900 *Mathematical Problems*.

The axioms are *implicit definitions*.

Axioms and existence

“In we want to investigate a given system of axioms according to the principles above, we must distribute the combinations of the objects taken as primitive into two classes, that of entities and that of nonentities, with the axioms playing the role of prescriptions that the partition must satisfy.” 1905 *On the foundations of logic and arithmetic*.

Where existence comes from?

“In the case before us, where we are concerned with the axioms of real numbers in arithmetic, the proof of the compatibility of the axioms is at the same time the proof of the mathematical existence of the complete system of real numbers or of the continuum.” 1900 *Mathematical Problems*.

Independence

“Upon closer consideration the question arises *whether, in any way, certain statements of single axioms depend upon one another, and whether the axioms may not therefore contain certain parts in common, which must be isolated if one wishes to arrive at a system of axioms that shall be altogether independent of one another.*” 1900
Mathematical Problems.

Second period

The concept of axiom starts to change

“In order to investigate a subfield of a science, one bases it on the smallest possible number of principles, which are to be as simple, intuitive, and comprehensible as possible, and which one collects together and sets up as axioms. Nothing prevents us from taking as axioms propositions which are provable, or which we believe are provable.” 1922 *The new grounding of mathematics*.

“Certain formulas which serve as building blocks for the formal structure of mathematics are called *axioms*”. 1922 *The new grounding of mathematics*.

The beginning of proof theory

“The axioms and provable theorems [...] are the images of the thoughts that make up the usual procedure of traditional mathematics; but they are not themselves the truth in any absolute sense. Rather, the absolute truths are the insights that my proof theory furnishes into the provability and the consistency of these formal systems.” 1923 *The logical foundations of mathematics*.

Ideas behind proof theory:

- 1 The whole of mathematics is formalizable, so that it becomes a repository of formulas.
- 2 There exists a *metamathematics* capable of handling in a meaningful way the meaningless formulae of formalized mathematics.

Axioms as meaningless formulas

“Certain of the formulas correspond to mathematical axioms. The rules whereby the formula are derived from one another correspond to material deduction. Material deduction is thus replaced by a formal procedure governed by rules. The rigorous transition from a naïve to a formal treatment is effected, therefore, both for the axioms (which, though originally viewed naïvely as basic truth, have been long treated in modern axiomatics as mere relations between concepts) and for the logical calculus (which originally was supposed to be merely a different language)”. 1925 *On the infinite*.

The winning of formalism? No

“This program already affects the choice of axioms for our proof theory”. 1923 *The logical foundations of mathematics*.

Foundational axioms arithmetic and logical in character. Their validity again rests on intuition.

Against logicism

“This circumstance corresponds to a conviction I have long maintained, namely, that a simultaneous construction of arithmetic and formal logic is necessary because of the close connection and inseparability of arithmetical and logical truth”. 1922 *The new grounding of mathematics*.

Intuition

Hilbert and intuition (first period)

“To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians”. 1900 *Mathematical Problems*.

Axioms:

- ① they give meaning to signs, through their intuitive character, and so
- ② they link intuition to mathematical practice.

Contextual intuition is not always safe

“we do not habitually follow the chain of reasoning back to the axioms in arithmetical, any more than in geometrical discussions. On the contrary we apply, especially in first attacking a problem, a rapid, unconscious, not absolutely sure combination, trusting to a certain arithmetical feeling for the behavior of the arithmetical symbols, which we could dispense with as little in arithmetic as with the geometrical imagination in geometry”. 1900 *Mathematical Problems*.

Intuition in the second period

“Kant taught [...] that mathematics treats a subject matter which is given independently of logic. Mathematics therefore can never be grounded solely on logic. [...] As a further precondition for using logical deduction and carrying out logical operations, something must be give in conception, viz. certain extralogical concrete objects which are intuited as directly experienced prior to all thinking”. 1925 *On the infinite*.

A priori knowledge

“Whoever wishes nevertheless to deny that the laws of the world come from experience must maintain that besides deduction and experience there is a third source of knowledge. Philosophers have in fact maintained – and Kant is the classical representative of this standpoint – that besides logic and experience we have a certain *a priori* knowledge of reality”. 1930 *Logic and the knowledge of the nature*.

The axioms of proof theory describe *a priori* knowledge

“The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds”. 1928 *The foundation of mathematics*.

Intuition is evident

“The subject matter of mathematics is, in accordance with this theory [i.e. the accordance between symbols and our perceptive structure], the concrete symbols themselves whose structure is immediately clear recognizable”. 1925 *On the infinite*.

Intuition and evidence are collapsed.

Parsons describes intuitive knowledge in Leibniz

“Knowledge is intuitive if it is *clear*, i.e. it gives the means for recognizing the object it concern, *distinct*, i.e. one is in a position to enumerate the marks or features that distinguish of one’s concept, *adequate*, i.e. one’s concept is completely analyzed down to primitives, and finally one has an immediate grasp of all these elements.” 1995 *Platonism and mathematical intuition in Kurt Gödel’s thought*.

Foundational axiom:

- 1 they express the accordance between intuitive knowledge and formal signs, and so
- 2 they give certitude and meaning to all mathematics, also the ideal one.

Gödel's theorems and Hilbert's program

Gödel's theorems end Hilbert's program (of the second period)

“Since finitary mathematics is defined as the mathematics of *concrete intuition*, this seems to imply that *abstract concepts* are needed for the proof of consistency of number theory ... By abstract concepts, in this context, are meant concepts which do not have as their content properties or relations of *concrete objects* (such as combinations of symbols), but rather of *thought structures* or *thought contents* (e.g. proofs, meaningful propositions, and so on), where in the proofs of propositions about these mental objects insights are needed which are not derived from a reflection upon the combinatorial (space-time) properties of symbols.” 1972 *On the extension of finitary mathematics which has not yet been used.*

Indeed Gödel's theorems say that there cannot be a set of foundational axioms for mathematics, that deserves this name.

Hilbert and the systems as open systems

“If geometry is to serve as a model for the treatment of physical axioms, we shall try first by a small number of axioms to include as large a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories” 1900 *Mathematical Problems*.

This mirrors the idea of the axiomatic method as a deepening in the foundation of a science.

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How to find new axioms, for Gödel?

“a complete solution of these problems [e.g. the continuum hypothesis] can be obtained only by a more profound analysis (than mathematics is accustomed to give) of the meaning of the terms occurring in them (such as “set”, “one-to-one correspondence”, etc.) and of the axioms underlying their use” 1964 *What is Cantor's continuum problem?*.

- 1 Profound analysis – Deepening the foundations
- 2 Analysis the meaning of the axioms underlying the use of the terms– Axiomatic investigation of the conceptual content of the symbols
- 3 New axioms as the result of the analysis – New axioms as a result of the deepening the foundations

Criteria for new axioms: Hilbert

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Criteria for new axioms: Gödel

“There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory”. 1964
What is Cantor's continuum problem?

Hilbert and contemporary set theory

“Thus the development of mathematical science as a whole takes place in two ways that constantly alternate: on the one hand we drive new provable formulae from the axioms by formal inferences; on the other, we adjoin new axioms and prove their consistency by contentual inference”. 1923 *The logical foundations of mathematics*.