

From Proof Nets to the Hyperfinite Factor

Mémoire de Master 2

Master de "Logique Mathématique et Fondements de l'Informatique" (LMFI)

Directrice de mémoire : Claudia Faggian

SEILLER Thomas

UNIVERSITE PARIS 7 - DENIS DIDEROT

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Claudia Faggian
Damiano Mazza
Pierre-Louis Curien
Chantal Berline

Introduction

The purpose of this work is to better understand Jean-Yves Girard's article *Geometry of interaction V : logic in the hyperfinite factor*.

In order to do that, we restrict our attention to the multiplicative fragment, and gradually develop a simplified version, where the computational objects (*designs*) are operators in a von Neumann algebra.

To move to operators, we start from the original version of geometry of interaction based on permutations. We present this geometry of interaction in the spirit of GoIV, considering permutations in their matrix representation. In this setting, we are able to :

- understand orthogonality as a reformulation of the longtrips criterion, which is a correctness criterion for proofnets
- analyze the key notion of trace, and its relation with the acyclicity of permutations

This geometry of interaction will then allow us to study the following notions of GoIV :

1. subjective truth
2. orthogonality (or polarity)
3. wagers, a sort of truth-value

The contributions of this report are :

- In the model of MLL based on permutations, we define a notion of truth that is close to the notion introduced in GoIV
- We study the adjunction of GoIV, and give an interpretation of it as a way of counting loops in the plugging of permutations.

Plan of sections

Section 1 We will first recall some basic results on multiplicative proofnets such as the longtrips criterion, and the notion of orthogonality between permutations it induces. We will then briefly present some well-known results on von Neumann algebras that will be useful to understand the sections 3 and 4.

Section 2 We will introduce the notion of plugging permutations and a result that connects the trace of a permutation matrix and the cycles of its associated permutation. We will then present the permutation-based geometry of interaction, and, besides the definitions, some results on the connectives we introduce. We will then show that we can retrieve a categorical model of MLL from this geometry of interaction. Finally, we will show how we can define a notion of truth in this model.

Section 3 We will explain how we can introduce a notion of truth that is subjective, depending on the choice of a base. We will then show how, by considering the loops appearing in the plugging of permutations, we can give an interpretation of the fundamental adjunction of geometry of interaction

$$\det(1 - F.A \oplus B) = \det(1 - F.A)\det(1 - [F]A)$$

Our interpretation will allow us to consider the coefficient $\det(1 - F.A)$ as a sort of truth-value that corresponds to the number of inner loops induced by the plugging of F and A . With the introduction of this truth-value we will be able to give a generalization of the basic notion of design.

Section 4 With the simplified definition of designs we obtained in section 3, we will be able to revisit the construction of GoIV and give a simplified version for the multiplicatives.

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1 Preliminaries

1.1 Multiplicative Proofnets (Under construction)

1.1.1 Definitions (Under construction)

We recall briefly the definition of a multiplicative proofnet. Proofnets are the natural deduction of multiplicative linear logic. We first define the following nodes : axiom-nodes and cut-nodes (fig.1) ; and $\wp$ -nodes and $\otimes$ -nodes (fig.2).

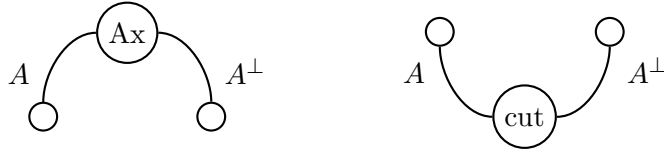


Figure 1: Axiom-node and Cut-node

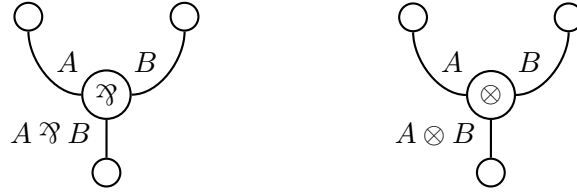


Figure 2: $\wp$ -node and $\otimes$ -node

We will call the upper edges *premise-edges* and the others *conclusion edges*.

Definition 1.1 A proof structure is a graph $\Theta = (V, E)$ constructed from the nodes from fig.1 and fig.2. The vertices (the elements of V) are labelled from the set $\{\wp, \otimes, Ax, Cut\}$ and the edges (the elements of E) are labelled with formulas.

Unfortunately, a proof structure does not always correspond to a proof obtained from the sequent calculus (ie. a correct proof). For instance, it is clear that the following proof does not correspond to a correct proof.

This is the reason why we need a criterion that allows us to characterize proof structures coming from real sequent calculus proofs, and proof structures that do not correspond to real proofs.

1.1.2 The Longtrips-criterion (under construction)

The longtrip criterion was the first correctness criterion to be enunciated, and is given in [Gir87].

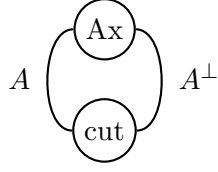


Figure 3: Incorrect proof-structure

For a proof structure $G = (V, E)$, we define *switchings*. It is, for every $\wp$ -node and $\otimes$ -node in G , a choice between L and R . We first consider that every vertex $v_A \in V$ is the juxtaposition of the two distinct directed vertexes $v_A^\uparrow$ and $v_A^\downarrow$ that goes opposite ways. Each switching S induces a way to go from one of these vertex to another, given by the figures 4, 5, and 6.

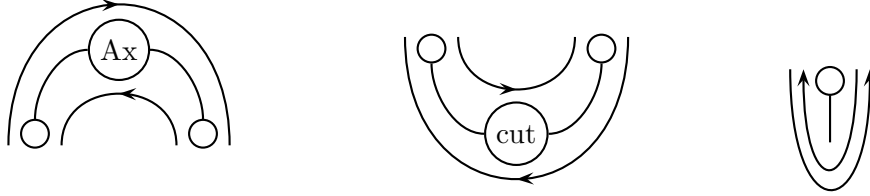


Figure 4: axiom, cut and conclusion switchings



Figure 5: $\otimes$ switchings

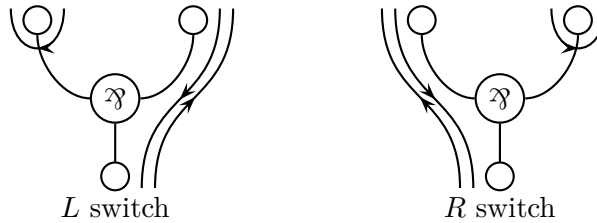


Figure 6: $\wp$ switchings

A *path* is a sequence of these directed vertexes. A *trip* is a minimal cyclic path, i.e. a path $u_1, \dots, u_k$ such that $u_1 = u_k$ and $u_i \neq u_j$ otherwise (obviously, for $i \neq j$). A trip will be called a *longtrip* when it contains every directed vertex, i.e. is of length $2 \times \text{Card}(V)$.

Theorem 1.1 *The Longtrips-criterion (LT-criterion)*

Let Θ be a proof structure. The followings are equivalent :

1. *Every switching S of Θ induces a longtrip*
2. *Θ corresponds to a sequent calculus proof*

1.1.3 Orthogonality (Under construction)

We follow the idea of [Gir88].

We can, from the LT-criterion, define a notion of orthogonality. Indeed, for all proof structure Θ we define the permutation σ_Θ that is induced by the switching of the axioms of Θ . We then define counter-permutations :

Definition 1.2 *A counter-permutation induced by a switching S of a proof structure Θ is the permutation on the atoms corresponding to the paths between atoms in Θ^* along S , where Θ^* is obtained from Θ by erasing the axiom-nodes.*

We present in appendix A p.41 an example of a proofnet and its associated permutations.

From this definition we obtain a clear reformulation of theorem 1.1, which is that the product of the permutations σ_Θ and σ_S is a cycle (of length $2n$, where $2n$ is the number of atoms).

We can then define a notion of *duality* on the set of permutations :

Definition 1.3 *We say that two permutations σ, τ are polar (denoted by $\sigma \perp \tau$) when the product $\sigma\tau$ is cyclic.*

And we have the following result :

Theorem 1.2 *Let Θ be a proof structure. the followings are equivalent :*

1. *Θ satisfies the LT-criterion*
2. *$\sigma_\Theta \perp \sigma_S$ for all switchings S induced by Θ*

1.2 Von Neumann Algebras (Under construction)

We will present some basic results on von Neumann algebras, in order to introduce some concepts that will be important for the section on geometry of interaction. We will be succinct and won't give proofs, but we refer to one of the followings for further reading : [Dix81] which is our main reference, but also [Tak01], [Mur90] or the classicals [Kad83]-[Kad86].

1.2.1 Definition of a von Neumann algebra (Under construction)

Let $\mathbb{H}$ be a complex Hilbert space, and $\mathcal{B}(\mathbb{H})$ the set of continuous linear operators of $\mathbb{H}$ into $\mathbb{H}$. This set possesses the structure of an algebra over the field of complex numbers.

Let M be any subset of $\mathcal{B}(\mathbb{H})$. We shall name the *commutant* of M , denoted by M' , the set of the elements of $\mathcal{B}(\mathbb{H})$ that commute with all the elements of M . For example, $(\mathcal{B}(\mathbb{H}))'$ is the set of scalar operators. We shall put $M'' = (M')'$ the *bicommutant* of M .

There is an *adjoint* operation on $\mathcal{B}(\mathbb{H})$. If $s \in \mathcal{B}(\mathbb{H})$, we will always denote the adjoint of s by s^* . We have :

$$(s + t)^* = s^* + t^* \quad (\lambda s)^* = \bar{\lambda} s^* \quad (st)^* = t^* s^* \quad s^{**} = s$$

We thus see that $\mathcal{B}(\mathbb{H})$ can be regarded as an involutive algebra, or $*$ -algebra.

Definition 1.4 A von Neumann algebra in $\mathbb{H}$ is a $*$ -subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathbb{H})$ such that $\mathcal{A}'' = \mathcal{A}$.

Definition 1.5 A factor is a von Neumann algebra such that its center $\mathcal{Z} = \mathcal{A} \cap \mathcal{A}'$ contains only the scalar operators.

In particular, $\mathcal{B}(\mathbb{H})$ is a von Neumann algebra, and is a factor.

1.2.2 Comparison of projections and dimension (Under construction)

We recall that a *projection* is an operator p verifying $p^2 = p$.

An *hermitian* is a operator $x \in \mathcal{A}$ such that $x = x^*$.

An operator $x \in \mathcal{A}$ is said to be *unitary* when $xx^* = x^*x = 1$.

An element $x \in \mathcal{A}$ is *positive* when x is hermitian, and $(u(x)|x) \geq 0$ for all x .

We will denote by $\mathcal{A}^+$ the set of positive hermitians of $\mathcal{A}$.

Definition 1.6 Let $\mathcal{A}$ be a von Neumann algebra, and e and f two projections of $\mathcal{A}$. e and f are said to be *equivalent* (noted $e \sim f$) if there exists in $\mathcal{A}$ a partial isometry u with e as final projection and f as final projection : $u^*u = e$ and $uu^* = f$.

We will write $e \prec f$ when there exists a projection g such that $g \sim e$ and $g \leq f$.

Proposition 1.3 If we have $e \prec f$ and $f \prec e$, we have $e \sim f$.

Proposition 1.4 Let $\mathcal{A}$ be a factor, and e and f two projections of $\mathcal{A}$. Then either $e \prec f$ or $f \prec e$.

Definition 1.7 Let $\mathcal{A}$ be a von Neumann algebra, and e a projection of $\mathcal{A}$. We say that e is finite if the algebra $\mathcal{A}_e = e\mathcal{A}e$ is finite.

We will say that a projection is *infinite* when it is not finite. An infinite projection p can also be characterized by the existence of another projection $q < p$ such that $p \sim q$.

Definition 1.8 Let $\mathcal{A}$ be a von Neumann algebra. A trace on $\mathcal{A}^+$ is a function ϕ defined on $\mathcal{A}^+$, taking non-negative, possibly infinite, real values, possessing the following properties :

1. If $s, t \in \mathcal{A}^+$, $\phi(s + t) = \phi(s) + \phi(t)$
2. If $s \in \mathcal{A}^+$ and $\lambda \in \mathbb{R}^+$, $\phi(\lambda s) = \lambda\phi(s)$
3. If $s \in \mathcal{A}^+$ and $u \in \mathcal{A}$ is unitary, $\phi(usu^{-1}) = \phi(s)$

Definition 1.9 Let $\mathcal{A}$ be a factor, and M the set of its projections. A relative dimension on M is defined to be the restriction to M of a trace on $\mathcal{A}^+$.

Proposition 1.5 Let $\mathcal{A}$ be a factor, M the set of its projections, and D a relative dimension on M .

1. If e is a finite projection of $\mathcal{A}$, $D(e) < \infty$.
2. If e is an infinite projection of $\mathcal{A}$, $D(e) = \infty$.
3. If e and f are finite projections such that $D(e) = D(f)$ (resp. $D(e) \leq D(f)$), we have $e \sim f$ (resp. $e \prec f$).

Proposition 1.6 Characterization of factor types

Let $\mathcal{A}$ be a factor, M the set of its projections, D a relative dimension on M and $\mathcal{D}$ the set of values taken by D .

1. If $\mathcal{A}$ is of type I_n , n finite (resp. $n = \infty$), we can, multiplying D by a suitable scalar, arrange that $\mathcal{D} = \{0, 1, \dots, n\}$ (resp. $\mathcal{D} = \{0, 1, \dots, \infty\}$).
2. If $\mathcal{A}$ is of type II_1 , we can, multiplying D by a suitable scalar, arrange that $\mathcal{D} = [0, 1]$.
3. If $\mathcal{A}$ is of type II_∞ , we have $\mathcal{D} = [0, \infty]$.
4. If $\mathcal{A}$ is of type III , $\mathcal{D} = \{0, \infty\}$.

Proposition 1.7 Let $\mathcal{A}$ be a von Neumann algebra. For $\mathcal{A}$ to be continuous (of type II), it is necessary and sufficient that every projection of $\mathcal{A}$ be the sum of two equivalent disjoint projections.

Definition 1.10 *A factor $\mathcal{A}$ is said to be hyperfinite when it satisfies one of the following conditions :*

1. *$\mathcal{A}$ is of type I_p , with $p < \infty$*
2. *$\mathcal{A}$ is finite, and is the von Neumann algebra generated by an increasing sequence of factors which are of the type $I_1, I_2, I_4, I_8, \dots, I_{2^n}, \dots$.*

Theorem 1.8 *Two continuous hyperfinite factors are isomorphic.*

We will denote by $\mathcal{H}$ the hyperfinite factor of type II_1 , and by $\mathcal{R}$ the hyperfinite¹ factor of type II_∞ defined as $\mathcal{R} = \mathcal{H} \otimes \mathcal{B}(\mathbb{H})$.

¹Not following our definition. But we gave the simplest definition possible, found in [Dix81]. In fact, a factor can be said to be hyperfinite without being finite. The details can be found in the references given at the beginning of the section, page 8.

2 Proofs as permutations

In this section, we will expose an interpretation of proofs as permutations, which comes from the longtrip criterion (section 1.1.2), and the notion of orthogonality exposed in [Gir88] which we recalled in section 1.1.3. We will present the three levels, from the level of geometry of interaction to the level of truth. But we will first present some results on permutations and permutation matrices.

2.1 Permutations and matrices

In the first section, we will explain the idea of *plugging* permutations, and how we can formalize this notion by considering the matrices of permutations. Then, in the second section, we will explain the relations between the trace of a permutation matrix A and the cycles of the permutation represented by A .

2.1.1 Plugging permutations

We will first present an example of what we have in mind when talking about the plugging of two permutations.

Let σ (fig.7) and τ (fig.8) be two permutations of $\{1, \dots, 10\}$ and $\{5, \dots, 10\}$ respectively. The result of the plugging of these two permutations is the permutation over $\{1, \dots, 10\}$ shown in figure 9, induced by the plugging shown in the same figure.

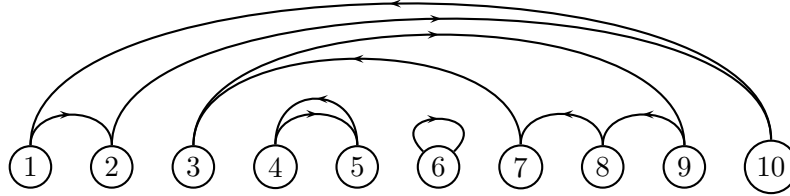


Figure 7: Permutation σ

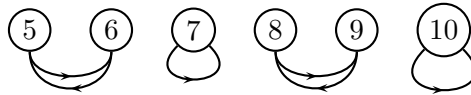


Figure 8: Permutation τ

We would like, however, to be able to define the resulting permutation from the two permutations σ and τ . This can be done in two ways : considering the permutations, or considering the permutations' matrices. The first solution does not really interest us, and we refer to [Gir07a] (section ... p.), where it is presented.

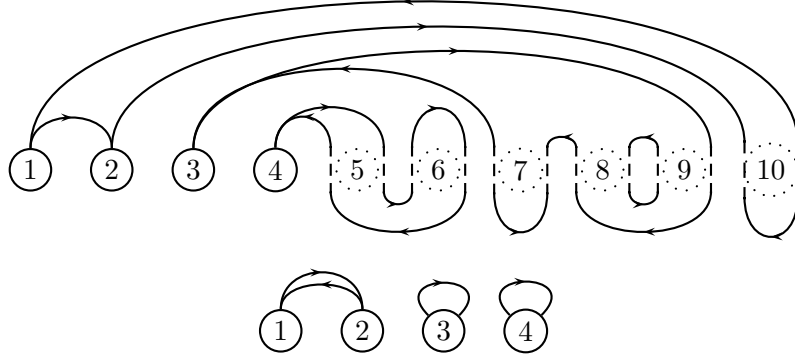


Figure 9: Plugging of σ and τ

The second solution, however, is of great interest, and we will now give the general definition.

Definition 2.1 (Plugging) Let σ and τ be two permutations over the sets $X \cup Y$ and $Y \cup Z$, such that $X \cap Z = \emptyset$. Let $A_\sigma = \begin{pmatrix} A_{X,X} & A_{Y,X} \\ A_{X,Y} & A_{Y,Y} \end{pmatrix}$ and $B_\tau = \begin{pmatrix} B_{Y,Y} & B_{Z,Y} \\ B_{Y,Z} & B_{Z,Z} \end{pmatrix}$ be the matrices representing σ and τ , decomposed along X, Y and Y, Z respectively. We define the plugging of σ and τ as the permutation represented by the matrix H with

$$H = \begin{pmatrix} H_{X,X} & H_{Z,X} \\ H_{X,Z} & H_{Z,Z} \end{pmatrix} \quad (1)$$

where

$$\begin{cases} H_{X,X} &= A_{X,X} + A_{X,Y}B_{Y,Y}(1 - A_{Y,Y}B_{Y,Y})^{-1}A_{Y,X} \\ H_{Z,X} &= A_{X,Y}(1 - B_{Y,Y}A_{Y,Y})^{-1}B_{Y,Z} \\ H_{X,Z} &= B_{Z,Y}(1 - A_{Y,Y}B_{Y,Y})^{-1}A_{Y,X} \\ H_{Z,Z} &= B_{Z,Z} + B_{Z,Y}A_{Y,Y}(1 - B_{Y,Y}A_{Y,Y})^{-1}B_{Y,Z} \end{cases}$$

In order to explain the algebraic expression, we will consider for instance the case of $H_{X,X}$. We will first make a remark which will help us understand the definition of $H_{X,X}$.

We will show how $H_{X,X}$ is equal to the expression that gives us the paths between X and X in the plugging of A and B . First, there are the paths belonging to A , of length 1, and given by the (sub-)matrix $A_{X,X}$. Then, we have paths of length 3, going from X to Y in A , then from Y to Y in B , and back from Y to X in A , which are given by the expression $A_{X,Y}B_{Y,Y}A_{Y,X}$. Then there are paths of length 5, of length 7, etc.

Since the length of such paths is potentially unbounded, we can express the result as

$$A_{X,X} + \sum_{k=0}^{\infty} A_{X,Y}B_{Y,Y}(A_{Y,Y}B_{Y,Y})^k A_{Y,X} \quad (2)$$

which can be rewritten as :

$$A_{X,X} + A_{X,Y}B_{Y,Y}(1 - A_{Y,Y}B_{Y,Y})^{-1}A_{Y,X} \quad (3)$$

which is exactly the term $H_{X,X}$.

We give now a definition of plugging in the particular case $Z = \emptyset$, which can be deduced from definition 2.1

Definition 2.2 *Let σ and τ be two permutations on the sets $X \cup Y$ and Y .*

Let $F = \begin{pmatrix} F_{X,X} & F_{Y,X} \\ F_{X,Y} & F_{Y,Y} \end{pmatrix}$ be the matrix of σ decomposed along X, Y and B the matrix of τ . We define the permutation matrix² $[F]A$ as the result of the plugging of σ and τ by

$$A_{X,X} + A_{X,Y}B_{Y,Y}(1 - A_{Y,Y}B_{Y,Y})^{-1}A_{Y,X} \quad (4)$$

2.1.2 Cycles and Trace

We will here explain the relation between the number — and length — of the cycles defined by a permutation and the trace of the powers of its permutation matrix.

Proposition 2.1 *Let σ be a permutation, A its associated matrix, and $k \in \mathbb{N}$. If $Tr(A^k) = 0$, then σ has no cycles of length k .*

Remark 1 *We can see a permutation matrix as the adjacency matrix of a directed graph. In that case, the product of two permutations matrix A and B (whose graphs are denoted by $\mathcal{G}_A$ and $\mathcal{G}_B$) gives us the adjacency matrix of the graph whose edges are pairs (a, b) of edges of A and B . In fact, it defines the graph of the paths of length 2 such that the first step is in $\mathcal{G}_A$ and the second in $\mathcal{G}_B$.*

PROOF.

The fact that the permutation σ defines a cycle of length k is exactly equivalent to the existence of a path of length k in $\mathcal{G}_A$ that makes a loop. Or, the paths of length k in $\mathcal{G}_A$ between vertices v_i and v_j are given by the coefficient $a_{i,j}^{(k)}$ of the matrix A^k . The existence of a cycle of length k thus induces that there is a diagonal coefficient $\delta = a_{i,i}^{(k)}$ of A^k such that $\delta \neq 0$. Moreover, since we are working on matrices with only positive coefficients, if $\delta \neq 0$, then $\delta > 0$, and we get that if $Tr(A^k) = 0$ the permutation σ has no cycles of length k .

□

We can even precise this result, and give an exact expression of the trace of the powers of a permutation matrix depending on the cycles of the permutation it represents. It can be deduced from the proposition 3.4 p.30.

²Anticipating on the notation.

2.2 Level -3

We will give now the definitions of level -3, which will be the basic definitions used in the remaining subsections of section 2. We will consider two important types of objects :

designs which represent proofs

conducts that represent formulas

2.2.1 Designs and conducts

Every set we consider will be a finite subset of $\mathbb{N} \cup \mathbb{N}^2$ (i.e. an element of $\mathcal{P}_f(\mathbb{N} \cup \mathbb{N}^2)$) such that $\text{Card}(X) = n_X$. We will work on permutations or on their representation as matrices indifferently. We will denote the identity matrix of size k by I_k or by 1 when the context is clear, and the set $\{1, \dots, k\}$ by $[k]$ (with $[0] = \emptyset$).

Definition 2.3 (Designs) A design will be a pair $\mathbf{a} = (X, A)$, where :

1. $X \in \mathcal{P}_f(\mathbb{N} \cup \mathbb{N}^2)$ is called the carrier.
2. A is a permutation on X , the plot of $\mathbf{a}$.

We recall that a permutation $\sigma \in \mathfrak{S}_X$ is cyclic iff it verifies

1. $\sigma^{n_X} = \text{Id}_{\mathfrak{S}_X}$
2. σ doesn't have subcycles, i.e. $\forall k \in [n_X - 1], \forall x \in X, \sigma^k(x) \neq x$

Following the idea of orthogonality given by the LT-criterion (definition 1.3), we can give a notion of duality :

Definition 2.4 (Duality) Two designs $\mathbf{a} = (X, A)$ and $\mathbf{b} = (X, B)$ will be said to be polar, denoted by $\mathbf{a} \perp \mathbf{b}$, when the permutation AB is cyclic.

Seeing A and B as matrices, we can reformulate the last definition :

Alternative Definition 2.4.1 Two designs $\mathbf{a} = (X, A)$ and $\mathbf{b} = (X, B)$ will be said to be polar, denoted by $\mathbf{a} \perp \mathbf{b}$, when

1. $(AB)^{n_X} = I_{n_X}$
2. For all $k \in [n_X - 1]$, we have $\text{Tr}((AB)^k) = 0$

Proposition 2.2 The definitions 2.4 and 2.4.1 are equivalent.

PROOF.

This is a direct consequence of proposition 2.1 page 14 and the definition of cyclicity.

□

Definition 2.5 (Conducts) A subset $\mathbf{A}$ of $\mathfrak{S}_X$ equal to its bipolar $\mathbf{A}^{\perp\perp}$ will be called a conduct of carrier X .

2.2.2 Connectives

Definition 2.6 Let $\mathbf{a} = (X, A)$ and $\mathbf{b} = (Y, B)$ be two designs of disjoint carriers, i.e. $X \cap Y = \emptyset$. We define the tensor product of $\mathbf{a}$ and $\mathbf{b}$ by :

$$\mathbf{a} = (X \cup Y, A \dot{+} B)$$

where $A \dot{+} B$ denotes the union of the permutations A and B as permutations over the set $X \cup Y$, which corresponds to the following operation on matrices

$$A \dot{+} B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

With this definition, we can now define the tensor product on conducts.

Definition 2.7 (Tensor Product) Let $\mathbf{A}$ and $\mathbf{B}$ be two conducts of respective carriers X and Y , such that $X \cap Y = \emptyset$. We define the conduct $\mathbf{A} \otimes \mathbf{B}$ of carrier $X \cup Y$ by :

$$\mathbf{A} \otimes \mathbf{B} = \{\mathbf{a} \otimes \mathbf{b} \mid \mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}\}^{\downarrow \downarrow}$$

We have the following results :

Theorem 2.3 The tensor product has the following properties :

Commutativity : $\mathbf{A} \otimes \mathbf{B} = \mathbf{B} \otimes \mathbf{A}$

Associativity : $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$

Neutral element : $\mathbf{A} \otimes \top = \mathbf{A}$, where $\top = \{(\emptyset, 0)\} = \top^{\downarrow \downarrow}$

PROOF.

Commutativity and associativity come directly from the definition of $\otimes$ on designs, and the restriction to $X \cap Y = \emptyset$.

Since there is only one permutation over $\emptyset$, which is the empty permutation 0, $\top$ is clearly a conduct, and its neutrality is immediate.

□

Proposition 2.4 (Internal Completeness of $\otimes$) Let $\mathbf{A}$ and $\mathbf{B}$ be two conducts of respective carriers X and Y such that $X \cap Y = \emptyset$. We have that

$$\mathbf{A} \otimes \mathbf{B} = \{\mathbf{a} \otimes \mathbf{b} \mid \mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}\}$$

PROOF.

Let $\mathbf{A}, \mathbf{B}$ be two conducts of respective carriers X, Y , with $X \cap Y = \emptyset$. We want to show that any design in $\mathbf{A} \otimes \mathbf{B}$ can be seen as the union of two disjoint permutations, one over X , and the other over Y .

First, let $\mathbf{A} \odot \mathbf{B} = \{\mathbf{a} \otimes \mathbf{b} \mid \mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}\}$. We have that $(\mathbf{A} \odot \mathbf{B})^\perp = (\mathbf{A} \otimes \mathbf{B})^\perp$.

Let $\mathbf{a}' = (X, A')$ and $\mathbf{b}' = (Y, B')$ be designs in $\mathbf{A}^\perp$ and $\mathbf{B}^\perp$, and $(x, y) \in X \times Y$. We define $A' \square_{(x,y)} B' = \begin{pmatrix} x & y \end{pmatrix} (A' \dot{+} B')$ the composition of the transposition $\begin{pmatrix} x & y \end{pmatrix}$, and the permutation $A' \dot{+} B'$. Let $\mathbf{c} = \mathbf{a}' \square_{(x,y)} \mathbf{b}' = (X \cup Y, A' \square_{(x,y)} B')$.

We have that for any $\mathbf{a} = (X, A) \in \mathbf{A}$ and $\mathbf{b} = (Y, B) \in \mathbf{B}$, the permutations $A'A$ and $B'B$ are cyclic. Then, since $X \cup Y = \emptyset$, the permutation $(A' \dot{+} B')(A \dot{+} B)$ defines exactly two cycles, one over X , and the other over Y . Since $x \in X$, and $y \in Y$, we have that $(A' \square_{(x,y)} B')(A \dot{+} B)$ is cyclic, and we have that $\mathbf{a}' \square_{(x,y)} \mathbf{b}' \in (\mathbf{A} \otimes \mathbf{B})^\perp$, for every $\mathbf{a}' \in \mathbf{A}^\perp$, $\mathbf{b}' \in \mathbf{B}^\perp$, and pair $(x, y) \in X \times Y$.

Now, let $\mathbf{c}' = (X \cup Y, C) \in \mathbf{A} \otimes \mathbf{B}$ such that C is not the union of two disjoint permutations, one over X , and the other over Y . We have that there exists an element $y \in Y$ whose image under C is an element $x \in X$. Let $\mathbf{a}' = (X, A') \in \mathbf{A}^\perp$ and $\mathbf{b}' = (Y, B') \in \mathbf{B}^\perp$ be two designs, and $\mathbf{c} = \mathbf{a}' \square_{(x,y)} \mathbf{b}' \in (\mathbf{A} \otimes \mathbf{B})^\perp$. Then, the permutation $C(A' \square_{(x,y)} B')$ clearly contains a 1-cycle, since $(A' \square_{(x,y)} B')(x) = y$, and $C(y) = x$. This is a contradiction, since $\mathbf{c}'$ should be polar to any $\mathbf{c} \in (\mathbf{A} \otimes \mathbf{B})^\perp$.

□

We now define the application on designs, which will allow us to define the linear implication on conducts. The definition of application is a simple reformulation of definition 2.2 p.14.

Definition 2.8 (Application) *Given two designs $\mathbf{a} = (X, A)$ and $\mathbf{f} = (X \cup Y, F)$, with $X \cap Y = \emptyset$, where $F = \begin{pmatrix} F_{X,X} & F_{Y,X} \\ F_{X,Y} & F_{Y,Y} \end{pmatrix}$ when decomposed along X and Y , we can define the application as :*

$$[\mathbf{f}]\mathbf{a} = (Y, F_{Y,Y} + F_{X,Y}A(I_{n_X} - F_{X,X}A)^{-1}F_{Y,X})$$

which is only defined if $I_{n_X} - F_{X,X}A$ is invertible.

Definition 2.9 (Linear Implication) *Let $\mathbf{A}$ and $\mathbf{B}$ be two conducts of respective carriers X and Y , such that $X \cap Y = \emptyset$. We define the conduct $\mathbf{A} \multimap \mathbf{B}$ of carrier $X \cup Y$ by :*

$$\mathbf{A} \multimap \mathbf{B} = \{\mathbf{f} \mid \forall \mathbf{a} \in \mathbf{A}, [\mathbf{f}]\mathbf{a} \in \mathbf{B}\}$$

Proposition 2.5 *The set $\mathbf{A} \multimap \mathbf{B}$ defined in definition 2.9 is a conduct.*

PROOF.

It is a consequence of the proof of theorem 2.6. By looking at the proof closely, one can see that we actually prove the following result

$$(\mathbf{A} \otimes \mathbf{B}^\perp)^\perp \subset \mathbf{A} \multimap \mathbf{B} \subset (\mathbf{A} \otimes \mathbf{B}^\perp)^\perp$$

which gives us $\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^\perp)^\perp$ which is clearly a conduct.

□

Remark 2 Suppose that F , A , and B' are permutations from designs in $\mathbf{A} \multimap \mathbf{B}$, $\mathbf{A}$, and $\mathbf{B}^\perp$. It is clear that $([F]A)B'$ and $F(A \dot{+} B')$ are just two different ways of expressing the same operation. Indeed, in the first case we begin by plugging A on F , and then plugging B' on the resulting permutation, while in the second case we are plugging A and B' simultaneously on F .

Theorem 2.6 The definitions are dual :

$$\mathbf{A} \otimes \mathbf{B}^\perp = (\mathbf{A} \multimap \mathbf{B})^\perp$$

PROOF.

If $f = (X \cup Y, F) \in \mathbf{A} \multimap \mathbf{B}$, then, by definition, for any $\mathbf{a} = (X, A) \in \mathbf{A}$ we have that $[f]\mathbf{a} \in \mathbf{B}$. So for every $\mathbf{b}' = (Y, B') \in \mathbf{B}^\perp$ we have that $([F]A)B'$ is cyclic. But then $F(A \dot{+} B')$ is also cyclic (see remark 2), and $\mathbf{a} \otimes \mathbf{b}' \sim f$. And $\mathbf{A} \multimap \mathbf{B} \subset (\mathbf{A} \otimes \mathbf{B}^\perp)^\perp$, i.e. $\mathbf{A} \otimes \mathbf{B}^\perp \subset (\mathbf{A} \multimap \mathbf{B})^\perp$.

We now take $f \in (\mathbf{A} \otimes \mathbf{B}^\perp)^\perp$, and $\mathbf{c} = \mathbf{a} \otimes \mathbf{b} \in \mathbf{A} \otimes \mathbf{B}^\perp$ (we use the internal completeness of $\otimes$ here). $[f]\mathbf{a}$ is defined, since the existence of an inner loop would contradict the cyclicity of FC . Moreover, we have that $([F]A)B'$ is cyclic for every $\mathbf{b}' = (Y, B') \in \mathbf{B}^\perp$, since for every such $\mathbf{b}'$, the product $\mathbf{a} \otimes \mathbf{b}$ is an element of $\mathbf{A} \otimes \mathbf{B}^\perp$. We deduce that $f \in \mathbf{A} \multimap \mathbf{B}$. We then have that $(\mathbf{A} \otimes \mathbf{B}^\perp)^\perp \subset \mathbf{A} \multimap \mathbf{B}$, from which we get the second inclusion $(\mathbf{A} \multimap \mathbf{B})^\perp \subset (\mathbf{A} \otimes \mathbf{B}^\perp)^\perp = \mathbf{A} \otimes \mathbf{B}^\perp$.

□

2.2.3 Localisation and Faxes

One of the most important particularity of geometry of interaction is locativity. Indeed, each design has a *locus*, an address, which is given by its carrier. Two designs (conducts) that could be representing the same proof (formula) will therefore not be equal, but only isomorphic. For instance, if we take the simple conducts $\{(1, 2)\}$ and $\{(3, 4)\}$, they are representing the same moves, but they are not the same object, since not on the same carrier.

We can see that a function $\mathbf{X} \multimap \mathbf{X}$ is not defined at level -3, since the carriers of $\mathbf{A}$ and $\mathbf{B}$ must be disjoint in order to define $\mathbf{A} \multimap \mathbf{B}$. In fact an

identity function cannot exist since it is not doing anything, while a design is defined by its interaction with the others designs, or with the elements of the set it is defined on. The only thing we can do at level -3 is copying a conduct in another place, which is equivalent to saying that $\mathbf{X}$ and $\mathbf{X}'$ are isomorphic. This kind of objects are the delocalisations, and they will allow us to define the $\mathfrak{Fax}$ which is the interpretation of the axiom.

Let ϕ be a bijection between the carrier X of $\mathbf{A}$ onto a disjoint carrier Y . A delocalisation ϕ of $\mathbf{A}$ is defined as the conduct $\phi(\mathbf{A}) = \{\phi\sigma\phi^{-1} \mid \sigma \in \mathbf{A}\}$. We can now define the $\mathfrak{Fax}$, which will represent the axioms. The faxes are defined, for every X, Y such that $X \cap Y = \emptyset$ and $Card(X) = Card(Y)$, as : $\mathfrak{Fax}_{X,Y} = (X \cup Y, \begin{pmatrix} 0 & I_{n_X} \\ I_{n_X} & 0 \end{pmatrix})$. They are elements of $\mathbf{A} \multimap \phi(\mathbf{A})$ where ϕ is a bijection from X onto Y .

2.3 Level -2

We will now retrieve a categorical model of MLL from our level -3 interpretation of proofs. But for that, we will need to consider the conducts *up to delocalisation*. We can here see where the levels -2 and -3 differ. Indeed, whereas conducts are defined on a particular carrier, and are defined as a permutation over this particular set, it is just the essence (the idea) of the permutation that interests us in level -2, which means that we will work with a permutation, without taking into account the set over which it is defined. In a way, we will be working with only the moves described by the permutation.

We will work with conducts on a standard carrier, which will be an initial segment of $\mathbb{N}$, and we will need to define delocalisations in order to do that.

We define $\phi : \begin{cases} \mathbb{N} \rightarrow \{1\} \times \mathbb{N} \\ k \mapsto (1, k) \end{cases}$ and $\psi : \begin{cases} \mathbb{N} \rightarrow \{2\} \times \mathbb{N} \\ k \mapsto (2, k) \end{cases}$

We will also need a way to associate with any conduct its standard conduct. For that, we define θ as the function that associates, to any conduct over $(\{1\} \times [n]) \cup (\{2\} \times [m])$, its delocalisation of carrier $[n+m]$ (respecting the lexicographical order).

2.3.1 Definition of the category $\mathbb{C}_{\mathfrak{S}}$

Definition 2.10 (The category $\mathbb{C}_{\mathfrak{S}}$) *The category $\mathbb{C}_{\mathfrak{S}}$ is defined by :*

$$Obj_{\mathbb{C}_{\mathfrak{S}}} = \{S \subset \mathfrak{S}_{[n]} \mid n \in \mathbb{N}, S^{\perp\perp} = S\}$$

$$Mor_{\mathbb{C}_{\mathfrak{S}}}(A, B) = \phi(A) \multimap \psi(B)$$

We still have to define the composition of morphisms, and verify that it is actually associative. In order to define $g \circ f$ where $f \in \phi(A) \multimap \psi(B)$ and $g \in \phi(B) \multimap \psi(C)$, we would need first to consider delocalisations f'

and g' of f and g such that $f' \in \phi(A) \multimap \xi(B)$ and $g \in \xi(B) \multimap \psi(C)$ where $\xi(B)$ has a carrier disjoint from $\phi(A)$ and $\psi(C)$. However, since the representations of f and g as matrices aren't modified by this operation, we will define the application directly on this said representations.

Definition 2.11 (Composition) *Let $f \in \phi(A) \multimap \psi(B)$ and $g \in \phi(B) \multimap \psi(C)$. We define the composition as the permutation induced by the plugging of f with g . We recall that, when the expressions are invertible, we can define it on the matrices as follows.*

Let $f = \begin{pmatrix} F_{A,A} & F_{B,A} \\ F_{A,B} & F_{B,B} \end{pmatrix}$ and $g = \begin{pmatrix} G_{B,B} & G_{C,B} \\ G_{B,C} & G_{C,C} \end{pmatrix}$. We define $h = g \circ f$ by :

$$h = \begin{pmatrix} H_{A,A} & H_{C,A} \\ H_{A,C} & H_{C,C} \end{pmatrix} \quad (5)$$

with

$$\begin{cases} H_{A,A} &= F_{A,A} + F_{A,B}G_{B,B}(1 - F_{B,B}G_{B,B})^{-1}F_{B,A} \\ H_{C,A} &= F_{A,B}(1 - G_{B,B}F_{B,B})^{-1}G_{B,C} \\ H_{A,C} &= G_{C,B}(1 - F_{B,B}G_{B,B})^{-1}F_{B,A} \\ H_{C,C} &= G_{C,C} + G_{C,B}F_{B,B}(1 - G_{B,B}F_{B,B})^{-1}G_{B,C} \end{cases}$$

Proposition 2.7 (Composition is well-defined)

Composition is well defined. In other words, if $f \in \phi(A) \multimap \psi(B)$ and $g \in \phi(B) \multimap \psi(C)$, no loops appear in the plugging of f and g , and the expressions $F_{B,B}G_{B,B}$ and $G_{B,B}F_{B,B}$ are therefore invertible.

PROOF.

Let $\mathbf{f} = (F, f) \in \mathbf{A} \multimap \mathbf{B}$ and $\mathbf{g} = (G, g) \in \mathbf{B} \multimap \mathbf{C}$ be two designs, and let us suppose that in the plugging of f and g a loop appears. It is easy to see that this loop can only appear on the subset $F \cap G$ which we will call X .

Since $\mathbf{f} \in \mathbf{A} \multimap \mathbf{B}$, if we take a design $\mathbf{a} \in \mathbf{A}$ we can obtain the design $[\mathbf{f}]\mathbf{a} \in \mathbf{B}$, and this operation won't change the sub-permutation on X creating the loop.

Similarly, since $\mathbf{g} \in \mathbf{B} \multimap \mathbf{C}$, we have that $\mathbf{g} \in \mathbf{C}^\perp \multimap \mathbf{B}^\perp$. By taking $\mathbf{c}' \in \mathbf{C}^\perp$, we can then obtain $[\mathbf{g}]\mathbf{c}' \in \mathbf{B}^\perp$. As in the case of $[\mathbf{f}]\mathbf{a}$, the sub-permutation on X creating the loop will be conserved by the application.

Or, we know that plugging $[\mathbf{f}]\mathbf{a}$ and $[\mathbf{g}]\mathbf{c}'$ results in a cyclic permutation. We then deduce that the loop appearing on X is in fact cyclic on X . But this gives us that the permutation f (resp. g) is in fact the union of two disjoint permutations, one over X , and the other over $F \setminus X$ (resp. $G \setminus X$). Then f (resp. g) is a tensor products of designs of $\mathbf{A}$ and $\mathbf{B}$ (resp. $\mathbf{B}$ and $\mathbf{C}$), which is impossible unless $\mathbf{A} = \top = \mathbf{C}$.

□

Proposition 2.8 (Associativity of composition) *The composition is associative, and we have that, for $f \in \phi(X) \multimap \psi(Y)$,*

$$f \circ \mathfrak{F}\mathfrak{a}\mathfrak{r}_{\phi(X),\psi(X)} = \mathfrak{F}\mathfrak{a}\mathfrak{r}_{\phi(Y),\psi(Y)} \circ f = f$$

PROOF.

Associativity. We have three permutations, σ, τ, ρ over the sets $W \cup X$, $X \cup Y$, and $Y \cup Z$, where X, Y are disjoint. Since we are plugging σ on $\tau|_X$ and ρ on $\tau|_Y$, the fact that we are plugging σ first or τ first is of no importance and changes nothing to the result.

Identity. To verify the equality, one has just to remember that $\mathfrak{F}\mathfrak{a}\mathfrak{r}_{\phi(A),\psi(A)}$ is defined as $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. If we replace the corresponding submatrices in the expression 5, we get the matrix of f .

If $H = \begin{pmatrix} F_{X,X} & F_{Y,X} \\ F_{X,Y} & F_{Y,Y} \end{pmatrix} \circ \mathfrak{F}\mathfrak{a}\mathfrak{r}_{\phi(X),\psi(X)}$, the term $H_{X,X}$ is equal to

$$Fax_{\phi(X),\phi(X)} + Fax_{\phi(X),\psi(X)} F_{X,X} (1 - Fax_{\psi(X),\psi(X)} F_{X,X})^{-1} Fax_{\psi(X),\phi(X)}$$

which is equal to $F_{X,X}$ since $\begin{cases} Fax_{\phi(X),\phi(X)} = Fax_{\psi(X),\psi(X)} = 0 \\ Fax_{\phi(X),\psi(X)} = Fax_{\psi(X),\phi(X)} = 1 \end{cases}$

We verify as easily the result for the other coefficients.

□

Theorem 2.9 $\mathbb{C}_{\mathfrak{S}}$ is a category.

PROOF.

This is clear from proposition 2.8.

□

2.3.2 A model of MLL

We first recall the definition of a $*$ -autonomous category.

Definition 2.12 *A monoidal category is a category $\mathbb{K}$ with a bifunctor $\otimes : \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{K}$, a (left and right) unit $\top \in \text{Obj}_{\mathbb{K}}$, verifying $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. In addition, some diagrams concerning associativity and the unit must commute (see [McL98] for a complete definition).*

It is said to be symmetric when we have $A \otimes B \cong B \otimes A$.

Definition 2.13 *A $*$ -autonomous category $\mathbb{K}$ is a symmetric monoidal category $(\mathbb{K}, \otimes, \top)$ with a contravariant closed functor $(-)^* : \mathbb{K} \rightarrow \mathbb{K}$ such that*

1. $A \cong A^{**}$ for all $A \in \text{Obj}_{\mathbb{K}}$

2. *There is a natural bijection $\mathbb{K}[A \otimes B, C] \cong \mathbb{K}[A, (B \otimes C^*)^*]$.*

It is well-known that a $*$ -autonomous category yields a model of MLL ([See89]). We shall now build a $*$ -autonomous structure on $\mathbb{C}_{\mathfrak{S}}$. After giving the definition of the functors $\otimes$ and $(-)^*$, we shall check a few of the required properties (those that are most interesting to us), leaving the others to the reader.

We will just define the functors $\bar{\otimes}$, $(-)^*$ and $\bar{\multimap}$, and verify that

1. $A \bar{\otimes} B \cong B \bar{\otimes} A$
2. $\phi(A \bar{\otimes} B) \multimap \psi(C) \cong \phi(A) \multimap \psi(B \bar{\multimap} C)$

Definition 2.14 *We define the functor $\otimes$ as :*

$$A \bar{\otimes} B = \theta(\phi(A) \otimes \psi(B)) \quad (6)$$

$$f \bar{\otimes} g = \begin{pmatrix} f_{A,A} & 0 & f_{B,A} & 0 \\ 0 & g_{C,C} & 0 & g_{D,C} \\ f_{A,B} & 0 & f_{B,B} & 0 \\ 0 & f_{C,D} & 0 & f_{D,D} \end{pmatrix} \quad (7)$$

where $A, B, C, D \in \text{Obj}_{\mathbb{C}_{\mathfrak{S}}}$ and $(f, g) \in \text{Mor}_{\mathbb{C}_{\mathfrak{S}}}(A, B) \times \text{Mor}_{\mathbb{C}_{\mathfrak{S}}}(C, D)$.

We see that we can define $(-)^* : A \mapsto A^{\downarrow}$. Indeed, the operation gives us a conduct on the same carrier as A , so we get an objet of $\mathbb{C}_{\mathfrak{S}}$. We have, by definition of $\mathbb{C}_{\mathfrak{S}}$, that $A^{**} = A$.

Using theorems 2.6 and 2.3, and the fact that a conduct is equal to its bipolar, we get that

$$\begin{aligned} \phi(\mathbf{A}) \multimap \psi(\mathbf{B}) &= (\phi(\mathbf{A}) \otimes \psi(\mathbf{B}^{\downarrow}))^{\downarrow} \\ &= (\psi(\mathbf{B}^{\downarrow}) \otimes \phi(\mathbf{A}^{\downarrow\downarrow}))^{\downarrow} = \psi(\mathbf{B}^{\downarrow}) \multimap \phi(\mathbf{A}^{\downarrow}) \end{aligned}$$

So, with the function $\chi : \begin{cases} (1, k) \mapsto (2, k) \\ (2, k) \mapsto (1, k) \end{cases}$ we can define the delocalisation $f^* = \chi(f) \in (\phi(B^*) \multimap \psi(A^*))$, for $f \in (\phi(A) \multimap \psi(B))$. We verify easily that $f^{**} = f$.

Definition 2.15 *We define the functor $\bar{\multimap}$ by :*

$$A \bar{\multimap} B = (A \bar{\otimes} B^*)^*$$

Proposition 2.10

$$A \bar{\otimes} B \cong B \bar{\otimes} A$$

PROOF.

We have $A \bar{\otimes} B = \theta(\phi(A) \otimes \psi(B))$ and $B \bar{\otimes} A = \theta(\phi(B) \otimes \psi(A))$ such that A has carrier $[n]$ and B on carrier $[m]$. We will show that there exists permutations Z of $\phi(A \bar{\otimes} B) \multimap \psi(B \bar{\otimes} A)$ and Y of $\phi(B \bar{\otimes} A) \multimap \psi(A \bar{\otimes} B)$ that correspond to what we are looking for.

Indeed, we have $A \bar{\otimes} B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $B \bar{\otimes} A = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$. If we take the permutation Z of $\phi(A \bar{\otimes} B) \multimap \psi(B \bar{\otimes} A)$ that exchanges $\phi([n])$ and $\psi(\{m+1, \dots, m+n\})$, and $\phi(\{n+1, \dots, n+m\})$ and $\psi([m])$, we will have that $[Z](\phi(A \bar{\otimes} B)) = \psi(B \bar{\otimes} A)$ which is exactly what we need. The permutation Y is defined as the permutation that exchanges $\phi([m])$ and $\psi(\{n+1, \dots, n+m\})$, and $\phi(\{m+1, \dots, m+n\})$ and $\psi([n])$. It verifies $[Y](\phi(B \bar{\otimes} A)) = \psi(A \bar{\otimes} B)$.

□

Remark 3 *In fact, we have $Z = \chi(Y)$ and $Y = \chi(Z)$. We could have just remarked that since $[Z](\psi(B \bar{\otimes} A)) = \phi(A \bar{\otimes} B)$, we had $[\chi(Z)](\phi(B \bar{\otimes} A)) = \psi(A \bar{\otimes} B)$.*

Proposition 2.11

$$\phi(A \bar{\otimes} B) \multimap \psi(C) \cong \phi(A) \multimap \psi(B \multimap C)$$

PROOF.

Let A, B, C be three objects of the category, of respective carriers $[m], [n], [k]$. Let $\mathcal{D} = \phi(A \bar{\otimes} B) \multimap \psi(C)$ and $\mathcal{E} = \phi(A) \multimap \psi(B \multimap C)$, and X and Y be defined as

$$\begin{aligned} X &= \{(1, 1), \dots, (1, m), (1, m+1), \dots, (1, m+n), (2, 1), \dots, (2, k)\} \\ Y &= \{(1, 1), \dots, (1, m), (2, 1), \dots, (2, n), (2, n+1), \dots, (2, n+k)\} \end{aligned}$$

If $\mathfrak{d} \in \mathcal{D}$, then $\mathfrak{d} = (X, D)$, where D is a permutation over X . Similarly, $\mathfrak{e} \in \mathcal{E}$ can be written $\mathfrak{e} = (Y, E)$. Let now $\xi : X \rightarrow Y$ be the function defined by

$$\xi((i, j)) = \begin{cases} (i, j) & \text{when } i = 1 \text{ and } j \leq m \\ (2, j - m) & \text{when } i = 1 \text{ and } j > m \\ (2, j + n) & \text{when } i = 2 \end{cases}$$

This function is clearly a bijection between X and Y . We define then $\Phi : \mathcal{D} \rightarrow \mathcal{E}$ as $\Phi(\mathfrak{d}) = (Y, \xi D \xi^{-1})$, where $\mathfrak{d} = (X, D)$. We can see that $\xi D \xi^{-1}$ defines a permutation over Y . Let now $\Psi(\mathfrak{e}) = (X, \xi^{-1} E \xi)$ for $\mathfrak{e} = (Y, E) \in \mathcal{E}$. We have that $\Psi = \Phi^{-1}$, and we have defined an isomorphism between $\mathcal{D}$ and $\mathcal{E}$.

□

Theorem 2.12 *The category $\mathbb{C}_{\mathcal{E}}$ is a model of MLL.*

2.4 Level -1

We can now retrieve a notion of truth defined inside our model. This notion is quite easy to understand, since we want for a design to represent a correct proof when it corresponds to the permutation associated to axiom-links. Such a permutation is a symmetry without any fixed point.

Definition 2.16 (Correctness) *We say that a design $\mathfrak{a} = (X, A)$ is correct when it verifies*

1. $A^2 = Id_X$
2. $\forall x \in X, A(x) \neq x$

Remark 4 *We could replace the second condition by $Tr(A) = 0$.*

Definition 2.17 (Truth) *A conduct is true when it contains a correct design.*

Theorem 2.13 *The two conducts $\mathbf{A}$ and $\mathbf{A}^\perp$ cannot both be true.*

PROOF.

First, we remark the fact that a conduct over a carrier X contains a correct design implies that n_X is even.

If $\mathfrak{a} = (X, A) \in \mathbf{A}$ and $\mathfrak{b} = (X, B) \in \mathbf{A}^\perp$ are both correct, then we have $\epsilon(A) = \epsilon(B) = (-1)^{\frac{n_X}{2}}$, where $\epsilon(\tau)$ denotes the signature of the permutation τ . Indeed, both are symmetries on X without fixed points. But, since we restricted ourselves to the case where n_X is even, we have that a cyclic permutation over X has a signature equal to -1 , so $\epsilon(AB) = -1$ whereas $\epsilon(A)\epsilon(B) = (-1)^{n_X} = 1$, which is contradictory.

□

3 Towards Geometry of Interaction

In this section, we will show how we can extend our model in order to get a simplified version (for the multiplicatives only, presented in section 4) of geometry of interaction V (presented in [Gir08], and referred to as GoIV from now on). In order to do that, we will consider permutations over a finite subset of a base of an hilbert space $\mathbb{H}$ (section 3.1). This will give us the opportunity to explain the notion of subjective truth (section 3.2) by extending the set of matrices we are working on. Finally we will show (section 3.3) how we can obtain the notion of duality defined in [Gir08] from our cyclicity-based definition (definitions 2.4 and 2.4.1).

3.1 Considering matrices

As a first step towards geometry of interaction, we considered the permutations as matrices, since [Gir08] makes use of operators. But until now, we have considered permutations over finite subsets of $\mathbb{N}$ or $\mathbb{N}^2$.

The second step is to consider permutation matrices over finite subsets of a particular basis of a Hilbert Space. We can easily see that it changes nothing to the results of section 2.

We will now work in a basis $(e_i)_{i \in \mathbb{N}^*}$ of a Hilbert Space $\mathbb{H}$.

We can define a partial trace over the elements of $\mathcal{B}(\mathbb{H})$ by $Tr(u) = \sum_{k=1}^{\infty} (ue_k | e_k)$.

The set over which the permutations are defined, is a finite subset of this basis, given by a finite projection (diagonal in the basis we are working on), i.e. a diagonal matrix $\mathbf{p}$ such that $Tr(\mathbf{p}) < \infty$ and $\mathbf{p}^2 = \mathbf{p}$.

This gives us a new definition of designs :

Definition 3.1 (Designs) *A design is a pair $\mathbf{a} = (\mathbf{p}, A)$ where*

1. $\mathbf{p}$ is a finite projection diagonal in the base (e_i)
2. $A \in \mathbf{p}\mathcal{B}(\mathbb{H})\mathbf{p}$ is a permutation matrix diagonal over a finite subset of (e_i) (i.e. is written as a permutation matrix in the basis (e_i))

For now, the notion of duality isn't really modified, since we already remarked that we can define it directly on the matrices by considering the trace (see the alternative definition 2.4.1 p.15).

This change of viewpoint does not modify the results of section 2, but will allow us to define the notion of subjective truth, and shade a new light on the notions of duality and designs that appear in [Gir08].

3.2 Viewpoints and Subjective Truth

We would like to modify our notion of truth (see definition 4.7) to take into account the fact that we considered more matrices. In fact, we would like

to say of a design that it is *true with respect to a certain basis $\mathcal{B}$* when the matrix represents a permutation over the basis $\mathcal{B}$.

In order to do that, we will need to enlarge the set of matrices we are working with. Thus, we give a more general notion of designs. To introduce the notion of subjective truth, we will extend the set of matrices we are working with in order to get the set of all permutation matrices. These matrices won't be necessarily permutations over the same base of the Hilbert Space $\mathbb{H}$.

Definition 3.2 (Generalized designs) *A design is a pair $\mathbf{a} = (\mathbf{p}, A)$ with*

1. *$\mathbf{p}$ is a finite projection*
2. *$A \in \mathbf{p}\mathcal{B}(\mathbb{H})\mathbf{p}$ is a permutation matrix over a finite subset of any base of $\mathbb{H}$*

We will now explain how the idea of considering matrices that are matrices of permutation, but not necessarily on the same bases, will allow us to introduce the notion of subjective truth. Since our matrices won't necessarily be representing permutations over the same basis, we must first verify that our notion of polarity doesn't need to be changed. However, the definition we use (definition 2.4.1 p.15) does not depend upon the basis, since the trace of a matrix is basis-independant, and so is the equality $(AB)^k = 1$.

3.2.1 Viewpoints

We want to say of a design that it is true with respect to a basis $\mathcal{B}$ when it is a permutation matrix *over a finite subset of basis $\mathcal{B}$* , and verifying conditions of definition 2.16. In order to achieve this, we need to introduce *viewpoints*. We will consider the subalgebras of $\mathcal{M}_n(\mathbb{C})$ that are defined as follow

$$\mathcal{P}_{\mathcal{B}} = \{A \in \mathcal{M}_n(\mathbb{C}) \mid A \text{ is diagonal in the base } \mathcal{B}\}$$

Proposition 3.1 *Let $\mathcal{B}$ be a base of $\mathbb{C}^n$. The subalgebra $\mathcal{P}_{\mathcal{B}}$ is a maximal abelian subalgebra of $\mathcal{M}_n(\mathbb{C})$.*

PROOF.

The fact that it is a commutative subalgebra is clear.

We will now prove that it is indeed maximal. Let $\mathcal{B}$ be a base. We will work with matrices written in the base $\mathcal{B}$. We first remark that the matrices $E_{i,j} = (\delta_{k,l}^{i,j})$ where $\delta_{k,l}^{i,j} = \begin{cases} 1 & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}$ form a basis of the algebra $\mathcal{M}_n(\mathbb{C})$, and that $(E_{i,i})$ is a basis of $\mathcal{P}_{\mathcal{B}}$. If there was a commutative algebra $\mathcal{P}'$ strictly containing $\mathcal{P}_{\mathcal{B}}$, then we could find a matrix E_{i_0,j_0} , with $i_0 \neq j_0$, commuting with every element of $\mathcal{P}_{\mathcal{B}}$.

Or, we can easily see that $E_{i,j}E_{k,l} = \begin{cases} 0 & \text{if } j \neq k \\ E_{i,l} & \text{otherwise} \end{cases}$ and we get that $E_{i_0,j_0}E_{j_0,j_0} = E_{i_0,j_0}$ and $E_{j_0,j_0}E_{i_0,j_0} = 0$, which shows that $\mathcal{P}'$ is not commutative.

□

The notion of *maximal abelian subalgebras* is a notion we will use in Geometry of Interaction, where they are called *viewpoints*.

3.2.2 A new definition of truth

In order to say that a matrix is a permutation matrix over the base $\mathcal{B}$, we need to be able to say that it has only one non zero coefficient in each row and column.

Remark 5 *Until now, we were only considering permutation matrices over the same basis, i.e. matrices that had only 0 and 1 as coefficients. An additional condition $A = A^*$ is necessary in order to make the distinction between the two matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$.*

We won't really be interested in the algebra $\mathcal{P}_{\mathcal{B}}$ itself, but in its *normalizer* $\mathcal{N}(\mathcal{P}_{\mathcal{B}})$, i.e. the matrices A such that $A\mathcal{P}_{\mathcal{B}}A^* \subset \mathcal{P}_{\mathcal{B}}$. These matrices, when verifying $A = A^*$, have the distinctive feature we are looking for, since they have only one coefficient non equal to zero in each column and each row.

Proposition 3.2 *Let $\mathcal{P}_{\mathcal{B}}$ be the maximal abelian subalgebra defined from the base $\mathcal{B}$. If $A = A^* \in \mathcal{M}_n(\mathbb{C}) \in \mathcal{N}(\mathcal{P}_{\mathcal{B}})$, then it has only one non zero coefficient in each column and in each row.*

PROOF.

Let $\mathcal{B}$ be a base of $\mathbb{H}$. We will consider all the matrices to be written in the basis $\mathcal{B}$. Using notations of the proof of proposition 3.1, we have that $(E_{i,i})$ is a base of $\mathcal{P}_{\mathcal{B}}$.

If $A = (a_{i,j})$ is a matrix in $\mathcal{N}(\mathcal{P}_{\mathcal{B}})$, then we have that $AE_{i,i}A^* = AE_{i,i}A$ is diagonal, for every $i \in [n]$. But, denoting $AE_{i,i}A$ by $D_i = (d_{k,l}^i)$, we have that $d_{k,l}^i = a_{k,i}a_{i,l}$. Since D_i must be diagonal, we have that $a_{k,i}a_{i,l} = 0$ for $k \neq l$.

Now, suppose that $a_{i_0,j_0} \neq 0$. Then, by considering D_{i_0} and D_{j_0} , we get that $a_{i_0,j_0}a_{j_0,k} = 0$ when $k \neq i_0$ and that $a_{l,i_0}a_{i_0,j_0} = 0$ for $l \neq j_0$. Which gives us, considering that $a_{i,j} = a_{j,i}^*$, that every coefficient (different from a_{i_0,j_0}) in the same row or in the same column as a_{i_0,j_0} is equal to 0.

□

We are now able to define the notion of subjective truth :

Definition 3.3 (Subjective Truth) *A design $(\mathbf{p}, A)$ will be correct w.r.t. $\mathcal{B}$ (or³ w.r.t. $\mathcal{P}_{\mathcal{B}}$) when*

1. $A^2 = I$ and $A = A^*$
2. $Tr(A) = 0$
3. $A \in \mathcal{N}(\mathcal{P}_{\mathcal{B}})$

3.3 Towards GoIV duality

The notion of duality introduced in [Gir08] poses two conditions. If $\mathbf{a}$ and $\mathbf{b}$ are two designs, and their matrices are denoted by A and B , then the two conditions are :

1. A restriction on the spectral radius of AB : $\rho(AB) < 1$.
2. A condition based on the definition of an "inner product" on designs

We will, in this section, comment on the two conditions separately (sections 3.3.1 and 3.3.2).

3.3.1 The spectral radius

When n approaches infinity, which is what happens when considering operators in the hyperfinite factor, the condition $(AB)^n = 1$ gives us $AB = 1$ which is too restrictive. On the other hand, a condition on the spectral radius would behave nicely. The disappearance of the first condition will be explained by remark 7 in the next section.

Proposition 3.3 *The condition $(AB)^n = 1$ implies that $\rho(AB) = 1$.*

PROOF.

Considering AB in a well-chosen base, we can write it as a triangular matrix. Indeed, every matrix in $\mathcal{M}_n(\mathbb{C})$ can be written as a triangular matrix, since $\mathbb{C}$ is algebraically closed. Written in this particular form, the elements of the diagonal are the eigenvalues $(\lambda_i)_{i \in [n]}$ of AB , and we know that $(AB)^n$ is also triangular, with diagonal elements λ_i^n for all $i \in [n]$. The spectrum $Sp(AB)$ of AB is therefore such that $\forall \lambda \in Sp(AB), \lambda^n = 1$, and $\rho(AB) = 1$.

□

³To stay close to the terminology of [Gir08].

We can deduce that when A and B are polar, the spectral radius of AB verifies $\rho(AB) = 1$. But the requirement appearing in [Gir08] is that $\rho(AB) < 1$. We will see in section 3.4 that this difference comes from a new generalization of designs.

Remark 6 *The condition $\rho(AB) = 1$ isn't equivalent to $(AB)^n = 1$. In fact, every permutation matrix D verifies $\rho(D) = 1$, and not every D such that $\rho(D) = 1$ is a permutation matrix. The condition therefore extends our primary notion of duality, in the case where we are considering more matrices, and not only matrices coming from permutations, which will be the case in geometry of interaction.*

The extension of the duality notion isn't problematic since every pair of formerly dual designs is still dual with the new definition.

3.3.2 Acyclicity

First, we remark that we can write, when $\rho(AB) < 1$,

$$-\log(\det(1 - AB)) = \sum_{k=1}^{\infty} \frac{\text{Tr}((AB)^k)}{k} \quad (8)$$

which looks a lot like our second requirement for duality.

Indeed, we said that the second requirement can be written as $\forall k < n, \text{Tr}((AB)^k) = 0$. But, since AB is a permutation matrix, we have that $\text{Tr}(AB) \geq 0$ since every coefficient appearing in AB is positive. This gives us the equivalence

$$\begin{aligned} \forall k < n, \text{Tr}((AB)^k) = 0 &\Leftrightarrow \sum_{k=1}^{n-1} \text{Tr}((AB)^k) = 0 \\ &\Leftrightarrow \sum_{k=1}^{n-1} \frac{\text{Tr}((AB)^k)}{k} = 0 \end{aligned}$$

which gives us equation 8 when n approaches infinity.

Remark 7 *When n approaches infinity, which is necessary in order to get operators in a factor of type II, the condition of acyclicity, which depends on n , becomes an acyclicity condition. This is the reason why the first condition disappears, since it was linked to the existence of a cycle.*

But we still have a problem, since the existence of a single k -cycle in AB makes the expression diverge when n approaches infinity. That's because we are considering permutation matrices, whose coefficients are equal to 1 or 0. For instance, suppose we have only one 1-cycle in AB , then it is clear that the right side of equation 8 diverges. Indeed, since $\text{Tr}((AB)^k) \geq 1$ for

all $k \geq 1$, we would get that $\sum_{k=1}^{\infty} \frac{1}{k} \leq \sum_{k=1}^{\infty} \frac{\text{Tr}((AB)^k)}{k}$. We can easily verify that the existence of a longer cycle would give us the same inequality.

This is where the condition $\rho(AB) < 1$ appears, since it is necessary to the convergence. We will need a new definition of designs in order to get the condition $\rho(AB) < 1$. This will be introduced in section 3.5, but we will first say a word about the adjunction.

3.4 Adjunction

In this section, we will follow an intuition that shades a new light on the founding equation⁴ of GoIV

$$\det(1 - [F]A.B)\det(A - F.A) = \det(1 - F.A \dot{+} B) \quad (9)$$

3.4.1 Counting loops

We saw, in fig.9 p.13, that sometimes the plugging of two permutations can create a inner loop that prevents us from computing the algebraic expression, since the inversion of the term $1 - F_{X,X}A$ is no longer possible. But, we also saw that a permutation can still be defined as we did in our example. We can actually modify our definition to take into account the inner loops, and define the application even when such a loop appears.

The expression defined in the last section (3.3.2) gives us a way to count the number of cycles defined by a permutation. We already saw that the trace of a permutation matrix is closely linked to the cycles of the permutation it represents (see section 2.1.2). We expose now a stronger result than the one we had earlier.

Defining $\text{ldet}_k(1 - A) = \sum_{k=1}^{n-1} \frac{\text{Tr}(A^k)}{k}$, we have that

Theorem 3.4 *Let A be a permutation matrix, representing the permutation σ over a set of cardinal n . Let $f_i(\sigma)$ denote the number of cycles of length i in σ . We have*

$$\text{ldet}_n(1 - A) = \sum_{i=0}^{n-1} f_i(\sigma) + \Delta(\sigma, n)$$

PROOF.

In the proof, we will make a distinction between a *loop* which is a sequence $a_1, \dots, a_k = a_1$ and a *cycle* which is a minimal loop, i.e. such that every a_i, a_j for $i \neq j$ in $[k-1]$ are distinct.

We already saw that if A is a permutation matrix, the existence of a non zero diagonal element in $\text{Tr}(A^k)$ is equivalent to the existence of a k -cycle in the permutation σ represented by A . Moreover, since A has only coefficients 0 or 1, the diagonal element in $\text{Tr}(A^k)$ can only be equal to 1. Now, let ν be a k -loop. We have two possibilities :

⁴Following earlier notation, we denote the operation $\oplus$ by $\dot{+}$.

1. The k -cycle ν is a cycle, i.e. is not induced by a d -cycle where d divides k . In this case, we have that exactly k non zero diagonal elements appear in A^k , one for each vertex composing the k -cycle.
2. The k -loop ν comes from a d -cycle, where d divides k . Then, we have that exactly d non zero diagonal elements in A^k are induced by ν .

We then have that $\frac{\text{Tr}(A^k)}{k}$ is equal to the number of k -cycles, to which we add the number of d -cycles, where d divides k , divided by $\frac{k}{d}$. This gives us

$$\sum_{k=1}^{n-1} \frac{\text{Tr}(A^k)}{k} = \sum_{k=1}^{n-1} f_k(\sigma) + \sum_{k=1}^{n-1} \sum_{d|k, d \neq k} \frac{d \cdot f_d(\sigma)}{k} \quad (10)$$

Definig $\Delta(\sigma, n) = \sum_{k=1}^{n-1} \sum_{d|k, d \neq k} \frac{d \cdot f_d(\sigma)}{k}$ we have the result.

□

We can also see that when defining the application, with $\mathfrak{f} = (X \cup Y, F)$ and $\mathfrak{a} = (X, A)$ such that $\text{Card}(X) = m$ and $\text{Card}(Y) = n$, the expression $\text{ldet}_m(1 - F.A)$ "counts"⁵ the number of inner loops due to the plugging of F with A , while the expression $\text{ldet}_n(1 - [F]A.B)$ counts the number of loops appearing when plugging $[F]A$ with B . Adding these two expressions, we would hope to get the number of loops appearing in the plugging of F with A and B simultaneously.

That is why we would like to obtain an equality like⁶

$$\text{ldet}_n(1 - [F]A.B) + \text{ldet}_m(1 - F.A) = \text{ldet}_{n+m}(1 - F.A \dot{+} B) \quad (11)$$

3.4.2 Problems

This equation, unfortunately, is false because the terms ldet_k doesn't *exactly* count the number of loops. This is because of the coefficient $\Delta(\sigma)$ of proposition 3.4 that this equality does not hold, since the value we compute depends on the length of the cycles (see proof of proposition 3.4 to get the exact value of $\Delta(\sigma)$).

For instance, suppose that A and F are the matrices corresponding to the permutations σ, τ presented in the figures 7 p.12 and 8 p.12, and B is the matrix of the permutation $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \in \mathfrak{S}_{[4]}$. Then the plugging of F with $A \dot{+} B$ induces a cycle of length 2, between A and F , and another of length 8. On the other hand, the plugging of $[F]A$ and B gives us a single cycle of

⁵It does not give exactly the number of cycles, but it is closely linked to it, so we will make an abuse of langage and say it *counts* the number of cycles.

⁶ $A \dot{+} B$ denotes the operation on matrices used in definition 2.6.

length 4.

We then have⁷ that $ldet_8(1 - F.A\dot{+}B) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + 1$, whereas

$$ldet_6(1 - FA) + ldet_4(1 - [F]A.B) = (1 + \frac{1}{2}) + (0)$$

However, when n, m approaches infinity this gives us the correct equation

$$-log(det(1 - [F]A.B)) - log(det(1 - FA)) = -log(det(1 - F.A\dot{+}B)) \quad (12)$$

which is equivalent to equation 9.

In fact, we have that, when n approaches infinity, the term $\Delta(\sigma, n)$ in proposition 3.4 page 30 will not depend on n anymore, and only on σ . In fact, this term, equal to $\sum_{k=1}^{n-1} \sum_{d|k, d \neq k} \frac{d.f_i(\sigma)}{k}$ will be replaced with $\sum_{k=1}^{\infty} \sum_{i=2}^{\infty} \frac{f_k(\sigma)}{i}$, and equation 10 can be rewritten (when A is a permutation matrix)

$$\sum_{k=1}^{\infty} \frac{Tr(A^k)}{k} = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{f_k(\sigma)}{i} \quad (13)$$

which is clearly divergent if only one of the $f_k(\sigma)$ is not null.

Once again, this equation is of no use if F, A, B are permutation matrices, since the existence of even a single loop makes the expressions diverge. We are forced to change our notion of designs.

3.5 Improving definition of designs

We will now consider the case where the component A of a design $(\mathbf{p}, A)$ is an operator in a factor of type II , i.e. the case where n approaches infinity.

3.5.1 Getting convergence

We need to change our operators since the consideration of permutation matrices makes some important expressions diverge. For instance, the expression $ldet(1 - A) = -log(det(1 - A))$ can take only two values (0 or ∞) when A is a permutation matrix.

Another problem is that in the algebraic expression defining the application (definition 2.8), the expression $1 - F_{X,X}A$ is not invertible when inner loops appear in the plugging of F and A . Since the definition works well, we would like to keep it, and the only way to do that is to modify the operators we are working on.

The keystone here is that a permutation matrix has non-zero coefficients equal to 1 (its norm is equal to 1), and that it makes the expression $\sum_{k=1}^{\infty} (F_{X,X}A)^k$ diverge when the plugging of F with A induces a loop.

In order to have a converging sum, we will not consider F and A anymore,

⁷The first four terms $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ come from the 2-cycle.

but $\frac{F}{\phi}$ and $\frac{A}{\alpha}$, where ϕ and α are real numbers such that⁸ $\alpha\phi > 1$. This will give us that⁹ $\|\frac{F}{\phi}\frac{A}{\alpha}\| < 1$, and

1. $1 - \frac{F_{X,X}}{\phi}\frac{A}{\alpha}$ can be invertible even if a loop appears in the plugging of F and A
2. $ldet(1 - \frac{F}{\phi}\frac{A}{\alpha})$ is not always equal to 0 or ∞ anymore.

Which real numbers ? If we look closely to the equation 9 we can see that the factor $det(1 - F.A)$ which corresponds to the number of inner loops was hidden until now, since its value was equal to 1 (since it is $e^{-ldet(1-F.A)}$) when the application was defined, and the application was undefined in any other case. We can then consider it as a sort of truth value. But since we are working with cologarithms, 0 will mean "true". The idea will be to take $\phi = e^f$ and $\alpha = e^a$, where f and a correspond to the "truth value" — the wager — of $\mathbf{f}$ and $\mathbf{a}$.

3.5.2 Truth-values

Considering the wager as a kind of truth value, we would like for the wager of a design containing 1-cycles to be non zero (i.e. false). The first idea, following the notion of truth we defined earlier, would be to define the wager of $\mathbf{a} = (\mathbf{p}, a, A)$ as the trace of A and replacing the condition $Tr(A) = 0$ by $a = 0$.

However, given a design, we cannot decide the value of its wager, since even a *correct* design in our sense (i.e. verifying $Tr(A) = 0$) could come from an application with inner loops. We will then have to define a design as a 3-tuple $\mathbf{a} = (\mathbf{p}, a, A)$, where a is the wager of $\mathbf{a}$, and where A is a permutation matrix divided by e^a , and extend our notions of duality and truth.

Definition 3.4 (Designs) *A design is a 3-tuple $\mathbf{a} = (\mathbf{p}, a, A)$ where*

1. $\mathbf{p}$ is a finite projection
2. $a \in \mathbb{R} \cup \{\infty\}$ is the wager of $\mathbf{a}$
3. $A \in \mathcal{R}_{\mathbf{p}} = \mathbf{p}\mathcal{R}\mathbf{p}$ is the plot of $\mathbf{a}$

Remark 8 *In a way, we could see it as the number of daemons¹⁰ used in a ludics' design.*

⁸We would like to say that they are greater than 1, but we don't have any real reasons for that since it's *their product only* that needs to be greater than 1 for the expression to converge.

⁹ $\| - \|$ denotes the norm of the operators.

¹⁰It is an incorrect axiom, defined in ludics, see [Gir01] or [Gir07a].

In our simple model of section 2, the trace of the matrix A of the design $\mathfrak{a} = (X, A)$ could be considered as its wager. It works well with the tensor product, since $Tr(A \dot{+} B) = Tr(A) + Tr(B)$, but is not easily defined in the case of application.

The notion of truth will also be modified. In fact, for a design $\mathfrak{a} = (\mathbf{p}, a, A)$ to be correct, the condition $Tr(A) = 0$ we had before will be replaced with the condition $a = 0$. But we need to modify our notions of duality, and redefine the operations on designs that are necessary to define the connectives.

3.5.3 Duality and connectives

We want our notion of duality to be consistant with our former notion.

Remark 9 *The operator A in a design $\mathfrak{a} = (\mathbf{p}, a, A)$ is not always a permutation matrix divided by the right coefficient e^a . And there are no permutation matrices in a factor of type II. But the particular designs $\mathfrak{a} = (\mathbf{p}, a, \frac{A}{e^a})$ where A is a unitary operator, which is what is closest to a permutation matrix since a permutation matrix is a unitary operator, are the objects that come directly from the former definitions of designs, and are the designs that really interest us.*

We will therefore consider two designs $\mathfrak{a} = (\mathbf{p}, a, A)$ and $\mathfrak{b} = (\mathbf{p}, b, B)$, keeping in mind that, for us, A and B come from permutation matrices (unitary operators) divided by e^a and e^b .

Then, our condition on the spectral radius becomes $e^{a+b}\rho(AB) = 1$, i.e. $\rho(AB) = e^{-a-b}$. But remember that we wanted that $e^a e^b > 1$. Since the operation of verifying the polarity is the same as considering the result of the application between an object of $\mathbf{A}$ and an object of $\mathbf{A}^\sim$, we will ask for the same condition to be verified. And we get the condition $\rho(AB) < 1$. Moreover, if we had that $\rho(AB) = 1$, then the value $ldet(1 - AB)$ would be equal to 0 or diverge.

This condition verified, our sum $ldet(1 - AB) = \sum_{k=1}^{\infty} \frac{Tr((AB)^k)}{k}$ won't diverge whenever there exists a loop and it can take values other than 0 or ∞ . Since we want an "infinite cycle", our cyclicity condition, which has been replaced with an acyclicity condition, becomes $ldet(1 - AB) = 0$. But then, since $\rho(AB) = e^{-a-b} < 1$, we cannot have $a + b = 0$, and we get that $a + b + ldet(1 - AB) \neq 0$.

Remark 10 *We can see now that if we say that $\mathfrak{a} \perp \mathfrak{b}$ when $a + b + ldet(1 - AB) \neq 0$, then we are just extending our primary notion of polarity : every pair of polar designs stay polar according to our new definition.*

It is quite clear that the wager of a tensor product $\mathfrak{a} \otimes \mathfrak{b}$ will just be the sum of the wagers of $\mathfrak{a}$ and $\mathfrak{b}$, since there are no applications involved in the operation.

For the application $[\mathfrak{f}]\mathfrak{a}$, the value of the wager will be given by the sum of the wagers of $\mathfrak{a}$ and $\mathfrak{f}$, to which we will add the wager associated to the application : $ldet(1 - F.A)$.

4 GoIV for the multiplicatives

Now that we explained how we can generalise our simple model based on permutations in order to get notions of duality and subjective truth that approach the ones that appear in [Gir08], we will present a simplified version of geometry of interaction for the multiplicatives. We will no longer restrict ourselves to certain operators, as we did in the last section.

4.1 Discussion on the factor type

Following what we did in section 2, we would hope to work in the von Neumann algebra $\mathcal{B}(\mathbb{H})$, which is a von Neumann algebra of type I_∞ , when considering only the multiplicatives. Indeed, the geometry of interaction, as presented in [Gir08], uses the hyperfinite factor of type II_∞ , denoted by $\mathcal{R}$, which is a less familiar frame.

It is important to note that we need to work in a space where, as in the factor $\mathcal{R}$, the identity is an *infinite* projection. It is the key to delocalisation. Indeed, designs are given with a finite projection $\mathbf{p}$ (the carrier) which corresponds to the *locus*¹¹ (or address) of its base, and a infinite identity will assure us of the possibility of finding, for any projection $\mathbf{q}$, a finite projection $\mathbf{q}'$, equivalent to $\mathbf{q}$, disjoint from $\mathbf{p}$. It is particularly important when defining¹² the tensor product (definition 4.4 p.37) since we consider designs with disjoint projections.

However, a small but fundamental difference will arise concerning the nature of the $\mathfrak{F}\mathfrak{a}\mathfrak{x}$ defined in a factor of type I and the $\mathfrak{F}\mathfrak{a}\mathfrak{x}$ defined in a factor of type II (see section 4.5 p.39). And the notion of duality cannot be defined using $ldet_k$ since the equation 12 p.32, which is fundamental for us to get the duality of the two multiplicative connectives, does not hold when considering its finite version (equation 11 p.32). We are therefore forced to work in the factor $\mathcal{R}$.

We will use the notation $\mathcal{R}_{\mathbf{p}} = \mathbf{p}\mathcal{R}\mathbf{p}$, which is a factor of type II_1 whose trace is not normalised (i.e. in which the trace of the identity isn't necessarily equal to 1).

4.2 Designs and conducts

Definition 4.1 A design is a triplet $\mathbf{a} = (\mathbf{p}, a, A)$, where :

1. $\mathbf{p} \in \mathcal{B}(\mathbb{H})$ is a projection, called the carrier of the design
2. $a \in \mathbb{R} \cup \{\infty\}$ is the wager.
3. $A \in \mathcal{R}_{\mathbf{p}}$ is the plot of $\mathbf{a}$.

¹¹The term comes from ludics, see [Gir01] or [Gir03].

¹²It is in fact necessary if we want to get totality of the tensor product *up to delocalisation*, i.e. be able to define, for any $\mathbf{a}$ and $\mathbf{b}$ a tensor product $\phi(\mathbf{a}) \otimes \psi(\mathbf{b})$.

We now need a definition of duality.
We recall that $\text{ldet}(1 - AB) = \sum_{k=1}^{\infty} \frac{\text{Tr}((AB)^k)}{k}$.

Definition 4.2 We define an "inner product" on designs, where $\mathbf{a} = (\mathbf{p}, a, A)$ and $\mathbf{b} = (\mathbf{q}, b, B)$:

$$\ll \mathbf{a} \mid \mathbf{b} \gg = a + b + \text{ldet}(1 - AB)$$

Two designs $\mathbf{a} = (\mathbf{p}, a, A)$ and $\mathbf{b} = (\mathbf{p}, b, B)$ on the same carrier $\mathbf{p}$ will be said to be polar when :

$$\begin{cases} \ll \mathbf{a} \mid \mathbf{b} \gg \neq 0 \\ \rho(AB) < 1 \end{cases}$$

We will denote it by $\mathbf{a} \perp \mathbf{b}$.

We can then define, for every set $\mathbf{A}$ of designs of carrier $\mathbf{p}$, the set of designs polar to $\mathbf{A}$: $\mathbf{A}^\perp = \{\mathbf{b} \mid \forall \mathbf{a} \in \mathbf{A}, \mathbf{a} \perp \mathbf{b}\}$.

Definition 4.3 A set $\mathbf{C}$ of designs of carrier $\mathbf{p}$ equal to its bipolar $\mathbf{C}^{\perp\perp}$ will be called a conduct.

4.3 Connectives

We will first define the tensor product and the applications for designs, so that we can later define the connectives (tensor product and linear implication) on the conducts.

Definition 4.4 Let $\mathbf{a} = (\mathbf{p}, a, A)$ and $\mathbf{b} = (\mathbf{q}, b, B)$ be two designs such that the two projections $\mathbf{p}$ and $\mathbf{q}$ are disjoint. We define :

$$\mathbf{a} \otimes \mathbf{b} = (\mathbf{p} + \mathbf{q}, a + b, A \dot{+} B)$$

Since $A \in \mathcal{R}_{\mathbf{p}}$ and $B \in \mathcal{R}_{\mathbf{q}}$, we need to consider them as elements of $\mathcal{R}_{\mathbf{p}+\mathbf{q}}$ in order to sum them, which is the equivalent of the operation on permutations (see definition 2.6). That is why we will denote this operation as $\dot{+}$.

Definition 4.5 Let $\mathbf{f} = (\mathbf{p} + \mathbf{q}, f, F)$ and $\mathbf{a} = (\mathbf{p}, a, A)$ be two designs and let $F = \begin{pmatrix} F_{1,1} & F_{1,2} \\ F_{2,1} & F_{2,2} \end{pmatrix}$ be the decomposition of F with respect to $\mathbf{p}$ and $\mathbf{q}$. Firstly, we define the following operation :

$$[F]A = F_{2,2} + F_{2,1}A(1 - F_{1,1}A)^{-1}F_{1,2}$$

We then define the application as :

$$[\mathbf{f}]\mathbf{a} = (\mathbf{q}, \ll \mathbf{a} \mid \mathbf{f} \gg, [F]A)$$

The application is therefore defined only if $(1 - F_{1,1}A)$ is invertible (in $\mathcal{R}_{\mathbf{p}}$).

We can now define the connectives on the conducts :

Definition 4.6 Let $\mathbf{p}, \mathbf{q}$ be two disjoint projections of $\mathcal{R}$. Let $\mathbf{A}, \mathbf{B}$ be conducts of respective carriers $\mathbf{p}, \mathbf{q}$. We define :

$$\mathbf{A} \odot \mathbf{B} = \{\mathbf{a} \otimes \mathbf{b} | \mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}\}$$

and its associated conduct $\mathbf{A} \otimes \mathbf{B} = (\mathbf{A} \odot \mathbf{B})^{\sim \perp}$ of carrier $\mathbf{p} + \mathbf{q}$.

We also define

$$\mathbf{A} \multimap \mathbf{B} = \{\mathbf{f} | \forall \mathbf{a} \in \mathbf{A}, [\mathbf{f}]\mathbf{a} \in \mathbf{B}\}$$

which is a conduct of carrier $\mathbf{p} + \mathbf{q}$.

We now have two theorems. We won't give the proofs, since they are just a particular case of the equivalent theorems in [Gir08].

Theorem 4.1 The tensor product has the following properties :

1. **Neutrality** : $\mathbf{A} \otimes \top = \mathbf{A}$, where $\top = \{(0, 0, 0)\}^{\sim \perp} = \top$
2. **Commutativity** : $\mathbf{A} \otimes \mathbf{B} = \mathbf{B} \otimes \mathbf{A}$
3. **Associativity** : $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$

Theorem 4.2 The two connectives are adjoint :

$$(\mathbf{A} \otimes \mathbf{B})^{\sim \perp} = \mathbf{A} \multimap \mathbf{B}^{\sim \perp}$$

4.4 Truth

The definition of truth makes use of a *viewpoint*, which is a maximal abelian subalgebra $\mathcal{P}$ of $\mathcal{R}$.

Definition 4.7 A design $\mathbf{a} = (\mathbf{p}, a, A)$ is said to be successful¹³ with respect to $\mathcal{P}$ when :

1. $\mathbf{a}$ is wager-free, ie. $a = 0$
2. A is a partial symmetry ($A^3 = A = A^*$)
3. A belongs to the normaliser of $\mathcal{P}$: $APA \subset \mathcal{P}$

A conduct is true (with respect to $\mathcal{P}$) when it contains a successful design.

Remark 11 The condition 2 is a little more general than the one we had in definitions 2.16 p.24 and 3.3 p.28, but this is due to the fact that the carrier $\mathbf{p}$ of a design $\mathbf{a}$ isn't necessarily minimal, and that there may exist¹⁴ a decomposition $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ where $\mathbf{p}_1$ and $\mathbf{p}_2$ are disjoint, such that $\mathbf{p}_1 A \mathbf{p}_2 = 0$.

¹³As for all the others definitions, this has been simplified, and we refer to [Gir08] for the complete definition.

¹⁴This leads to the consideration of *incarnations*, see [Gir08].

4.5 Faxes and factor type

The last thing we need to introduce is the $\mathfrak{Fax}$ which is one of the most important design. If $\mathbf{A}$ is a conduct of carrier $\mathbf{p}$ and $\phi \in \mathcal{B}(\mathbb{H})$ is a partial isometry of $\mathbf{p}$ onto $\mathbf{q}$ disjoint from $\mathbf{p}$, it belongs to the conduct $\mathbf{A} \multimap \phi(\mathbf{A})$ and is defined as $\mathfrak{Fax} = (\mathbf{p} + \mathbf{q}, 0, \phi \dot{+} \phi^*)$. The $\mathfrak{Fax}$ is of type $\mathbf{A}^\perp \mathfrak{V} \phi(\mathbf{A})$ and can be seen as an axiom.

Considering the $\mathfrak{Fax}$ we can see that working within an algebra of type I_∞ is not equivalent to working within an algebra of type II_∞ . Indeed, in the factor $\mathcal{R}$, the continuity of the trace on projections is in fact related to the possibility of η -expanding any $\mathfrak{Fax}$ as we want. The best way to see it is to consider the $\mathfrak{Fax}$ in ludics. It is defined¹⁵ as :

$$\frac{\begin{array}{c} \vdots \mathfrak{Fax}_{\xi' \star i, \xi \star i} \\ \dots \quad \xi' \star i \vdash \xi \star i \quad \dots \end{array} \quad (\xi', I)}{\vdash \xi', \xi \star I} \quad (\xi', I) \quad \dots \quad (\xi, \mathcal{P}_f(\mathbb{N}))$$

$$\frac{\vdash \xi', \xi \star I}{\xi \vdash \xi'}$$

and can be seen as a generalized and infinite¹⁶ η -expansion. The $\mathfrak{Fax}$ in GoI V must be thought as the fax of ludics, and the continuity of the trace on projections is the key to subaddressing. Indeed, we can consider that the projection of a design is its address. Since every projection can be decomposed as the sum of an arbitrarily large number of disjoint equivalent projections, we can always find a number of subaddresses as large as we want.

If we take the tensor product of two Faxes, of disjoint carriers, in the respective conducts $\mathbf{A} \multimap \phi(\mathbf{A})$ and $\mathbf{B} \multimap \psi(\mathbf{B})$, $\mathfrak{Fax}_p = (\mathbf{p} + \mathbf{p}', 0, \phi + \phi^*)$ and $\mathfrak{Fax}_q = (\mathbf{q} + \mathbf{q}', 0, \psi + \psi^*)$, we get :

$$\begin{aligned} \mathfrak{Fax}_p \otimes \mathfrak{Fax}_q &= (\mathbf{p} + \mathbf{p}' + \mathbf{q} + \mathbf{q}', 0, (\phi + \phi^*) \dot{+} (\psi + \psi^*)) \\ &= (\mathbf{p} + \mathbf{q} + \mathbf{p}' + \mathbf{q}', 0, (\phi \dot{+} \psi) + (\phi \dot{+} \psi)^*) \end{aligned}$$

which is a Fax of carrier $\mathbf{p} + \mathbf{q} + \mathbf{p}' + \mathbf{q}'$, which belongs to the conduct $\mathbf{A} \otimes \mathbf{B} \multimap \mathbf{A} \otimes \mathbf{B}$ ¹⁷. We can see that it generalizes easily, and given disjoint projections $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathcal{R}$, and Faxes $\mathfrak{Fax}_i$ in $\mathbf{A} \multimap \phi_i(\mathbf{A})$, we have that¹⁸ $\bigotimes_{i=1}^n \mathfrak{Fax}_i$ belongs to $\bigotimes_{i=1}^n \mathbf{A}_i \multimap \bigotimes_{i=1}^n \phi_i(\mathbf{A}_i)$. This is a very important thing, because this is how we can η -expand a $\mathfrak{Fax}$.

In a factor of type I , however, the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is a $\mathfrak{Fax}$, cannot

¹⁵See [Gir01] or [Gir03] for more details on ludics.

¹⁶Infinite in height and in width, even if only a finite-width sub-design is used in practise.

¹⁷It is not so strange, considering that $\mathfrak{V}$ is reversible. You have to think of $A \multimap \phi(A)$ as a notation for $A^\perp \phi(A)$.

¹⁸We need that the images of p_i under ϕ_i are disjoint to consider the tensor product.

be η -expanded. This is due to the existence of a *minimal* projection in factors of type I .

A A proofnet and its permutations

Let us take the following proof :

$$\begin{array}{c}
\frac{}{\vdash A, A^\perp} \text{Ax} \quad \frac{}{\vdash B, B^\perp} \text{Ax} \\
\hline
\vdash A, B, A^\perp \otimes B^\perp \quad \frac{}{\vdash C, C^\perp} \text{Ax} \\
\hline
\vdash A, C^\perp, B \otimes C, A^\perp \otimes B^\perp \quad \otimes \\
\hline
\vdash A, B \otimes C, C^\perp \wp (A^\perp \otimes B^\perp) \quad \wp \\
\hline
\vdash A \wp (B \otimes C), C^\perp \wp (A^\perp \otimes B^\perp) \quad \wp \\
\hline
\vdash (A \wp B) \otimes C \multimap A \wp (B \otimes C) \quad \multimap^{19}
\end{array}$$

The corresponding proof structure Θ is :

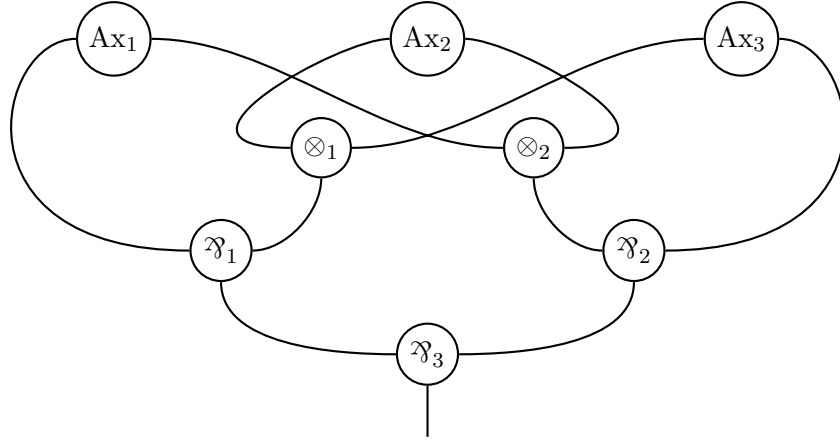


Figure 10: ProofNet of $\vdash (A \wp B) \otimes C \multimap A \wp (B \otimes C)$

Let us take for instance the following switching :

$$\begin{array}{ll}
S(\wp_1) = L & S(\otimes_1) = R \\
S(\wp_2) = R & S(\otimes_2) = L \\
S(\wp_3) = R &
\end{array}$$

We have the two induced graphs (fig.11 and fig.12).

Both these graphs give us a permutation over the atoms and negations of atoms (represented by the integers $1, \dots, 6$). We also give the representation of these permutations by matrices (fig.13 and 14).

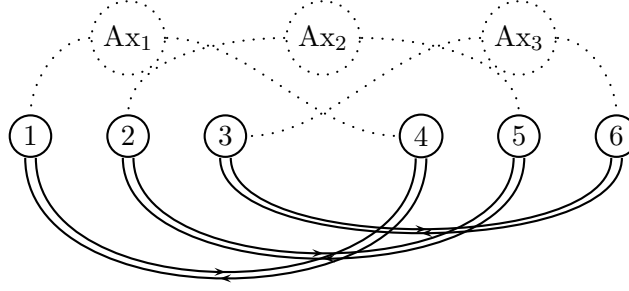


Figure 11: Axiom-nodes induced graph

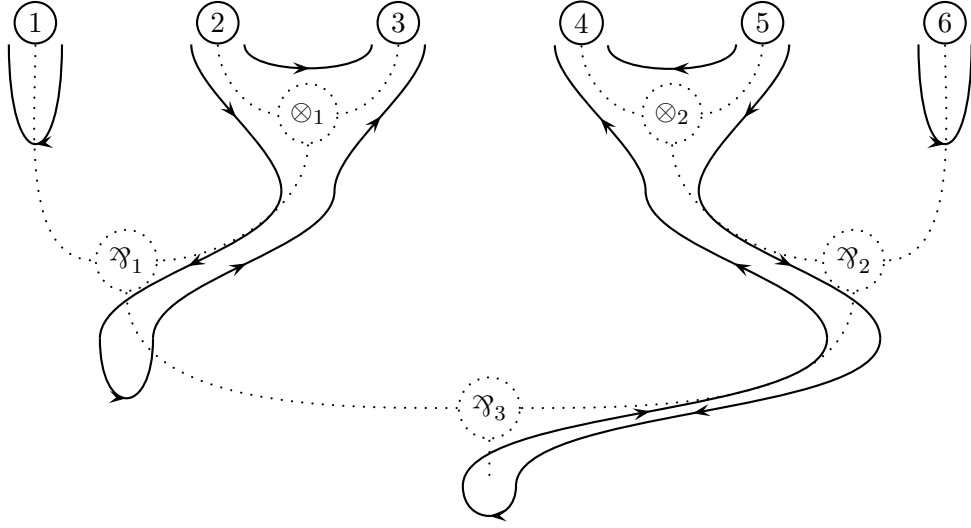


Figure 12: Connective-nodes induced graph

$$\sigma_{\Theta} = (1, 4)(2, 5)(3, 6)$$

$$\Theta^{\bullet} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 13: Permutation σ_{Θ} , and matrix $\Theta^{\bullet}$

$$\sigma_S = (2, 3)(4, 5)$$

$$\Sigma^\bullet = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 14: Permutation σ_S , and matrix $\Sigma^\bullet$

The composition of the permutations σ_Θ and σ_S gives us :

$$\begin{aligned} \pi = \sigma_\Theta \circ \sigma_S &= (1, 4)(2, 5)(3, 6)(2, 3)(4, 5) \\ &= (1, 4, 2, 6, 3, 5) \end{aligned}$$

which is cyclic, since

$$\begin{aligned} \pi^2 &= (1, 2, 3)(4, 6, 5) \\ \pi^3 &= (1, 6)(2, 5)(3, 4) \\ \pi^4 &= (1, 3, 2)(4, 5, 6) \\ \pi^5 &= (1, 5, 3, 6, 2, 4) \\ \pi^6 &= Id_{\mathfrak{S}_6} \end{aligned}$$

We can see that the product of the matrices $\Theta^\bullet$ and $\Sigma^\bullet$ gives us a matrix Π corresponding to π , and such that $\Pi^6 = I_6$ and $\Pi^i \neq I_6$ for $1 \leq i \leq 5$.

$$\begin{aligned} \Pi = \Theta^\bullet \Sigma^\bullet &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The powers of Π are presented in fig.15. We can see that each power corresponds to the corresponding power of the permutation π .

$$\begin{aligned}
\Pi &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \Pi^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\
\Pi^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \Pi^4 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
\Pi^5 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} & \Pi^6 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Figure 15: Powers of the matrix Π

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