

A Correspondence between Maximal Abelian Sub-Algebras and Linear Logic Fragments

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We show a correspondence between a classification of maximal abelian sub-algebras (MASAs) proposed by Jacques Dixmier (Dix54) and fragments of linear logic. We expose for this purpose a modified construction of Girard's hyperfinite geometry of interaction (Gir11). The expressivity of the logic soundly interpreted in this model is dependent on properties of a MASA which is a parameter of the interpretation. We also unveil the essential role played by MASAs in previous geometry of interaction constructions.

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1. Introduction

1.1. *Geometry of Interaction.*

This is a research program initiated by Girard (Gir89b) a year after his seminal paper on linear logic (Gir87a). Its aim is twofold: define a semantics of proofs that accounts for the dynamics of cut-elimination, and then construct realizability models for linear logic around this semantics of cut-elimination.

The first step for defining a *GoI model*, i.e. a construction that fulfills the geometry of

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interaction program, is to describe the set of mathematical objects $\mathcal{O}$ that will represent the proofs, together with a binary operation on this set of objects $:: : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$ that will represent the cut-elimination procedure. This function should satisfy one key property: associativity, i.e. $(a :: f) :: b = a :: (f :: b)$. This property is a reformulation of the Church-Rosser property: If f represents a function taking two arguments a, b , one can evaluate the function $f :: a$ on the argument b or equivalently evaluate the function $f :: b$ on the argument a . Both this evaluation strategies should yield the same result, i.e. the operation $::$ is associative. The data of the set $\mathcal{O}$ together with the associative binary operation $::$ constitute the first step in defining a GoI model, as it provides the semantics of cut-elimination¹.

The majority of works dealing with geometry of interaction content themselves with this part of the geometry of interaction program. Although this first part of the GoI program has been a rich source of dynamical interpretation of proofs (algebras of clauses, token machines, interaction machine), it is a crucial mistake to think the second part unimportant. On the contrary, it is the second part of the program that makes it a very innovative and strong tool, both from a technical and a philosophical point of view. The reconstruction of logic from the dynamical interpretation of proofs is a step further into the Curry-Howard correspondence: one reconstructs the logic of programs. Indeed, the set $\mathcal{O}$ represents a set of programs and the operation $::$ describes how these programs compose and evaluate. By choosing a set $\perp$ of “bad programs”, such as infinite loops, one can then built the logic that naturally arises from the notion of programs described by $\mathcal{O}$, $::$ and the set $\perp$. This is done in two steps, the first being the definition the notion of types — subsets of $\mathcal{O}$. Then every operation $\odot$ on programs which allows to construct, define, new programs $a \odot b$ from two programs a and b , rises to an operation on types: $A \odot B$ is defined as the set of $a \odot b$ for all a in A and all b in B . The set of bad programs $\perp$ defines a notion of *negation*: a program a' has type $A^\perp$ if and only if for all program a of type A one has $a :: a' \in \perp$. As a consequence, the types and connectives are only descriptions of the structure of the set of programs considered. Let us notice that work in this direction have been directed at obtaining models of linear logic and therefore the notion of types is restricted to biorthogonally closed sets because linear negation is involutive; this is however a choice of design and not a requirement. For the same reasons, the connectives defined and studied in these models are those of linear logic, although many others may be considered.

1.2. Geometry of Interaction and Maximal Abelian Sub-Algebras.

A major result in the geometry of interaction program was obtained by Girard about ten years ago. In previous work, he had described a way of representing cut-elimination by the so-called *execution formula* $\text{Ex}(A, B)$ between two operators A, B . This formula is actually an explicit solution to a functional equation involving A and B , the *feedback equation*, but this explicit solution is defined as an infinite series whose convergence can be insured only when the product AB is nilpotent. Using techniques of operator algebras, Girard

¹ Obviously, one needs this setting to satisfy additional properties in order to interpret full linear logic. We describe here the minimal requirements for obtaining a GoI model, which could be a model of multiplicative linear logic only.

showed that this explicit solution admits a “continuation”, i.e. can be extended so as to be defined on the whole unit ball of a von Neumann algebra. More precisely, the solution $\text{Ex}(A, B)$ exists and is unique even when the product AB is not nilpotent when A, B are operators of norm at most 1; moreover, $\text{Ex}(A, B)$ is an operator of norm at most 1 in the von Neumann algebra generated by A, B . This implies that for every von Neumann algebra $\mathfrak{M}$, there exists an operation $\text{Ex}(\cdot, \cdot)$ which extends the execution formula on the unit ball $\mathfrak{M}_1$ of $\mathfrak{M}$. This means that the couple $(\mathfrak{M}_1, \text{Ex}(\cdot, \cdot))$ fulfills the first part of the GoI program as it yields the set $\mathcal{O} = \mathfrak{M}_1$ and an associative operation $\text{Ex}(\cdot, \cdot) : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$. The first constructions proposed by Girard were therefore based on a smaller set of operators, a subset of the unit ball of $\mathcal{L}(\mathbb{H})$, the algebra of all continuous linear maps on a Hilbert space $\mathbb{H}$. The “pole” $\perp$ was then defined as the set of nilpotent operators, i.e. Girard defined two operators to be *orthogonal* if and only if their product is nilpotent. After he worked out the general solution to the feedback equation, Girard defined a geometry of interaction in the hyperfinite factor of type II_1 — the hyper finite GoI model, where the pole $\perp$ is chosen based on the determinant: two operators A, B are orthogonal when $\det(1 - AB) \neq 0, 1$.

This work stems from an attempt to obtain soundness results for the hyperfinite GoI model. As one can define in this model an exponential connective satisfying the “functorial promotion rule”, one would expect a soundness result for at least *Elementary Linear Logic*, a exponentially-constrained fragment of linear logic which characterizes elementary time functions. It turns out however that the interpretation of proofs depends on the choice of a *Maximal Abelian Sub-Algebra (MASA)* $\mathfrak{A}$ of the algebra $\mathfrak{M}$, and that one can interpret more or less expressive fragments of linear logic according to the “size” of the algebra generated by the normalizer of $\mathfrak{A}$: if this algebra is $\mathfrak{A}$ itself (the minimal case) then no non-trivial interpretation of proofs exists, if this algebra is $\mathfrak{M}$ (the maximal case) we can interpret soundly elementary linear logic. Although it might seem at first glance a specific feature of the hyperfinite GoI model, it turns out a choice of MASA for the interpretation of proofs was already done in earlier GoI models (Gir89a; Gir88; Gir95a). However, this choice was done implicitly in the definition of the model, and it did not affect the expressivity of the interpretation since the algebra generated by the normalizer of MASA in a type *I* algebra $\mathfrak{M}$ is always equal to $\mathfrak{M}$. The passage from a type *I* algebra to a type *II* algebra thus only put into light the crucial rôle played by the MASAs, thanks to the rich variety of such sub-algebras in type *II* factors.

1.3. Outline of the Paper

We will first give an overview of some of the theory of von Neumann algebras. Although we do not expect the reader to learn and feel familiar with the theory from reading this short section, we hope it will give him a broad idea of the richness of the subject and a few intuitions about the objects it studies. This also gives us the opportunity to define notions and state results that will be used in the following sections.

In a second section, we offer a historical overview of the various GoI models defined by Girard. This overview gives us the opportunity to give a homogeneous point of view on those since we define all the models using operator theory. It is moreover the occasion to pinpoint the implicit choice of a MASA which was made in earlier constructions; this

choice which was unimportant in these constructions but will be crucial in the hyperfinite GoI model has seemingly never been noticed before.

We then motivate the notion of *subjective truth* which appears in Girard's hyperfinite GoI. We try to explain why it is necessary to have a notion of truth that depends on the choice of a MASA. We then define a variant of Girard's hyperfinite GoI model, which we call the *matricidal GoI model*. This variant of Girard's model makes a more explicit connection with MASAs and will be used to prove the main theorem of the paper. We show how a satisfying notion of truth depending on a MASA can be defined for this model and relate it with the notion of truth defined by Girard in his hyperfinite GoI model.

The last section is then concerned with the proof of the main theorem of the paper. We show that the expressivity of the fragment of linear logic one can soundly interpret is in direct correlation with the classification of the MASA proposed by Dixmier (Dix54). This shows, in particular, that non-trivial interpretation exists if the MASA is *singular*, and that any exponential connective can be interpreted if the MASA is *regular* — therefore one can soundly interpret elementary linear logic. This section shows also that in the intermediate case — that of *semi-regular* MASAs — one can at least interpret multiplicative-additive linear logic but no general statement can be made concerning the interpretation of exponential connectives.

2. von Neumann Algebras and MASAs

2.1. First Definitions and Results

The theory of von Neumann algebras, under the name of “rings of operators”, was first developed by Murray and von Neumann in a series of seminal papers (vN30; MvN36; MvN37; vN38; vN40; MvN43; vN43; vN49).

2.1.1. The double commutant theorem. A normed $*$ -algebra is a normed algebra endowed with an antilinear isometric involution $(\cdot)^*$ which reverses the product:

$$(t^*)^* = t \quad \|t^*\| = \|t\| \quad (t+u)^* = t^* + u^* \quad (\lambda t)^* = \bar{\lambda} t^* \quad (tu)^* = u^* t^*$$

A normed $*$ -algebra is a C^* -algebra when it moreover satisfies the C^* -identity $\|t^*t\| = \|t\|^2$.

We denote by $\mathcal{L}(\mathbb{H})$ the $*$ -algebra of continuous (or equivalently, bounded) linear maps from the Hilbert space $\mathbb{H}$ to itself. This algebra can be endowed with the three following topologies:

- The norm topology, for which a net (T_λ) converges toward 0 when the net $\|T_\lambda\|$ converges to 0 in $\mathbb{C}$;
- The strong operator topology (SOT) which is the topology of pointwise convergence when $\mathbb{H}$ is considered endowed with its norm topology: a net (T_λ) converges toward 0 when for all $\zeta \in \mathbb{H}$, the net $(\|T_\lambda \zeta\|)$ converges towards 0 in $\mathbb{C}$;
- The weak operator topology (WOT) which is the topology of pointwise convergence when $\mathbb{H}$ is considered endowed with its weak topology: a net (T_λ) converges toward 0 when for all $\zeta, \eta \in \mathbb{H}$, the net $(\langle T_\lambda \zeta, \eta \rangle)$ converges towards 0 in $\mathbb{C}$;

Definition 1 (von Neumann algebra). A von Neumann algebra is a $*$ -sub-algebra of $\mathcal{L}(\mathbb{H})$ which is closed for the strong operator topology (SOT).

We now explain Murray and von Neumann’s fundamental “double commutant theorem”. Pick $M \subset \mathcal{L}(\mathbb{H})$. We define the commutant of M (in $\mathcal{L}(\mathbb{H})$) as the set $M' = \{x \in \mathcal{L}(\mathbb{H}) \mid \forall m \in M, mx = xm\}$. We will in general omit to precise the ambient algebra and denote abusively M' the commutant of M if the context is sufficiently clear. We will denote by M'' the bi-commutant $(M')'$ of M .

The following theorem is the keystone of the von Neumann algebras theory. It is particularly elegant, since it shows an equivalence between a purely algebraic notion — being equal to its bi-commutant — and a purely topological notion — being closed for the strong operator topology.

Theorem 2 (Double Commutant Theorem, von Neumann (vN30)). *Let M be a $*$ -sub-algebra of $\mathcal{L}(\mathbb{H})$ such that $1_{\mathcal{L}(\mathbb{H})} \in M$. Then M is a von Neumann algebra if and only if $M = M''$.*

Remark 3. Since the strong operator topology (SOT) is weaker than the norm topology, a von Neumann algebra is also closed for the norm topology, and is also a C^* -algebra. Moreover, since M , as a von Neumann algebra, is the commutant of a set of operators, it necessarily contains the identity operator in $\mathcal{L}(\mathbb{H})$, and consequently is a unital C^* -algebra. One can therefore define the continuous spectral calculus for operators in M .

2.1.2. Direct Integrals. Let $\mathfrak{M}$ be a von Neumann algebra. We define the *center* of $\mathfrak{M}$ as the von Neumann algebra $\mathfrak{Z}(\mathfrak{M}) = \mathfrak{M} \cap \mathfrak{M}'$.

Definition 4 (Factor). A *factor* is a von Neumann algebra $\mathfrak{M}$ whose center is trivial, i.e. such that $\mathfrak{Z}(\mathfrak{M}) = \mathbb{C} \cdot 1_{\mathcal{L}(\mathbb{H})}$.

The study of von Neumann algebras can be reduced to the study of factors. This is one of the most important results of the theory, which is due to von Neumann (vN49): he showed that every von Neumann algebra can be written as a *direct integral* of factors. A direct integral is a direct sum over a continuous index set, in the same way an integral is a sum over a continuous index set. A complete exposition of this result in the first book of the Takesaki series (Tak01), Section IV.8, page 264.

Here are the main ideas. If $\mathfrak{A}$ is not a factor, its center $\mathfrak{Z}(\mathfrak{A})$ is a non-trivial commutative von Neumann algebra (i.e. different from $\mathbb{C}$). Suppose now that $\mathfrak{Z}(\mathfrak{A})$ is a *diagonal* algebra, i.e. that there exists a countable set I (which could be finite) and a family $(p_i)_{i \in I}$ of pairwise disjoint minimal projections such that $\sum_{i \in I} p_i = 1$. Then the algebras $p_i \mathfrak{A}$ are factors, and one has $\mathfrak{A} = \bigoplus_{i \in I} p_i \mathfrak{A}$. However, in the general case, the center $\mathfrak{Z}(\mathfrak{A})$ does not need to be a diagonal algebra, and it can contain a *diffuse* sub-algebra, i.e. a sub-algebra that does not have minimal projections. Then it is necessary to consider a continuous version of the direct sum: the direct integral.

Definition 5. Let $(X, \mathcal{B}, \mu)$ be a measured space. A family $(\mathbb{H}_x)_{x \in X}$ of Hilbert spaces is *measurable* over $(X, \mathcal{B}, \mu)$ when there exists a countable partition $(X_i)_{i \in I}$ of X such that for all $i \in I$:

$$\exists \mathbb{K}, \forall x \in X_i, \mathbb{H}_x = \mathbb{K}$$

where $\mathbb{K}$ is either equal to $\mathbf{C}^n$ ($n \in \mathbf{N}$) or equal to $\ell^2(\mathbf{N})$.

A section $(\xi_x)_{x \in X}$ ($\xi_x \in \mathbb{H}_x$) is measurable when its restriction to each element X_n of the partition is measurable.

Definition 6. Let $(\mathbb{H}_x)_{x \in X}$ be a measurable family of Hilbert spaces over a measured space $(X, \mathcal{B}, \lambda)$. The *direct integral* $\int_X^\oplus \mathbb{H}_x d\lambda(x)$ is the Hilbert space whose elements are equivalence classes of measurable sections modulo almost everywhere equality, and the scalar product is defined by:

$$\langle (\xi_x)_{x \in X}, (\zeta_x)_{x \in X} \rangle = \int_X^\oplus \langle \xi_x, \zeta_x \rangle d\lambda(x)$$

In the same way commutative C^* -algebras are exactly the algebras of continuous functions from locally compact Hausdorff spaces to $\mathbf{C}$ (this is Gelfand's theorem, (Gel41)), one can show that every commutative von Neumann algebra can be identified with the algebra $L^\infty(X, \mathcal{B}, \lambda)$ of essentially bounded measurable functions on a measured space $(X, \mathcal{B}, \lambda)$.

Theorem 7. Let $\mathfrak{A}$ be a commutative von Neumann algebra. There exists a measurable family of Hilbert spaces $(\mathbb{H}_x)_{x \in X}$ over a measured space $(X, \mathcal{B}, \lambda)$ such that $\mathfrak{A}$ is unitarily equivalent to the algebra $L^\infty(X)$ acting on the Hilbert space $\int_X^\oplus \mathbb{H}_x d\lambda(x)$.

We will not define here neither the notion of measurable family of von Neumann algebras, nor the one of direct integrals of von Neumann algebras. We only state the fundamental theorem mentioned above.

Theorem 8 (von Neumann (vN49), Takesaki (Tak03a), Theorem IV.8.21 page 275). *Every von Neumann algebra can be written as a direct integral of factors.*

2.1.3. Classification of factors. The study of factors led to a classification based on the study of the set of projections and their isomorphisms (partial isometries). We recall that a projection is an operator p such that $p = p^* = p^2$ (this is sometimes referred to as “orthogonal projections”). Since M is a sub-algebra of $\mathcal{L}(\mathbb{H})$ for a given Hilbert space $\mathbb{H}$, the projections in $\Pi(\mathfrak{M})$ are in particular projections in $\mathcal{L}(\mathbb{H})$. As such, they are in correspondence with subspaces of $\mathbb{H}$: the projection p corresponds to the closed subspace $p\mathbb{H}$. If $\mathfrak{M}$ is a von Neumann algebra, we will denote by $\Pi(\mathfrak{M})$ the set of projections in $\mathfrak{M}$. Two projections p, q are *disjoint* when $pq = 0$, translating the fact that the two corresponding closed subspaces $p\mathbb{H}$ and $q\mathbb{H}$ are disjoint. Moreover, the set $\Pi(\mathfrak{M})$ is endowed with a partial ordering inherited from the inclusion of subspaces: $p \leq q$ if and only if $pq = p$ if and only if $p\mathbb{H} \subset q\mathbb{H}$.

Now, the idea of Murray and von Neumann (MvN36) was to consider an equivalence relation on the set of projections. This equivalence relation depends on the algebra $\mathfrak{M}$ and translates the fact that $\mathfrak{M}$ contains an isomorphism between the corresponding subspaces. Namely, they define the equivalence as follows: two projections p, q are *Murray von Neumann equivalent* in $\mathfrak{M}$, noted $p \sim_{\mathfrak{M}} q$, when there exists an element $u \in \mathfrak{M}$ such that $uu^* = p$ and $u^*u = q$. Notice that this implies that u is a partial isometry.

The partial ordering $\leq$ then induces a partial ordering $\preceq_{\mathfrak{M}}$ on the equivalence classes of projections in $\mathfrak{M}$, i.e. on the set $\Pi(\mathfrak{M})/\sim_{\mathfrak{M}}$.

Remark 9. As we explained above, $p \leq q$ means that $p\mathbb{H}$ is a closed subspace of $q\mathbb{H}$. The fact that $p \sim_{\mathfrak{M}} q$ translates the fact that $p\mathbb{H}$ and $q\mathbb{H}$ are *inner (w.r.t $\mathfrak{M}$) isomorphic*, i.e. there exists an isomorphism between them which is an element of $\mathfrak{M}$, or in other terms, the fact that they are isomorphic is witnessed by an element of $\mathfrak{M}$. Consequently, the fact that $p \lesssim_{\mathfrak{M}} q$ translates the idea that $p\mathbb{H}$ is *inner isomorphic* to a closed subspace of $q\mathbb{H}$, and therefore that $p\mathbb{H}$ is somehow *smaller* than $q\mathbb{H}$ in the sense that an element of $\mathfrak{M}$ witnesses the fact that it is smaller.

Definition 10. A projection p in a von Neumann algebra $\mathfrak{M}$ is *infinite* (in $\mathfrak{M}$) when there exists $q < p$ (i.e. a proper sub-projection) such that $q \sim_{\mathfrak{M}} p$. A projection is *finite* (in $\mathfrak{M}$) when it is not infinite (in $\mathfrak{M}$).

Proposition 11 (Takesaki (Tak01), Proposition V.1.3 page 291 and Theorem V.1.8 page 293). *Let $\mathfrak{M}$ be a von Neumann algebra. Then $\mathfrak{M}$ is a factor if and only if the relation $\lesssim_{\mathfrak{M}}$ is a total ordering.*

To state the following theorem, we will use a slight variant of the usual notion of order type: we distinguish the element denoted by ∞ from any other element, considering that ∞ represents a class of infinite projections. For instance, $\{0, 1\}$ and $\{0, \infty\}$ should be considered as distinct since the first does not contain infinite elements contrarily to the second.

Proposition 12 (Type of a Factor). *Let $\mathfrak{M}$ be a factor. We will say that:*

- $\mathfrak{M}$ is of type I_n when $\lesssim_{\mathfrak{M}}$ has the same order type as $\{0, 1, \dots, n\}$;
- $\mathfrak{M}$ is of type I_{∞} when $\lesssim_{\mathfrak{M}}$ has the same order type as $\mathbf{N} \cup \{\infty\}$;
- $\mathfrak{M}$ is of type II_1 when $\lesssim_{\mathfrak{M}}$ has the same order type as $[0, 1]$;
- $\mathfrak{M}$ is of type II_{∞} when $\lesssim_{\mathfrak{M}}$ has the same order type as $\mathbf{R}_{\geq 0} \cup \{\infty\}$;
- $\mathfrak{M}$ is of type III when $\lesssim_{\mathfrak{M}}$ has the same order type as $\{0, \infty\}$, i.e. all non-zero projections are infinite.

Moreover, $\lesssim_{\mathfrak{M}}$ cannot be of another order type as the ones listed above.

Proof. We refer once again to the first volume of Takesaki's series (Tak01), Corollary V.1.20 page 297. $\square$

We can show that a factor of type I_n is isomorphic to $\mathfrak{M}_n(\mathbf{C})$, the algebra of square matrices of size $n \times n$ with complex coefficients. A factor of type I_{∞} is isomorphic to $\mathcal{L}(\mathbb{H})$, where $\mathbb{H}$ is an infinite-dimensional Hilbert space.

Definition 13. A *trace* τ on a von Neumann algebra $\mathfrak{M}$ is a function from $\mathfrak{M}^+$ into $[0, \infty]$ satisfying:

- 1 $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathfrak{M}^+$;
- 2 $\tau(\lambda x) = \lambda \tau(x)$ for all $x \in \mathfrak{M}^+$ and all $\lambda \geq 0$;
- 3 $\tau(x^*x) = \tau(xx^*)$ for all $x \in \mathfrak{M}$.

We will moreover say that τ is:

- *faithful* if $\tau(x) > 0$ for all $x \neq 0$ in $\mathfrak{M}^+$;
- *finite* when $\tau(1) < \infty$;
- *semi-finite* when for all element x in $\mathfrak{M}^+$ there exists $y \in \mathfrak{M}^+$ such that $x - y \in \mathfrak{M}^+$ and $\tau(y) < \infty$;
- *normal* when $\tau(\sup\{x_i\}) = \sup\{\tau(x_i)\}$ for all increasing bounded net $\{x_i\}$ in $\mathfrak{M}^+$.

Theorem 14 (Takesaki (Tak01), Theorem V.2.6 page 312). *If $\mathfrak{M}$ is a finite factor (i.e. the identity is a finite projection), then there exists a faithful normal finite trace τ . Moreover, every other faithful normal finite trace ρ is proportional to τ .*

If $\mathfrak{M}$ is of type II_1 , we will refer to the unique faithful normal finite trace tr such that $\text{tr}(1) = 1$ as the normalized trace.

Remark 15. Since the set of positive operators in $\mathfrak{M}$ generates the von Neumann algebra $\mathfrak{M}$, a finite trace τ extends uniquely to a positive linear form on $\mathfrak{M}$ that we will abusively write τ as well. In particular, every operator a in a type II_1 factor has a finite trace.

In order to define the notion of hyperfiniteness we need to define yet another topology on $\mathcal{L}(\mathbb{H})$, the so-called σ -weak topology. This definition is based upon the notion of weak* topology: if X is a space and X^* is its dual, then the weak* topology on X^* is defined as the topology of pointwise convergence on X . To define the σ -weak topology on $\mathcal{L}(\mathbb{H})$ as a weak* topology, we moreover need to see $\mathcal{L}(\mathbb{H})$ as the dual space of some other space. This is a well-known result which can be found in standard textbooks: the algebra $\mathcal{L}(\mathbb{H})$ is the dual of the space of *trace-class operators* that we will denote $\mathcal{L}(\mathbb{H})_*$ and which is itself the dual space of the algebra of compact operators.

Definition 16. Let $\mathbb{H}$ be a Hilbert space. The σ -weak topology on $\mathcal{L}(\mathbb{H})$ is defined as the weak* topology induced by the predual $\mathcal{L}(\mathbb{H})_*$ of $\mathcal{L}(\mathbb{H})$.

Remark 17. If $\mathbb{H}$ is a infinite-dimensional separable Hilbert space, $\mathcal{L}(\mathbb{H})$ embeds into $\mathcal{L}(\mathbb{H} \otimes \mathbb{H})$ through the morphism $x \mapsto x \otimes 1$. One can show that the restriction of the weak operator topology (WOT) on $\mathcal{L}(\mathbb{H} \otimes \mathbb{H})$ coincides with the σ -weak topology on $\mathcal{L}(\mathbb{H})$.

Definition 18. A von Neumann algebra $\mathfrak{M}$ is *hyperfinit* if there exists a directed family $\mathfrak{M}_i$ of finite-dimensional *-sub-algebras of $\mathfrak{M}$ such that $\cup_i \mathfrak{M}_i$ is dense in $\mathfrak{M}$ for the σ -weak topology.

Theorem 19 (Takesaki (Tak03b), Theorem XIV.2.4 page 97). *Two hyperfinite type II_1 factors are isomorphic. We will write $\mathfrak{R}$ the unique hyperfinite type II_1 factor.*

Theorem 20 (Takesaki (Tak03b), Theorem XVI.1.22 page 236). *Two hyperfinite type II_∞ factors are isomorphic. In particular, they are isomorphic to the tensor product $\mathfrak{R}_{0,1} = \mathcal{L}(\mathbb{H}) \otimes \mathfrak{R}$.*

2.1.4. Sakai's Theorem and W^* -algebras. We have defined above the von Neumann algebras as sub-algebras of $\mathcal{L}(\mathbb{H})$ where $\mathbb{H}$ is a separable Hilbert space. We therefore defined a von Neumann algebra as a “concrete” algebra, i.e. as a set of operators acting on a given space. As it is the case with C^* -algebras, which can be defined either concretely as a norm-closed sub-algebra of $\mathcal{L}(\mathbb{H})$ or abstractly as an involutive Banach algebra satisfying the C^* -identity, there exists an abstract definition of von Neumann algebras. This important result is due to Sakai.

Definition 21. Let $\mathfrak{M}$ be a von Neumann algebra. The *pre-dual* $\mathfrak{M}_*$ of $\mathfrak{M}$ is the set of linear forms² which are continuous for the σ -weak topology (Definition 16).

² We recall that a *linear form on a vector space* V is a linear map from V into $\mathbb{C}$, i.e. an element of the dual of V . When V is a topological vector space, the elements of the topological dual of V are therefore the continuous linear forms.

Proposition 22 (Takesaki (Tak01), Theorem II.2.6, page 70). *Let $\mathfrak{M}$ be a von Neumann algebra. There exists an isometric isomorphism between $\mathfrak{M}$ and $(\mathfrak{M}_*)^*$ — the dual (as a Banach space) of the pre-dual of $\mathfrak{M}$.*

The reciprocal statement was proved by Sakai (Sak71) and gives an exact characterization of von Neumann algebras among C^* -algebras. A proof can be found in Takesaki (Tak01), Theorem 3.5, page 133, and Corollary 3.9, page 135.

Theorem 23. *A C^* -algebra $\mathfrak{A}$ is a von Neumann algebra if and only if there exists a Banach algebra B such that $\mathfrak{A}$ is the dual of B : $\mathfrak{A} = B^*$. The algebra B is moreover unique.*

One can then define von Neumann algebras *abstractly*, i.e. as an abstract algebra *vs* as an algebra of operators acting on a specific space. Such abstract algebras can then be *represented* as algebras of operators.

Definition 24. A *representation* of a von Neumann algebra $\mathfrak{M}$ is a couple $(\mathbb{H}, \pi)$ where $\pi : \mathfrak{M} \rightarrow \mathcal{L}(\mathbb{H})$ is a C^* -algebra homomorphism. If the homomorphism π is injective, we say the associated representation is *faithful*.

2.1.5. The Standard Representation. One of the major results in the theory of von Neumann algebras is that every such algebra has a “standard representation”, i.e. a representation that satisfies a number of important properties. Namely, once realized that von Neumann algebras can be defined in an abstract way, the next step is to identify them with particularly satisfying concrete algebras. A proof of the following result can be found in Takesaki (Tak03a), Section IX.1, page 142.

Theorem 25 (Haagerup (Haa75)). *Let $\mathfrak{M}$ be a von Neumann algebra. Then there exists a Hilbert space $\mathbb{H}$, a von Neumann algebra $\mathfrak{S} \subset \mathcal{L}(\mathbb{H})$, an isometric antilinear involution $J : \mathbb{H} \rightarrow \mathbb{H}$ and a cone $\mathfrak{P}$ closed under $(\cdot)^*$ such that:*

- $\mathfrak{M}$ and $\mathfrak{S}$ are isomorphic;
- $J\mathfrak{M}J = \mathfrak{M}'$;
- $JaJ = a^*$ for all $a \in \mathfrak{Z}(\mathfrak{M})$;
- $Ja = a$ for all $a \in \mathfrak{P}$;
- $aJaJ\mathfrak{P} = \mathfrak{P}$ for all $a \in \mathfrak{M}$.

The tuple $(\mathfrak{S}, \mathbb{H}, J, \mathfrak{P})$ is called the standard form of the algebra $\mathfrak{M}$.

Let us work out the case of a von Neumann algebra $\mathfrak{M}$ endowed with a faithful normal semi-finite trace. In this case, we can describe a quite easy construction of the standard form of $\mathfrak{M}$. We first define the ideal $\mathfrak{n}_\tau = \{x \in \mathfrak{M} \mid \tau(x^*x) < \infty\}$ (notice that in the case of a finite algebra $\mathfrak{n}_\tau = \mathfrak{M}$). We then consider the linear form $(\cdot, \cdot)$ on $\mathfrak{M}$ defined by:

$$(x, y) = \tau(y^*x)$$

From the linearity of the trace and the anti-linearity of the involution, we can show that it is a sesquilinear form. Moreover, since x^*x is a positive operator, we know that $\tau(x^*x) \geq 0$. Therefore, this defined a scalar product on $\mathfrak{n}_\tau$, and we can now define the Hilbert space $L^2(\mathfrak{M}, \tau)$ as the completion of $\mathfrak{n}_\tau$ ($\mathfrak{M}$ when the algebra is finite) for the norm defined by $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$.

One can then show that for every element $a \in \mathfrak{M}$ and every $x \in \mathfrak{n}_\tau$,

$$\begin{aligned}\|ax\|_2 &\leq \|a\|\|x\|_2 \\ \|xa\|_2 &\leq \|a\|\|x\|_2\end{aligned}$$

We then denote by π_τ (resp. π'_τ) the representation of $\mathfrak{M}$ onto $L^2(\mathfrak{M}, \tau)$ by left (resp. right) multiplication.

We then notice that the operation $(\cdot)^*$ defines an isometry on $\mathfrak{n}_\tau$ for the norm $\|\cdot\|_2$. It thus extends to an antilinear involution $J : L^2(\mathfrak{M}, \tau) \rightarrow L^2(\mathfrak{M}, \tau)$. One then shows that:

- π_τ (resp. π'_τ) is a faithful representation (resp. antirepresentation³);
- $\pi_\tau(\mathfrak{M})' = \pi'_\tau(\mathfrak{M})$ and $\pi'_\tau(\mathfrak{M})' = \pi_\tau(\mathfrak{M})$;
- $J\pi_\tau(a)J = \pi'_\tau(a^*)$ for all $a \in \mathfrak{M}$.

2.2. Maximal Abelian Sub-Algebras

The purpose of this paper is to exhibit a remarkable correspondence between a classification of maximal abelian sub-algebras and fragments of linear logic. We will therefore need a number of definitions and results about such sub-algebras. The purpose of this section is to provide those together with a number of intuitions that should help the reader grasps some subtleties of the theory. After defining what exactly is a maximal abelian sub-algebra, we will start by explaining the classification of such in type I factors, the simpler case. We will then go on with the case of type II algebras which is more involved.

2.2.1. MASAs in type I factors.

Definition 26. Let $\mathfrak{M}$ be a von Neumann algebra. A *maximal abelian sub-algebra* (MASA) $\mathfrak{A}$ of $\mathfrak{M}$ is a von Neumann sub-algebra of $\mathfrak{M}$ such that for all intermediate sub-algebra $\mathfrak{B}$, i.e. $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{M}$, if $\mathfrak{B}$ is abelian then $\mathfrak{A} = \mathfrak{B}$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are MASAs in a von Neumann algebra $\mathfrak{M}$, they can be "isomorphic" in three different ways:

- they can be isomorphic as von Neumann algebras — this is the weakest notion;
- there can exist an automorphism Φ of $\mathfrak{M}$ such that $\Phi(\mathfrak{A}) = \mathfrak{B}$; we then say that $\mathfrak{A}$ and $\mathfrak{B}$ are conjugated;
- there can exist a unitary operator⁴ $u \in \mathfrak{M}$ such that $u\mathfrak{A}u^* = \mathfrak{B}$ — this is the strongest notion; we then say that $\mathfrak{A}$ and $\mathfrak{B}$ are unitarily equivalent.

Let us quickly discuss the finite-dimensional case. We fix $\mathbb{H}$ a finite-dimensional Hilbert space of dimension $k \in \mathbb{N}$. Then $\mathcal{L}(\mathbb{H})$ is isomorphic to the algebra of $k \times k$ matrices. Picking a basis $\mathcal{B} = (b_1, \dots, b_k)$ of $\mathbb{H}$, one can define the sub-algebra $\mathfrak{D}_{\mathcal{B}}$ of $\mathcal{L}(\mathbb{H})$ containing all diagonal matrices in the basis $\mathcal{B}$. This algebra is obviously commutative, and it is moreover maximal as a commutative sub-algebra of $\mathcal{L}(\mathbb{H})$: if $\mathfrak{A}$ is a commutative sub-algebra of $\mathcal{L}(\mathbb{H})$ containing $\mathfrak{D}_{\mathcal{B}}$, then $\mathfrak{A} = \mathfrak{D}_{\mathcal{B}}$. A more involved argument shows that any maximal

³ An antirepresentation is a representation that inverses multiplication: $\pi'_\tau(xy) = \pi'_\tau(y)\pi'_\tau(x)$.

⁴ We recall that a unitary operator is an operator u such that $uu^* = u^*u = 1$.

abelian sub-algebra of $\mathcal{L}(\mathbb{H})$ is the diagonal algebra induced by a basis; this result is also a direct corollary of Proposition 27. These algebras $\mathcal{D}_{\mathcal{B}}$ where $\mathcal{B}$ is a basis of $\mathbb{H}$ are clearly pairwise isomorphic, as it suffices to map bijectively the bases one onto the other. They are in fact unitarily equivalent, as such a bijection induces a unitary operator. This shows that the distinctions we just made are useless in the finite-dimensional case: all MASAs are unitarily equivalent.

We will now state a classification result about maximal abelian sub-algebras of $\mathcal{L}(\mathbb{H})$, which gives a complete answer to the classification problem of MASAs in type I factors. This theorem will be preceded by a proposition showing that all *diffuse* MASAs in $\mathcal{L}(\mathbb{H})$ are unitarily equivalent; this will be of use later on, as those MASAs of a type II factor $\mathfrak{M} \subset \mathcal{L}(\mathbb{H})$ which are also MASAs of $\mathcal{L}(\mathbb{H})$ are necessarily diffuse.

Proposition 27 (Sinclair and Smith (SS08)). *Let $\mathfrak{A}$ be a MASA of $\mathcal{L}(\mathbb{H})$ which do not have (non-zero) minimal projections — we say in this case that $\mathfrak{A}$ is a diffuse MASA. Then there exists a unitary $U : \mathbb{H} \rightarrow L^2([0, 1])$ such that $U\mathfrak{A}U^* = L^\infty([0, 1])$.*

Theorem 28 (Sinclair and Smith (SS08)). *Let $\mathfrak{A}$ be a MASA in $\mathcal{L}(\mathbb{H})$. Then:*

- *either $\mathfrak{A}$ is unitarily equivalent to $L^\infty([0, 1])$ (diffuse case);*
- *either $\mathfrak{A}$ is unitarily equivalent to $\mathcal{D}$, a diagonal algebra (discrete case);*
- *either $\mathfrak{A}$ is unitarily equivalent to $\mathcal{D} \oplus L^\infty([0, 1])$, where $\mathcal{D}$ is a diagonal algebra (mixed case);*

Things are therefore clear concerning the MASAs in $\mathcal{L}(\mathbb{H})$, as the previous theorem provides a complete classification of those. In the case of von Neumann algebras of type II_1 however, things are more complicated and such a complete classification does not exist in spite of the numerous works on the subject.

2.2.2. Dixmier's Classification. We now begin the discussion about MASAs of type II_1 von Neumann algebras by explaining Dixmier's classification (Dix54), which considers the algebra generated by the *normalizer* of the MASA. Let us stress that this classification is not exhaustive. This presentation of Dixmier's classification will also give us the opportunity to state some results that will be of use in the rest of the paper.

Definition 29 (Normalizer). Let $\mathfrak{M}$ be a von Neumann algebra, and $\mathfrak{A}$ a von Neumann sub-algebra of $\mathfrak{M}$. We will denote by $N_{\mathfrak{M}}(\mathfrak{A})$ the *normalizer* of $\mathfrak{A}$ in $\mathfrak{M}$ which is defined as:

$$N_{\mathfrak{M}}(\mathfrak{A}) = \{u \in \mathfrak{M} \mid u \text{ unitaire, } u\mathfrak{A}u^* = \mathfrak{A}\}$$

We will denote by $\mathcal{N}_{\mathfrak{M}}(\mathfrak{A})$ the von Neumann algebra generated by $N_{\mathfrak{M}}(\mathfrak{A})$.

Definition 30 (Normalizing Groupoid). Let $\mathfrak{M}$ be a von Neumann algebra and $\mathfrak{A}$ be a von Neumann sub-algebra of $\mathfrak{M}$. We will denote by $G_{\mathfrak{M}}(\mathfrak{A})$ the *normalizing groupoid* of $\mathfrak{A}$ in $\mathfrak{M}$ which is defined as:

$$G_{\mathfrak{M}}(\mathfrak{A}) = \{u \in \mathfrak{M} \mid uu^*u = u, uu^* \in \mathfrak{A}, u^*u \in \mathfrak{A}, u\mathfrak{A}u^* \subset \mathfrak{A}\}$$

We will denote by $\mathcal{G}_{\mathfrak{M}}(\mathfrak{A})$ the von Neumann algebra generated by $G_{\mathfrak{M}}(\mathfrak{A})$.

Definition 31 (Dixmier Classification). Let $\mathfrak{M}$ be a factor, and $\mathfrak{P}$ a MASA in $\mathfrak{M}$. We distinguish three cases:

- 1 if $\mathcal{N}_{\mathfrak{M}}(\mathfrak{P}) = \mathfrak{M}$, we say that $\mathfrak{P}$ is *regular* (or *Cartan*);
- 2 if $\mathcal{N}_{\mathfrak{M}}(\mathfrak{P}) = \mathfrak{K}$, where $\mathfrak{K}$ is a factor distinct from $\mathfrak{M}$, we say that $\mathfrak{P}$ is *semi-regular*;
- 3 if $\mathcal{N}_{\mathfrak{M}}(\mathfrak{P}) = \mathfrak{P}$, we say that $\mathfrak{P}$ is *singular*.

The following three results can be found in the literature. The first two propositions can be found along with their proofs in Sinclair and Smith book (SS08) about MASAs in finite factors. The third is a quite recent generalization (Chi07) of a result which was previously known to hold for singular MASAs.

Theorem 32 (Dye, (Dye63)). *Let $\mathfrak{M}$ be a von Neumann algebra with a faithful normal trace, and $\mathfrak{A}$ a MASA in $\mathfrak{M}$. Then the set $G_{\mathfrak{M}}(\mathfrak{A})$ is contained in the sub-vector space of $\mathfrak{M}$ generated by $N_{\mathfrak{M}}(\mathfrak{A})$.*

Corollary 33. *Under the hypotheses of the preceding theorem, the von Neumann algebras $\mathcal{N}_{\mathfrak{M}}(\mathfrak{A})$ and $\mathcal{G}_{\mathfrak{M}}(\mathfrak{A})$ are equal.*

Theorem 34 (Jones and Popa, (JP82)). *Let $\mathfrak{M}$ be a type II_1 factor, and $\mathfrak{A}$ a MASA in $\mathfrak{M}$. Let $p, q \in \mathfrak{A}$ be projections of equal trace. Then, if $\mathcal{N}_{\mathfrak{M}}(\mathfrak{A})$ is a factor, there exists a partial isometry $v_0 \in G_{\mathfrak{M}}(\mathfrak{A})$ such that $p = v_0 v_0^*$ and $q = v_0^* v_0$.*

Theorem 35 (Chifan, (Chi07)). *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be type II_1 factors. For $i = 1, 2$, let $\mathfrak{A}_i$ be a MASA in $\mathfrak{M}_i$. Then:*

$$\mathcal{N}_{\mathfrak{M}_1 \otimes \mathfrak{M}_2}(\mathfrak{A}_1 \otimes \mathfrak{A}_2) = \mathcal{N}_{\mathfrak{M}_1}(\mathfrak{A}_1) \otimes \mathcal{N}_{\mathfrak{M}_2}(\mathfrak{A}_2)$$

Theorem 36 (Connes, Feldman and Weiss (CFW81)). *Let $\mathfrak{A}, \mathfrak{B}$ be two regular MASAs of the hyperfinite factor $\mathfrak{R}$ of type II_1 . Then $\mathfrak{A}$ and $\mathfrak{B}$ are unitarily equivalent.*

2.2.3. Pukansky's Invariant. Pukansky (Puk60) defined a numerical invariant for MASAs $\mathfrak{A}$ of a type II_1 factor which is based on the type I decomposition of $(\mathfrak{A} \cup J\mathfrak{A}J)'$ where J is the anti linear isometry $Jx = x^*$ on $L^2(\mathfrak{N})$. This algebra, as the commutant of an abelian algebra is of type I, and therefore can be decomposed as a sum of factors of type I_n (where n might be equal to ∞). The Pukansky invariant is then essentially the set of all value of n that appear in this decomposition.

Let $\mathfrak{A}$ be a MASA in a factor $\mathfrak{N}$ of type II_1 endowed with a faithful normal trace τ , and let J be the anti linear isometry $Jx = x^*$ onto $L^2(\mathfrak{N})$ and ξ the vector of $L^2(\mathfrak{N})$ corresponding to the identity in $\mathfrak{N}$. We define $e_{\mathfrak{A}}$ as the projection of $L^2(\mathfrak{N})$ onto $L^2(\mathfrak{A})$. We will write $\mathfrak{B}_{\mathfrak{A}}$ the commutative algebra generated by $\mathfrak{A} \cup J\mathfrak{A}J$. The following lemma justifies the definition of Pukansky's invariant.

Lemma 37 (Sinclair et Smith (SS08), Chapter 7). *Let $\mathfrak{N}$ be a type II_1 factor represented onto $L^2(\mathfrak{N})$ and $\mathfrak{A}$ a MASA in $\mathfrak{N}$. Then $e_{\mathfrak{A}} \in \mathfrak{B}_{\mathfrak{A}}$ and $e_{\mathfrak{A}}$ is a central projection — i.e. a projection onto the center of the algebra — in $\mathfrak{B}'_{\mathfrak{A}}$.*

Definition 38. Let $\mathfrak{A}$ be a MASA in a factor $\mathfrak{N}$ of type II_1 . We define the *Pukansky invariant* $\text{Puk}(\mathfrak{A}, \mathfrak{N})$ of $\mathfrak{A}$ in $\mathfrak{N}$ — usually denoted by $\text{Puk}(\mathfrak{A})$ when the context is clear — as the set of all natural numbers $n \in \mathbf{N} \cup \{\infty\}$ such that $(1 - e_{\mathfrak{A}})\mathfrak{B}'_{\mathfrak{A}}$ has a non-zero type I_n part.

In ambiguous cases, we will include a reference to $\mathfrak{N}$ in the notation and write $\text{Puk}(\mathfrak{A}, \mathfrak{N})$.

By removing the projection $e_{\mathfrak{A}}$ from $\mathfrak{B}'_{\mathfrak{A}}$, we are erasing the part $\mathfrak{B}'_{\mathfrak{A}} e_{\mathfrak{A}} = \mathfrak{B}_{\mathfrak{A}} e_{\mathfrak{A}}$ which is abelian for all MASA $\mathfrak{A}$. This allows for a better invariant since its inclusion would add

the integer 1 to all Pukansky invariants, rendering impossible the distinction between MASAs of invariant $\{2\}$ and those of invariant $\{1, 2\}$.

The Pukansky invariant satisfies that if $\mathfrak{A}$ and $\mathfrak{B}$ are two unitarily equivalent MASAs in a factor $\mathfrak{M}$ of type II_1 , then $\text{Puk}(\mathfrak{A}) = \text{Puk}(\mathfrak{B})$. However, the reciprocal statement is not true. One can even find four MASAs $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ in the type II_1 hyperfinite factor with equal invariants (all equal to $\{1\}$) where $\mathfrak{A}$ is regular, $\mathfrak{B}$ is semi-regular, $\mathfrak{C}$ is singular, and $\mathfrak{D}$ lies outside of Dixmier's classification. The Pukansky invariant is nonetheless very useful and some results about it will be used in this paper.

The four following theorems can be found in the book by Sinclair and Smith (SS08).

Proposition 39. *Let $\mathfrak{N}$ be a type II_1 factor and $\mathfrak{A}$ be a MASA in $\mathfrak{N}$. If $\mathfrak{A}$ is regular, then $\text{Puk}(\mathfrak{A}) = \{1\}$.*

Proposition 40. *Let $\mathfrak{N}$ be a type II_1 factor and $\mathfrak{A}$ be a MASA in $\mathfrak{N}$. The following statements are equivalent:*

- $\mathfrak{A}$ is a MASA in $\mathcal{L}(L^2(\mathfrak{N}))$;
- $\text{Puk}(\mathfrak{A}) = \{1\}$.

Proposition 41. *Let $\mathfrak{N}$ be a type II_1 factor and $\mathfrak{A}$ be a MASA in $\mathfrak{N}$.*

- *If $\text{Puk}(\mathfrak{A}) \subset \{2, 3, 4, \dots, \infty\}$, then $\mathfrak{A}$ is singular.*
- *If $\mathcal{M}(\mathfrak{A}) \neq \mathfrak{A}$, then $1 \in \text{Puk}(\mathfrak{A})$.*

Proposition 42. *Let $\mathfrak{A}$ (resp. $\mathfrak{B}$) be a MASA in a factor $\mathfrak{M}$ (resp. $\mathfrak{N}$) of type II_1 . Then:*

$$\text{Puk}(\mathfrak{A} \otimes \mathfrak{B}) = \text{Puk}(\mathfrak{A}) \cup \text{Puk}(\mathfrak{B}) \cup \text{Puk}(\mathfrak{A})\text{Puk}(\mathfrak{B})$$

where $\text{Puk}(\mathfrak{A})\text{Puk}(\mathfrak{B}) = \{a \times b \mid a \in \text{Puk}(\mathfrak{A}), b \in \text{Puk}(\mathfrak{B})\}$.

We have stated above that one can find four MASAs $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ of the hyperfinite factor $\mathfrak{R}$ of type II_1 that all have the same Pukansky invariant and such that $\mathfrak{A}$ is regular, $\mathfrak{B}$ is semi-regular, $\mathfrak{C}$ is singular, and $\mathfrak{D}$ lies outside of Dixmier classification (we will say that $\mathfrak{D}$ is *non-Dixmier-classifiable*). The regular MASAs are necessarily of Pukansky invariant $\{1\}$, and one can show that such MASAs exist in the type II_1 hyperfinite factor $\mathfrak{R}$ (for instance by considering a construction of the hyperfinite factor as a crossed product (Definition 94), as explained in Sinclair and Smith book (SS08)). This gives the existence of a regular MASA $\mathfrak{A}$ in $\mathfrak{R}$ of Pukansky invariant $\{1\}$.

On the other hand, Stuart White (Whi06) showed that the so-called *Tauer MASAs* all have as Pukansky invariant the singleton $\{1\}$. And it is known that there exists singular Tauer MASAs (WS07) and semi-regular Tauer MASAs (Whi06) in the hyperfinite factor $\mathfrak{R}$. This gives the existence of a semi-regular MASA $\mathfrak{B}$ and a singular MASA $\mathfrak{C}$ such that $\text{Puk}(\mathfrak{B}) = \text{Puk}(\mathfrak{C}) = \{1\}$.

Lastly, let us show that the existence of singular MASAs with Pukansky invariant equal to $\{1\}$ implies the existence of non-Dixmier-classifiable MASAs whose Pukansky invariant is equal to the singleton $\{1\}$. Indeed, if $\mathfrak{A}$ is a MASA with $\text{Puk}(\mathfrak{A}) = \{1\}$, we can consider $\mathfrak{A} \otimes \mathfrak{Q}$ where $\mathfrak{Q}$ is a regular MASA (thus $\text{Puk}(\mathfrak{Q}) = \{1\}$) of $\mathfrak{R} \otimes \mathfrak{R}$. We then have that $\text{Puk}(\mathfrak{A} \otimes \mathfrak{Q}) = \{1\}$ by Proposition 42, and moreover, by Theorem 35, we have:

$$\mathcal{N}_{\mathfrak{R} \otimes \mathfrak{R}}(\mathfrak{A} \otimes \mathfrak{Q}) = \mathcal{N}_{\mathfrak{R}}(\mathfrak{A}) \otimes \mathcal{N}_{\mathfrak{R}}(\mathfrak{Q}) = \mathfrak{A} \otimes \mathfrak{R}$$

But the center of $\mathfrak{A} \otimes \mathfrak{R}$ is equal to $\mathfrak{A} \otimes \mathbf{C}$ since $\mathfrak{A}$ is commutative and the commutant of a

tensor product is equal to the tensor product of the commutants (a result due to Tomita (Tom67)). Thus $\mathfrak{A} \otimes \mathfrak{R}$ is not a factor, which implies that $\mathfrak{A} \otimes \mathfrak{Q}$ is neither regular nor semi-regular. Since $\mathfrak{A} \otimes \mathfrak{Q}$ is obviously not equal to $\mathfrak{A} \otimes \mathfrak{R}$, we know that $\mathfrak{A} \otimes \mathfrak{Q}$ is not singular: it is therefore non-Dixmier-classifiable. Eventually, as $\mathfrak{R} \otimes \mathfrak{R}$ is isomorphic to $\mathfrak{R}$, it is enough to chose such an isomorphism ϕ to define $\mathfrak{D} = \phi(\mathfrak{A} \otimes \mathfrak{Q})$ a MASA in $\mathfrak{R}$ which is non-Dixmier-classifiable and such that $\text{Puk}(\mathfrak{D}) = \{1\}$.

3. Geometry of Interaction

In this section, we review Girard's GoI models. This section has two distinct aims. The first is to offer a presentation of those constructions where the role of MASAs is shown explicitly. Indeed, MASAs played a role in all of Girard's GoI models, even though they were implicitly used through the choice of a specific basis of the Hilbert space in consideration. The second is to review Girard's GoI model in the hyperfinite factor (Gir11) since it is the starting point of our study.

3.1. First Constructions: Nilpotency

3.1.1. Multiplicative Connectives and Exponentials. The first construction of a GoI model (Gir89a) was already using operator algebras as the notion of partial isometries provides a natural generalization of permutations. Indeed, it is necessary to deal with infinite objects in order to represent exponential connectives, and finite permutations can naturally be replaced by permutations of a basis of an separable infinite-dimensional Hilbert space. This actually consists in working with partial isometries in the normalizing groupoid of a fixed MASA, although this point of view was unknown to Girard at the time. We will therefore present Girard's first GoI model under this novel perspective.

Let us start by choosing a separable infinite-dimensional Hilbert space $\mathbb{H}$, and a MASA $\mathfrak{A}$ in $\mathcal{L}(\mathbb{H})$. We will suppose that $\mathbb{H} = \ell^2(\mathbb{N})$ and that $\mathfrak{A}$ is the MASA of diagonal operators in the basis $(\delta_{i,n})_{n \in \mathbb{N}}$. We can then define operators⁵ $1, r \in G_{\mathcal{L}(\mathbb{H})}(\mathfrak{A})$ such that $rr^* + 11^* = 1$ and $r^*r = 1^*1 = 1$. We will chose here $r((x_n)_{n \in \mathbb{N}}) = (x_{2n})_{n \in \mathbb{N}}$ and $1((x_n)_{n \in \mathbb{N}}) = (x_{2n+1})_{n \in \mathbb{N}}$. If π is a projection in $\mathfrak{A}$ it is immediate that $r\pi r^*$ (respectively $1\pi 1^*$) is a projection in $\mathfrak{A}$, thus $r\mathfrak{A}r^* \subset \mathfrak{A}$ (respectively $1\mathfrak{A}1^* \subset \mathfrak{A}$) since $\mathfrak{A}$ is generated by its projections. Moreover, r and 1 are partial isometries and the projections rr^* and 11^* are in $\mathfrak{A}$. We have thus checked that r and 1 are indeed elements of $G_{\mathcal{L}(\mathbb{H})}(\mathfrak{A})$.

If $u \in \mathcal{L}(\ell^2(\mathbb{N}))$, we will write $r(u)$ (resp. $1(u)$) the operator rur^* (resp. $1u1^*$).

We will restrict in the following to elements in $G(\mathfrak{A})$. We now define a notion of orthogonality based on nilpotency.

Definition 43. Two operators u, v in $G(\mathfrak{A})$ are orthogonal — denoted by $u \perp v$ — when uv is nilpotent, i.e. when there exists an integer n such that $(uv)^n = 0$.

This notion of orthogonality allows one to define types as bi-orthogonally closed sets.

⁵ We recall that $G_{\mathcal{L}(\mathbb{H})}(\mathfrak{A})$ is the normalizaing groupoid of $\mathfrak{A}$ (Definition 30).

Definition 44 (Types). A *type* is a set of elements in $G(\mathfrak{A})$ which is bi-orthogonally closed, i.e. a set $T \subset G(\mathfrak{A})$ such that $T^{\perp\perp} = T$.

The construction of the tensor product is performed using r and l which *internalize* the direct sum of Hilbert spaces. Indeed, the Hilbert space $\mathbb{H}$ satisfies $\mathbb{H} \cong \mathbb{H} \oplus \mathbb{H}$.

Definition 45. If A, B are types, the tensor product of A and B is defined as:

$$A \otimes B = \{r(u) + l(v) \mid u \in A, v \in B\}^{\perp\perp}$$

We will write $u \odot v = r(u) + l(v)$. Using Lemma 107 and Lemma 108 one can show that if u, v are in $G_{\mathcal{L}(\mathbb{H})}(\mathfrak{A})$ then $u \odot v \in G_{\mathcal{L}(\mathbb{H})}(\mathfrak{A})$.

We also define the linear implication. It is defined by the so-called *execution formula*.

Definition 46. Let u, v be operators in $G(\mathfrak{A})$ such that $u \perp r(v)$. The execution of u and $r(v)$, denoted by $u :: r(v)$, is defined as:

$$u :: r(v) = (1 - rr^*)(1 - ur(v))^{-1}(1 - rr^*)$$

Since $u \perp r(v)$, the inverse of $1 - ur(v)$ always exists and can be computed as the series $\sum_{i=0}^{\infty} (ur(v))^i$. One can show that⁶ if u and $r(v)$ are in $G_{\mathcal{L}(\mathbb{H})}(\mathfrak{A})$, then $u :: d(v) \in G_{\mathcal{L}(\mathbb{H})}(\mathfrak{A})$.

One then shows that the following property, called the *adjunction*, holds; it ensures that one can interpret soundly the connectives of linear logic⁷.

Proposition 47 (Adjunction). *If u, v, w are elements of $G(\mathfrak{A})$, then:*

$$f \perp u \odot v \Leftrightarrow (f \perp r(u)) \wedge (f :: r(u) \perp l(v))$$

Theorem 48. *If A, B are types, we define the set $A \multimap B$ as*

$$A \multimap B = \{f \in G(\mathfrak{A}) \mid \forall u \in A, \exists v \in B, f :: d(u) = g(v)\}$$

This set is a type and satisfies the following:

$$A \multimap B = (A \otimes B^{\perp})^{\perp}$$

Proof. Let $f \in A \multimap B$, $u \in A$ and $v' \in B^{\perp}$. Then $f :: r(u) = l(v)$ where v is an element of B , thus $f \perp r(u)$ and $f :: r(u) \perp l(v')$. Thus $f \perp u \odot v'$, and we have $f \in (A \otimes B^{\perp})^{\perp}$.

Conversely, if $f \in (A \otimes B^{\perp})^{\perp}$, then $f \perp u \odot v'$ for all $u \in A, v' \in B^{\perp}$, and therefore $f :: r(u)$ is defined and equal to $l(v)$ for v in B . Thus $f \in A \multimap B$.

We conclude that $A \multimap B = (A \otimes B^{\perp})^{\perp}$, which implies that $A \multimap B$ is a type. $\square$

In order to define exponential connectives, one uses an internalisation of the tensor product of Hilbert spaces. Indeed, since $\mathbb{H}$ is separable and infinite-dimensional, it satisfies $\mathbb{H} \cong \mathbb{H} \otimes \mathbb{H}$. We thus chose such an isomorphism β and an internalization of the associativity: an operator called t . For instance, one can use the bijection $\beta : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ defined by $(n, m) \mapsto 2^n(2m + 1) - 1$. This bijection β induces a unitary $u_{\beta} : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ by defining $(\delta_{i,n}, \delta_{i,m})_{n,m \in \mathbf{N}} \mapsto (\delta_{i, \beta(m,n)})$ on basis elements. We can then define an internalization of the tensor product: if u, v are elements of $G(\mathfrak{A})$, we define $u \bar{\otimes} v = u_{\beta}(u \otimes v)u_{\beta}^*$.

⁶ We will not present a proof here, but it would be similar to the proof of Proposition 118.

⁷ One can find detailed explanation in the author's previous work on interaction graphs (Sei12; Sei14a).

$$\begin{array}{c}
\frac{}{\vdash \square, A, A^\perp} \text{Ax} \\
\frac{\vdash [\Theta], \Gamma, A, B}{\vdash [\Theta], \Gamma, A \wp B} \wp \\
\frac{\vdash [\Theta_1], \Delta, A \quad \vdash [\Theta_2], \Gamma, A^\perp}{\vdash [\Theta_1, \Theta_2, A, A^\perp], \Gamma, \Delta} \text{Cut} \\
\frac{\vdash [\Theta_1], \Gamma, A \quad \vdash [\Theta_2], \Delta, B}{\vdash [\Theta_1, \Theta_2], \Gamma, \Delta, A \otimes B} \otimes
\end{array}$$

Figure 1. Sequent Calculus with Explicit Cuts

Naturally, $(u \bar{\otimes} v) \bar{\otimes} w = u_\beta((u \bar{\otimes} v) \otimes w) u_\beta^* = u_\beta((u_\beta(u \otimes v) u_\beta^*) \otimes w) u_\beta^*$ is not equal to $u \bar{\otimes} (v \bar{\otimes} w) = u_\beta(u \otimes (u_\beta(v \otimes w) u_\beta^*)) u_\beta^*$. There exists however a unitary t which internalizes the associativity, namely the operator induced by the map:

$$\begin{array}{ccc}
\mathbf{N} & \rightarrow & \mathbf{N} \\
\gamma: \beta(\beta(p, q), r) & \mapsto & \beta(p, \beta(q, r))
\end{array}$$

Definition 49. Let $u \in G(A)$. We define $!u$ as the internalization of $1 \otimes u$, i.e. as $1 \bar{\otimes} u$.

The definition of the exponential by $1 \bar{\otimes} u$ boils down to replacing u by an infinite (countable) numbers of copies of itself. Indeed, $1 \otimes u \in \mathcal{L}(\mathbb{H} \oplus \mathbb{H})$ is equal to $\sum_{i \in \mathbf{N}} e_i \otimes u$, where (e_i) is a basis of $\mathbb{H}$.

3.1.2. Interpretation of Proofs. In his paper (Gir89a), Girard defined the interpretation of proofs as matrices in $\mathcal{M}_n(\mathcal{L}(\ell^2(\mathbf{N})))$, where n is the number of formulas in the conclusion (taking into account the cut formulas that appear in the conclusion in the particular sequent calculus he considers). We will here present only the interpretation of the multiplicative fragment (MLL). Figure 1 shows the derivation rules of the system considered. Formulas are those of multiplicative linear logic, and sequence are of the form $\vdash [\Delta], \Gamma$ where $\Delta = A_1, A_1^\perp, A_2, A_2^\perp, \dots, A_k, A_k^\perp$ is the multiset of cut formulas. Girard then defines the interpretation of a proof as a couple $(\pi^\bullet, \sigma_\pi)$, where $\pi^\bullet$ is a partial isometry in $\mathcal{M}_n(\mathcal{L}(\ell^2(\mathbf{N})))$ (more precisely in $\mathcal{M}_n(G_{\mathcal{L}(\ell^2(\mathbf{N}))}(\mathfrak{A}))$) which represents the proof π , and σ_π is a partial symmetry $\mathcal{M}_n(\mathcal{L}(\ell^2(\mathbf{N})))$ (more precisely in $\mathcal{M}_n(G_{\mathcal{L}(\ell^2(\mathbf{N}))}(\mathfrak{A}))$) which represents the set of cut rules in π .

As opposed to Girard, we will define directly the interpretation of proofs as elements of $\mathcal{L}(\ell^2(\mathbf{N}))$ by internalizing the algebra of matrices, i.e. by working modulo the isomorphism between $\mathfrak{M}_n(\mathcal{L}(\ell^2(\mathbf{N})))$ and $\mathcal{L}(\ell^2(\mathbf{N}))$. We will represent a sequent by the $\wp$ of the formulas it is composed of. The two projections rr^* and ll^* are equivalent in the sense of Murray and von Neumann: the partial isometry $a = lr^*$ satisfies $aa^* = (lr^*)(lr^*)^* = lr^*rl^* = ll^*$ and $a^*a = rr^*$. It will be used to represent axioms.

Let $\vdash [\Delta], \Gamma$ be a sequent. Each formula A in $\Delta \cup \Gamma$ can be assigned an *address*, i.e. a sequence of r and l describing the projection onto the subspace corresponding to A . If A and $A^\perp$ are two formulas are the addresses p_1, p_2 respectively, we can define a partial isometry $p_2 p_1^*$ between those (constructed from the partial isometries r and l) which we will denote by $\tau(p_1, p_2)$. Notice that $\tau(r, l) = a$.

Definition 50 (Representation of Proofs). We define the representation $(\pi^\bullet, \sigma_\pi)$ of a proof π inductively:

- if π is an axiom rule, we define $\pi^\bullet = a + a^*$ and $\sigma_\pi = 0$;

- if π is obtained from π_1 and π_2 by applying a $\otimes$ rule, we define $\pi^\bullet = \pi_1^\bullet \odot \pi_2^\bullet$ and $\sigma_\pi = \tau(\sigma_1) + \tau(\sigma_2)$;
- if π is obtained from π_1 by applying a $\wp$ rule, then $\pi^\bullet = \pi_1^\bullet$ and $\sigma_\pi = \sigma_{\pi_1}$;
- if π is obtained from π_1 and π_2 by applying a Cut rule between formulas at the addresses p_1, p_2 , we define $\pi^\bullet = \pi_1^\bullet \odot \pi_2^\bullet$ and $\sigma_\pi = \tau(\sigma_1) + \tau(\sigma_2) + \tau(p_1, p_2) + \tau(p_2, p_1)$.

One can show that if π is a proof with cuts and π' is the cut-free proof obtained from applying the cut-elimination procedure on π , the operators $\pi^\bullet$ and σ_π are orthogonal and the result of the execution formula $\text{Ex}(\pi^\bullet, \sigma_\pi) = \sum_{i=0}^{\infty} (\pi^\bullet \sigma_\pi)^i$ is equal to $(\pi')^\bullet$.

3.1.3. Weak Nilpotency. The GoI model we partially exposed allows one to interpret system F . In order to extend the model to the full pure lambda-calculus, Girard replaced the notion of nilpotency by a weaker notion, namely *weak nilpotency*, i.e. point wise nilpotency.

Definition 51. An operator u is *weakly nilpotent* if u^n weakly converges to 0.

The main difficulty in this work consists in showing that the execution formula $\text{Ex}(u, \sigma)$ is still well-defined when $u\sigma$ is only *weakly nilpotent* but not necessarily nilpotent. In the previous construction, the nilpotency of $u\sigma$ ensured the convergence of the series $\sum_{i=0}^{\infty} (u\sigma)^i$, and therefore the fact that the execution formula was well-defined. The case when $u\sigma$ is only weakly nilpotent is more delicate. Indeed, if $u\sigma$ is weakly nilpotent, the operator $1 - u\sigma$ need not be invertible. Girard showed (Gir88) that it however admits an unbounded inverse ρ defined on a dense subspace of $\mathbb{H}$. Moreover, since the operators considered are all partial isometries in the normalizing groupoid of a given MASA $\mathfrak{A}$, the operators $(u\sigma)^k$ are partial isometries of disjoint domains and codomains and their sum is again a partial isometry in the normalizing groupoid⁸ of $\mathfrak{A}$. From this, one can show that the restriction of ρ to the subspace $(1 - \sigma^2)\mathbb{H}$ is a bounded operator, which yields the following proposition.

Proposition 52 (Girard (Gir88)). *If $u\sigma$ is weakly nilpotent, the execution $\text{Ex}(u, \sigma)$ is well-defined.*

3.1.4. Additive Connectives. The definition of additive connectives appeared in the next GoI model (Gir95a), using the notion of *dialect*. This notion allows one to encode *private information* in the operators, i.e. information which has no consequence on the interaction with other operators. To get a good intuition, one may compare this to the usual notion of *state* for abstract machines: while the current state of a machine M have an impact on how this machine will transition at the next step, it will modify the behavior of another machine that may compute the input or compute on the output of M . This is translated when composing machines by taking a product of the sets of states; similarly here, composition will be dealt with by taking a tensor product — an operation which amounts to a product of the bases.

We therefore replace operators acting on a Hilbert space $\mathbb{H}$ by operators acting on the Hilbert space $\mathbb{H} \otimes \mathbb{H}$, where the first copy of $\mathbb{H}$ is *public*, while the second (the dialect) is *private*. This will translate mainly when considering the composition of two such operators,

⁸ This is justified by Lemma 108 and Lemma 107.

when we will consider their dialects to be disjoint: an operator will necessarily act as the identity on the dialect of the other. To enforce operators to act as identities on the dialect of another, one uses the tensor product $\mathbb{H} \otimes \mathbb{H}$: if u, v are operators on $\mathbb{H} \otimes \mathbb{H}$, we can extend them as operators $u^\dagger, v^\ddagger$ acting on $\mathbb{H} \otimes (\mathbb{H} \otimes \mathbb{H})$. They are then considered as operators with two dialects (the dialect of u and the dialect of v) but which act non-trivially only on one of them (they are extended by the identity on the second dialect, defining for instance $u^\dagger = u \otimes 1$). Through an internalization of the tensor product, one can then consider that the couple of two dialects as a single dialect: the operator $u^\dagger v^\ddagger$ can be seen as acting on $\mathbb{H} \otimes \mathbb{H}$.

Girard's article *Geometry of Interaction III: Accomodating the Additives* (Gir95a) interprets proofs as operators in a C^* -algebra, as it was the case in the previous models described above. This algebra is however described by Girard as an algebra of clauses. For homogeneity reasons, and because the presentation as an algebra of clauses hides one again the dependency of the construction on the choice of a MASA, we will present it here by using operators. This presentation is a small variation on the presentation one can find in Duchesne's thesis (Duc09), although the latter once again hides the role of MASAs behind a fixed choice of basis.

Let us chose once again the Hilbert space $\mathbb{H} = \ell^2(\mathbf{N})$, and the MASA $\mathfrak{A}$ of $\mathcal{L}(\ell^2(\mathbf{N}))$ defined as the algebra of diagonal operators in the basis $(\delta_{i,n})_{n \in \mathbf{N}}$. We will consider (disjoint) sums of operators of the form $u \otimes p$ — where p is a projection and u a partial isometry — in the normalizing groupoid of $\mathfrak{A} \otimes \mathfrak{A}$, a MASA in $\mathcal{L}(\mathbb{H} \otimes \mathbb{H})$.

Definition 53. Let u be an operator. We say that u is a *GoI operator* when $u = \sum_{i \in I} u_i \otimes p_i$, where for all $i \in I$, u_i is a partial isometry in $G(\mathfrak{A})$ and p_i is a projection in $\mathfrak{A}$. We will moreover impose that $\sum_{i \in I} p_i \sim 1$ and $p_i \sim p_j$ (for all i, j) where $\sim$ represents the Murray and von Neumann equivalence.

Remark 54. If u is a GoI operator, then $u \in G_{\mathcal{L}(\mathbb{H} \otimes \mathbb{H})}(\mathfrak{A} \otimes \mathfrak{A})$.

Definition 55. Let $u = \sum_{i \in I} u_i \otimes p_i$ and $v = \sum_{j \in J} v_j \otimes q_j$ be two GoI operators. We define $u^\dagger$ as the oerator $\sum_{i \in I} u_i \otimes (p_i \bar{\otimes} 1)$. We define similarly $v^\ddagger$ as the operator $\sum_{j \in J} v_j \otimes (1 \bar{\otimes} q_j)$.

Definition 56. If u and v are two GoI operators, we will say they are orthogonal when $u^\dagger v^\ddagger$ is nilpotent. We will say that u and v are weakly orthogonal when the product $u^\dagger v^\ddagger$ is weakly nilpotent.

We will now use once again the partial isometries r, l introduced earlier in this section. However, r and l are operators in $\mathcal{L}(\mathbb{H})$ while GoI operators are elements in $\mathcal{L}(\mathbb{H} \otimes \mathbb{H})$. We thus extend these operators to operators in $\mathcal{L}(\mathbb{H} \otimes \mathbb{H})$ in order to take dialects into account. We will write $\hat{r}$ (respectively $\hat{l}$) the operator $r \otimes 1$ (respectively $1 \otimes l$) and we will write $\bar{r}(u)$ (respectively $\bar{l}(u)$) the operator $\hat{r} u \hat{r}^*$ (respectively $\hat{l} u \hat{l}^*$).

Definition 57. Let u, v be GoI operators. Then:

$$\begin{aligned}\bar{r}(u^\dagger) &= \sum_{i \in I} r(u_i) \otimes (p_i \bar{\otimes} 1) \\ &= \sum_{i \in I} \sum_{j \in J} r(u_i) \otimes (p_i \bar{\otimes} q_j) \\ \bar{l}(v^\ddagger) &= \sum_{j \in J} l(v_j) \otimes (1 \bar{\otimes} q_j) \\ &= \sum_{j \in J} \sum_{i \in I} l(v_j) \otimes (p_i \bar{\otimes} q_j)\end{aligned}$$

The operator $u \odot v = \bar{r}(u^\dagger) + \bar{l}(v^\ddagger)$ is therefore a GoI operator equal to:

$$\sum_{i \in I} \sum_{j \in J} (r(u_i) + l(v_j)) \otimes (p_i \bar{\otimes} q_j)$$

Once again, one can define a (weak) type as a (weakly) biorthogonally closed set of GoI operators.

Definition 58. Let A, B be two (weak) types, we define their tensor product as:

$$A \otimes B = \{u \odot v \mid u \in A, v \in B\}^{\perp\perp}$$

Definition 59. Let u, v be orthogonal GoI operators. We define the execution of u and $\bar{r}(v)$ as the GoI operators $u :: \bar{r}(v) = (1 - \hat{r}\hat{r}^*) \sum_{i \geq 1} (u^\dagger \bar{r}(v^\ddagger))^i u (1 - \hat{r}\hat{r}^*)$.

The fact that this is a GoI operator comes from the following computation:

$$\begin{aligned}u :: \bar{r}(v) &= (1 - \hat{r}\hat{r}^*) \sum_{i \geq 1} (u^\dagger \bar{r}(v^\ddagger))^i u (1 - \hat{r}\hat{r}^*) \\ &= (1 - \hat{r}\hat{r}^*) \sum_{i \geq 1} \left(\left(\sum_{k=0}^n \sum_{l=0}^m u_k \otimes (p_k \bar{\otimes} q_l) \right) \left(\sum_{k'=0}^n \sum_{l'=0}^m d(v_{l'}) \otimes (p_{k'} \bar{\otimes} q_{l'}) \right) \right)^i u (1 - \hat{r}\hat{r}^*) \\ &= (1 - \hat{r}\hat{r}^*) \sum_{i \geq 1} \left(\sum_{k=0}^n \sum_{l=0}^m (u_k v_l) \otimes (p_k \bar{\otimes} q_l) \right)^i \left(\sum_{k=0}^n \sum_{l=0}^m u_k \otimes (p_k \bar{\otimes} q_l) \right) (1 - \hat{r}\hat{r}^*) \\ &= \sum_{k=0}^n \sum_{l=0}^m \left((1 - \hat{r}\hat{r}^*) \sum_{i \geq 1} (u_k d(v_l))^i u_k (1 - p p^*) \right) \otimes (p_k \bar{\otimes} q_l)\end{aligned}$$

Proposition 60. Let u, v be orthogonal GoI operators. Then:

$$u \perp (v \odot w) \Leftrightarrow (u \perp \bar{r}(v^\ddagger)) \wedge (u :: \bar{r}(v^\ddagger)) \perp \bar{l}(w)$$

Theorem 61. Let A, B be (weak) types. Then

$$A \multimap B = \{f \mid \forall u \in A, \exists v \in B, f :: \bar{r}(u) = \bar{l}(v)\}$$

is a (weak) type and $A \multimap B = (A \otimes B^\perp)^\perp$.

Definition 62. Let u, v be GoI operators. We define

$$u \& v = (1 \otimes p)u(1 \otimes p)^* + (1 \otimes q)v(1 \otimes q)^*$$

If A, B are (weak) types, we define

$$A \& B = \{u \& v \mid u \in A, v \in B\}^{\perp\perp}$$

Definition 63. Let u, v be GoI operators. We say that u is a variant of v — written $u \sim v$ — when there exists a partial isometry $w \in G(\mathfrak{L})$ such that $u = (1 \otimes w)v(1 \otimes w)^*$.

Proposition 64. Let u, v, w be GoI operators such that $u \sim v$. Then $u \perp w$ if and only if $v \perp w$. Moreover, $u :: w \sim v :: w$.

3.2. Hyperfinite GoI

3.2.1. Locativity. Between the GoI models explained above and the hyperfinite GoI model (Gir11), Girard introduced *ludics* (Gir01). If the constructions of ludics may appear at first sight quite different from the constructions of GoI models, both constructions are, in a sense, exactly the same⁹. Indeed, the constructions differ only from their starting point: when GoI models are built upon an abstraction of proof nets (or rather proof structures) (NPS14), ludics is built upon an abstraction of (focalized) MALL sequent calculus derivations (with a modified axiom rule (NPS14; Cur06)).

One can show that a formula A est prouvable in MALL if and only if a specific formula $A^\sharp$ is provable in a system MALL_{foc} . This formula $A^\sharp$ is a normal form of A obtained by using distributivity isomorphisms. The system MALL_{foc} uses the fact that all provable sequent has a *focalized proof*, i.e. a proof alternating between a sequence of reversible (negative) rules of maximal length, and a sequence of non reversibles (positive) rules introducing the positive connectives of a single formula (thus the choice of terminology) of maximal length. This sequent calculus possesses an axiom rule and two schemes of rules: a negative scheme and a positive scheme — each representing the possible sequences of reversible or non-reversible rules.

Ludics is then an abstraction of this sequent calculus: we first replace the axiom rule by a rule $\boxtimes$. This rule $\boxtimes$ in ludics introduce only positive sequent and therefore can never introduce a sequent of the form $\vdash A, A^\perp$; it consequently never corresponds to the application of an axiom rule. This is counterbalanced by the consideration of infinite derivation trees: a correct sequent — such as $A \vdash A$ — will then be introduced by a sort of infinite η -expansion named the *fax*. The second abstraction consist in replacing formulas by addresses — finite sequences of integers. We already considered a notion of address in our presentation of the interpretation of proofs in Girard’s first GoI model.

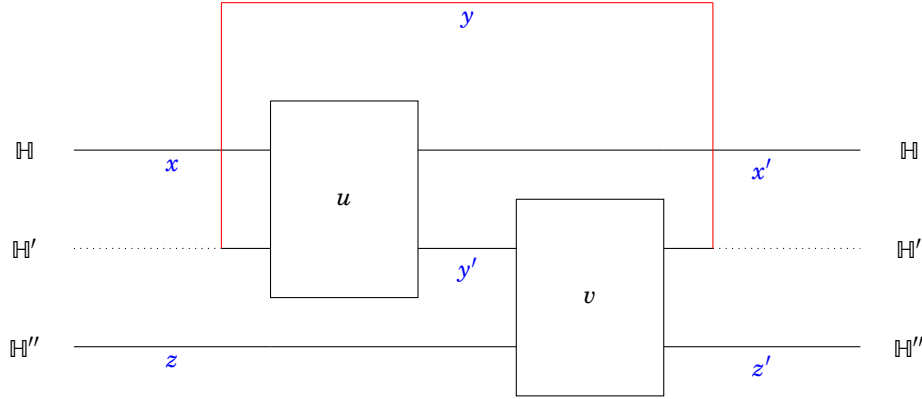
We will not here detail the constructions of ludics, but we will stress this *locative* aspect. If Girard’s first GoI models were already locative — an address was then a sequence of symbols r and l — it only became explicit after the introduction of ludics. In particular, in the first GoI models, the tensor product was always defined because its was defined through adequate *delocations*: conjugating the left-hand element by l and the right-hand by l . This allows in particular to consider operators that always act on the same space: the Hilbert space $\ell^2(\mathbf{N})$. In Girard’s hyperfinite GoI model (Gir11), the operators considered are elements of an algebra $\mathfrak{R}_{0,1}$ of type II_∞ , but act only on a *finite* subspace (finite from the point of view of the algebra, i.e. the projection onto the subspace is finite in the algebra).

⁹ In particular, we have shown that our construction of Interaction Graphs based on graphings unifies Girard’s GoI models (Sei14c). We believe that the general framework of graphings will yield a special case Girard’s ludics.

Then, the objects under study are given together with a projection $p \in \mathfrak{A}_{0,1}$ — the *location* — and an operator u such that $pup = u$. A consequence of the locative approach is that some operations are only partially defined — as the tensor product. It is however possible to retrieve total constructions by working *modulo delocations*; for instance, this is how one builds categories from GoI models (Sei12; Sei14a).

3.2.2. The Feedback Equation. The feedback equation is the operator-theoretic counterpart of the cut-elimination procedure. Hence a solution to the feedback equation is the equivalent of the normal form a proof net that may contain cuts. This equation is stated as follows: if u, v are operators acting on the Hilbert spaces $\mathbb{H} \oplus \mathbb{H}'$ and $\mathbb{H}' \oplus \mathbb{H}''$ respectively, a *solution to the feedback equation* is an operator w acting on $\mathbb{H} \oplus \mathbb{H}''$ and such that $w(x \oplus z) = x' \oplus z'$ as long as there exist $y, y' \in \mathbb{H}'$ satisfying:

$$\begin{aligned} u(x \oplus y) &= x' \oplus y' \\ v(y' \oplus z) &= y \oplus z' \end{aligned}$$



Let us write p, p', p'' the projections onto the subspaces $\mathbb{H}, \mathbb{H}', \mathbb{H}''$ respectively. The execution formula $u :: v = (p + p''v) \sum_{i \geq 0} (uv)^i (up + p'')$, when it is defined, yields a solution to the feedback equation involving u and v . More generally, the formula $(p + p''v)(1 - uv)^{-1}(up + p'')$, when defined, describes a solution to the feedback equation.

Girard studied, in the paper *Geometry of Interaction IV: the Feedback Equation* (Gir06) an extension of this solution. Indeed, he showed that as long as u, v are hermitians of norm at most 1, the solution $(p + p''v)(1 - uv)^{-1}(up + p'')$ defines a partial functional application which can be extended to be defined for all couples of hermitian operators in the unit ball 1. Moreover, this extension is the unique such extension that preserves some properties.

3.2.3. The Determinant. The hyperfinite GoI model no longer use the orthogonality defined by nilpotency but considers a more involved notion defined through the determinant of operators. In order to motivate this change, we consider G, H, F three square matrices of respective dimensions $n \times n$, $m \times m$ and $(n + m) \times (n + m)$. We can write F as a block matrix as follows:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

where F_{11} (respectively F_{22}) is a square matrix of dimension $n \times n$ (respectively $m \times m$). We will write $G \oplus H$ the square matrix of dimension $(n + m) \times (n + m)$ defined as:

$$G \oplus H = \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix}$$

One can then notice that when $1 - F_{11}G$ is invertible (1 is here the identity matrix of dimension $n \times n$), the computation of the determinant of $1 - F(G \oplus H)$ involves the execution formula $Ex(F, G)$:

$$\begin{aligned} & \det(1 - F(G \oplus H)) \\ &= \begin{vmatrix} 1 - F_{11}G & -F_{12}H \\ -F_{21}G & 1 - F_{22}H \end{vmatrix} \\ &= \begin{vmatrix} 1 - F_{11}G & -F_{12}H + (1 - F_{11}G)C \\ -F_{21}G & 1 - F_{22}H - F_{21}GC \end{vmatrix} \quad (C = (1 - F_{11}G)^{-1}F_{12}H) \\ &= \begin{vmatrix} 1 - F_{11}G & 0 \\ -F_{21}G & 1 - Ex(F, G).H \end{vmatrix} \\ &= \det(1 - F_{11}G) \det(1 - Ex(F, G).H) \end{aligned}$$

However, in order to interpret exponential connectives one has to consider operators acting on infinite-dimensional spaces. This is why the hyperfinite GoI model takes place in a von Neumann algebra of type II. Indeed, the existence of a trace in factors of type II_1 allows one to define a generalization of the determinant.

This new GoI model thus considers operators in a particular algebra: the type II_∞ hyperfinite factor. In fact, as already mentioned, the operators considered will belong to a sub-algebra $p\mathfrak{A}_{0,1}p$, where p is a finite projection. This amounts to say that we are working with operators in the type II_1 hyperfinite factor embedded in the type II_∞ hyperfinite factor; the latter being used only to ensure that one does not run out of locations.

In a type II_1 factor, as explained in the previous section, there exists a trace. It is therefore possible to define a generalization of the determinant of matrices by using the identity $\det(\exp(A)) = \exp(\text{tr}(A))$ which is, in finite dimensions, satisfied for all matrix A .

Indeed, if A is a matrix with complex coefficients, we can suppose it is in upper triangular form. The determinant of $\exp(A)$ is then the product of the exponentials of eigenvalues of A : $\det(\exp(A)) = \prod_i \exp(\lambda_i)$ and therefore $\det(\exp(A)) = \exp(\sum_i \lambda_i)$. This shows that $\det(\exp(A)) = \exp(\text{tr}(A))$.

In the case of a factor of type II_1 , with its normalized trace Tr , we can define for all invertible operator A :

$$\det(A) = e^{Tr(\log(|A|))}$$

This generalization of the determinant was introduced by Fuglede and Kadison (FK52) who showed that it can be extended to all operators, though not in a unique way. They also show a number of properties satisfied by the determinant, among which the following that will be useful in this paper:

- $\det$ is multiplicative: $\det(AB) = \det(A)\det(B)$;
- for all A $\det(A) < \text{rad}(A)$, where $\text{rad}(A)$ is the spectral radius of A .

In the construction of the GoI model, Girard actually uses a generalization of the notion of trace: the so-called pseudo-traces. A *pseudo-trace* is an hermitian ($\alpha(u) = \overline{\alpha(u^*)}$), tracial ($\alpha(uv) = \alpha(vu)$), faithful and normal (σ -weakly continuous) linear form.

Definition 65. If α is a pseudo-trace on $\mathfrak{A}$, and tr a trace on $\mathfrak{R}$, we can define for all invertible $A \in \mathfrak{R} \otimes \mathfrak{A}$:

$$\det_{\text{tr} \otimes \alpha}(A) = e^{\text{tr} \otimes \alpha(\log(|A|))}$$

Proposition 66. If ϕ is a $*$ -isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, then for all invertible $A \in \mathfrak{R} \otimes \mathfrak{A}$:

$$\det_{\text{tr} \otimes (\alpha \circ \phi^{-1})}(\text{Id} \otimes \phi(A)) = \det_{\text{tr} \otimes \alpha}(A)$$

Proof. Let A be invertible in $\mathfrak{R}_{0,1} \otimes \mathfrak{A}$. Then, using the definition of the determinant and the fact that $\text{Id} \otimes \phi$ commutes with functional calculus, we have:

$$\begin{aligned} \det_{\text{tr} \otimes (\alpha \circ \phi^{-1})}(\text{Id} \otimes \phi(A)) &= \exp(\text{tr} \otimes (\alpha \circ \phi^{-1})(\log(|\text{Id} \otimes \phi(A)|))) \\ &= \exp(\text{tr} \otimes (\alpha \circ \phi^{-1})(\text{Id} \otimes \phi(\log(|A|)))) \\ &= \exp(\text{tr} \otimes \alpha(\log(|A|))) \end{aligned}$$

Thus $\det_{\text{tr} \otimes (\alpha \circ \phi^{-1})}(\text{Id} \otimes \phi(A)) = \det_{\text{tr} \otimes \alpha}(A)$. $\square$

Lemma 67. If A is nilpotent, then $\text{rad}(A) = 0$.

Proof. We have $\text{rad}(A) = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$. Since A is nilpotent, of degree k for instance, we have that $\|A^n\| = 0$ for all $n \geq k$. Thus $\text{rad}(A) = 0$. $\square$

Lemma 68. If A is nilpotent, then $P(A) = \sum_{i=1}^k \alpha_k A^k$ is nilpotent.

Proof. The minimal degree of A in $P(A)^i$ is equal to A^i . Thus $P(A)^i = 0$ for $i \geq k$. $\square$

Proposition 69. If A is nilpotent, then $\det(1 + A) = 1$.

Proof. We will denote by k the degree of nilpotency of A . Since A is nilpotent, $\text{rad}(A) = 0$. Pick $\lambda \in \text{Spec}(1 + A)$. By definition, $\lambda \cdot 1 - 1 - A$ is non-invertible, which means that $(\lambda - 1) \cdot 1 - A$ is non-invertible, i.e. $\lambda - 1 \in \text{Spec}(A)$. This implies that $\lambda = 1$ since the spectrum of A is reduced to $\{0\}$. This implies that $\text{rad}(1 + A) \leq 1$ and therefore that $\det(1 + A) \leq 1$.

Moreover, $(1 + A)^{-1} = \sum_{i=0}^{k-1} (-A)^i = 1 + \sum_{i=1}^{k-1} (-A)^i$. By the preceding lemma we know that $B = \sum_{i=1}^{k-1} (-A)^i$ is nilpotent, and therefore $\det(1 + B) \leq 1$ using the same argument as before. Since $\det(1 + B) = \det((1 + A)^{-1}) = (\det(1 + A))^{-1}$, we conclude that $\det(1 + A) = 1$. $\square$

3.3. The Hyperfinite GoI

We will here present a modified version of Girard's hyperfinite GoI model. The modification concerns mainly the additive connectives for which we will follow the constructions detailed in our work on interaction graphs (Sei14a); as it was explained in the cited work, the construction of additives thus obtained is much more satisfactory than the construction detailed in Girard's paper (Gir11). The results shown do not depend on this choice, but it will allow us to consider a sequent calculi for elementary linear logic we already studied (Sei13). We will call this construction *Girard's hyperfinite GoI model*. We will also present a slight alteration of this construction (namely, with a different orthogonality) later on;

this second construction will be used to obtain the main result of the paper, after which we will clarify its relation to Girard's hyperfinite GoI model.

The constructions are based on two essential properties, as explained in the author's previous work (Sei14a; Sei14c):

- the associativity of execution (Gir06);
- the “adjunction” (the next theorem) which relates the execution and the *measurement* between operators

$$\llbracket u, v \rrbracket_{\text{hyp}} = -\log(\det(1 - uv))$$

Theorem 70 (Girard (Gir11)). *Let u, v, w be three hermitian operators in the unit ball of the hyperfinite factor of type II_1 , and $u :: v$ the solution to the feedback equation involving u and v . Then:*

$$\llbracket u, v + w \rrbracket_{\text{hyp}} = \llbracket u, v \rrbracket_{\text{hyp}} + \llbracket u :: v, w \rrbracket_{\text{hyp}}$$

3.3.1. Multiplicatives.

Definition 71. A *hyperfinite project* is a tuple $\mathfrak{a} = (p, a, \mathfrak{A}, \alpha, A)$, where:

- p is a finite projection in $\mathfrak{R}_{0,1}$, the *carrier* of $\mathfrak{a}$;
- $a \in \mathbf{R} \cup \{\infty\}$ is called the *wager* of $\mathfrak{a}$;
- $\mathfrak{A}$ is a finite von Neumann algebra of type I, the *dialect* of $\mathfrak{a}$;
- α is a pseudo-trace on $\mathfrak{A}$;
- $A \in p\mathfrak{R}_{0,1}p \otimes \mathfrak{A}$ is a hermitian operator of norm at most 1.

Using Girard's notation, we will write $\mathfrak{a} = a \cdot + \cdot \alpha + A$. When the dialect is equal to $\mathbf{C}$, we will denote by $1_{\mathbf{C}}$ the “trace” of $x \mapsto x$.

If $A \in \mathfrak{R}_{0,1} \otimes \mathfrak{A}$ and $B \in \mathfrak{R}_{0,1} \otimes \mathfrak{B}$, we will write $A^{\dagger \mathfrak{B}}$ and $B^{\ddagger \mathfrak{A}}$ (usually simplified as $A^{\dagger}$ and $B^{\ddagger}$) the operators in $\mathfrak{R}_{0,1} \otimes \mathfrak{A} \otimes \mathfrak{B}$ defined as:

$$\begin{aligned} A^{\dagger \mathfrak{B}} &= A \otimes 1_{\mathfrak{B}} \\ B^{\ddagger \mathfrak{A}} &= (\text{Id}_{\mathfrak{R}_{0,1}} \otimes \tau)(B \otimes 1_{\mathfrak{A}}) \end{aligned}$$

where τ is the isomorphism $\mathfrak{B} \otimes \mathfrak{A} \rightarrow \mathfrak{A} \otimes \mathfrak{B}$.

Definition 72. Let $\mathfrak{a} = a \cdot + \cdot \alpha + A$ and $\mathfrak{b} = b \cdot + \cdot \beta + B$ be two hyperfinite projects. Then $\mathfrak{a} \mathbin{\lrcorner} \mathfrak{b}$ when:

$$\llbracket \mathfrak{a}, \mathfrak{b} \rrbracket_{\text{hyp}} = a\beta(1_{\mathfrak{B}}) + \alpha(A_{\mathfrak{A}})b - \log(\det_{\text{tr} \otimes \alpha \otimes \beta}(1 - A^{\dagger}B^{\ddagger})) \neq 0, \infty$$

If $\mathbf{A}$ is a set of hyperfinite projects, we will write $A^{\lrcorner} = \{\mathfrak{b} \mid \forall \mathfrak{a} \in \mathbf{A}, \mathfrak{a} \mathbin{\lrcorner} \mathfrak{b}\}$ and $A^{\lrcorner \lrcorner} = (A^{\lrcorner})^{\lrcorner}$.

Definition 73. Let p be a finite projection in $\mathfrak{R}_{0,1}$. A *conduct* of carrier p is a set $\mathbf{A}$ of hyperfinite projects of carrier p such that $\mathbf{A} = \mathbf{A}^{\lrcorner \lrcorner}$.

Definition 74. If $\mathfrak{a}, \mathfrak{b}$ are hyperfinite projects of disjoint carrier, the tensor product of $\mathfrak{a}$ and $\mathfrak{b}$ is defined as the hyperfinite project of carrier $p_{\mathfrak{a}} + p_{\mathfrak{b}}$ defined as

$$\mathfrak{a} \otimes \mathfrak{b} = a\beta(1_{\mathfrak{B}}) + \alpha(1_{\mathfrak{A}})b \cdot + \cdot \alpha \otimes \beta + A^{\dagger} + B^{\ddagger}$$

Definition 75. If $\mathbf{A}, \mathbf{B}$ are conducts of disjoint carrier, their tensor product is defined as the conduct:

$$\mathbf{A} \otimes \mathbf{B} = \{\mathfrak{a} \otimes \mathfrak{b} \mid \mathfrak{a} \in \mathbf{A}, \mathfrak{b} \in \mathbf{B}\}^{\lrcorner \lrcorner}$$

Proposition 76. Let $\mathbf{A}, \mathbf{B}$ be two conducts of disjoint carrier. Then:

$$(\mathbf{A} \otimes \mathbf{B}^\perp)^\perp = \{f \mid \forall a \in \mathbf{A}, f :: a \in \mathbf{B}\}$$

Definition 77. If $\mathbf{A}, \mathbf{B}$ are conducts of disjoint carrier, we define:

$$\mathbf{A} \multimap \mathbf{B} = \{f \mid \forall a \in \mathbf{A}, f :: a \in \mathbf{B}\}$$

Theorem 78.

$$\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^\perp)^\perp$$

3.3.2. Additives.

Lemma 79 (Variants). Let a be a hyperfinite project in a conduct $\mathbf{A}$, and $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ a $*$ -isomorphism. Then the hyperfinite project $a^\phi = a \cdot + \cdot \alpha \circ \phi^{-1} + \text{Id} \otimes \phi(A)$ is an element of $\mathbf{A}$. We will say that a^ϕ is a variant of a .

Proof. Let c be a hyperfinite project whose carrier is equal to the carrier of a . Then:

$$\begin{aligned} \ll a^\phi, c \gg_{\text{hyp}} &= \alpha \gamma(1_{\mathfrak{C}}) + \alpha \circ \phi^{-1}(1_{\mathfrak{B}})c - \log(\det(1 - (\text{Id} \otimes \phi(A))^{\dagger \mathfrak{C}} C^{\ddagger \mathfrak{B}})) \\ &= \alpha \gamma(1_{\mathfrak{C}}) + \alpha(1_{\mathfrak{A}})c - \log(\det(1 - \text{Id} \otimes \phi \otimes \text{Id}(A^{\dagger \mathfrak{C}}) C^{\ddagger \phi(\mathfrak{A})})) \\ &= \alpha \gamma(1_{\mathfrak{C}}) + \alpha(1_{\mathfrak{A}})c - \log(\det(1 - \text{Id} \otimes \phi \otimes \text{Id}(A^{\dagger \mathfrak{C}}) C^{\ddagger \mathfrak{A}})) \end{aligned}$$

Finally, since $\det(1 - A) = \det(1 - \psi(A))$ for all isomorphism ψ , we obtain $\ll a^\phi, c \gg_{\text{hyp}} = \ll a, c \gg_{\text{hyp}}$. We deduce that for all $c \in \mathbf{A}^\perp$, $a^\phi \perp c$, and therefore $a^\phi \in \mathbf{A}$. $\square$

Definition 80. Let a, b be hyperfinite projects of equal carrier p , and $\lambda \in \mathbf{R}$. We define $a + \lambda b$ as the hyperfinite project $a + \lambda b \cdot + \cdot \alpha \oplus \lambda \beta + A \oplus B$, of dialect $\mathfrak{A} \oplus \mathfrak{B}$ and carrier p .

Definition 81. A conduct has the *inflation property* when for all $a \in \mathbf{A}$, and all $\lambda \in \mathbf{R}$, the hyperfinite project $a + \lambda o$ belongs to $\mathbf{A}$, where o is the project $0 \cdot + \cdot 1_{\mathfrak{C}} + 0$ whose carrier is equal to the carrier of a .

The following proposition shows that this definition is equivalent to the one used by Girard (Gir11).

Proposition 82. If $\mathbf{A}$ has the inflation property, then for all element $a = (p, a, \mathfrak{A}, \alpha, A)$ in $\mathbf{A}$, for all finite von Neumann algebra $\mathfrak{B}$ and all injective $*$ -morphism $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$, the hyperfinite project $a^\phi = (p, a, \mathfrak{B}, \beta, \text{Id} \otimes \phi(A))$ such that $\beta \circ \phi = \alpha$ is an element of $\mathbf{A}$.

Proof. Let p be the projection which is the image of the identity through ϕ . Then $\text{Id} \otimes \phi(A) = p \text{Id} \otimes \phi(A) p$. Moreover,

$$\beta(1_{\mathfrak{B}}) = \beta(p + (1_{\mathfrak{B}} - p)) = \beta(p) + \beta(1_{\mathfrak{B}} - p) = \beta(\phi(1_{\mathfrak{A}})) + \beta(1_{\mathfrak{B}} - p) = \alpha_{1_{\mathfrak{A}}} + \beta(1 - p)$$

Let $c \in \mathbf{A}^\perp$. We notice that:

$$\begin{aligned}
 \det(1 - \text{Id} \otimes \phi \otimes \text{Id}(A^\dagger)C^{\dagger\mathfrak{B}}) &= \det(1 - (1 \otimes p \otimes 1)\text{Id} \otimes \phi \otimes \text{Id}(A^\dagger)(1 \otimes p \otimes 1)C^{\dagger\mathfrak{B}}) \\
 &= \det(1 - \text{Id} \otimes \phi \otimes \text{Id}(A^\dagger)(1 \otimes p \otimes 1)C^{\dagger\mathfrak{B}}(1 \otimes p \otimes 1)) \\
 &= \det(1 - \text{Id} \otimes \phi \otimes \text{Id}(A^\dagger)C^{\dagger p \mathfrak{B} p}) \\
 &= \det(1 - \text{Id} \otimes \phi \otimes \text{Id}(A^\dagger)C^{\dagger \phi(\mathfrak{A})}) \\
 &= \det(1 - \text{Id} \otimes \phi \otimes \text{Id}(A^\dagger)C^{\dagger \mathfrak{A}})) \\
 &= \det(1 - A^\dagger C^{\dagger \mathfrak{A}})
 \end{aligned}$$

We can now compute $\ll c, a^\phi \gg_{\text{hyp}}$:

$$\begin{aligned}
 \ll c, a^\phi \gg_{\text{hyp}} &= c\beta(1_{\mathfrak{B}}) + a\gamma(1_{\mathfrak{C}}) - \log(\det(1 - (\text{Id} \otimes \phi(A))^\dagger C^{\dagger \mathfrak{B}})) \\
 &= c(\alpha(1_{\mathfrak{A}}) + \lambda) + a\gamma(1_{\mathfrak{C}}) - \log(\det(1 - (\text{Id} \otimes \phi \otimes \text{Id}(A^\dagger))C^{\dagger \mathfrak{B}})) \\
 &= c(\alpha(1_{\mathfrak{A}}) + \lambda) + a\gamma(1_{\mathfrak{C}}) - \log(\det(1 - A^\dagger C^{\dagger \mathfrak{A}})) \\
 &= c(\alpha(1_{\mathfrak{A}}) + \lambda) + a\gamma(1_{\mathfrak{C}}) - \log(\det(1 - (A \oplus 0)^\dagger C^{\dagger \mathfrak{A} \oplus \mathfrak{C}})) \\
 &= \ll c, a + \lambda o \gg_{\text{hyp}}
 \end{aligned}$$

We just showed that $\ll c, a^\phi \gg_{\text{hyp}} = \ll c, a + \lambda o \gg_{\text{hyp}}$. Since $\mathbf{A}$ has the inflation property and $a \in \mathbf{A}$, we have that $\ll c, a + \lambda o \gg_{\text{hyp}} \neq 0, \infty$ for all $c \in \mathbf{A}^\perp$. Thus $a^\phi \perp c$ for all $c \in \mathbf{A}^\perp$, which implies that $a^\phi \in \mathbf{A}$. $\square$

Definition 83. A *dichology* is a conduct $\mathbf{A}$ such that both $\mathbf{A}$ and $\mathbf{A}^\perp$ have the inflation property. A dichology $\mathbf{A}$ is *proper* when both $\mathbf{A}$ and $\mathbf{A}^\perp$ are non-empty.

Definition 84. Let a be a hyperfinite project of carrier p , and q a projection such that $pq = 0$. We define a_{p+q} as the hyperfinite project $a \cdot + \cdot \alpha + (A + 0)$ of carrier $p + q$.

If $\mathbf{A}$ is a conduct of carrier p , we define $\mathbf{A}_{p+q} = \{a_{p+q} \mid a \in \mathbf{A}\}^{\perp \perp}$.

Definition 85. Let $\mathbf{A}, \mathbf{B}$ be two conducts of respective disjoint carriers p, q . We define:

$$\begin{aligned}
 \mathbf{A} \& \mathbf{B} &= ((\mathbf{A}^\perp)_{p+q})^\perp \cap ((\mathbf{B}^\perp)_{p+q})^\perp \\
 \mathbf{A} \oplus \mathbf{B} &= ((\mathbf{A}_{p+q})^{\perp \perp} \cup (\mathbf{B}_{p+q})^{\perp \perp})^{\perp \perp}
 \end{aligned}$$

Proposition 86. If $\mathbf{A}, \mathbf{B}$ are dichologies of disjoint carriers, the conducts $\mathbf{A} \otimes \mathbf{B}, \mathbf{A} \& \mathbf{B}, \mathbf{A} \oplus \mathbf{B}$ and $\mathbf{A} \multimap \mathbf{B}$ are dichologies.

Proposition 87 (Distributivity). For any behaviors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and delocations ϕ, ψ, θ, ρ of $\mathbf{A}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ respectively, there is a project distr in the behavior

$$((\phi(\mathbf{A}) \multimap \theta(\mathbf{B})) \& (\psi(\mathbf{A}) \multimap \rho(\mathbf{C}))) \multimap (\mathbf{A} \multimap (\mathbf{B} \& \mathbf{C}))$$

Definition 88. Given two hyperfinite projects $a = a \cdot + \cdot \alpha + A$ and $b = b \cdot + \cdot \beta + B$, we define the hyperfinite project $a + b$:

$$a + b = a + b \cdot + \cdot \alpha \oplus \beta + A \oplus 0 + 0 \oplus B$$

Lemma 89. If $\mathbf{A}, \mathbf{B}$ are proper dichologies, then $\mathbf{A} + \mathbf{B} = \{a_{p+q} + b_{p+q} \mid a \in \mathbf{A}, b \in \mathbf{B}\}$ is such that $\mathbf{A} + \mathbf{B} \subset \mathbf{A} \& \mathbf{B}$.

3.3.3. Exponentials. Exponential connectives are defined through the notion of *perennialization*. We will not justify this definition nor explain why it indeed yields exponential connectives, the interested reader can find those in our paper on exponentials in interaction graphs (Sei13). We will here only briefly describe the particular perennialization used by Girard (Gir11).

Definition 90. A perennialization is an isomorphism $\Phi : \mathfrak{A}_{0,1} \otimes \mathfrak{A} \rightarrow \mathfrak{A}_{0,1}$.

Definition 91. A hyperfinite project $\mathfrak{a} = a \cdot + \cdot \alpha + A$ is *balanced* when $a = 0$, $\mathfrak{A}$ is a finite factor of type I, and α is the normalized trace on $\mathfrak{A}$. If $\mathfrak{a}$ is balanced with dialect $\mathfrak{M}_k(\mathbf{C})$, and $\theta : \mathfrak{M}_k(\mathbf{C}) \rightarrow \mathfrak{A}$ is a trace-preserving $*$ -isomorphism, we will abusively write $\mathfrak{a}^\theta$ as the “project¹⁰” $a \cdot + \cdot \text{tr} + \text{Id} \otimes \theta(A)$, where tr is the normalized trace on $\mathfrak{A}$.

Definition 92. If $\mathbf{A}$ is a dichology and Φ a perennialization, we define $\#_\Phi \mathbf{A}$ as the set:

$$\#_\Phi \mathbf{A} = \{!_\Phi \mathfrak{a}^\theta = 0 \cdot + \cdot 1_{\mathbf{C}} + \Phi(\text{Id} \otimes \theta(A)) \mid \mathfrak{a} \in \mathbf{A} \text{ balanced}, \theta : \mathfrak{A} \rightarrow \mathfrak{A} \text{ trace-preserving } * \text{-iso}\}$$

We can then define the conducts $!_\Phi \mathbf{A} = (\#_\Phi \mathbf{A})^{\perp\perp}$ and $?_\Phi \mathbf{A} = (\#_\Phi (\mathbf{A}^\perp))^{\perp}$.

The morphism used in Girard’s hyperfinite Goi model is defined from a group action. The group is chosen so as to possess a number of properties: it is an infinite conjugacy class (I.C.C.) and amenable group which contains the free monoid on two elements. At a first glance, the existence of such a group is not clear, as the typical example of a non-amenable group is the free group on two generators.

Let us denote by $\mathbf{Z}^{|\mathbf{Z}|}$ the group of almost-everywhere null functions $\mathbf{Z} \rightarrow \mathbf{Z}$ with point wise sum. We can then define an action of the group $\mathbf{Z}$ on $\mathbf{Z}^{|\mathbf{Z}|}$ by translation: we define $\alpha : \mathbf{Z} \rightarrow \text{Aut}(\mathbf{Z}^{|\mathbf{Z}|})$ by $\alpha(p) : (x_n)_{n \in \mathbf{Z}} \mapsto (x_{n+p})_{n \in \mathbf{Z}}$. We now consider the group $\mathfrak{G}$ defined as the semi-direct product, or crossed product, of $\mathbf{Z}^{|\mathbf{Z}|}$ by $\mathbf{Z}$ along the action α . Elements of $\mathfrak{G}$ are couples $((x_n)_{n \in \mathbf{Z}}, p)$ where the first element is in $\mathbf{Z}^{|\mathbf{Z}|}$ and the second in $\mathbf{Z}$, and the composition is defined as:

$$((x_n)_{n \in \mathbf{Z}}, p) \cdot ((y_n)_{n \in \mathbf{Z}}, q) = ((x_n)_{n \in \mathbf{Z}} + (\alpha(p)((y_n)))_{n \in \mathbf{Z}}, p + q) = ((x_n + y_{n+p})_{n \in \mathbf{Z}}, p + q)$$

As a semi-direct product of amenable groups, $\mathfrak{G}$ is an amenable group. It is moreover I.C.C. since, if $x = ((x_n), p)$ is different from $((0), 0)$, the conjugacy class of x contains the elements $((\delta_{n,0}), k)^{-1} x ((\delta_{n,0}), k)$ for all $k \in \mathbf{N}$. But $((\delta_{n,0}), k)^{-1} = ((-\delta_{n,-k}), -k)$, and therefore $((\delta_{n,0}), k)^{-1} x ((\delta_{n,0}), k) = ((x_{n-k} + \delta_{n,p} - \delta_{n,-k})_{n \in \mathbf{Z}}, p)$. Thus the conjugacy class of x is infinite since those elements are pairwise distinct.

Lastly, one can find a copy of the free monoid on two elements in $\mathfrak{G}$. Let us first define $\mathbf{a} = ((\delta_{n,0})_n, 0)$ and $\mathbf{b} = ((0)_n, 1)$. We can then compute

$$((a_k)_k, p) \mathbf{b} = ((a_k)_k, p + 1) \quad ((a_k)_k, p) \mathbf{a} = ((a_k + \delta_{k,0})_k, p)$$

We can use these equalities to show:

$$\mathbf{a}^{p_k} \mathbf{b}^{q_k} \mathbf{a}^{p_{k-1}} \mathbf{b}^{q_{k-1}} \dots \mathbf{a}^{p_1} \mathbf{b}^{q_1} = ((\bar{p}_n)_n, \sum_{i=1}^k q_i)$$

¹⁰ It is not exactly a hyperfinite project since its dialect is not an algebra of type I.

where $\bar{p}_n = p_i$ when $n = \sum_{j=1}^i q_j$, and $\bar{p}_n = 0$ otherwise. This show that the submonoid generated by $\mathbf{a}$ and $\mathbf{b}$ in $\mathfrak{G}$ is free.

For instance, the word $\mathbf{a}^2\mathbf{b}^1\mathbf{a}^{48}\mathbf{b}^2$ is equal to $((x_n), 3)$ where (x_n) is the sequence defined by $x_2 = 48$, $x_3 = 2$ and $x_n = 0$ for all $n \neq 2, 3$.

This shows that:

Proposition 93 (Girard (Gir11)). *The group $\mathbf{Z}^{\mathbb{Z}} \rtimes \mathbf{Z}$ is amenable, I.C.C. and contains the free monoid on two generators.*

The definition of the perennialization used by Girard (Gir11) is built on the *crossed product algebra* which generalizes the semi-direct product of group.

Definition 94 (Crossed product). Let $\mathcal{M} \subset \mathcal{B}(\mathbb{H})$ be a von Neumann algebra, G a locally compact group, and α an action of G on $\mathcal{M}$. Let $\mathbb{K} = L^2(G, \mathbb{H})$ be the Hilbert space of square-summable $\mathbb{H}$ -valued functions on G . We define representations π_α of $\mathcal{M}$ and λ of G on $\mathbb{K}$ as follows

$$\begin{aligned} (\pi_\alpha(x).\xi)(g) &= (\alpha(g)^{-1}(x)\xi)(g) \\ (\lambda(g).\xi)(h) &= \xi(g^{-1}h) \end{aligned}$$

Then the von Neumann algebra on $\mathbb{K}$ generated by $\pi_\alpha(\mathcal{M})$ and $\lambda(G)$ is called the crossed product of $\mathcal{M}$ by α and is denoted by $\mathcal{M} \rtimes_\alpha G$.

Now, if A is an operator in $\mathfrak{R}_{0,1} \otimes \mathfrak{R}$, we use the fact that there exists an isomorphism between $\mathfrak{R}$ and $\otimes_{\omega \in \mathbf{aM}} \mathfrak{R}$ to obtain an operator $\bar{A}$ in $\mathfrak{R}_{0,1} \otimes \mathfrak{R}^M$. This operator embeds as an element $\pi_\alpha(\bar{A})$ of the crossed product algebra $\mathfrak{R}_{0,1} \otimes \mathfrak{R}^\mathfrak{G} \rtimes \mathfrak{G}$. Since $\mathfrak{G}$ is I.C.C. and amenable, the crossed product $\mathfrak{R}^\mathfrak{G} \rtimes \mathfrak{G}$ is isomorphic to $\mathfrak{R}$. Moreover, $\mathfrak{R}_{0,1} \otimes \mathfrak{R}$ is isomorphic to $\mathfrak{R}_{0,1}$, and we can thus find an isomorphism Ψ from $\mathfrak{R}_{0,1} \otimes \mathfrak{R}^\mathfrak{G} \rtimes \mathfrak{G}$ into $\mathfrak{R}_{0,1}$. Defining $\Omega(A) = \Psi(\pi_\alpha(\bar{A}))$, we easily check that Ω defines an injective morphism from $\mathfrak{R}_{0,1} \otimes \mathfrak{R}$ to $\mathfrak{R}_{0,1}$.

4. Subjective Truth and Matricial GoI

4.1. Success and Bases

In geometry of Interaction, as in the theory of proof structures (Gir87a), or in game semantics (HO00) or in classical realizability (Kri01; Kri09), one needs to characterize those elements which correspond to proofs: proof nets (i.e. satisfying the correctness criterion), winning strategies, or proof-like terms. In GoI models, these “proof-like terms”, or *winning strategies* are called *successful projects*. In previous GoI models a successful project was defined as a partial symmetry. This definition was quite satisfying, but some of its important properties relied on the fact that the model depended on a chosen MASA $\mathfrak{A}$, i.e. it relied on the fact that the constructions were basis-dependent (i.e. operators are chosen in the normalizing groupoid of $\mathfrak{A}$ only).

In Girard’s hyperfinite model, constructions are no longer basis-dependent: the operator considered are no longer restricted to those elements that are in the normalizing groupoid of a MASA, but can be any hermitian operator of norm at most 1. By going to this more general setting, defining successful projects as partial symmetries is no longer satisfying.

The reason for this is quite easy to understand. Indeed, a satisfying notion of success should verify two essential properties. The first of these is that it should “compose”, i.e. the execution of two successful projects should be a successful projects. The second is that it should be “coherent”, i.e. two orthogonal projects cannot be simultaneously successful.

Since we are no longer restricted to operators in a chosen normalizing groupoid, the definition of successful projects as partial symmetries now lacks these two essential properties. This can be illustrated by easy examples on matrices (to obtain examples in the hyperfinite factor, use your favorite embedding). For instance, if S and T are the matrices defined as:

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad v = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 & 0 \\ -\sqrt{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

One can check that u, v are partial symmetries: it is obvious for u , and the following computation shows it for v .

$$vv^* = v^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}^2$$

However, their product is not a partial isometry (hence not a partial symmetry), which shows that the notion do not compose.

$$uv = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 & 0 \\ -\sqrt{\frac{1}{2}} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ 0 & 0 & 0 \end{pmatrix}$$

Moreover, the computation of the determinant of $1 - uv$ shows that the notion is not coherent.

$$\begin{vmatrix} 1 - \sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & 1 - \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & 0 & 1 \end{vmatrix} = (1 - \sqrt{\frac{1}{2}})^2 \neq 0, \infty$$

In order to obtain a good notion of successful project, we will have to restrict ourselves to a class of partial symmetries which is closed for sum and composition. As shown in Lemma 108 and Lemma 107, sums and products of partial isometries in the normalizing groupoid of a MASA $\mathfrak{A}$ is again a partial isometry¹¹ in the normalizing groupoid of $\mathfrak{A}$. That is, if u and v are partial symmetries in $G(\mathfrak{A})$, then $u :: v$ is a partial symmetry in $G(\mathfrak{A})$.

In the finite-dimensional case, this amounts to choosing a basis. Indeed, the complete classification of MASAs in $\mathcal{L}(\mathbb{H})$ (Theorem 28) shows that when $\mathbb{H}$ is of finite dimension the MASAs of $\mathcal{L}(\mathbb{H})$ are exactly the diagonal MASAs: the set of diagonal matrices in a fixed basis. One can therefore define a *subjective* notion of successful projects, i.e. a notion

¹¹ In the case of the sum, one has to impose a condition on domains and codomains.

a success that depends on the choice of a basis. An operator is then *successful w.r.t. $\mathcal{B}$* when it is a partial symmetry in the normalizing groupoid of the algebra $\mathfrak{D}_{\mathcal{B}}$ of diagonal operators in the basis $\mathcal{B}$. The composition of such partial symmetries can be shown to be itself a partial symmetry in the normalizing groupoid of $\mathfrak{D}_{\mathcal{B}}$ and the definition of success is therefore consistent with the execution. However, we are still unable to show the coherence of this definition: given two partial symmetries u, v in $G(\mathfrak{D}_{\mathcal{B}})$, the logarithm of the determinant of $1 - uv$ is not necessarily equal to 0 or ∞ . Once again, it is enough to consider matrices to illustrate this fact, and we will give an example with 2×2 matrices. Let u and v be the following matrices:

$$u = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\det(1 - uv) = 4$, i.e. $-\log(\det(1 - uv)) \neq 0, \infty$.

The issue here arises from the fact that one cannot distinguish between the identity and its opposite, i.e. the definition do not exclude negative coefficients. The solution proposed by Girard (Gir11) is to consider a notion of success that depends on a representation of the algebra: a successful operator will then have its operators u defined as the operator induced from a measure-preserving transformation on a measured space. We expose the precise definition in subsection 4.5, but we will first study another way to bypass the problem just exposed. It corresponds to an old version of Girard's hyperfinite GoI model, which we will refer to as the *matricidal* GoI model, in which orthogonality is slightly modified. In this GoI model, it is possible to keep the notion of successful projects as partial symmetries in the normalizing groupoid of a MASA $\mathfrak{A}$ since the change of orthogonality bypasses the issue with coherence. This GoI model will be related to the matricidal GoI model later on.

4.2. Matricial GoI

The matricidal GoI model is based on the same notion of projects as the hyperfinite GoI model. The two constructions essentially differ on the measurement $\ll \cdot, \cdot \gg$ which is used to defines the orthogonality relation. Notice that all constructions on hyperfinite projects are the same in both models.

Definition 95. A *dialectal operator* of carrier $p^* = p^2 = p \in \mathfrak{R}_{0,1}$ and dialect $\mathfrak{A}$ a finite von Neumann algebra of type I is a couple (A, α) where:

- 1 $A^* = A \in p\mathfrak{R}_{0,1}p \otimes \mathfrak{A}$ is an hermitian operator such that $\|A\| \leq 1$;
- 2 α is a pseudo-trace on $\mathfrak{A}$.

For all von Neumann algebras $\mathfrak{A}, \mathfrak{B}$ we define the isomorphisms:

$$\begin{aligned} (\cdot)^{\dagger_{\mathfrak{B}}} &: \mathfrak{R}_{0,1} \otimes \mathfrak{A} \rightarrow \mathfrak{R}_{0,1} \otimes \mathfrak{A} \otimes \mathfrak{B} \\ (\cdot)^{\ddagger_{\mathfrak{A}}} &: \mathfrak{R}_{0,1} \otimes \mathfrak{B} \rightarrow \mathfrak{R}_{0,1} \otimes \mathfrak{A} \otimes \mathfrak{B} \end{aligned}$$

Definition 96 (ldet). Let $A \in \mathfrak{R}_{0,1} \otimes \mathfrak{A}$ be a dialectal operator of norm strictly less than 1,

let tr be a trace on $\mathfrak{R}_{0,1}$ and α be a pseudo-trace on the dialect $\mathfrak{A}$. We define:

$$\text{ldet}(1 - A) = \sum_{k=1}^{\infty} \frac{\text{tr} \otimes \alpha(A^k)}{k}$$

Definition 97. We define the measurement between two dialectal operators A, B of respective carriers p, q and dialects $\mathfrak{A}, \mathfrak{B}$ as:

$$\llbracket A, B \rrbracket_{\text{mat}} = \begin{cases} \text{ldet}(1 - A^{\dagger_{\mathfrak{B}}} B^{\ddagger_{\mathfrak{A}}}) & \text{when } \text{rad}(A^{\dagger_{\mathfrak{B}}} B^{\ddagger_{\mathfrak{A}}}) < 1 \\ \infty & \text{otherwise} \end{cases}$$

Lemma 98. Let $u, v \in \mathfrak{R}_{0,1} \otimes \mathfrak{A}$ and α be a pseudo-trace on $\mathfrak{A}$. Then, supposing the series converge:

$$\text{ldet}(1 - (u + v - uv)) = \text{ldet}(1 - u) + \text{ldet}(1 - v)$$

Proof. Supposing the series converge:

$$\begin{aligned} \text{ldet}(1 - (u + v)) &= -\text{tr}(\log((1 - u)(1 - v))) \\ &= -\text{tr}(\log(1 - u) + \log(1 - v)) \\ &= -\text{tr}(\log(1 - u)) - \text{tr}(\log(1 - v)) \end{aligned}$$

Thus $\text{ldet}(1 - (u + v - uv)) = \text{ldet}(1 - u) + \text{ldet}(1 - v)$. $\square$

Lemma 99. Chose $u \in \mathfrak{R}_{0,1} \otimes \mathfrak{A}$, a trace tr on $\mathfrak{R}_{0,1}$ and a pseudo-trace α on an dialect $\mathfrak{A}$. For all dialect $\mathfrak{B}$ and pseudo-trace β on $\mathfrak{B}$, we have:

$$\text{ldet}(1 - u \otimes 1_{\mathfrak{B}}) = \beta(1_{\mathfrak{B}}) \text{ldet}(1 - u)$$

Proof. By definition:

$$\begin{aligned} \text{ldet}(1 - u \otimes 1_{\mathfrak{B}}) &= \sum_{k=1}^{\infty} \frac{\text{tr} \otimes \alpha \otimes \beta((u \otimes 1_{\mathfrak{B}})^k)}{k} \\ &= \sum_{k=1}^{\infty} \frac{\text{tr} \otimes \alpha \otimes \beta(u^k \otimes 1_{\mathfrak{B}})}{k} \\ &= \sum_{k=1}^{\infty} \frac{\text{tr} \otimes \alpha(u^k) \beta(1_{\mathfrak{B}})}{k} \\ &= \beta(1_{\mathfrak{B}}) \sum_{k=1}^{\infty} \frac{\text{tr} \otimes \alpha(u^k)}{k} \end{aligned}$$

$\square$

Definition 100. Two dialectal operators $(A, \alpha), (A', \alpha')$ are universally equivalent of for all dialectal operator (B, β) , we have $\llbracket (A, \alpha), (B, \beta) \rrbracket_{\text{mat}} = \llbracket (A', \alpha'), (B, \beta) \rrbracket_{\text{mat}}$.

Lemma 101 (Variants). Let (A, α) be a dialectal operator of dialect $\mathfrak{A}$, and $\varphi : \mathfrak{A} \rightarrow \mathfrak{C}$ a unital isomorphism of von Neumann algebras. Then $((\text{Id}_{\mathfrak{R}_{0,1}} \otimes \varphi)(A), \alpha \circ \varphi^{-1})$ is universally equivalent to (A, α) .

Proof. Let (B, β) be a dialectal operator. Since $B^{\ddagger_{\mathfrak{C}}} = (\text{Id}_{\mathfrak{R}_{0,1}} \otimes \varphi)(B^{\ddagger_{\mathfrak{A}}})$, since $\text{Id}_{\mathfrak{R}_{0,1}} \otimes \varphi$ is

a unital isomorphism of von Neumann algebras and since isomorphisms between C^* -algebras are isometric, we obtain:

$$\begin{aligned} \|[(\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi)(A)]^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{C}}\| &= \|(\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi)(A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}})\| \\ &= \|A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}}\| \end{aligned}$$

Similarly, we show that for all k :

$$\|[(\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi)(A)]^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{C}}\|^k = \|(A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}})^k\|$$

Consequently, since $\text{rad}(u) = \lim_{k \rightarrow \infty} \|u\|^{\frac{1}{k}}$ we obtain that:

$$\text{rad}([(\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi)(A)]^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{C}}) = \text{rad}(A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}})$$

Moreover, if $\text{rad}(A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}}) < 1$:

$$\begin{aligned} &\text{l det}(1 - [(\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi)(A)]^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{C}}) \\ &= \text{l det}(1 - (\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi \otimes 1_{\mathfrak{B}})(A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}})) \\ &= \sum_{k=1}^{\infty} \frac{\text{tr}([(\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi \otimes 1_{\mathfrak{B}})(A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}})]^k)}{k} \\ &= \sum_{k=1}^{\infty} \text{tr}((\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi \otimes 1_{\mathfrak{B}})(\frac{[A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}}]^k}{k})) \\ &= \sum_{k=1}^{\infty} (\text{tr}_{\mathfrak{H}_{0,1}} \otimes (\alpha \circ \varphi^{-1}) \otimes \beta)((\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi \otimes 1_{\mathfrak{B}})(\frac{[A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}}]^k}{k})) \\ &= \sum_{k=1}^{\infty} (\text{tr}_{\mathfrak{H}_{0,1}} \otimes \alpha \otimes \beta)(\frac{[A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}}]^k}{k}) \\ &= \text{l det}(1 - A^{\dagger \mathfrak{B}} B^{\ddagger \mathfrak{A}}) \end{aligned}$$

Finally, we have just shown that $\|((\text{Id}_{\mathfrak{H}_{0,1}} \otimes \varphi)(A), \alpha \circ \varphi^{-1}), (B, \beta)\|_{\text{mat}} = \|(A, \alpha), (B, \beta)\|_{\text{mat}}$. $\square$

Definition 102. Let $(A, \alpha), (B, \beta)$ be two dialectal operators of carrier $p + r, r + q$ and dialects $\mathfrak{A}, \mathfrak{B}$ such that $\text{rad}(A, B) < 1$. The execution of A and B is defined as the dialectal operator $(A :: B, \alpha \otimes \beta)$ of carrier $p + q$ and dialect $\mathfrak{A} \otimes \mathfrak{B}$ where $A :: B$ is defined as:

$$A :: B = (pA^{\dagger \mathfrak{B}} + q)(1 - B^{\ddagger \mathfrak{A}} A^{\dagger \mathfrak{B}})^{-1}(p + B^{\ddagger \mathfrak{A}} q)$$

When $r = 0$, we have $B^{\ddagger \mathfrak{A}} A^{\dagger \mathfrak{B}} = 0$, and we will write

$$A \cup B = A :: B = pA^{\dagger \mathfrak{B}} p + qB^{\ddagger \mathfrak{A}} q = A^{\dagger \mathfrak{B}} + B^{\ddagger \mathfrak{A}}$$

Proposition 103 (Adjunction, Girard (Gir07)). *Let $(F, \phi), (G, \gamma), (H, \rho)$ be dialectal operators of respective carriers p, q, r such that $qr = 0$. Then:*

$$\|F, G \cup H\|_{\text{mat}} = \rho(1_{\mathfrak{H}})\|F, G\|_{\text{mat}} + \|H, F :: G\|_{\text{mat}}$$

Definition 104. If (A, α) and (B, β) are dialectal operators of equal carrier p and respective dialects $\mathfrak{A}, \mathfrak{B}$, and if $\lambda \in \mathbf{R}$ is a real number, one can define $\lambda(A, \alpha) + (B, \beta)$ as the dialectal operator $(A + B, \lambda\alpha \oplus \beta)$ of carrier p and dialect $\mathfrak{A} \oplus \mathfrak{B}$.

Definition 105 (Projets). A hyperfinite project is a couple $\mathfrak{a} = (a, (A, \alpha))$ where (A, α) is a dialectal operator and $a \in \mathbf{R} \cup \{\infty\}$ is the wager.

Definition 106 (Variants). Let $(A, \alpha), (B, \beta)$ be dialectal operators. If there exists $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ an isomorphism such that $(B, \beta) = (\text{Id}_{\mathfrak{R}_{0,1}} \otimes \phi(A), \alpha \circ \phi^{-1})$, we will say that (B, β) is a variant of (A, α) .

We can now define the measurement $\ll \mathfrak{a}, \mathfrak{b} \gg_{\text{mat}}$ between hyperfinite projects as:

$$\ll \mathfrak{a}, \mathfrak{b} \gg_{\text{mat}} = \alpha(1_{\mathfrak{A}})b + \beta(1_{\mathfrak{B}})a + \ll A, B \gg_{\text{mat}}$$

We then follow the constructions of the hyperfinite GoI model described above to define multiplicative connectives, behaviors, additive connectives, and exponential connectives. Indeed, the key properties used in the constructions of connectives of Linear Logic are the following the associativity of execution and the adjunction, both of which hold in the two constructions. These two slightly different GoI models actually only differ on their measurement between operators: $\ll A, B \gg_{\text{hyp}} = -\log(\det(1 - AB))$ in the hyperfinite GoI model and $\ll A, B \gg_{\text{mat}}$ in the matricidal GoI model.

4.3. A Few Technical Lemmas

Lemma 107 (Products of Partial Isometries). *Let u, v be partial isometries, both in the normalizing groupoid $G_{\mathfrak{N}}(\mathfrak{P})$ of a MASA $\mathfrak{P}$ of a von Neumann algebra $\mathfrak{N}$. Then uv is a partial isometry, and uv is in $G_{\mathfrak{N}}(\mathfrak{P})$.*

Proof. Since u and v are in the normalizing groupoid of $\mathfrak{P}$, we have that $p = u^*u \in \mathfrak{P}$ and $q = vv^* \in \mathfrak{P}$. Moreover, since $\mathfrak{P}$ is commutative, $pq = qp$. Thus:

$$(uv)(uv)^*(uv) = uvv^*u^*uv = uqp v = upqv = uu^*uvv^*v = uv$$

We just showed that $(uv)(uv)^*(uv) = uv$, and therefore uv is a partial isometry. Finally, since the projections $uv(uv)^*$ and $(uv)^*uv$ are elements of $\mathfrak{P}$ and $uv\mathfrak{P}(uv)^* \subset \mathfrak{P}$, uv is in the normalizing groupoid of $\mathfrak{A}$. $\square$

Lemma 108 (Sums of Partial Isometries). *Let u, v be partial isometries in the normalizing groupoid $G_{\mathfrak{R}_{0,1}}(\mathfrak{P})$ of a MASA $\mathfrak{P}$ in a von Neumann algebra $\mathfrak{N}$. If $uv^* = u^*v = 0$, then the sum $u + v$ is a partial isometry and belongs to $G_{\mathfrak{R}_{0,1}}(\mathfrak{P})$.*

Proof. We supposed that $(uv^*)^* = vu^* = 0$, which implies that $(u^*v)^* = v^*u = 0$. We can then compute:

$$(u + v)(u + v)^*(u + v) = (u + v)(u^* + v^*)(u + v) = uu^*u + vv^*v = u + v$$

This shows that $u + v$ is a partial isometry. We now have to show that it is in the normalizing groupoid of $\mathfrak{P}$. The projections uu^*, u^*u, vv^*, v^*v are all elements of $\mathfrak{P}$, and commute one with another. If a is an element of $\mathfrak{P}$:

$$(u + v)a(u + v)^* = uau^* + vav^*$$

Since u, v are in the normalizing groupoid of $\mathfrak{P}$, we conclude from this that $(u + v)a(u + v)^*$ is the sum of two elements in $\mathfrak{P}$, which is again an element in $\mathfrak{P}$. This implies that $(u +$

$$v)\mathfrak{P}(u+v)^* \subseteq \mathfrak{P}.$$

Finally, the projections $(u+v)(u+v)^* = uu^* + vv^*$ and $(u+v)^*(u+v) = u^*u + v^*v$ are also in $\mathfrak{P}$ as sums of elements in $\mathfrak{P}$. $\square$

Lemma 109. *Let A be an operator in a factor $\mathfrak{M}$, and tr be a trace on $\mathfrak{M}$. If A is nilpotent, $\text{tr}(A) = 0$.*

Proof. Let N be the degree of nilpotency of A . We define by induction a sequence of projections $(p_i)_{i=1}^N$ and a sequence of operators $(A_i)_{i=1}^N$ as follows:

- $A_1 = A$ and p_1 is the projection onto the closure of the range of A_1 ;
- $A_{i+1} = A_i p_i$ and p_{i+1} is the projection onto the closure of the range of A_{i+1} .

Notice that for all i , the projections satisfy $p_i p_{i+1} = p_i$ since $p_i \geq p_{i+1}$. A simple induction shows that $A_{i+1} = A_i p_i$ is nilpotent of degree $N - i$:

- First, notice that $A_2^{N-1} = (A_1 p_1)^{N-1} = A^{N-1} p_1$ since $p_i A_i = A_i$ by definition of p_i . Since p_1 is the projection onto the closure of the range of A , each element of the form $p_1 x$ is the limit of a sequence $(A y_i)_{i=0}^\infty$, i.e. $A^{N-1} p_1 x$ is the limit of the sequence $(A^N y_i)_{i=0}^\infty$ by continuity of A . As A is nilpotent of degree N , the sequence is equal to 0 everywhere, whence $A^{N-1} p_1 = 0$, and $A_2 = (A_1 p_1)$ is nilpotent of degree $N - 1$.
- Then we know by induction that A_i is nilpotent of degree $N - i + 1$ and that p_i is defined as the projection onto the closure of the range of A_i . We use the same argument and show that $A_{i+1} = A_i p_i$ is nilpotent of degree $N - i$ since $(A_i p_i)^{N-i} = A_i^{N-i} p_i = 0$.

As a consequence, A_N is nilpotent of degree 1, i.e. $A_N = 0$.

We now use the “traciality” of the trace, i.e. that $\text{tr}(AB) = \text{tr}(BA)$ and the fact that $A_i = p_i A_i$ to show that $\text{tr}(A) = \text{tr}(A_N)$:

$$\text{tr}(A) = \text{tr}(A_1) = \text{tr}(p_1 A_1) = \text{tr}(A_1 p_1) = \text{tr}(A_2) = \text{tr}(p_2 A_2) = \text{tr}(A_2 p_2) = \text{tr}(A_3) = \dots = \text{tr}(A_N)$$

Since $A_N = 0$, we conclude that $\text{tr}(A) = \text{tr}(A_N) = 0$. $\square$

Lemma 110. *Let $\mathfrak{A} = \mathfrak{M}_k(\mathbb{C})$ be a matrix algebra, and α a pseudo-trace on $\mathfrak{A}$. There exists a real number λ such that $\alpha = \lambda \text{tr}$ where tr is the normalized trace (i.e. $\text{tr}(1) = 1$) on $\mathfrak{A}$.*

Proof. Let us fix $\pi_1, \dots, \pi_k$ a set of minimal projections of $\mathfrak{A} = \mathfrak{M}_k(\mathbb{C})$ such that $\sum_{j=1}^k \pi_j = 1$. Since the projections π_j are equivalent in the sense of Murray and von Neumann, one can find for all $1 \leq j \leq k$ a partial isometry u_j such that $u_j u_j^* = \pi_j$ and $u_j^* u_j = \pi_1$. Using the “traciality” of α we obtain:

$$\alpha(\pi_j) = \alpha(u_j u_j^*) = \alpha(u_j^* u_j) = \alpha(\pi_1)$$

We now write $\lambda = k \times \alpha(\pi_1)$.

We now show that $\alpha(p) = \lambda \text{tr}(p)$ for all projection p in $\mathfrak{M}_k(\mathbb{C})$. If p is such a projection, there exists a partial isometry w between p and a sum $\sum_{j=1}^m \pi_j$ where j is an integer between 1 and k . The “traciality” of α and tr then respectively imply that $\alpha(p) = \alpha(\sum_{j=1}^m \pi_j)$ and $\text{tr}(p) = \text{tr}(\sum_{j=1}^m \pi_j)$. We thus obtain:

$$\alpha(p) = \alpha\left(\sum_{j=1}^m \pi_j\right) = \sum_{j=1}^m \alpha(\pi_j) = \sum_{j=1}^m \lambda \text{tr}(\pi_j) = \lambda \text{tr}\left(\sum_{j=1}^m \pi_j\right) = \lambda \text{tr}(p)$$

The two linear forms α and λtr are therefore equal on the set of all projections and continuous: they are therefore equal on the whole algebra since the latter is generated by its projections. We are left with showing that λ is a real number. This is a straightforward consequence of the equality $\overline{\alpha(a)} = \alpha(a^*)$: if a is self-adjoint, then $\alpha(a) \in \mathbf{R}$; in particular, a projection π_1 is self-adjoint and $\text{tr}(\pi_1) = 1/k$ is in $\mathbf{R}$. Finally, $\lambda = \alpha(\pi_1)/\text{tr}(\pi_1)$ is a real number. $\square$

Lemma 111. *Let $\mathfrak{A} = \bigoplus_{i=1}^l \mathfrak{M}_{k_i}(\mathbf{C})$ be a direct sum of matrix algebras, and α a pseudo-trace on $\mathfrak{A}$. There exists a family $(\lambda_i)_{i=1}^l$ of real numbers such that $\alpha = \bigoplus_{i=1}^l \lambda_i \text{tr}_{\mathfrak{M}_{k_i}(\mathbf{C})}$.*

Proof. We write p_i ($i = 1, \dots, l$) the projection of $\mathfrak{A}$ onto the algebra $\mathfrak{M}_{k_i}(\mathbf{C})$. If $a \in \mathfrak{A}$, we have $a = \bigoplus_{i=1}^l a_i$, from which we obtain that $\alpha(a) = \sum_{i=1}^l \alpha(a_i)$. We therefore write α_i the restriction of α to the algebra $\mathfrak{M}_{k_i}(\mathbf{C})$, i.e. $\alpha_i = \alpha(p_i \iota(x) p_i)$ where $\iota : \mathfrak{M}_{k_i}(\mathbf{C}) \rightarrow \mathfrak{A}$ is the canonical injection. Since α is a pseudo-trace, α_i is a pseudo-trace on $\mathfrak{M}_{k_i}(\mathbf{C})$ and, by Lemma 110 there exists a real number λ_i such that $\alpha_i = \lambda_i \text{tr}_{\mathfrak{M}_{k_i}(\mathbf{C})}$. Finally, for all $a \in \mathfrak{A}$, we have:

$$\alpha(a) = \sum_{i=1}^l \alpha(a_i) = \sum_{i=1}^l \alpha_i(a_i) = \sum_{i=1}^l \lambda_i \text{tr}_{\mathfrak{M}_{k_i}(\mathbf{C})}(a_i) = \left(\bigoplus_{i=1}^l \lambda_i \text{tr}_{\mathfrak{M}_{k_i}(\mathbf{C})} \right)(a)$$

Which concludes the proof. $\square$

Lemma 112. *If a dialectal operator (A, α) is nilpotent, then $\text{ldet}(1 - A) = 0$.*

Proof. By the equality $\text{rad}(A) = \liminf_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$ we have that $\text{rad}(A) = 0$ since $\|A^k\| = 0$ for all big enough k (i.e. k greater than the nilpotency degree N of A). Thus $\text{ldet}(1 - A)$ is computed as the series $\sum_{i=1}^{\infty} \frac{\text{tr} \otimes \alpha(A^i)}{i}$, which we can immediately simplify as $\sum_{i=1}^N \frac{\text{tr} \otimes \alpha(A^i)}{i}$. Let us pick an integer k in $\{1, 2, \dots, N\}$, and suppose that $\text{tr} \otimes \alpha(A^k) \neq 0$. By Lemma 111 the pseudo-trace α can be written as $\alpha = \bigoplus_{i=1}^a \alpha_i \text{tr}_{d_i}$ where α_i are real numbers and the traces tr_{d_i} are normalized traces on type I_n factors. Moreover, A can be written as a direct sum $\bigoplus_{i=1}^a A_i$. Since A is nilpotent, each A_i is nilpotent. Therefore, if we prove that $\text{tr} \otimes \text{tr}_{d_i}(A_i) = 0$, we will be able to conclude. But $\text{tr} \otimes \text{tr}_{d_i}$ is a trace (not merely a pseudo-trace), and this is a consequence of Lemma 109. $\square$

4.4. Subjective Truth in the Matricial GoI Model

Definition 113 (Outlook). An outlook is a MASA in $\mathfrak{R}_{0,1}$.

Definition 114 (Promising Project). An hyperfinite project $\alpha = a \cdot + \cdot \alpha + A$ is *promising w.r.t the outlook $\mathfrak{P}$* if:

- **Dialect.** The dialect $\mathfrak{A}$ is a finite factor, i.e. a matrix algebra;
- **Pseudo-Trace.** α is the normalized trace on $\mathfrak{A}$;
- **Wager.** α is wager-free: $\alpha = 0$;
- **Symmetry.** A is a partial symmetry in the normalizing groupoid $\mathfrak{P} \otimes \mathfrak{Q}$, where $\mathfrak{Q}$ is a MASA in $\mathfrak{A}$;
- **Traces.** for all projection $\pi \in \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{A}}(\mathfrak{Q})$, $\text{tr}(\pi A) = 0$.

We remark here that the last “trace condition” is an addition to Girard’s condition of successful projects (see Remark 121). This is however a quite natural condition as a successful or promising project should be understood as the representation of a set of axiom links in a proof nets, i.e. it should be a symmetry without any fixpoints. Even though the alternative definition of success without this additional condition would be satisfactory (it would satisfy coherence and compositionality), it does not convey the intuitions coming from proof nets.

Remark 115. If α is a promising project, we can suppose that $\mathfrak{A}$ is a factor. Indeed, we already shows that if ϕ is an injective morphism from $\mathfrak{A}$ into $\mathfrak{B}$, then α^ϕ is universally equivalent to α in the sens that $\ll \alpha, b \gg = \ll \alpha^\phi, b \gg$ for all hyperfinite project b . Thus, $\alpha \in \mathbf{A}$ if and only if $\alpha^\phi \in \mathbf{A}$.

Definition 116. A conduct $\mathbf{A}$ is *correct w.r.t. the outlook* $\mathfrak{A}$ if there exists a hyperfinite project $\alpha \in \mathbf{A}$ which is promising w.r.t. $\mathfrak{A}$.

We now check that the notion of promising project satisfies the essential properties: compositionnality and coherence.

Proposition 117 (Coherence). *Let $\mathfrak{P}$ be an outlook. The two conducts $\mathbf{A}$ and $\mathbf{A}^\perp$ cannot both contain a promising project w.r.t. $\mathfrak{P}$.*

Proof. Suppose that f, g are promising hyperfinite projects in $\mathbf{A}$ and $\mathbf{A}^\perp$ respectively.

We will show that $\text{ldet}(1 - F^{\dagger\otimes} G^{\ddagger\otimes}) = 0$. Since $f \perp g$, we know that $\rho(F^{\dagger\otimes} G^{\ddagger\otimes}) < 1$. In other terms:

$$\liminf \| (F^{\dagger\otimes} G^{\ddagger\otimes})^k \|^{1/k} < 1 \quad (1)$$

Let $\mathfrak{P}_{\mathfrak{F}}$ be a MASA in $\mathfrak{F}$ and $\mathfrak{P}_{\mathfrak{G}}$ a MASA in $\mathfrak{G}$ such that F and G are in the normalizing groupoids of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}}$ and $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{G}}$ respectively.

Since F and G are partial symmetries in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}}$ and $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{G}}$ respectively, it is clear that $F^{\dagger\otimes}$ and $G^{\ddagger\otimes}$ are again partial symmetries, and we can show¹² that they are both in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$, a MASA of $\mathfrak{R}_{0,1} \otimes \mathfrak{F} \otimes \mathfrak{G}$. Moreover, they are *a fortiori* partial isometries, and Lemma 107 ensures that $F^{\dagger\otimes} G^{\ddagger\otimes}$ is a partial isometry in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$. One easily shows in this way that for all $k \in \mathbf{N}$, $(F^{\dagger\otimes} G^{\ddagger\otimes})^k$ is a partial isometry. Since the norm of a partial isometry is necessarily equal to 0 or 1, we deduce that for all $k \in \mathbf{N}$, $\|(F^{\dagger\otimes} G^{\ddagger\otimes})^k\| = 0$ or $\|(F^{\dagger\otimes} G^{\ddagger\otimes})^k\| = 1$. Using Equation 1, we conclude that there exists $N \in \mathbf{N}$ such that $\|(F^{\dagger\otimes} G^{\ddagger\otimes})^N\| = 0$, i.e. $(F^{\dagger\otimes} G^{\ddagger\otimes})^N = 0$. Finally, this shows that $F^{\dagger\otimes} G^{\ddagger\otimes}$ is nilpotent, which implies that $\text{ldet}(1 - F^{\dagger\otimes} G^{\ddagger\otimes}) = 0$ by Lemma 112. $\square$

The following proposition shows that the notion of promising project is compositional. The statement contains however a small condition on the types, as we need to use elements in the orthogonal types in the proof. These conditions are however non-restrictive as they exclude a case of composition of proofs that do not arise in the interpretation of sequent calculus.

¹² Using the fact that the unit of a von Neumann algebra $\mathfrak{N}$ is contained in all MASA of $\mathfrak{N}$.

Proposition 118 (Compositionality). *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be conducts such that $\mathbf{A}$ and $\mathbf{C}^\perp$ are non-empty. If $f \in \mathbf{A} \multimap \mathbf{B}$ and $g \in \mathbf{B} \multimap \mathbf{C}$ are promising hyperfinite projects w.r.t. the outlook $\mathfrak{A}$, then $f :: g$ is a promising hyperfinite project w.r.t. $\mathfrak{A}$ in the conduct $\mathbf{A} \multimap \mathbf{C}$.*

Proof. Let $f = 0 \cdot + \cdot \phi + F$ and $g = 0 \cdot + \cdot \psi + G$ be the promising projects w.r.t. $\mathfrak{A}$, respectively in $\mathbf{A} \multimap \mathbf{B}$ of carrier $p + q$ and in $\mathbf{B} \multimap \mathbf{C}$ of carrier $q + r$. Let $h = f :: g = \text{ldet}(1 - F^\dagger \circ G^{\ddagger \mathfrak{A}}) \cdot + \cdot \phi \otimes \psi + F :: G$ be the hyperfinite project obtained as the execution of f and g .

- **Dialect.** It is clear that the dialect $\mathfrak{F} \otimes \mathfrak{G}$ is a finite factor, since it is the tensor product of two finite factors $\mathfrak{F}$ and $\mathfrak{G}$.
- **Pseudo-Trace.** Since ϕ and ψ are the normalized traces on $\mathfrak{F}$ and $\mathfrak{G}$ respectively, the “pseudo-trace” $\phi \otimes \psi$ is the normalized trace on $\mathfrak{F} \otimes \mathfrak{G}$.
- **Wager.** Suppose that $F^\dagger \circ G^{\ddagger \mathfrak{A}}$ is not nilpotent. By Lemma 107, it is a partial isometry, which implies that its spectral radius is either equal to 1 or 0. However, all the iterates $(F^\dagger \circ G^{\ddagger \mathfrak{A}})^n$ are non-zero partial isometries, thus of norm equal to 1. Therefore, the spectral radius of $F^\dagger \circ G^{\ddagger \mathfrak{A}}$, which is equal to $\liminf_{n \rightarrow \infty} \|(F^\dagger \circ G^{\ddagger \mathfrak{A}})^n\|^{\frac{1}{n}}$, is necessarily equal to 1. Moreover, the norm of $F^\dagger \circ G^{\ddagger \mathfrak{A}}$ is also equal to 1 since it is a non-zero partial isometry. Since $\text{rad}(F^\dagger \circ G^{\ddagger \mathfrak{A}}) = 1$, the measurement $\llbracket F, G \rrbracket_{\text{mat}}$ is equal to ∞ . Since $\mathbf{A}, \mathbf{C}^\perp$ are non-empty, we can chose $a \in \mathbf{A}$ and $c \in \mathbf{C}^\perp$ and consider $f :: a$ and $g :: c$ which are respectively in $\mathbf{B}$ and $\mathbf{B}^\perp$. Since they are orthogonal, we have that the measurement $\llbracket f :: a, g :: c \rrbracket_{\text{mat}}$ is different from 0 and ∞ . But $\llbracket f :: a, g :: c \rrbracket_{\text{mat}} = \llbracket f, (g :: c) \otimes a \rrbracket_{\text{mat}} = \llbracket f, (g \otimes a) :: c \rrbracket_{\text{mat}} = \llbracket f \otimes c, g \otimes a \rrbracket_{\text{mat}}$. Thus $\llbracket f \otimes c, g \otimes a \rrbracket_{\text{mat}} \neq \infty$. This implies however that $F^\dagger \circ G^{\ddagger \mathfrak{A}}$ has spectral radius strictly less than 1 since if λ is in the spectrum of $F^\dagger \circ G^{\ddagger \mathfrak{A}}$, it is also in the spectrum of $(F^\dagger \circ G^{\ddagger \mathfrak{A}})^\dagger \circ (G^{\ddagger \mathfrak{A}} \circ F)^\dagger \circ (G^{\ddagger \mathfrak{A}} \circ F)^\dagger \circ (G^{\ddagger \mathfrak{A}} \circ F)^\dagger$. This is a contradiction, and we can conclude that $F^\dagger \circ G^{\ddagger \mathfrak{A}}$ is nilpotent. We can now conclude that $\text{ldet}(1 - F^\dagger \circ G^{\ddagger \mathfrak{A}}) = 0$ by Lemma 112.
- **Symmetry.** We will abusively denote by p, r the tensor products $p \otimes 1_{\mathfrak{F}} \otimes 1_{\mathfrak{G}}$ and $r \otimes 1_{\mathfrak{F}} \otimes 1_{\mathfrak{G}}$ in order to simplify the expression and make the computations more readable. Since $F^\dagger \circ G^{\ddagger \mathfrak{A}}$ is nilpotent, we have:

$$\begin{aligned}
 F :: G &= (pF^\dagger \circ G^{\ddagger \mathfrak{A}} + r)(1 - G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^{-1}(p + G^{\ddagger \mathfrak{A}} r) \\
 &= (pF^\dagger \circ G^{\ddagger \mathfrak{A}} + r) \left(\sum_{k=0}^{N-1} (G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^k \right) (p + G^{\ddagger \mathfrak{A}} r) \\
 &= \sum_{k=0}^{N-1} pF^\dagger \circ G^{\ddagger \mathfrak{A}} (G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^k p + \sum_{k=1}^{N-1} r(G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^k p \\
 &\quad + \sum_{k=0}^{N-2} pF^\dagger \circ G^{\ddagger \mathfrak{A}} (G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^k G^{\ddagger \mathfrak{A}} r + \sum_{k=0}^{N-1} r(G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^k G^{\ddagger \mathfrak{A}} r \\
 &= \sum_{k=0}^{N-1} pF^\dagger \circ G^{\ddagger \mathfrak{A}} (G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^k p + \sum_{k=0}^{N-1} r(G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^k G^{\ddagger \mathfrak{A}} r \\
 &\quad + \sum_{k=1}^{N-1} (r(G^{\ddagger \mathfrak{A}} F^\dagger \circ G^{\ddagger \mathfrak{A}})^k p + p(F^\dagger \circ G^{\ddagger \mathfrak{A}})^k r)
 \end{aligned}$$

We define:

$$\begin{aligned} t_k &= F^{\dagger\mathfrak{G}}(G^{\ddagger\mathfrak{F}}F^{\dagger\mathfrak{G}})^k \\ t'_k &= (G^{\ddagger\mathfrak{F}}F^{\dagger\mathfrak{G}})^k G^{\ddagger\mathfrak{F}} \\ s_k &= (G^{\ddagger\mathfrak{F}}F^{\dagger\mathfrak{G}})^k \\ s'_k &= (F^{\dagger\mathfrak{G}}G^{\ddagger\mathfrak{F}})^k \end{aligned}$$

Using the fact that F, G are partial symmetries in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}}$ and $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{G}}$ respectively and that $p, r \in \mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$, we show that the three terms t_k, t'_k and $s_k + s'_k$ are partial isometries in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$ (using Lemma 107) and are hermitian (since F and G are). Therefore, those are partial symmetries in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$.

- We have that $(pt_k p)^3 = pt_k p$ since the projections t_k^2 are subprojections of p , and this implies that $(pt_k pt_k)^2 = pt_k pt_k$. Since $pt_k pt_k = (pt_k pt_k)^*$, we obtain, composing by p on the right:

$$pt_k pt_k = t_k pt_k p \quad (2)$$

$$pt_k pt_k = pt_k pt_k p \quad (3)$$

- We show similarly that:

$$rt'_k rt_k = rt'_k rt'_k r \quad (4)$$

- One can also compute the following:

$$(ps'_k r + rs_k p)^3 = ps'_k rs_k ps'_k r + rs_k ps'_k rs_k$$

Since $(ps'_k r + rs_k p)^3 = ps'_k r + rs_k p$, composing by p on the left (resp. on the right) and r on the right (resp. on the left), we obtain $ps'_k r = ps'_k rs_k ps'_k r$ (resp. $rs_k p = rs_k ps'_k rs_k$). This implies:

$$\begin{aligned} (rs_k ps'_k)^2 &= rs_k ps'_k \\ (ps'_k rs_k)^2 &= ps'_k rs_k \end{aligned}$$

We then show that:

$$\begin{aligned} rs_k ps'_k + ps'_k rs_k &= (rs_k ps'_k + ps'_k rs_k)^* \\ &= (rs_k ps'_k)^* + (ps'_k rs_k)^* \\ &= s_k ps'_k r + s'_k rs_k p \end{aligned}$$

Composing by p (resp. r) on the right, and using the fact that $Gp = 0$ (resp. $Fr = 0$), we obtain:

$$rs_k ps'_k r = rs_k ps'_k \quad (5)$$

$$ps'_k rs_k p = ps'_k rs_k \quad (6)$$

Using Equation 3, Equation 4, Equation 5 and Equation 6, we show that the product of two distinct terms is always equal to zero. Since $pr = rp = 0$, we have five cases to consider:

– $(pt_i p)(pt_j p)$, with $i \neq j$.

We can suppose that $i < j$, since the case $j < i$ reduces to the case $i < j$ by considering the adjoints. We then have:

$$\begin{aligned} (pt_i p)(pt_j p) &= pt_i pt_j p \\ &= (pt_i p F^{\dagger \mathfrak{E}} (G^{\ddagger \mathfrak{F}} F^{\dagger \mathfrak{E}})^j p \\ &= (pt_i pt_i)(G^{\ddagger \mathfrak{F}} F^{\dagger \mathfrak{E}})^{j-i} p \\ &= (pt_i pt_i p) s_{j-i} p \end{aligned}$$

Since $p G^{\ddagger \mathfrak{F}} = 0$, we obtain $p s_{j-i} = 0$, and finally $(pt_i p)(pt_j p) = 0$.

– $(rt'_i r)(rt'_j r)$, with $i \neq j$.

This case is similar to the previous one. We treat the case $j < i$:

$$\begin{aligned} ((rt'_i r)(rt'_j r))^* &= (rt'_j r)(rt'_i r) \\ &= rt'_j rt'_i r \\ &= rt'_j r (G^{\ddagger \mathfrak{F}} F^{\dagger \mathfrak{E}})^i G^{\ddagger \mathfrak{F}} r \\ &= (rt'_j rt'_j)(F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^{i-j} r \\ &= (rt'_j rt'_j r) s'_{i-j} r \end{aligned}$$

Since $r F^{\dagger \mathfrak{E}} = 0$, we have $r s'_{i-j} = 0$, and thus $(rt'_i r)(rt'_j r) = 0$.

– $(pt_i p)(ps'_j r)$.

If $j \leq i$, we have:

$$\begin{aligned} (pt_i p)(ps'_j r) &= pt_i ps'_j r \\ &= pt_{i-j} (s_j ps'_j r) \\ &= pt_{i-j} (rs_j ps'_j r) \end{aligned}$$

If $i < j$, we have:

$$\begin{aligned} (pt_i p)(ps'_j r) &= pt_i ps'_j r \\ &= p F^{\dagger \mathfrak{E}} (G^{\ddagger \mathfrak{F}} F^{\dagger \mathfrak{E}})^i p (F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^j r \\ &= p F^{\dagger \mathfrak{E}} (G^{\ddagger \mathfrak{F}} F^{\dagger \mathfrak{E}})^i p (F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^i F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}} (F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^{j-i-1} r \end{aligned}$$

Since $F^{\dagger \mathfrak{E}}$ and $G^{\ddagger \mathfrak{F}}$ are hermitian and in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{E}}$, and since r is an element of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{E}}$, we show that the operator

$$F^{\dagger \mathfrak{E}} (G^{\ddagger \mathfrak{F}} F^{\dagger \mathfrak{E}})^i p (F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^i F^{\dagger \mathfrak{E}}$$

is an element of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{E}}$. It then commutes with p which is itself an element of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{E}}$, an abelian algebra. Finally:

$$\begin{aligned} (pt_i p)(ps'_j r) &= p (F^{\dagger \mathfrak{E}} (G^{\ddagger \mathfrak{F}} F^{\dagger \mathfrak{E}})^i p (F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^i F^{\dagger \mathfrak{E}}) G^{\ddagger \mathfrak{F}} (F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^{j-i-1} r \\ &= (F^{\dagger \mathfrak{E}} (G^{\ddagger \mathfrak{F}} F^{\dagger \mathfrak{E}})^i p (F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^i F^{\dagger \mathfrak{E}}) p G^{\ddagger \mathfrak{F}} (F^{\dagger \mathfrak{E}} G^{\ddagger \mathfrak{F}})^{j-i-1} r \end{aligned}$$

Since $p G^{\ddagger \mathfrak{F}} = 0$, we have that $(pt_i p)(ps'_j r) = 0$.

– $(rt'_i r)(rs_j p)$.

Similarly, in the case $j \leq i$, we have:

$$\begin{aligned}
 (rt'_i r)(rs_j p) &= rt'_i rs_j p \\
 &= rG^{\dagger\mathfrak{F}} s'_i rs_j p \\
 &= rG^{\dagger\mathfrak{F}} s'_{i-j} (s'_j rs_j p) \\
 &= rG^{\dagger\mathfrak{F}} s'_{i-j} (ps'_j rs_j p)
 \end{aligned}$$

If $i < j$, we obtain:

$$\begin{aligned}
 (rt'_i r)(rs_j p) &= rt'_i rs_j p \\
 &= r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^i G^{\dagger\mathfrak{F}} r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^{j-i} p \\
 &= rG^{\dagger\mathfrak{F}} (F^{\dagger\mathfrak{G}} G^{\dagger\mathfrak{F}})^i r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^i G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}} (G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^{j-i-1} p
 \end{aligned}$$

Once again, we can show that $G^{\dagger\mathfrak{F}} (F^{\dagger\mathfrak{G}} G^{\dagger\mathfrak{F}})^i r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^i G^{\dagger\mathfrak{F}}$ is an element of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$ and therefore commutes with r . Thus:

$$\begin{aligned}
 (rt'_i r)(rs_j p) &= r(G^{\dagger\mathfrak{F}} (F^{\dagger\mathfrak{G}} G^{\dagger\mathfrak{F}})^i r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^i G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}} (G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^{j-i-1} p \\
 &= (G^{\dagger\mathfrak{F}} (F^{\dagger\mathfrak{G}} G^{\dagger\mathfrak{F}})^i r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^i G^{\dagger\mathfrak{F}}) r F^{\dagger\mathfrak{G}} (G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^{j-i-1} p
 \end{aligned}$$

Since $rF^{\dagger\mathfrak{G}} = 0$, we have $(rt'_i r)(rs_j p) = 0$.

– $(rs_i p + ps'_i r)(rs_j p + ps'_j r)$, with $i \neq j$.

We suppose, without loss of generality, that $i < j$. Then:

$$\begin{aligned}
 (rs_i p + ps'_i r)(rs_j p + ps'_j r) &= rs_i ps'_j r + ps'_i rs_j p \\
 &= (rs_i ps'_i) s'_{j-i} r + (ps'_i rs_i) s_{j-i} p \\
 &= (rs_i ps'_i r) s'_{j-i} r + (ps'_i rs_i p) s_{j-i} p
 \end{aligned}$$

Since $rF = 0 = pG$, we have that $rs'_{j-i} = ps_{j-i} = 0$, and therefore $(rs_i p + ps'_i r)(rs_j p + ps'_j r) = 0$.

Finally, we showed that $F :: G$ is a sum of partial isometries that satisfy the hypotheses of Lemma 108. We then deduce that $F :: G$ is a partial isometry in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$.

— **Traces.** Let¹³ $\pi \in \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{G}})$ be a projection. We can compute $\text{tr}(\pi A)$:

$$\begin{aligned}
 \text{tr}(\pi F :: G) &= \text{tr}(\pi[(pF^{\dagger\mathfrak{G}} + r)(1 - G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^{-1}(p + G^{\dagger\mathfrak{F}} r)]) \\
 &= \text{tr}(\pi[\sum_{k=0}^{N-1} pF^{\dagger\mathfrak{G}} (G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^k p]) + \text{tr}(\pi[\sum_{k=0}^{N-1} r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^k G^{\dagger\mathfrak{F}} r]) \\
 &\quad + \text{tr}(\pi[\sum_{k=1}^{N-1} r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^k p]) + \text{tr}(\pi[\sum_{k=0}^{N-1} p(F^{\dagger\mathfrak{G}} G^{\dagger\mathfrak{F}})^k r]) \\
 &= \sum_{k=0}^{N-1} \text{tr}(\pi[pF^{\dagger\mathfrak{G}} (G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^k p]) + \sum_{k=0}^{N-1} \text{tr}(\pi[r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^k G^{\dagger\mathfrak{F}} r]) \\
 &\quad + \sum_{k=1}^{N-1} \text{tr}(\pi[r(G^{\dagger\mathfrak{F}} F^{\dagger\mathfrak{G}})^k p]) + \sum_{k=1}^{N-1} \text{tr}(\pi[p(F^{\dagger\mathfrak{G}} G^{\dagger\mathfrak{F}})^k r])
 \end{aligned}$$

¹³ Notice that $\mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R} \otimes \mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}) = \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{G}})$ by Theorem 35.

It is enough to show that $\text{tr}(\pi p t_k p) = \text{tr}(\pi r t'_k r) = 0$ and $\text{tr}(\pi r s_k p) = \text{tr}(\pi p s'_k r) = 0$ for all $1 \leq k \leq N - 1$.

$$- \text{tr}(\pi p t_k p) = 0 \text{ et } \text{tr}(\pi r t'_k r) = 0$$

We suppose, without loss of generality, that $\pi p = \pi$, i.e. that π is a subjection of p . Then:

$$\text{tr}(\pi p t_k p) = \begin{cases} \text{tr}(F^{\dagger\otimes} (G^{\ddagger\otimes} F^{\dagger\otimes})^{\frac{k}{2}} \pi (F^{\dagger\otimes} G^{\ddagger\otimes})^{\frac{k}{2}}) & \text{si } k \equiv 0[2] \\ \text{tr}(G^{\ddagger\otimes} (F^{\dagger\otimes} G^{\ddagger\otimes})^{\frac{k-1}{2}} (F^{\dagger\otimes} \pi F^{\dagger\otimes}) (G^{\ddagger\otimes} F^{\dagger\otimes})^{\frac{k-1}{2}}) & \text{si } k \equiv 1[2] \end{cases}$$

Since $F^{\dagger\otimes}$ and $G^{\ddagger\otimes}$ are in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$, we can show they are elements of $\mathfrak{N}(\mathfrak{P}_{p+q+r} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{G}}))$. Indeed:

$$\begin{aligned} \mathfrak{N}(\mathfrak{P}_{p+q+r} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}) &= \mathfrak{N}(\mathfrak{P}_{p+q+r}) \otimes \mathfrak{N}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}(\mathfrak{P}_{\mathfrak{G}}) \\ &\subset \mathfrak{N}(\mathfrak{P}_{p+q+r}) \otimes \mathfrak{N}(\mathfrak{N}(\mathfrak{P}_{\mathfrak{F}})) \otimes \mathfrak{N}(\mathfrak{N}(\mathfrak{P}_{\mathfrak{G}})) \\ &\subset \mathfrak{N}(\mathfrak{P}_{p+q+r} \otimes \mathfrak{N}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}(\mathfrak{P}_{\mathfrak{G}})) \end{aligned}$$

Thus, since π is a projection in $\mathfrak{P}_{p+q+r} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{G}})$, we deduce that the terms $(G^{\ddagger\otimes} F^{\dagger\otimes})^{\frac{k}{2}} \pi (F^{\dagger\otimes} G^{\ddagger\otimes})^{\frac{k}{2}}$ et $(F^{\dagger\otimes} G^{\ddagger\otimes})^{\frac{k-1}{2}} (F^{\dagger\otimes} \pi F^{\dagger\otimes}) (G^{\ddagger\otimes} F^{\dagger\otimes})^{\frac{k-1}{2}}$ represent projections in $\mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{G}})$.

Since $\text{tr}(Fv) = 0$ for all projection $v \in \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}})$, we can show that $\text{tr}(F^{\dagger\otimes} \mu) = 0$ for all projection $\mu \in \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{R}$ and therefore for all projection μ in the algebra $\mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{G}})$. Similarly, $\text{tr}(G^{\ddagger\otimes} \mu) = 0$ for such a projection μ , and we can conclude that $\text{tr}(\pi p t_k p) = 0$.

The case $\text{tr}(\pi r t'_k r) = 0$ is treated in a similar fashion.

$$- \text{tr}(\pi r s_k p) = 0 \text{ et } \text{tr}(\pi p s'_k r) = 0$$

Since $p = \tilde{p} \otimes 1 \otimes 1$ and since $\mathfrak{P}$ is commutative, p is an element of the center $\mathfrak{Z}(\mathfrak{P} \otimes \mathfrak{R} \otimes \mathfrak{R})$ of $\mathfrak{P} \otimes \mathfrak{R} \otimes \mathfrak{R}$ and therefore commutes with every elements in the algebra $\mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{G}})$. Similarly, the projection r is an element of $\mathfrak{Z}(\mathfrak{P} \otimes \mathfrak{R} \otimes \mathfrak{R})$ and therefore commutes with every element in $\mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{F}}) \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P}_{\mathfrak{G}})$. Then, since $pr = rp = 0$:

$$\begin{aligned} \text{tr}(\pi r s_k p) &= \text{tr}(r \pi s_k p) = \text{tr}(p r \pi s_k) = 0 \\ \text{tr}(\pi p s'_k r) &= \text{tr}(p \pi s'_k r) = \text{tr}(r p \pi s'_k) = 0 \end{aligned}$$

Finally we have shown that the project $f::g$ is wager-free, normalized, that the operator $F::G$ is a partial symmetry in the normalizing groupoid of $\mathfrak{P} \otimes \mathfrak{P}_{\mathfrak{F}} \otimes \mathfrak{P}_{\mathfrak{G}}$, and that for all projection $\pi \in \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\mathfrak{P})$, we have $\text{tr}(\pi(F::G)) = 0$. This shows that it is promising w.r.t. the outlook $\mathfrak{P}$. $\square$

4.5. Hyperfinite GoI

4.5.1. Promising and Successful Projects We first recall the notion of truth considered by Girard (Gir11) which is based on the notion of *successful hyperfinite project*. We will then exhibit a correspondence between our notion of *promising project* and the latter. This will allow us to deduce some results about Girard's GoI model.

Definition 119 (Viewpoint). A *viewpoint* is a representation π of the algebra $\mathfrak{A}_{0,1}$ onto $L^2(\mathbf{R}, \lambda)$ where λ is the Lebesgue measure, which satisfies the following conditions:

- $L^\infty(\mathbf{R}, \lambda) \subset \pi(\mathfrak{A}_{0,1})$;
- $\forall A \subset \mathbf{R}, \text{tr}(\pi^{-1}(\chi_A)) = \lambda(A)$, where χ_A is the characteristic function of A .

A viewpoint is *faithful* when the representation π is faithful (Definition 24).

If $T : \mathbf{R} \rightarrow \mathbf{R}$ is a measure-preserving transformation, one can define the isometry $[T] \in \mathcal{L}(L^2(\mathbf{R}, \lambda))$:

$$[T] : f \in L^2(\mathbf{R}, \lambda) \mapsto f \circ T \in L^2(\mathbf{R}, \lambda)$$

That $[T]$ is an isometry comes from the fact that T is measure-preserving:

$$\begin{aligned} \langle [T]f, [T]g \rangle &= \int_{\mathbf{R}} ([T]f)(x) \overline{([T]g)(x)} d\lambda(x) \\ &= \int_{\mathbf{R}} f \circ T(x) \overline{g \circ T(x)} d\lambda(x) \\ &= \int_{\mathbf{R}} f \circ T(x) \overline{g \circ T(x)} d\lambda(x) \\ &= \int_{T(\mathbf{R})} f(x) \overline{g(x)} d\lambda(x) \\ &= \int_{\mathbf{R}} f(x) \overline{g(x)} d\lambda(x) \\ &= \langle f, g \rangle \end{aligned}$$

Suppose now given $U : X \rightarrow Y$ a measure-preserving transformation, with $X, Y \subset \mathbf{R}$ measurable subsets. We define, for all map $f \in L^2(\mathbf{R}, \lambda)$, $[U]f(x) = f \circ U(x)$ if $x \in X$ and $[U]f(x) = 0$ otherwise. The operator $[U]$ thus defined is a partial isometry. Indeed, if we write p the projection in $\mathcal{L}(L^2(\mathbf{R}, \lambda))$ induced by the characteristic map of Y , then for all $f, g \in pL^2(\mathbf{R}, \lambda)$,

$$\begin{aligned} \langle [U]f, [U]g \rangle &= \int_{\mathbf{R}} ([U]f)(x) \overline{([U]g)(x)} d\lambda(x) \\ &= \int_X ([U]f)(x) \overline{([U]g)(x)} d\lambda(x) \\ &= \int_X f \circ U(x) \overline{g \circ U(x)} d\lambda(x) \\ &= \int_Y f(x) \overline{g(x)} d\lambda(x) \\ &= \int_{\mathbf{R}} f(x) \overline{g(x)} d\lambda(x) \end{aligned}$$

Moreover, it is clear that for all $f, g \in (1-p)L^2(\mathbf{R}, \lambda)$ one has $\langle [U]f, [U]g \rangle = 0$.

Definition 120. A hyperfinite project $\alpha = 0 \cdot + \cdot \alpha + A$ of carrier p is *successful* w.r.t. a viewpoint π when:

- $\pi(p) \in L^\infty(\mathbf{R})$;
- α is the normalized trace on $\mathfrak{A}$;
- there exists a basis $e_1, \dots, e_n$ of the dialect $\mathfrak{A}$ such that $A = [f]$ where f is a partial measure-preserving bijection of $\mathbf{R} \times \{1, \dots, n\}$;

— the set $\{x \in \mathbf{R} \times \{1, \dots, n\} \mid f(x) = x\}$ is of null measure.

Remark 121. We added the last condition to the definition proposed by Girard (Gir11). This condition corresponds to the trace condition in our definition of promising projects (Definition 114).

Proposition 122. *All faithful viewpoint defines an outlook.*

Proof. Let π be a faithful viewpoint, then $L^\infty(\mathbf{R}, \lambda) \subset \pi(\mathfrak{R}_{0,1})$. Since $L^\infty(\mathbf{R}, \lambda)$ is a MASA in $\mathcal{L}(L^2(\mathbf{R}, \lambda))$, $L^\infty(\mathbf{R}, \lambda)$ is equal to its commutant in $\mathcal{L}(L^2(\mathbf{R}, \lambda))$. We deduce that the commutant of $L^\infty(\mathbf{R}, \lambda)$ in $\pi(\mathfrak{R}_{0,1})$ is included in $L^\infty(\mathbf{R}, \lambda)$. But, since the algebra is commutative, the converse inclusion is also satisfied. Thus $L^\infty(\mathbf{R}, \lambda)$ is a MASA in $\pi(\mathfrak{R}_{0,1})$. Let us now consider $\mathfrak{B} = \pi^{-1}(L^\infty(\mathbf{R}, \lambda))$. This sub algebra of $\mathfrak{R}_{0,1}$ is clearly commutative. Moreover, since $x \in \mathfrak{R}_{0,1}$ commutes with the elements in $\mathfrak{B}$, then $\pi(x)$ commutes with the elements in $L^\infty(\mathbf{R}, \lambda)$. From the maximality of the latter, we deduce that $\pi(x) \in L^\infty(\mathbf{R}, \lambda)$, and therefore $x \in \mathfrak{B}$. Thus $\mathfrak{B}$ is a MASA in $\mathfrak{R}_{0,1}$. $\square$

Proposition 123. *Let π be a faithful viewpoint, and $\mathfrak{B} = \pi^{-1}(L^\infty(\mathbf{R}, \lambda))$ the outlook defined by π . If $\alpha = 0 \cdot + \cdot tr + A$ is successful w.r.t. π , then it is promising w.r.t. $\mathfrak{B}$.*

Proof. One only needs to check that A belongs to the normalizing groupoid of $\mathfrak{B} \otimes \mathfrak{Q}$ for a MASA $\mathfrak{Q}$ in $\mathfrak{A}$. Since $\mathfrak{A}$ is a finite factor, we can suppose without loss of generality (modulo considering a variant of α) that it is equal to $\mathfrak{M}_n(\mathbf{C})$ for an integer $n \in \mathbf{N}$. The basis $e_1, \dots, e_n$ of the dialect such that $\pi(A) = [f]$ where f is a partial measure-preserving bijection on $\mathbf{R} \times \{1, \dots, n\}$, defines a MASA $\mathfrak{D}$ in $\mathfrak{M}_n(\mathbf{C})$, the algebra of diagonal operators in the basis $\{e_1, \dots, e_n\}$. Let p be a projection in $\mathfrak{B} \otimes \mathfrak{D}$. We know that $\pi(p) = [\chi_U]$ where U is a measurable subset of $\mathbf{R} \times \{1, \dots, n\}$. For all $\xi \in L^2(\mathbf{R}, \lambda)$, we obtain:

$$\pi(A)^* \pi(p) \pi(A) \xi = \pi(A) \pi(p) \xi \circ f = \pi(A) \xi \circ f \circ \chi_U = \xi \circ f \circ \chi_U \circ f^{-1}$$

But $f \circ \chi_U \circ f^{-1} = \chi_V$ where V is the measurable subset of $\mathbf{R} \times \{1, \dots, n\}$ defined by $V = f(U)$. Thus $\pi(A)^* \pi(p) \pi(A) \xi = [\chi_V] \xi = \pi(q) \xi$, where q is the projection $[\chi_V]$. Since q is in $\mathfrak{B} \otimes \mathfrak{D}$, we showed that $\pi(A)$ normalizes every projection in $\mathfrak{B} \otimes \mathfrak{D}$. We can conclude that $\pi(A) \in G(\mathfrak{B} \otimes \mathfrak{D})$ by using the fact that the projections in $\mathfrak{B} \otimes \mathfrak{D}$ generate the whole algebra. As a consequence, A is a partial symmetry in the normalizing groupoid of $\mathfrak{B} \otimes \mathfrak{D}$.

Lemma 124. *Let $\phi : X \rightarrow Y$ be a measure-preserving transformation such that $[\phi] \in \mathfrak{R}_{0,1}$. Then $\text{tr}(\omega[\phi]) = \lambda(\{x \in X \mid \phi(x) = x\}) \times \omega$.*

Proof. From the linearity of the trace, we have $\text{tr}(\omega[\phi]) = \omega \times \text{tr}([\phi])$. We are thus left to show that $\text{tr}([\phi]) = \lambda(\{x \in X \mid \phi(x) = x\})$ for all partial measure-preserving transformation. This result is shown by Girard in the annex of his paper on hyperfinite geometry of interaction (Gir11). $\square$

We now are left with showing that the trace condition holds. For this, we pick a projection p in $\mathfrak{B} \otimes \mathfrak{M}_n(\mathbf{C})$ and use Lemma 124 to show:

$$\text{tr}(pA) = \lambda(\{x \in U_p \times \{1, \dots, n\} \mid f(x) = x\}) \leq \lambda(\{x \in \mathbf{R} \times \{1, \dots, n\} \mid f(x) = x\})$$

Since α is successful, we then obtain $0 \leq \text{tr}(pA) \leq 0$, from which we can conclude. $\square$

4.5.2. Adapting a Few Results

Theorem 125. *Let π be a faithful viewpoint. The associated outlook $\mathfrak{A}$ is such that for all finite projection $p \in \mathfrak{A}$, $\text{Puk}(p\mathfrak{A}p) = \{1\}$.*

Proof. For all finite projection $p \in \mathfrak{A}$, the algebra $p\mathfrak{A}p$ is a MASA in $p\mathfrak{R}_{0,1}p$ by Lemma 129. Suppose there exists a projection p such that $\text{Puk}(p\mathfrak{A}p)$ contains at least one integer greater than 2, that we will denote by k . For all regular MASA $\mathfrak{Q}$ in $\mathfrak{R}$, $\text{Puk}(p\mathfrak{A}p \otimes \mathfrak{Q})$ contains the integer k by Proposition 42. We then deduce from Proposition 41 that $p\mathfrak{A}p \otimes \mathfrak{Q}$ is not a MASA in $\mathcal{L}(L^2(p\mathfrak{R}_{0,1}p \otimes \mathfrak{R}))$. But $\pi(\mathfrak{A}) = L^\infty(\mathbf{R})$ is a MASA in $\mathcal{L}(L^2(\mathbf{R}))$, and therefore $\pi(p\mathfrak{A}p) = L^\infty(X)$ for X a measurable subset of finite measure. We deduce that $\pi(p\mathfrak{A}p)$ is a MASA in $\mathcal{L}(L^2(X))$, and thus unitarily equivalent (we write u the said unitary) to a MASA in $\mathcal{L}(L^2(p\mathfrak{R}_{0,1}p \otimes \mathfrak{R}))$ (Proposition 27). The morphism $\phi : a \mapsto u\pi(a)u^*$ is then a morphism from $p\mathfrak{R}_{0,1}p \otimes \mathfrak{Q}$ into $\phi(p\mathfrak{R}_{0,1}p \otimes \mathfrak{Q})$ such that $\text{Puk}(\phi(\mathfrak{A})) = \{1\}$ which contradicts the fact that k belongs to $\text{Puk}(\mathfrak{A})$. $\square$

Proposition 126. *There exists regular viewpoints, semi-regular viewpoints, as well as non-Dixmier-classifiable viewpoints.*

Proof. We consider $\mathfrak{R}$ in its standard representation, and we fix $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ three MASAs in $\mathfrak{R}$ such that $\mathfrak{A}$ is regular, $\mathfrak{B}$ is semi-regular, $\mathfrak{C}$ is singular, and $\text{Puk}(\mathfrak{A}) = \text{Puk}(\mathfrak{B}) = \{1\}$. We recall that we showed this is possible at the end of Proposition 2.2.3.

Since $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ have their Pukansky invariant equal to the singleton $\{1\}$, they are MASAs in $\mathcal{L}(L^2(\mathfrak{R}))$ by Proposition 40. Moreover, they are diffuse since $\mathfrak{R}$ is of type II and there consequently exist unitaries u, v, w such that $u^*\mathfrak{A}u = L^\infty([0, 1])$, $v^*\mathfrak{B}v = L^\infty([0, 1])$ and $w^*\mathfrak{C}w = L^\infty([0, 1])$. We now pick a diagonal MASA $\mathfrak{D}$ in $\mathcal{L}(\mathbb{H})$ induced by a basis, and define the following MASAs in $\mathfrak{R} \otimes \mathcal{L}(\mathbb{H})$:

$$\mathfrak{E}_{\mathfrak{A}} = \mathfrak{A} \otimes \mathfrak{D} \quad \mathfrak{E}_{\mathfrak{B}} = \mathfrak{B} \otimes \mathfrak{D} \quad \mathfrak{E}_{\mathfrak{C}} = \mathfrak{C} \otimes \mathfrak{D}$$

The unitaries $u \otimes 1, v \otimes 1, w \otimes 1 : L^2(\mathfrak{R}) \otimes \mathcal{L}(\mathbb{H}) \rightarrow L^2([0, 1]) \otimes \mathcal{L}(\mathbb{H})$ define — modulo the isomorphism between $L^2([0, 1]) \otimes \mathcal{L}(\mathbb{H})$ and $L^2(\mathbf{R})$ — representations $\pi_{\mathfrak{A}} : x \mapsto (u \otimes 1)^*x(u \otimes 1)$, $\pi_{\mathfrak{B}} : x \mapsto (v \otimes 1)^*x(v \otimes 1)$ and $\pi_{\mathfrak{C}} : x \mapsto (w \otimes 1)^*x(w \otimes 1)$ of $\mathfrak{R}_{0,1}$ onto $L^2(\mathbf{R})$ such that:

$$\pi_{\mathfrak{A}}(\mathfrak{E}_{\mathfrak{A}}) = L^\infty(\mathbf{R}) \quad \pi_{\mathfrak{B}}(\mathfrak{E}_{\mathfrak{B}}) = L^\infty(\mathbf{R}) \quad \pi_{\mathfrak{C}}(\mathfrak{E}_{\mathfrak{C}}) = L^\infty(\mathbf{R})$$

We now show that the outlooks associated to $\pi_{\mathfrak{A}}$, $\pi_{\mathfrak{B}}$ and $\pi_{\mathfrak{C}}$, i.e. the MASAs $\mathfrak{E}_{\mathfrak{A}}$, $\mathfrak{E}_{\mathfrak{B}}$ and $\mathfrak{E}_{\mathfrak{C}}$, are respectively regular, semi-regular and non-Dixmier-classifiable. It is in fact a simple application of Theorem 35. In the case of $\mathfrak{E}_{\mathfrak{A}} = \mathfrak{A} \otimes \mathfrak{D}$, we have:

$$\mathcal{N}_{\mathfrak{R} \otimes \mathcal{L}(\mathbb{H})}(\mathfrak{A} \otimes \mathfrak{D}) = \mathcal{N}_{\mathfrak{R}}(\mathfrak{A}) \otimes \mathcal{N}_{\mathcal{L}(\mathbb{H})}(\mathfrak{D}) = \mathfrak{R} \otimes \mathcal{L}(\mathbb{H})$$

Thus $\mathfrak{E}_{\mathfrak{A}}$ is regular.

In the case of $\mathfrak{E}_{\mathfrak{B}} = \mathfrak{B} \otimes \mathfrak{D}$, we obtain:

$$\mathcal{N}_{\mathfrak{R} \otimes \mathcal{L}(\mathbb{H})}(\mathfrak{B} \otimes \mathfrak{D}) = \mathcal{N}_{\mathfrak{R}}(\mathfrak{B}) \otimes \mathcal{N}_{\mathcal{L}(\mathbb{H})}(\mathfrak{D}) = \mathfrak{N} \otimes \mathcal{L}(\mathbb{H})$$

Since $\mathfrak{B}$ is semi-regular, $\mathfrak{N}$ is a proper sub factor of $\mathfrak{R}$, and therefore $\mathfrak{N} \otimes \mathcal{L}(\mathbb{H})$ is a proper

sub factor of $\mathfrak{R} \otimes \mathcal{L}(\mathbb{H})$ (we use here a theorem¹⁴ due to Tomita stating that the commutant of a tensor product is the tensor product of the commutants). Thus $\mathfrak{E}_{\mathfrak{B}}$ is semi-regular.

In the case of $\mathfrak{E}_{\mathfrak{C}} = \mathfrak{C} \otimes \mathfrak{D}$, we have:

$$\mathcal{N}_{\mathfrak{R} \otimes \mathcal{L}(\mathbb{H})}(\mathfrak{C} \otimes \mathfrak{D}) = \mathcal{N}_{\mathfrak{R}}(\mathfrak{C}) \otimes \mathcal{N}_{\mathcal{L}(\mathbb{H})}(\mathfrak{D}) = \mathfrak{C} \otimes \mathcal{L}(\mathbb{H})$$

Since the commutant of $\mathfrak{C} \otimes \mathcal{L}(\mathbb{H})$ is equal to $\mathfrak{C} \otimes \mathbf{C}$, the center of $\mathfrak{C} \otimes \mathcal{L}(\mathbb{H})$ is equal to $\mathfrak{C} \otimes \mathbf{C}$. We deduce that $\mathcal{N}_{\mathfrak{R} \otimes \mathcal{L}(\mathbb{H})}(\mathfrak{E}_{\mathfrak{C}})$ is neither equal to $\mathfrak{E}_{\mathfrak{C}}$, neither a factor. Therefore $\mathfrak{E}_{\mathfrak{C}}$ is non-Dixmier-classifiable. $\square$

The question of the existence of singular viewpoints is still open. Indeed, the method used in the preceding proof do not apply for showing the existence of singular MASAs: by writing $\mathfrak{R}_{0,1} = \mathfrak{R} \otimes \mathcal{L}(\mathbb{H})$ and choosing a MASA of the form $\mathfrak{A} \otimes \mathfrak{D}$, we impose ourselves a certain “regularity”, since $\mathfrak{D}$ is necessarily a regular MASA ($\mathcal{L}(\mathbb{H})$ is of type $\{I\}$). The existence of singular MASAs with Pukansky invariant equal to $\{1\}$ in the hyperfinite factor $\mathfrak{R}$ of type II_1 obtained by White (WS07) suggests however that such viewpoints exist.

5. Dixmier’s Classification and Linear Logic

5.1. Singular MASAs

In this section, we show that every promising project w.r.t. an outlook $\mathfrak{P}$ which is a singular MASA in $\mathfrak{R}_{0,1}$ is trivial, i.e. its operator is equal to 0. We will first show two lemmas that will be of use afterwards. We will write $\mathfrak{A}_p$ the von Neumann algebra $p\mathfrak{A}p$ where p is a projection, i.e. the restriction of $\mathfrak{A}$ to the subspace corresponding to p .

Lemma 127. *Let $\mathfrak{A}$ be a MASA in a factor $\mathfrak{M}$, and let p be a projection in $\mathfrak{A}$. If $A \in \mathfrak{M}$ normalizes $\mathfrak{A}$, and $Ap = pA$, then A normalizes $\mathfrak{A}_p$.*

Proof. We pick x in $\mathfrak{A}_p \subset \mathfrak{A}$. Then $AxA^* = y \in \mathfrak{A}$ since A normalizes $\mathfrak{A}$. Moreover, $yp = Ax A^* p = Ax(pA)^* = Ax(Ap)^* = Ax p A^* = Ax A^* = y$, and a similar argument shows that $py = y$. Thus $y = py$ and $y \in \mathfrak{A}_p$. $\square$

Remark 128. This result implies that $pGN_{\mathfrak{M}}(\mathfrak{A})p \subset GN_{\mathfrak{M}_p}(\mathfrak{A}_p)$.

The following lemma is of particular importance since it will allow us to reduce our study to the case of a factor of type II_1 , and thus to use Chifan’s result (Theorem 35). The fact that $\mathfrak{A}$ is abelian is essential here. Indeed, one can even find finite-dimensional counter-examples in the case $\mathfrak{A}$ is a non-commutative singular von Neumann sub-algebra. For instance, the subfactor $\mathfrak{A} = \mathfrak{M}_2(\mathbf{C}) \oplus \mathbf{C}$ of $\mathfrak{M}_3(\mathbf{C})$ is singular : $\mathfrak{N}_{\mathfrak{M}_3(\mathbf{C})}(\mathfrak{A}) = \mathfrak{A}$. Picking the projection $p = 0 \oplus 1 \oplus 1$ in $\mathfrak{M}_3(\mathbf{C})$, we have that $\mathfrak{A}_p$ is not singular in $(\mathfrak{M}_3(\mathbf{C}))_p$ — it is even a regular sub-algebra — since $\mathfrak{N}_{\mathfrak{M}_2(\mathbf{C})}(\mathfrak{A}_p) = \mathfrak{M}_2(\mathbf{C})$.

Lemma 129. *Let $\mathfrak{A}$ be a MASA in a von Neumann algebra $\mathfrak{M}$, and p a projection in $\mathfrak{A}$. Then $\mathfrak{A}_p$ is a maximal abelian sub-algebra of $\mathfrak{M}_p$. Moreover, if $\mathfrak{A}$ is singular, $\mathfrak{A}_p$ is singular.*

¹⁴ This theorem can be found with its proof in Takesaki’s series (Tak01; Tak03a; Tak03b).

Proof. Let v be an element of $(\mathfrak{A}_p)'$, the commutant of $\mathfrak{A}_p$ in $\mathfrak{M}_p$, and let $a \in \mathfrak{A}$. We have $pa = pap$ and $(1-p)a = (1-p)a(1-p)$ since p and $(1-p)$ are elements of $\mathfrak{A}$ — which is commutative. Using the fact that $vp = pv = v$, we obtain:

$$\begin{aligned} va &= v(pa + (1-p)a) \\ &= vpa + v(1-p)a \\ &= vpa \\ &= papv \\ &= av \end{aligned}$$

As a consequence, $(\mathfrak{A}_p)' \subset (\mathfrak{A}')_p = \mathfrak{A}_p$ from the maximality of $\mathfrak{A}$. This gives us that $(\mathfrak{A}_p)' \subset \mathfrak{A}_p$, i.e. $\mathfrak{A}_p$ is a MASA in $\mathfrak{M}_p$.

Suppose now that $\mathfrak{A}$ is a singular MASA. Pick $u \in N_{\mathfrak{M}_p}(\mathfrak{A}_p)$. By definition, u is a unitary in $\mathfrak{M}_p$, meaning that $uu^* = u^*u = p$, and therefore $up = pu = u$. Let us define $v = u + (1-p)$, which is an element of $\mathfrak{M}$. Then, since $p(1-p) = 0$ and v is a unitary: $vv^* = (u + (1-p))(u^* + (1-p)) = uu^* + (1-p) = 1 = v^*v$, et $pvp = u$.

Let us now chose $x \in \mathfrak{A}$. Since x and $1-p$ are elements of $\mathfrak{A}$ and $\mathfrak{A}$ is commutative, $ux(1-p) = u(1-p)x = up(1-p)x = 0$. Similarly, we show that $(1-p)xu = 0$. This implies that $v xv^* = uxu^* + (1-p)x(1-p)$. But, since u normalizes $\mathfrak{A}$, $uxu^* = upxpu^*$ is an element y in $\mathfrak{A}$. Moreover, since $1-p$ is in $\mathfrak{A}$, the element $(1-p)x(1-p)$ also lives in $\mathfrak{A}$. Finally, $v xv^*$ is in $\mathfrak{A}$ and $v \in N_{\mathfrak{M}}(\mathfrak{A})$. Thus $u \in pN_{\mathfrak{M}}(\mathfrak{A})p$. But $N_{\mathfrak{M}}(\mathfrak{A}) \subset \mathfrak{N}_{\mathfrak{M}}(\mathfrak{A}) = \mathfrak{A}$ ($\mathfrak{A}$ is singular), and therefore $u \in \mathfrak{A}_p$.

Since $N_{\mathfrak{M}_p}(\mathfrak{A}_p) \subset \mathfrak{A}_p$ this gives us that $\mathfrak{N}_{\mathfrak{M}_p}(\mathfrak{A}_p) = \mathfrak{A}_p$. $\square$

Theorem 130 (Singular Outlooks and Soundness). *If $\mathfrak{P}$ is a singular MASA in $\mathfrak{R}_{0,1}$, then every promising project w.r.t. the outlook $\mathfrak{P}$ is trivial.*

Proof. Let $\alpha = (p, 0, \mathfrak{R}, tr, A)$ be a promising hyperfinite project w.r.t. $\mathfrak{P}$ a singular MASA in $\mathfrak{R}_{0,1}$.

We want to show that $A \in \mathfrak{P} \otimes \mathfrak{R}$. Since A is an element of $(\mathfrak{R}_{0,1})_p \otimes \mathfrak{R}$, we know that $A\bar{p} = \bar{p}A = A$ where $\bar{p}$ is the projection $p \otimes 1$. Since A is in the normalizing groupoid $GN_{\mathfrak{R}_{0,1} \otimes \mathfrak{R}}(\mathfrak{P} \otimes \Omega)$, it is also in $GN_{(\mathfrak{R}_{0,1})_p \otimes \mathfrak{R}}(\mathfrak{P}_p \otimes \Omega)$ by Lemma 127, and it is therefore an element of the algebra $\mathfrak{N}_{(\mathfrak{R}_{0,1})_p \otimes \mathfrak{R}}(\mathfrak{P}_p \otimes \Omega)$ by Theorem 32. Now, by Theorem 35, we obtain that $A \in \mathfrak{N}_{\mathfrak{R}_{0,1}}(\mathfrak{P}_p) \otimes \mathfrak{N}_{\mathfrak{R}}(\Omega)$. Since $\mathfrak{P}$ is singular in $\mathfrak{R}_{0,1}$, $\mathfrak{P}_p$ is singular in $(\mathfrak{R}_{0,1})_p$ by Lemma 129, which shows that $A \in \mathfrak{P}_p \otimes \mathfrak{N}_{\mathfrak{R}}(\Omega)$.

Since A is promising w.r.t. $\mathfrak{P}$ also implies that for all projections $\pi \in \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\Omega)$, we have $tr(\pi A) = 0$. Since the set of projections in $\mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\Omega)$ generates the algebra $\mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\Omega)$ and since the trace is continuous, we deduce that for all $B \in \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\Omega)$, we have $tr(BA) = 0$.

Finally, since $A^3 = A = A^*$, $A^2 = r$ is a projection, with $rA = Ar = A$. Since $A \in \mathfrak{P} \otimes \mathfrak{N}_{\mathfrak{R}}(\Omega)$, we can conclude that $tr(A^2) = 0$, i.e. $A^2 = 0$ and therefore $A = A^3 = 0$. $\square$

Remark 131. Without the additional condition (about projections) in the definition of promising projects, it would be easy to find non-trivial hyperfinite projects which are promising w.r.t. a singular MASA $\mathfrak{P}$ in $\mathfrak{R}_{0,1}$. Indeed, let p, q be two projections in $\mathfrak{P}$. Then the project $\alpha = (p + q, 0, 1, \mathbf{C}, p + q)$ would then clearly be promising $\mathfrak{P}$.

One might wonder however if this condition could be weakened, asking for instance that the trace of A be zero. This condition would not be sufficient, since for all projections $p, q \in \mathfrak{P}$ such that $\text{tr}(p) = \text{tr}(q)$, the hyperfinite project $\mathfrak{b}$ defined as $(p + q, 0, 1, \mathbf{C}, p - q)$ would then be promising as $\text{tr}(p - q) = \text{tr}(p) - \text{tr}(q) = 0$.

Another weaker condition would be: for all projections $\pi \in \mathfrak{P} \otimes \mathfrak{Q}$, $\text{tr}(\pi A) = 0$. However, the following project would then be promising w.r.t. $\mathfrak{P}$ when $p, q \in \mathfrak{P}$ are projections:

$$\mathfrak{c} = (p + q, 0, \text{tr}, \mathfrak{M}_2(\mathbf{C}), \begin{pmatrix} 0 & (p + q)_{\otimes 1_{\mathfrak{P}}} \\ (p + q)_{\otimes 1_{\mathfrak{P}}} & 0 \end{pmatrix})$$

All those projects may be considered as successful, so why do we want to exclude them? The reason can be found in the relationship between the GoI interpretation of proofs and the theory of proof nets. Indeed, as it is explained in both Girard and the author's work on the interpretation of multiplicatives (Gir87b; Sei12), the GoI interpretation of a proof corresponds to a representation of the axiom links of the corresponding proof net. As a consequence, a successful project should be understood intuitively as a set of axiom links, i.e. a partial symmetry not containing any fixed point — something translated by the fact that for all non-zero vector ξ the symmetry S satisfies $S\xi \neq \xi$. In this respect, the first projects considered above should therefore not be considered as successful as they obviously do not satisfy this property. The reason why last project should also not be considered as successful is, however, more involved as it is a symmetry not containing fixed points. However the vectors ξ and $S\xi$ in this case differ only from the *dialect*, i.e. the second projection of the vector. Thinking about proof nets again, this second projection, the dialect, corresponds to *slices* in additive proof nets (Gir95b). This last project represents, in this respect, an axiom link between a formula A in a slice s_1 and the same formula A in a different slice s_2 . The reader familiar with additive proof nets should now be convinced that such a project should not be successful, as it represents something which is not a valid axiom link.

5.2. Non-Singular MASAs

In this section, we consider chosen an outlook $\mathfrak{P}$ which is either a regular or a semi-regular MASA in $\mathfrak{R}_{0,1}$. We will show a full soundness result for the sequent calculus $\text{MALL}_{\mathbf{T},0}$ (Figure 2), i.e. we interpret formulas and sequents as conducts and proofs as hyperfinite projects and we show that for all proof π of a sequent $\vdash \Gamma$, the interpretation $\|\pi\|$ is a promising project which belongs in $\|\Gamma\|$.

5.2.1. The Sequent Calculi $\text{MALL}_{\mathbf{T},0}$ We will briefly define the sequent calculus $\text{MALL}_{\mathbf{T},0}$ for which we show a soundness result. This sequent calculus was defined in order to prove a soundness result for interaction graphs (Sei14a). In this earlier work, we took into account the locativity of the framework by defining a *localized sequent calculus* $\text{locMALL}_{\mathbf{T},0}$ for which formulas have a specific location and rules are subject to constraints on the locations of the formulas appearing in the sequents. This localized version of the sequent calculus is used in order to prove a soundness result more easily as it presupposes the locativity constraints of the GoI model. The soundness result for the usual non-localized calculus is then obtained by noticing that every formula, thus sequent, and every proof

$$\begin{array}{c}
\frac{}{\vdash X_i^\perp, X_i} \text{Ax} \\
\frac{\vdash A, \Delta \quad \vdash B, \Gamma}{\vdash A \otimes B, \Delta, \Gamma} \otimes \\
\frac{\vdash A_i, \Gamma}{\vdash A_0 \oplus A_1, \Gamma} \oplus^i \\
\frac{}{\vdash \top, \Gamma} \top
\end{array}
\qquad
\begin{array}{c}
\frac{\vdash A, \Delta \quad \vdash A^\perp, \Gamma}{\vdash \Delta, \Gamma} \text{Cut} \\
\frac{\vdash A, B, \Gamma}{\vdash A \wp B, \Gamma} \wp \\
\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \& \\
\text{No rules for } \mathbf{0}.
\end{array}$$

Figure 2. Sequent calculus $\text{MALL}_{\mathbf{T}, \mathbf{0}}$

can be “localized”, i.e. interpreted as a formula, sequent or proof of the localized calculus. We will here define directly localized interpretations of the non-localized sequent calculus in order to limit the space needed to show the results.

Let us fix $\mathcal{V} = \{X_i\}_{i,j \in \mathbf{N}}$ a set of variables.

Definition 132 (Formulas of $\text{MALL}_{\mathbf{T}, \mathbf{0}}$). The formulas of $\text{MALL}_{\mathbf{T}, \mathbf{0}}$ are defined by the following grammar:

$$F := X_i \mid X_i^\perp \mid F \otimes F \mid F \wp F \mid F \& F \mid F \oplus F \mid \mathbf{0} \mid \mathbf{T}$$

where the X_i are variables.

Definition 133 (Proofs of $\text{MALL}_{\mathbf{T}, \mathbf{0}}$). A proof of $\text{MALL}_{\mathbf{T}, \mathbf{0}}$ is a derivation obtained from the sequent calculus rules shown in Figure 2.

5.2.2. Interpretation of Formulas

Definition 134 (Delocations). Let p, q be projections in $\mathfrak{P}$. A *delocation* from p onto q is a partial isometry $\theta : p \rightarrow q$ such that $\theta \in GN_{\mathfrak{A}_{0,1}}(\mathfrak{P})$.

To interpret the sequent calculus, we will actually work with the MASA $\mathfrak{P} \oplus \mathfrak{P}$ of the algebra $\mathfrak{M}_2(\mathfrak{A}_{0,1})$ in order to distinguish a *primitive space* (the first component of the direct sum $\mathfrak{P} \oplus \mathfrak{P}$) and an *interpretation space* (the second component of the direct sum). Interpretations of proofs and formulas will be elements of the interpretation space, hence the interpretation will in fact take place in $\mathfrak{A}_{0,1}$, while the primitive space will be used in order to define correctly the syntax. The following proposition shows that, since the interpretations will be hyperfinite projects defined in the second component of the sum $\mathfrak{P} \oplus \mathfrak{P}$, the fact that they are promising w.r.t. $\mathfrak{P} \oplus \mathfrak{P}$ in $\mathfrak{M}_2(\mathfrak{A}_{0,1})$ implies that their restriction to $\mathfrak{A}_{0,1}$ (the second component) is promising w.r.t. $\mathfrak{P}$.

Proposition 135 (Restriction). *Let $a = (p, 0, tr, \mathfrak{A}, A)$ be a promising project w.r.t. $\mathfrak{P} \oplus \mathfrak{P} \subset \mathfrak{M}_2(\mathfrak{A}_{0,1})$ such that $p \leq 0 \oplus 1$. Then $A(0 \oplus 1) = (0 \oplus 1)A = A$, and a is a promising project w.r.t. $\mathfrak{P} \subset (\mathfrak{M}_2(\mathfrak{A}_{0,1}))_{0 \oplus 1} \simeq \mathfrak{A}_{0,1}$.*

Proof. It is clear that $\mathfrak{P}$ is a MASA in $(\mathfrak{M}_2(\mathfrak{A}_{0,1}))_{0 \oplus 1}$. The result is then a direct consequence of Lemma 127. $\square$

Let us now define variables. We pick a family of pairwise disjoint projections $(p_i)_{i \in \mathbf{N}}$.

The projections $p_i \oplus 0$ will be called the *primitive locations* of the variables, and one should think of this as our actual set of variables.

Definition 136 (Variable names). A *variable name* is an integer $i \in \mathbf{N}$ denoted by capital letters X, Y, Z , etc. A *variable* is a couple $X_\theta = (X, \theta)$ where X is a variable name, i.e. an integer i , and θ is a relocation of $p_i \oplus 0$ onto a projection $0 \oplus q_{X_\theta}$. The projection $0 \oplus q_{X_\theta}$ is referred to as the *location* of the variable, and we will sometimes allow ourselves to forget about the first component and simply write q_{X_θ} .

We now define the interpretation of formulas.

Definition 137 (Interpretation Basis). An *interpretation basis* is a map δ associating to each variable name $X = i$ a behavior $\delta(X)$ of carrier the primitive location p_i of X . This map extends to function $\tilde{\delta}$ which associates, to each variable X_θ , the behavior $\tilde{\delta}(X_\theta) = \theta(\delta(X))$ of carrier q_{X_θ} — the location of X_θ . This extension will be abusively referred to as a *interpretation basis*.

Definition 138 (Interpretation of Formulas). The interpretation $\|F\|_\delta$ of a formula F along the interpretation basis δ is defined inductively as follows:

- $F = X_\theta$. We define $\|F\|_\delta$ as the behavior $\tilde{\delta}(X_\theta)$ of carrier q_{X_θ} ;
- $F = X_\theta^\perp$. We define $\|F\|_\delta = (\|X_\theta\|_\delta)^\perp$, a behavior of carrier q_{X_θ} ;
- $F = A \star B$ ($\star \in \{\otimes, \wp, \&, \oplus\}$). We define $\|F\|_\delta = \|A\|_\delta \star \|B\|_\delta$, a behavior of carrier $p + q$, where p and q are the respective carriers of $\|A\|_\delta$ et $\|B\|_\delta$;
- $F = \mathbf{T}$ (resp. $F = \mathbf{0}$). We define $\|F\|_\delta = \mathbf{T}_0$ (resp. $\mathbf{0}_0$).

Definition 139 (Interpretation of Sequents). A sequent $\vdash \Gamma$ will be interpreted as the $\wp$ of formulas in Γ , denoted by $\wp \Gamma$.

5.2.3. Interpretation of proofs The introduction rule of the $\wp$ as well as the exchange rule will have a trivial interpretation, since premise and conclusion saquants are interpreted by the same behavior (due to locativity, the commutativity of $\wp$). Similarly, rules for $\oplus$ have a easy interpretation as it suffices to extend the carrier of the project interpreting the premise to define the interpretation of the conclusion. Moreover, the rule $\top_p$ has a straightforward interpretation as the hyperfinite project $(0, 0, 1_{\mathbf{C}}, \mathbf{C}, 0)$. Axioms will be easily interpreted by delocations, whose existence is ensured by Theorem 34. The case of cut has already been treated in Proposition 118, and we therefore only need to deal with the introduction rules of $\otimes$ and $\&$.

Given two hyperfinite projects f and g in the interpretations of the premises of an $\otimes$ introduction rule, we will define a hyperfinite project h in the interpretation of the conclusion. The operation that naturally comes to mind is to define this project as the tensor product of the projects f and g . It turns out that this interpretation of the $\otimes$ introduction rule is perfectly satisfactory: the following proposition shows that the project h defined as $f \otimes g$ is a project in the interpretation of the conclusion.

Proposition 140 (Interpretation of the Tensor Rule). Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be conducts of respective supports p_A, p_B, p_C, p_D . We have the following inclusion:

$$((\mathbf{A} \multimap \mathbf{B}) \otimes (\mathbf{C} \multimap \mathbf{D})) \subset ((\mathbf{A} \otimes \mathbf{C}) \multimap (\mathbf{B} \otimes \mathbf{D}))$$

Proof. We show that $(\mathbf{A} \otimes \mathbf{C}) \multimap (\mathbf{B} \otimes \mathbf{D})$ contains the conduct $(\mathbf{A} \multimap \mathbf{B}) \otimes (\mathbf{C} \multimap \mathbf{D})$, for all conducts $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$.

We denote by p_A, p_B, p_C and p_D the respective supports of the conducts $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$. Let $\mathfrak{f} \in \mathbf{A} \multimap \mathbf{B}$ and $\mathfrak{g} \in \mathbf{C} \multimap \mathbf{D}$ be the projects:

$$\begin{aligned}\mathfrak{f} &= (p_A + p_B, f, \phi, \mathfrak{F}, F) \\ \mathfrak{g} &= (p_C + p_D, g, \psi, \mathfrak{G}, G)\end{aligned}$$

We will show that for all projects $\mathfrak{a} = (p_A, a, \alpha, \mathfrak{A}, A)$ and $\mathfrak{c} = (p_C, c, \gamma, \mathfrak{C}, C)$ in $\mathbf{A}$ and $\mathbf{C}$ respectively, the result of the execution $(\mathfrak{f} \otimes \mathfrak{g}) :: (\mathfrak{a} \otimes \mathfrak{c})$ is an element of the conduct $\mathbf{B} \otimes \mathbf{D}$.

We have:

$$(\mathfrak{f} \otimes \mathfrak{g}) :: (\mathfrak{a} \otimes \mathfrak{c}) = (p_B + p_D, w, \nu, \mathfrak{N}, P)$$

where

$$\begin{aligned}w &= \phi \otimes \psi (1_{\mathfrak{F} \otimes \mathfrak{G}})(a\gamma(1_{\mathfrak{C}}) + c\alpha(1_{\mathfrak{A}})) \\ &\quad + \alpha \otimes \gamma (1_{\mathfrak{A} \otimes \mathfrak{C}})(f\psi(1_{\mathfrak{G}}) + g\phi(1_{\mathfrak{F}})) \\ &\quad + \text{ldet}(1 - (F^\dagger_{\mathfrak{F}} + G^\dagger_{\mathfrak{G}})^{\dagger_{\mathfrak{A} \otimes \mathfrak{C}}}(A^\dagger_{\mathfrak{C}} + C^\dagger_{\mathfrak{A}})^{\dagger_{\mathfrak{F} \otimes \mathfrak{G}}}) \\ \nu &= \phi \otimes \psi \otimes \alpha \otimes \gamma \\ \mathfrak{N} &= \mathfrak{F} \otimes \mathfrak{G} \otimes \mathfrak{A} \otimes \mathfrak{C} \\ P &= (F^\dagger_{\mathfrak{F}} + G^\dagger_{\mathfrak{G}})^{\dagger_{\mathfrak{A} \otimes \mathfrak{C}}} :: (A^\dagger_{\mathfrak{C}} + C^\dagger_{\mathfrak{A}})^{\dagger_{\mathfrak{F} \otimes \mathfrak{G}}}\end{aligned}$$

We are going to show that this project is a variant of the project $(\mathfrak{f} :: \mathfrak{a}) \otimes (\mathfrak{g} :: \mathfrak{c})$. We will then conclude that $(\mathfrak{f} \otimes \mathfrak{g}) :: (\mathfrak{a} \otimes \mathfrak{c}) \in (\mathbf{A} \otimes \mathbf{C}) \multimap (\mathbf{B} \otimes \mathbf{D})$ using Lemma 101.

- It is clear that $\mathfrak{N}$ is equal to the dialect of $(\mathfrak{f} :: \mathfrak{a}) \otimes (\mathfrak{g} :: \mathfrak{c})$ up to a commutativity isomorphism. Indeed, the dialect of $(\mathfrak{f} :: \mathfrak{a}) \otimes (\mathfrak{g} :: \mathfrak{c})$ is equal to $\mathfrak{F} \otimes \mathfrak{A} \otimes \mathfrak{G} \otimes \mathfrak{C}$ and the morphism $\phi = \text{Id}_{\mathfrak{F}} \otimes \tau \otimes \text{Id}_{\mathfrak{C}}$ is a isomorphism between this algebra and $\mathfrak{N}$;
- It is also clear that ν is equal to $\mu \circ \phi^{-1}$ where μ is the pseudo-trace of the project $(\mathfrak{f} :: \mathfrak{a}) \otimes (\mathfrak{g} :: \mathfrak{c})$;
- Since F and A are elements of $(\mathfrak{N}_{0,1})_{p_A + p_B}$ and G and C are elements of $(\mathfrak{N}_{0,1})_{p_C + p_D}$, we deduce from the pairwise disjointness of projections that:

$$P = (F^\dagger_{\mathfrak{F}})^{\dagger_{\mathfrak{A} \otimes \mathfrak{C}}} :: (A^\dagger_{\mathfrak{C}})^{\dagger_{\mathfrak{F} \otimes \mathfrak{G}}} + (G^\dagger_{\mathfrak{G}})^{\dagger_{\mathfrak{A} \otimes \mathfrak{C}}} :: (C^\dagger_{\mathfrak{A}})^{\dagger_{\mathfrak{F} \otimes \mathfrak{G}}}$$

Once again, this is equal, modulo ϕ , to the operator of the project $(\mathfrak{f} :: \mathfrak{a}) \otimes (\mathfrak{g} :: \mathfrak{c})$:

$$(F^\dagger_{\mathfrak{F}} :: A^\dagger_{\mathfrak{C}})^{\dagger_{\mathfrak{C} \otimes \mathfrak{G}}} + (G^\dagger_{\mathfrak{G}} :: C^\dagger_{\mathfrak{A}})^{\dagger_{\mathfrak{A} \otimes \mathfrak{F}}}$$

- Using the fact that $\phi \otimes \psi (1_{\mathfrak{F} \otimes \mathfrak{G}}) = \phi \otimes \psi (1_{\mathfrak{F}} \otimes 1_{\mathfrak{G}}) = \phi(1_{\mathfrak{F}})\psi(1_{\mathfrak{G}})$ and that $\alpha \otimes \gamma (1_{\mathfrak{A} \otimes \mathfrak{C}}) = \alpha(1_{\mathfrak{A}})\gamma(1_{\mathfrak{C}})$, we obtain:

$$\begin{aligned}w &= \gamma(1_{\mathfrak{C}})\psi(1_{\mathfrak{G}})(a\phi(1_{\mathfrak{F}}) + f\alpha(1_{\mathfrak{A}})) \\ &\quad + \alpha(1_{\mathfrak{A}})\phi(1_{\mathfrak{F}})(c\psi(1_{\mathfrak{G}}) + g\gamma(1_{\mathfrak{C}})) \\ &\quad + \text{ldet}(1 - (F^\dagger_{\mathfrak{F}} + G^\dagger_{\mathfrak{G}})^{\dagger_{\mathfrak{A} \otimes \mathfrak{C}}}(A^\dagger_{\mathfrak{C}} + C^\dagger_{\mathfrak{A}})^{\dagger_{\mathfrak{F} \otimes \mathfrak{G}}})\end{aligned}$$

Moreover, since $AG = 0$, Lemma 98 allows us to conclude:

$$\begin{aligned}
 & \text{ldet}(1 - (F^{\dagger\mathfrak{C}} + G^{\dagger\mathfrak{F}})^{\dagger\mathfrak{A}\otimes\mathfrak{C}}(A^{\dagger\mathfrak{C}} + C^{\dagger\mathfrak{A}})^{\dagger\mathfrak{F}\otimes\mathfrak{C}}) \\
 = & \text{ldet}(1 - ((F^{\dagger\mathfrak{C}})^{\dagger\mathfrak{A}\otimes\mathfrak{C}}(A^{\dagger\mathfrak{C}})^{\dagger\mathfrak{F}\otimes\mathfrak{C}} + (G^{\dagger\mathfrak{F}})^{\dagger\mathfrak{A}\otimes\mathfrak{C}}(C^{\dagger\mathfrak{A}})^{\dagger\mathfrak{F}\otimes\mathfrak{C}})) \\
 = & \text{ldet}(1 - (F^{\dagger\mathfrak{C}})^{\dagger\mathfrak{A}\otimes\mathfrak{C}}(A^{\dagger\mathfrak{C}})^{\dagger\mathfrak{F}\otimes\mathfrak{C}}) + \text{ldet}(1 - (G^{\dagger\mathfrak{F}})^{\dagger\mathfrak{A}\otimes\mathfrak{C}}(C^{\dagger\mathfrak{A}})^{\dagger\mathfrak{F}\otimes\mathfrak{C}}) \\
 = & \gamma(1_{\mathfrak{C}})\psi(1_{\mathfrak{F}})\text{ldet}(1 - F^{\dagger\mathfrak{A}}A^{\dagger\mathfrak{F}}) + \alpha(1_{\mathfrak{A}})\phi(1_{\mathfrak{F}})\text{ldet}(1 - G^{\dagger\mathfrak{C}}C^{\dagger\mathfrak{F}})
 \end{aligned}$$

from which we can conclude:

$$\begin{aligned}
 w = & \gamma(1_{\mathfrak{C}})\psi(1_{\mathfrak{F}})[a\phi(1_{\mathfrak{F}}) + f\alpha(1_{\mathfrak{A}}) + \text{ldet}(1 - F^{\dagger\mathfrak{A}}A^{\dagger\mathfrak{F}})] \\
 & + \alpha(1_{\mathfrak{A}})\phi(1_{\mathfrak{F}})[c\psi(1_{\mathfrak{C}}) + g\gamma(1_{\mathfrak{C}}) + \text{ldet}(1 - G^{\dagger\mathfrak{C}}C^{\dagger\mathfrak{F}})]
 \end{aligned}$$

which is the wager of the project $(f::a) \otimes (g::c)$.

We thus deduce that for all f, g, a, c , we have $(f \otimes g)::(a \otimes c) \in (\mathbf{A} \otimes \mathbf{C}) \multimap (\mathbf{B} \otimes \mathbf{D})$ by Lemma 101 page 31. Finally, we showed that $(\mathbf{A} \multimap \mathbf{B}) \otimes (\mathbf{C} \multimap \mathbf{D}) \subset (\mathbf{A} \otimes \mathbf{C}) \multimap (\mathbf{B} \otimes \mathbf{D})$. $\square$

We will now inexpert the introduction rule for $\&$. We will interpret a proof ending with a $\&$ introduction rule by the sum of the projects f_{p+q} and g_{p+q} , where f and g — of respective carriers p and q — are the interprétions of the sub-proofs whose conclusions are the premises of the $\&$ rule. In order to perform this operation, it is necessary to first delocalize the interpretations of the premises as the premises do not have disjoint locations. Once this relocation is done, we can define the project h as $\theta_1(f) \& \theta_2(g)$ — where θ_1 and θ_2 are the delocations just mentioned. We then apply the project implementing distributivity in order to superpose the contexts. We refer the reader to the interpretation of proofs of $MALL_{\mathbf{T},0}$ in interaction graphs (Sei14a) for a more thorough explanation of this.

Proposition 141 (Interpretation of the $\&$ Rule). *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be behaviors of respective pairwise disjoint carriers p_A, p_B, p_C , and let $\phi(\mathbf{A})$ be a relocation of $\mathbf{A}$ whose carrier is a projection disjoint from the projections $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Then for all delocations¹⁵ $\theta_1, \theta_2, \theta_3$ in $GN_{\mathfrak{R}_{0,1}}(\mathfrak{P})$, there exists a project $\mathfrak{With}$ in the behavior:*

$$((\mathbf{A} \multimap \mathbf{B}) \& (\phi(\mathbf{A}) \multimap \mathbf{C})) \multimap (\theta_1(\mathbf{A}) \multimap (\theta_2(\mathbf{B}) \& \theta_3(\mathbf{D})))$$

Moreover, $\mathfrak{With}$ is promising w.r.t. the outlook $\mathfrak{P}$.

Proof. We chose projections $p'_A \sim_{\Pi} p_A$, $p''_A \sim_{\Pi} p_A$, $p'_B \sim_{\Pi} p_B$ and $p'_C \sim_{\Pi} p_C$ which are pairwise disjoint and disjoint from the carriers of $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Then, since $\mathfrak{P}$ is either regular or semi-regular there exists by Theorem 34 partial isometries ϕ, θ_1, θ_2 and θ_3 in the normalizing groupoid $GN_{\mathfrak{R}_{0,1}}(\mathfrak{P})$:

$$\left\{ \begin{array}{lll} \phi & : & p_A \rightarrow p'_A \\ \theta_1 & : & p_A \rightarrow p''_A \\ \theta_2 & : & p_B \rightarrow p'_B \\ \theta_3 & : & p_C \rightarrow p'_C \end{array} \right.$$

¹⁵ Supposing of course that the carriers are pairwise disjoint.

We will write $p = p_A + p_B + p'_A + p_C$ et $p' = p''_A + p'_B + p'_C$, and we define $\mathfrak{k} = (p + p', 0, \kappa, \mathfrak{K}, K)$ where:

$$\begin{aligned}\kappa &= \frac{1_{\mathbf{C}} \oplus 1_{\mathbf{C}}}{2} \\ \mathfrak{K} &= \mathbf{C} \oplus \mathbf{C} \\ K &= (\theta_1 + \theta_1^* + \theta_2 + \theta_2^*) \oplus (\theta_1 \phi^* + \phi \theta_1^* + \theta_3 + \theta_3^*)\end{aligned}$$

We now show that $\mathfrak{k}$ is promising w.r.t. the outlook $\mathfrak{P}$ and an element of the behavior:

$$((\mathbf{A} \multimap \mathbf{B}) \& (\phi(\mathbf{A}) \multimap \mathbf{C})) \multimap (\theta_1(\mathbf{A}) \multimap (\theta_2(\mathbf{B}) \& \theta_3(\mathbf{C})))$$

Let us first show that it belongs to the latter behavior. We will write $q = p_A + p_B$ and $r = p'_A + p_C$.

— Let $\mathfrak{f} \in \mathbf{A} \& \mathbf{B}$:

$$\mathfrak{f} = (q + r, 0, \phi \oplus \gamma, \mathfrak{F}, F)$$

We can compute $\mathfrak{k} :: \mathfrak{f}$:

$$\mathfrak{w} = \mathfrak{k} :: \mathfrak{f} = (p', w, \xi, \mathfrak{W}, W)$$

where:

$$\begin{aligned}w &= 0 \\ \xi &= \frac{1 \oplus 1}{2} \otimes \phi \equiv \frac{\phi \oplus \phi}{2} \\ \mathfrak{W} &= (\mathbf{C} \oplus \mathbf{C}) \otimes \mathfrak{F} \equiv \mathfrak{F} \otimes \mathfrak{F}\end{aligned}$$

and

$$\begin{aligned}W &= K^{\dagger \mathfrak{F}} :: F^{\dagger \mathfrak{K}} \\ &= (\theta_1 + \theta_1^* + \theta_2 + \theta_2^*)^{\dagger \mathfrak{F}} F^{\dagger \mathfrak{K}} (\theta_1 + \theta_1^* + \theta_2 + \theta_2^*)^{\dagger \mathfrak{F}} \\ &\quad \oplus (\theta_1 \phi^* + \phi \theta_1^* + \theta_3 + \theta_3^*)^{\dagger \mathfrak{F}} F^{\dagger \mathfrak{K}} (\theta_1 \phi^* + \phi \theta_1^* + \theta_3 + \theta_3^*)^{\dagger \mathfrak{F}}\end{aligned}$$

We now use the fact that (see the proof of Proposition 87):

$$(\mathbf{A} \multimap \mathbf{B}) \& (\phi(\mathbf{A}) \multimap \mathbf{C}) = (\mathbf{A} \multimap (\mathbf{1}_{\mathbf{r}} \multimap \mathbf{B})) \cap (\phi(\mathbf{A}) \multimap (\mathbf{1}_{\mathbf{q}} \multimap \mathbf{C})) \quad (7)$$

We will also write W_1 (resp. W_2) the first (resp. the second) component of W ; i.e. $W = W_1 \oplus W_2$.

The proof follows the one of Proposition 87: we pick an element $\theta_1(\mathbf{a}) \in \theta_1(\mathbf{A})$ and we show that $\mathfrak{w} :: \mathbf{a} \in \theta_2(\mathbf{B}) \& \theta_3(\mathbf{C})$. We therefore consider $\mathfrak{w} :: \mathbf{a}$:

$$\mathfrak{w} :: \mathbf{a} = (p'_B + p'_C, m, (\phi \otimes \mathbf{a}) \oplus (\phi \otimes \mathbf{a}), \mathfrak{F} \otimes \mathfrak{A} \oplus \mathfrak{F} \otimes \mathfrak{A}, W_1^{\dagger \mathfrak{A}} :: A^{\dagger \mathfrak{F}} \oplus W_2^{\dagger \mathfrak{A}} :: A^{\dagger \mathfrak{F}})$$

where the wager m is computed as follows:

$$\begin{aligned}m &= \text{ldet}(1 - (W_1 \oplus W_2)^{\dagger \mathfrak{A}} A^{\dagger \mathfrak{F}} \oplus \mathfrak{F}) \\ &= \text{ldet}(1 - W_1^{\dagger \mathfrak{A}} A^{\dagger \mathfrak{F}} \oplus W_2^{\dagger \mathfrak{A}} A^{\dagger \mathfrak{F}} \oplus \mathfrak{F}) \\ &= \text{ldet}(1 - W_1^{\dagger \mathfrak{A}} A^{\dagger \mathfrak{F}}) + \text{ldet}(1 - W_2^{\dagger \mathfrak{A}} A^{\dagger \mathfrak{F}})\end{aligned}$$

We can then show that W_1 is an element of $\theta_1(\mathbf{A}) \multimap \theta_2(\mathbf{B})$ using the equality of the

conducts in Equation 7. Thus $\text{ldet}(1 - W_1^{\dagger\mathfrak{A}} A^{\ddagger\mathfrak{F}})$ is equal to zero since it is the wger of an element of $\theta_2(\mathbf{B})$. Moreover, the operator $B = W_1^{\dagger\mathfrak{A}} :: A^{\ddagger\mathfrak{F}}$ defines an elements of $\theta_2(\mathbf{B})$: $\mathfrak{b} = (p'_B, 0, \phi \otimes \alpha, \mathfrak{F} \otimes \mathfrak{A}, B) \in \theta_2(\mathbf{B})$.

A similar reasoning shows that $\mathfrak{c} = (p'_C, 0, \phi \otimes \alpha, \mathfrak{F} \otimes \mathfrak{A}, W_2^{\dagger\mathfrak{A}} :: A^{\ddagger\mathfrak{F}}) \in \theta_3(\mathbf{C})$. We thus obtain that $\mathfrak{w} :: \mathfrak{a} = \mathfrak{b} + \mathfrak{c} \in \theta_2(\mathbf{B}) \& \theta_3(\mathbf{C})$, which shows that $\mathfrak{k}$ is an element of the behavior $((\mathbf{A} \multimap \mathbf{B}) \& (\phi(\mathbf{A}) \multimap \mathbf{C})) \multimap (\theta_1(\mathbf{A}) \multimap (\theta_2(\mathbf{B}) \& \theta_3(\mathbf{D})))$.

- Since the project is constructed from delocations in the normalizing groupoid of $\mathfrak{P}$, it is clearly promising w.r.t. $\mathfrak{P}$.

□

Remark 142. The interpretation of the $\&$ introduction rule will therefore be defined as the relatively complex construction¹⁶ $\overline{\mathfrak{f} \& \mathfrak{g}} = [\mathfrak{With}](\theta_1(\mathfrak{f}) \& \theta_2(\mathfrak{g}))$. This construction should not hide however the simplicity of the underlying idea. Indeed, given two projects $\mathfrak{f} = (p + r, 0, \phi, \mathfrak{F}, F)$ and $\mathfrak{g} = (q + r, 0, \gamma, \mathfrak{G}, G)$, we are just constructing the project:

$$\overline{\mathfrak{f} \& \mathfrak{g}} = (p + q + r, 0, \phi \oplus \gamma, \mathfrak{F} \oplus \mathfrak{G}, F \oplus G)$$

5.2.4. Soundness In order to state and show the full soundness result, we first define the interpretation of proofs.

Definition 143 (Interpretations of Proofs). We inductively define the interpretation of a proof Π :

- if Π is an axiom rule introducing the sequent $\vdash X_\theta; X_\phi$, we define the interpretation $\Pi^\bullet$ as the project $(q_{X_\theta} + q_{X_\phi}, 0, tr, \mathfrak{A}, \theta\phi^* + \phi\theta^*)$;
- if Π is obtained by application of a rule $\mathfrak{A}$, or an exchange rule, to a proof Π_1 , we define $\Pi^\bullet = \Pi_1^\bullet$;
- if Π is obtained by applying a $\oplus$ rule to a proof Π_1 whose interpretation's carrier is p , then $\Pi^\bullet = (\Pi_1^\bullet)_{p+q}$ where q is the carrier of the interpretation of the introduced formula;
- if Π is obtained by applying a cut rule between two proofs Π_1 and Π_2 , then $\Pi^\bullet = \Pi_1^\bullet :: \Pi_2^\bullet$;
- if Π is obtained by applying a $\otimes$ introduction rule to the proofs Π_1 and Π_2 , then $\Pi^\bullet = \Pi_1^\bullet \otimes \Pi_2^\bullet$;
- if Π is obtained by the application of a $\&$ rule on the proofs Π_1 and Π_2 interpreted by projects $\Pi_1^\bullet$ and $\Pi_2^\bullet$, we then define $\Pi^\bullet = [\mathfrak{With}](\theta_1(\Pi_1^\bullet) \& \theta_2(\Pi_2^\bullet))$ where θ_1, θ_2 are delocations of $\Pi_1^\bullet$ and $\Pi_2^\bullet$ onto disjoint projections, and where $\mathfrak{With}$ is the project whose existence is ensured by Proposition 141.

Theorem 144 (Full Soundness). *Let π be a proof of the sequent $\vdash \Gamma$ in $MALL_{\mathbf{T},0}$, and δ an interpretation basis. Then the interpretation $\pi^\bullet$ of π is a promising project w.r.t. $\mathfrak{P}$ in the interpretation $\|\vdash \Gamma; \mathbf{A}\|_\delta$ of $\vdash \Gamma$.*

Proof. The proof is a simple induction. The base case is the simple observation that a fax constructed from partial isometries in the normalizing groupoid of $\mathfrak{P}$ is promising w.r.t. $\mathfrak{P}$. The induction steps are then consequences of Proposition 140 and Proposition 141, as well as Proposition 118. □

¹⁶ We recall that θ_1 and θ_2 are well-chosen delocations.

5.3. Regular MASAs

To interpret exponentials of Elementary Linear Logic (ELL), we will consider the construction proposed by Girard (Gir11) and exposed above. There is one major problem with this construction, however. Indeed, if a is a promising hyperfinite project w.r.t. the outlook $\mathfrak{P}$, it is clear that the hyperfinite project $!_{\Omega}a$ is promising w.r.t the outlook $\Omega(\mathfrak{P} \otimes \Omega)$ where Ω is a MASA in $\mathfrak{R}$. However, if it is obvious that $\Omega(\mathfrak{P} \otimes \Omega)$ is a MASA in $\mathfrak{R}_{0,1}$, it won't be true, in general, that $\Omega(\mathfrak{P} \otimes \Omega) = \mathfrak{P}$. As those are both MASAs in $\mathfrak{R}_{0,1}$, the two algebras $\Omega(\mathfrak{P} \otimes \Omega)$ and $\mathfrak{P}$ are diffuse abelian von Neumann algebras, thus isomorphic *as von Neumann algebras*. This is however too weak a result as this isomorphism is not in general realized by a unitary operator, a necessary condition for an adequate interpretation of the promotion rule.

Proposition 145. *Let a be a promising project w.r.t. the outlook $\mathfrak{P}$. Suppose that $\mathfrak{P}$ is a regular (or Cartan) MASA in $\mathfrak{R}_{0,1}$. Then there exists a partial isometry u such that $u\Omega(A)u^*$ is a partial symmetry in the normalizing groupoid of $\mathfrak{P}$.*

Proof. This proof relies on two hypotheses: the fact that the outlook is a regular MASA, and the fact that the operator A is an element of $p\mathfrak{R}_{0,1}p$, i.e. an element of a type II_1 hyperfinite factor. Indeed, since a is promising w.r.t. $\mathfrak{P}$, A is an element of the normalizing groupoid of $\mathfrak{P} \otimes \Omega$, where Ω is a maximal abelian sub-algebra of $\mathfrak{A}$ which is obviously “regular” as $\mathfrak{A}$ is a finite factor of type I. Since $\mathfrak{P}$ and Ω are regular, their tensor product is a regular MASA in $\mathfrak{R}_{0,1} \otimes \mathfrak{A}$, and therefore¹⁷ $\Omega(\mathfrak{P} \otimes \Omega)$ is a regular MASA in $\mathfrak{R}_{0,1}$.

Moreover, $A = pAp$ where p is a finite projection in $\mathfrak{P}$. Then $\Omega(A) = \Omega(p)\Omega(A)\Omega(p)$, which implies that $\Omega(A)$ is an element of the normalizing groupoid of $\Omega(p)\Omega(\mathfrak{P} \otimes \Omega)\Omega(p)$, which is a MASA in $\Omega(p)\mathfrak{R}_{0,1}\Omega(p)$ by Lemma 129. Let us pick p' a projection in $\mathfrak{R}_{0,1}$ with the same trace as p (and therefore with the same trace as $\Omega(p)$). We can then consider the regular MASA $p'\mathfrak{P}p'$ in $p\mathfrak{R}_{0,1}p$. Since p' and $\Omega(p)$ have equal traces, there exists a partial isometry u such that $uu^* = p'$ and $u^*u = \Omega(p)$. Then $u^*p\mathfrak{P}pu$ is a MASA in $\Omega(p)\mathfrak{R}_{0,1}\Omega(p)$. A result of Connes, Feldman and Weiss (Theorem 36) shows that two regular MASAs in the hyperfinite factor of type II_1 are unitarily equivalent. Therefore, there exists a unitary $v \in \Omega(p)\mathfrak{R}_{0,1}\Omega(p)$ such that $v^*(u^*p'\mathfrak{P}p'u)v = \Omega(p)\Omega(\mathfrak{P} \otimes \Omega)\Omega(p)$. Since the product uv is a partial isometry ($uv(uv)^*uv = uvv^*u^*uv = u\Omega(p)\Omega(p)v = uv$), we can show that $(uv)^*A(uv)$ is a partial symmetry in the normalizing groupoid of $\mathfrak{P}$:

$$\begin{aligned} v^*u^*Auvv^*u^*Auv &= v^*u^*A\Omega(p)u^*Auv = v^*u^*ApAuv = v^*u^*AAuv = v^*u^*Auv \\ (v^*u^*Auv)^* &= v^*u^*A^*uv = v^*u^*Auv \end{aligned}$$

The partial isometry uv is therefore the one we were after. $\square$

One can notice that the interpretations of the contraction and functorial promotion rules only use promising projects w.r.t. $\mathfrak{P}$. From this and the preceding proposition, one can easily show an extension of the soundness result stated above for the sequent calculi ELL_{pol} and ELL_{comp} considered in the author's work on interaction graphs (Sei13) as soon as the

¹⁷ We allow ourselves a small abuse of notations here, as $\mathfrak{A}$ is not a MASA in $\mathfrak{R}$. However, one can chose the embedding of $\mathfrak{A}$ into $\mathfrak{R}$ in such a way that the image of $\mathfrak{A}$ is a regular MASA in $\mathfrak{R}$.

outlook is a regular MASA. Let us notice that this proposition do not depend on the morphism Ω chosen to define the exponentials (the soundness result however depends on Ω since not all choices of morphisms would allow for the interpretation of functorial promotion).

It is then natural to ask oneself if the converse of this result holds, i.e. if the fact that $\mathfrak{P}$ is not regular implies that one cannot interpret (at least one) exponential connective. We won't answer fully to this question in this paper, but we will discuss it anyway.

Let us first consider the Pukansky invariant of the outlook $\mathfrak{P}$ and of the sub algebra $\Omega(\mathfrak{P} \otimes \Omega)$ (using the same notations as in the preceding proof). It is known¹⁸ that there exists singular MASAs in $\mathfrak{R}$ whose Pukansky invariant is included in $\{2, 3, \dots, \infty\}$, and the sub-algebra Ω satisfies $\text{Puk}(\Omega) = \{1\}$ since it is regular (Proposition 39). Using Proposition 42, we get that $\text{Puk}(\Omega(\mathfrak{P} \otimes \Omega))$ contains 1, and it is therefore impossible in this case that $\Omega(\mathfrak{P} \otimes \Omega)$ and $\mathfrak{P}$ be unitarily equivalent.

However, the Pukansky invariant of a semi-regular MASA is a subset of $\mathbf{N} \cup \{\infty\}$ that contains 1 (from Proposition 41). Then, by using Proposition 42, one shows that in this case $\text{Puk}(\Omega(\mathfrak{P} \otimes \Omega)) = \text{Puk}(\mathfrak{P})$. It is therefore not possible to show the reciprocal statement of Proposition 145 in this manner. We conjecture that there exists perennializations Ω and semi-regular outlooks $\mathfrak{P}$ such that (the equivalent of) Proposition 145 holds. We also conjecture that there exists perennializations Ω and semi-regular outlooks $\mathfrak{P}$ such that the (equivalent of) Proposition 145 does not hold. A more interesting question would be to know if for all perennialization (and therefore the one defined by Girard) there exists a semi-regular outlook $\mathfrak{P}$ such that the (equivalent of) Proposition 145 does not hold.

5.4. Conclusion

The results obtained in this section can be combined into the following theorem.

Theorem 146. *Let $\mathfrak{P}$ be a maimal abelian sub-algebra of $\mathfrak{R}_{0,1}$. Then:*

- *if $\mathfrak{P}$ is singular, there are no non-trivial interpretations of MALL (or ELL) by promising hyperfinite projects w.r.t. $\mathfrak{P}$;*
- *if $\mathfrak{P}$ is semi-regular, one can interpret soundly MALL by promising hyperfinite projects w.r.t. $\mathfrak{P}$;*
- *if $\mathfrak{P}$ is regular, one can interpret soundly ELL by promising hyperfinite projects w.r.t. $\mathfrak{P}$.*

6. Conclusion

This work shows a correspondence between the expressivity of the fragment of linear logic reconstructed from GoI techniques and a classification of maximal abelian sub-algebras proposed by Dixmier (Dix54). Indeed, it was known that the interpretation of linear logic proofs in GoI models depends on the choice of the algebra $\mathfrak{M}$ in which the GoI construction is performed, i.e. the hyperfinite factor $\mathfrak{R}_{0,1}$ in Girard's GoI5 (Gir11) or the algebra

¹⁸ White (Whi08) showed that all subset of $\mathbf{N} \cup \{\infty\}$ is the Pukansky invariant of a MASA in $\mathfrak{R}$.

$\mathcal{L}(\mathbb{H})$ of all operators on a separable infinite-dimensional Hilbert space $\mathbb{H}$ in earlier works (Gir89a; Gir88; Gir95a). We showed here that another algebra influence this interpretation of proofs, a maximal abelian sub-algebra $\mathfrak{A}$ of $\mathfrak{M}$: this algebra, which represents the choice of a basis was implicitly fixed in early GoI models but appears explicitly in the hyperfinite GoI model. Dixmier’s classification of MASAs specifies three particular types of such inclusions $\mathfrak{A} \subset \mathfrak{M}$ is a maximal abelian sub-algebra of a von Neumann algebra $\mathfrak{M}$. We showed here that the expressivity of the fragment of linear logic interpreted in the model is closely related to the type of the chosen sub-algebra $\mathfrak{A}$.

This work does not provide a complete correspondence, as Dixmier’s classification is not an exhaustive one. While we showed that any exponential connective can be interpreted when $\mathfrak{A}$ is a *regular* MASA and that no non-trivial interpretation exists when $\mathfrak{A}$ is a *singular* MASA, the intermediate case of *semi-regular* MASAs is not completely understood. Indeed we proved that one can *at least* interpret multiplicative-additive linear logic in the case, but some exponential connectives (although probably not all) might be interpreted in some cases. This opens the way for a more complete investigation of this case. A complicated approach would be to study the possible choices of semi-regular sub-algebras and understand how the exponential connectives behave with respect to them. We propose to study these questions using the systematic approach offered by Interaction Graphs (Sei12; Sei14a; Sei14c; Sei13). In particular, the use of *graphings* allows one to consider a construction of GoI models parametrized by the choice of a so-called *microcosm*: a microcosm is a monoid of measurable transformations of a measured space X , which defines a measurable equivalence relation, which in turn defines a couple of a von Neumann algebra $\mathfrak{M}$ and a MASA $\mathfrak{A}$ of $\mathfrak{M}$ by Feldman and Moore construction (FM77a; FM77b). It therefore offers a more practical approach of this problem as it allows for subtle distinctions on the MASAs considered, and these distinctions can be understood as restrictions of the computational principles allowed in the model (Sei14b).

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