

Interaction Graphs: The Additive Construction

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Interaction Graphs:

- ▶ Generalization of the first Gol (*Multiplicatives*);
- ▶ Gol for MALL;
- ▶ The construction is parametrized by a map and:
 - ▶ one choice of this parameter yields a combinatorial version of Girard's Gol5;
 - ▶ another choice will give a refined version of older Gols (where orthogonality is based on nilpotency);
- ▶ The construction relies on one property: the cyclic property;
- ▶ The additive construction gives an alternative presentation of idioms and pseudo-trace in Gol5.

Directed weighted graphs

Definition

A *directed weighted graph* is a tuple $G = (V^G, E^G, s^G, t^G, \omega^G)$, where V^G is the set of vertices, E^G is the set of edges, s^G and t^G are two functions from E^G to V^G , the *source* and *target* functions, and ω^G is a function $E^G \rightarrow]0, 1]$.

Remark

We will usually denote $G = (V^G, E^G)$ such a graph to simplify notations.

Plugging

Plugging corresponds to the recognition of a cut.

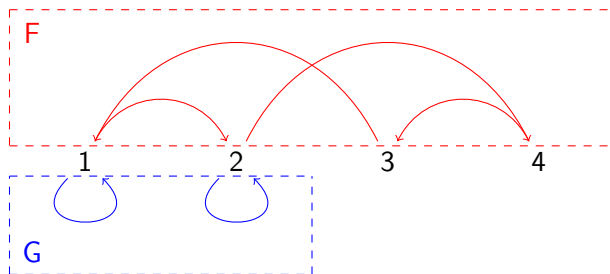
Definition

If $G = (V^G, E^G)$ and $H = (V^H, E^H)$ are two graphs, we define $G \boxplus H$ as the graph $G \uplus H = (V^G \cup V^H, E^G \uplus E^H)$ together with a coloring function δ such that

$$\delta(e) = \begin{cases} 0 & \text{if } e \in E^G \\ 1 & \text{if } e \in E^H \end{cases}$$

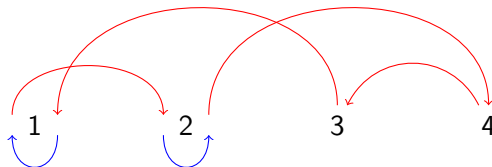
Execution

The execution of two graphs F, G is the graph of alternating paths from $V^F \Delta V^G$ to itself.



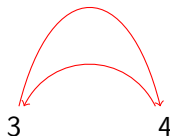
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Definition

We define the execution $F :: G$ between two graphs F and G as

$$F :: G = (V^F \Delta V^G, \text{Path}_{V^F \Delta V^G}^{\text{alt}}(F \square G))$$

where $\text{Path}^{\text{alt}}(V^F \Delta V^G)F \square G$ is the set of alternating paths in $F \square G$ whose source and target are in $V^F \Delta V^G$.

Remark

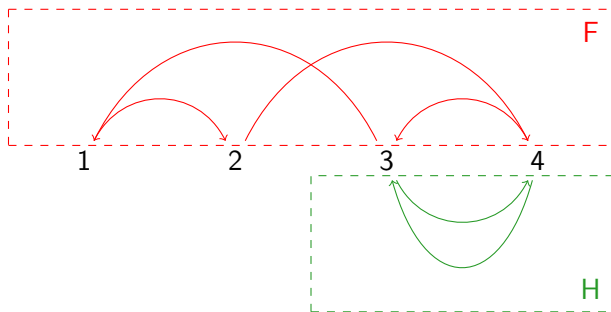
When $V^F \cap V^G = \emptyset$, we will use the notation $F \cup G$ for $F :: G$ (in this particular case, $F :: G = (V^F \cup V^G, E^F \uplus E^G)$).

Counting cycles

Sometimes, cycles can appear between two graphs.

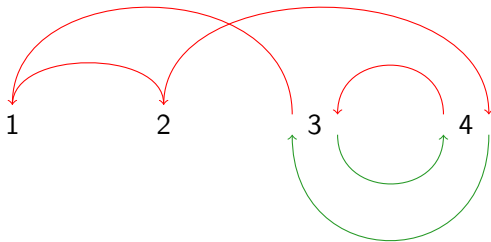
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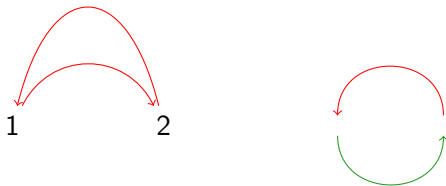
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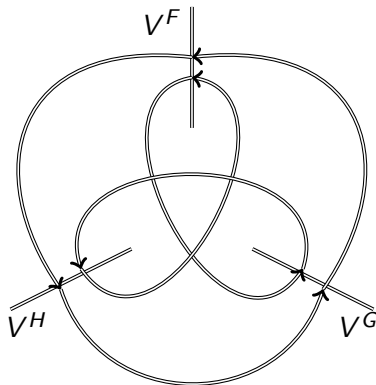
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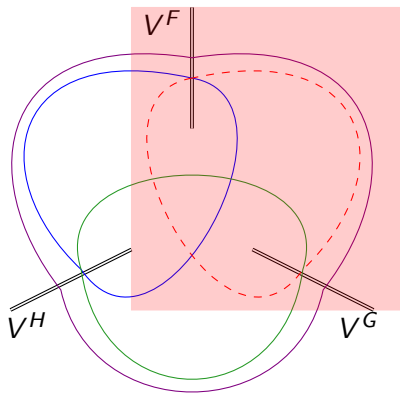
We will denote by $\text{cycl}(F, G)$ the set of alternating cycles in $F \square G$, i.e. cycles $\pi = \{e_i\}_{0 \leq i \leq n}$ such that:

- ▶ it is an alternating path: $\delta(e_i) \neq \delta(e_{i+1})$;
- ▶ the concatenation of π with itself is an alternating path:
 $\delta(e_0) \neq \delta(e_n)$.

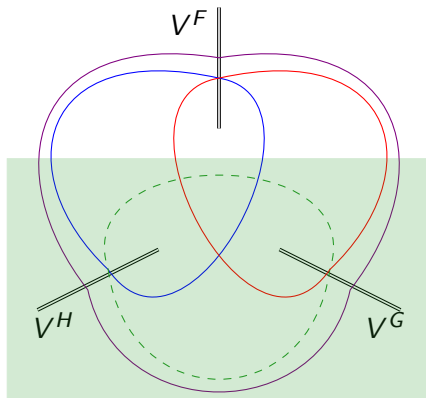
The cyclic (or trefoil) property



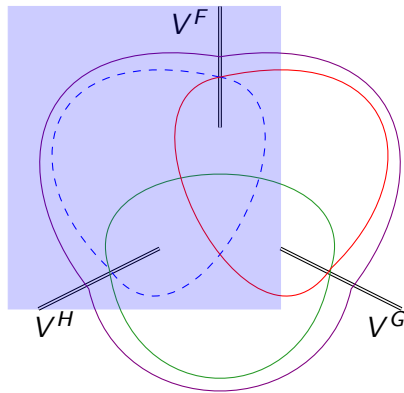
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Counting cycles

Proposition

Let F, G, H be graphs with $V^F \cap V^G \cap V^H = \emptyset$. Then:

$$\text{cycl}(F, G :: H) \uplus \text{cycl}(G, H) \cong \text{cycl}(F :: G, H) \uplus \text{cycl}(F, G)$$

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Corollary

Let F, G, H be graphs with $V^G \cap V^H = \emptyset$. Then:

$$\text{cycl}(F, G \cup H) \cong \text{cycl}(F, G) \uplus \text{cycl}(F :: G, H)$$

Measuring cycles

Based on this result, we can define a *measurement* of the interaction between two graphs.

Definition

Let $m :]0, 1] \rightarrow \mathbf{R}_{\geq 0} \cup \{\infty\}$. We define, for any two graphs F, G :

$$\llbracket F, G \rrbracket_m = \sum_{\pi \in \text{cycl}(F, G)} m(\omega(\pi))$$

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$$\llbracket F, G \cup H \rrbracket_m + \llbracket G, H \rrbracket_m = \llbracket F :: G, H \rrbracket_m + \llbracket F, G \rrbracket_m$$

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Corollary

Let F, G, H be graphs with $V^G \cap V^H = \emptyset$. Then:

$$\llbracket F, G \cup H \rrbracket_m = \llbracket F :: G, H \rrbracket_m + \llbracket F, G \rrbracket_m$$

Definition

A *sliced graph* G of carrier V^G is a formal weighted finite sum of graphs $G = \sum_{i \in I^G} \alpha_i^G G_i$ where for all $i \in I^G$, $V^{G_i} = V^G$ and $\alpha_i^G \in \mathbf{R}$.

We will denote by $\mathbf{1}_G$ the total weight of G : $\mathbf{1}_G = \sum_{i \in I^G} \alpha_i^G$.

We simply extend the execution and measurement by "linearity":

$$\begin{aligned} \left(\sum_{i \in I^G} \alpha_i^G G_i \right) :: \left(\sum_{i \in I^H} \alpha_i^H H_i \right) &= \sum_{i \in I^G} \sum_{j \in I^H} \alpha_i^G \alpha_j^H F_i :: G_j \\ \llbracket \sum_{i \in I^G} \alpha_i^G G_i, \sum_{i \in I^H} \alpha_i^H H_i \rrbracket &= \sum_{i \in I^G} \sum_{j \in I^H} \alpha_i^G \alpha_j^H \llbracket F_i, G_j \rrbracket \end{aligned}$$

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Proposition

Let F, G, H be sliced graphs such that $V^F \cap V^G \cap V^H = \emptyset$. Then:

$$\llbracket F :: G, H \rrbracket + \mathbf{1}_H \llbracket F, G \rrbracket = \llbracket F, G :: H \rrbracket + \mathbf{1}_F \llbracket G, H \rrbracket$$

Definition

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Corollary

Let F, G, H be sliced graphs such that $V^G \cap V^H = \emptyset$. Then:

$$\llbracket F :: G, H \rrbracket + \mathbf{1}_H \llbracket F, G \rrbracket = \llbracket F, G \cup H \rrbracket$$

Wagers

In order to deal with the additional term in the adjunction, we will associate a number in $\mathbf{R} \cup \{\infty\}$ to our graphs, and extend the execution and the measure to couples (f, F) :

$$\begin{aligned}(f, F) :: (g, G) &= (f\mathbf{1}_G + g\mathbf{1}_F + \llbracket F, G \rrbracket_m, F :: G) \\ \llbracket (f, F), (g, G) \rrbracket_m &= f\mathbf{1}_G + g\mathbf{1}_F + \llbracket F, G \rrbracket_m\end{aligned}$$

Then, if F, G, H are three graphs with $V^F \cap V^G \cap V^H = \emptyset$, we get, for any f, g, h :

$$\llbracket (f, F), (g, G) :: (h, H) \rrbracket_m = \llbracket (f, F) :: (g, G), (h, H) \rrbracket_m$$

And in the special case $V^G \cap V^H = \emptyset$, we have:

$$\llbracket (f, F), (g, G) \cup (h, H) \rrbracket_m = \llbracket (f, F) :: (g, G), (h, H) \rrbracket_m$$

Projects

Definition (Projects — Proofs)

A *project* is a couple $\alpha = (a, A)$ where $a \in \mathbf{R} \cup \{\infty\}$ is the *wager* and A is a graph. The finite set V_A of vertices of A will be called the *carrier* of α .

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Definition

Let α and $\flat$ be two projects. We define

$$\ll \alpha, \flat \gg_m = a\mathbf{1}_B + b\mathbf{1}_A + \ll A, B \gg_m$$

Conducts

Definition (Orthogonality)

Two projects a and b of same carrier are said to be *orthogonal* when $\ll a, b \gg_m \neq 0, \infty$.

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Two projects a and b of same carrier are said to be *orthogonal* when $\ll a, b \gg_m \neq 0, \infty$.

Definition (Conducts — Formulas)

A conduct is a set of projects (with same carrier) equal to its bi-orthogonal.

Multiplicatives

Definition (Cut)

Let f and g be two projects. We define the cut between f and g as

$$f :: g = (\llbracket f, g \rrbracket_m, F :: G)$$

Definition

Let $\mathbf{A}, \mathbf{B}$ be two conducts of disjoint carriers, we define

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= \{a \otimes b = a :: b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\sim\sim} \\ \mathbf{A} \multimap \mathbf{B} &= \{f \mid \forall a \in \mathbf{A}, f :: a \in \mathbf{B}\} \end{aligned}$$

Multiplicatives

Definition (Cut)

Let f and g be two projects. We define the cut between f and g as

$$f :: g = (\llbracket f, g \rrbracket_m, F :: G)$$

Proposition

$$(A \otimes B^\perp)^\perp = A \multimap B$$

Proof.

Direct consequence of the adjunction.



Denotational semantics

Definition

We define the category $\mathcal{G}raph_{MLL}$

$$\mathcal{O}bj = \{\mathbf{A} \mid \mathbf{A} = \mathbf{A}^{\perp\perp} \text{ of carrier } X_{\mathbf{A}} \in \mathcal{P}_f(\mathbb{N})\}$$

$$\mathcal{M}or[A, B] = \{f \in \psi_0(\mathbf{A}) \multimap \psi_1(\mathbf{B})\}$$

Proposition

We define functors from the connectives $\otimes$, $\multimap$ (with delocations) and the negation. With theses definitions, the category $\mathcal{G}raph_{MLL}$ is a $*$ -autonomous category in which $\otimes \neq \wp$ and $\mathbf{1} \neq \perp$.

Additives

Definition

If $\mathbf{A}$ is a conduct and V a set such that $V^A \cap V = \emptyset$, we define $\mathbf{A}_{\uparrow V} = \{a \otimes o_V \mid a \in \mathbf{A}\}^{\perp\perp}$.

Definition

If $\mathbf{A}, \mathbf{B}$ are two conducts of disjoint carriers, we define:

$$\begin{aligned}\mathbf{A} \oplus \mathbf{B} &= (\mathbf{A}_{\uparrow_{VB}} \cup \mathbf{B}_{\uparrow_{VA}})^{\perp\perp} \\ \mathbf{A} \&\mathbf{B} &= ((\mathbf{A}^{\perp})_{\uparrow_{VB}})^{\perp} \cap ((\mathbf{B}^{\perp})_{\uparrow_{VA}})^{\perp}\end{aligned}$$

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Proposition

$$\mathbf{A} \oplus \mathbf{B} = (\mathbf{A}^{\perp} \& \mathbf{B}^{\perp})^{\perp}$$

Additives

Definition

The sum and product by a scalar on graphs can be extended to projects:

$$\mathfrak{a} + \lambda \mathfrak{b} = (a + \lambda b, \sum_{i \in I^A} \alpha_i^A A_i + \sum_{i \in I^B} \lambda \alpha_i^B B_i)$$

Definition

A *behaviour* is a conduct such that:

- ▶ $\mathfrak{a} \in \mathbf{A} \Rightarrow \mathfrak{a} + \lambda \mathfrak{o} \in \mathbf{A}$;
- ▶ $\mathfrak{a} \in \mathbf{A}^\perp \Rightarrow \mathfrak{a} + \lambda \mathfrak{o} \in \mathbf{A}^\perp$;

Behaviors and connectives

Remark

The orthogonal of a behaviour is a behaviour.

Proposition

If A, B are behaviours, then $A \otimes B, A \multimap B, A \& B, A \oplus B$ are behaviours.

Additives

Definition

A behaviour is *proper* if:

- ▶ $\alpha \in \mathbf{A} \Rightarrow \alpha + \lambda 0 \in \mathbf{A}$;
- ▶ $\alpha = (a, A) \in \mathbf{A} \Rightarrow a = 0$;
- ▶ $\mathbf{A}$ is non empty.

Proposition

If $\mathbf{A}$ is a proper behaviour, then $\mathbf{A}^\perp$ is proper.

Proposition

A behaviour $\mathbf{A}$ is either proper, equal to $\mathbf{0}_{VA} = \emptyset$, or $\mathbf{A}^\perp = \mathbf{0}_{VA}$.

Additives

Definition

If $\mathbf{A}, \mathbf{B}$ are conducts, we define:

$$\mathbf{A} + \mathbf{B} = \{a \otimes o_{VB} + o_{VA} \otimes b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$$

Proposition

If $\mathbf{A}, \mathbf{B}$ are behaviours, $\mathbf{A} + \mathbf{B} \subset \mathbf{A} \& \mathbf{B}$.

Proposition

If $\mathbf{A}, \mathbf{B}$ are proper behaviours there exists $c \in (\mathbf{A}^\perp + \mathbf{B}^\perp)^\perp$ such that $c \notin \mathbf{A} \oplus \mathbf{B}$. Hence $(\mathbf{A} + \mathbf{B})^{\perp\perp} \subsetneq \mathbf{A} \& \mathbf{B}$.

Observational equivalence

Definition

If $\mathbf{A}$ is a conduct, we define the *observational equivalence* $\cong_{\mathbf{A}}$:

$$a \cong_{\mathbf{A}} b \Leftrightarrow \forall c \in \mathbf{A}^{\perp}, \ll a, c \gg_m = \ll b, c \gg_m$$

Proposition

If $\mathbf{A}, \mathbf{B}$ are behaviours, then for all $f \in \mathbf{A} \& \mathbf{B}$, there exists $a \in \mathbf{A}$ and $b \in \mathbf{B}$ such that $f \cong_{\mathbf{A} \& \mathbf{B}} a + b$.

Proposition

If $\mathbf{A}, \mathbf{B}$ are behaviours, then for all $f \in \mathbf{A} \oplus \mathbf{B}$, there exists $c \in \{a \otimes o_{VB}\} \cup \{b \otimes o_{VA}\}$ such that $f \cong_{\mathbf{A} \oplus \mathbf{B}} c$.

Denotational semantics for additives

The functor induced by the $\&$ is not a categorical product, however the following are all consequences of the cyclic property:

Proposition

*Let $f \in \mathbf{A} \multimap \mathbf{B}$ and $g \in \mathbf{A} \multimap \mathbf{C}$ be projects, and write $h = f \bar{\&} g$.
Then for all $b \in \mathbf{B}^\perp$, $f :: b \cong_{\mathbf{A}} h :: b \otimes o_{\mathbf{V}\mathbf{C}}$.*

Proposition

The observational equivalence defines a congruence on the category Graph_{MLL} .

Proposition

The quotient category Cond inherits the $*$ -autonomous structure of Graph_{MLL} , and the full subcategory Behav is a categorical model for MALL with additive units (without weakening and mix).

Graph Contraction

Definition (Contracting edges)

We define the operation $\hat{\cdot}$ that associates, to each weighted graph G , a simple weighted graph by replacing the set $E^G(v, w)$ of edges from v to w by a single edge which weight is equal to $\sum_{x \in E^G(v, w)} \omega(x)$.

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Theorem

Let $m(x) = -\log(1 - x)$. Then:

$$\llbracket F, G \rrbracket_m = \llbracket F, \hat{G} \rrbracket_m$$

From graphs to operators

- ▶ We restrict ourselves to graphs G such that the matrix of weights $\mathcal{M}(\widehat{G})$ have norm ≤ 1 .

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- ▶ We define an embedding Ψ associating to a graph F an operator in the hyperfinite factor of type II_∞ (by considering the matrix of weights of $\widehat{F}$), and extend it to an embedding of (graph) projects into Girard's Gol5 projects;
- ▶ We extend it to sliced graphs: for A a sliced graph, we consider the idiom $\bigoplus_{i \in I^A} \mathbf{C}$ and the pseudo-trace $\alpha : \bigoplus_{i \in I^A} x_i \mapsto \sum_{i \in I^A} \alpha_i^A x_i$.

Embedding in Gol5

With $m(x) = -\log(1 - x)$, we have :

Graphs		Gol5
$\llbracket a, b \rrbracket_m$	=	$\llbracket \Psi(a), \Psi(b) \rrbracket$
$\Psi(a \hat{+} b)$	=	$\Psi(a) :: \Psi(b)$
$\Psi(a + b)$	=	$\Psi(a) \& \Psi(b)$

As a consequence:

$$a \perp b \Leftrightarrow \Psi(a) \perp \Psi(b)$$

$$\Psi(a \otimes b) = \Psi(a) \otimes \Psi(b)$$

Nilpotence

One could also take $m(x) = \infty$ for all $x \in]0, 1]$. Then:

- ▶ The weights do not matter anymore, neither does the number of edges between two vertices;
- ▶ Two projects (a, A) and (b, B) are orthogonal if and only if:
 - ▶ $\mathcal{C}(A)\mathcal{C}(B)$ ($\mathcal{C}(\cdot)$ denotes the connectivity matrix) is nilpotent;
 - ▶ $a\mathbf{1}_B + b\mathbf{1}_A \neq 0, \infty$;
- ▶ We therefore get a geometry of interaction for MLL whose orthogonality is based on nilpotency, i.e. "old-style" Gol (with the addition of wagers) giving rise to a *-autonomous category;

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- ▶ We therefore get a geometry of interaction for MLL whose orthogonality is based on nilpotency, i.e. "old-style" Gol (with the addition of wagers) giving rise to a *-autonomous category;
- ▶ If we look at the additive construction, the behaviours are either empty or full (orthogonal of an empty conduct), so we get nothing more than a truth-valued model for MALL.