

Graphs of Interaction

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Graphs of Interaction:

- ▶ Generalization of the first Gol (*Multiplicatives*);
- ▶ Gol for MALL;
- ▶ The construction is parametrized by a map and:
 - ▶ one choice of this parameter yields a combinatorial version of Girard's Gol5;
 - ▶ another choice will give a refined version of older Gols (where orthogonality is based on nilpotency);

Directed weighted graphs

Definition

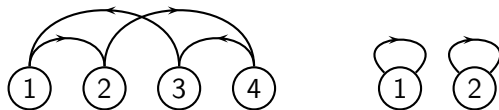
A *directed weighted graph* is a tuple $G = (V^G, E^G, s^G, t^G, \omega^G)$, where V^G is the set of vertices, E^G is the set of edges, s^G and t^G are two functions from E^G to V^G , the *source* and *target* functions, and ω^G is a function $E^G \rightarrow]0, 1]$.

Remark

We will usually denote $G = (V^G, E^G)$ such a graph to simplify notations.

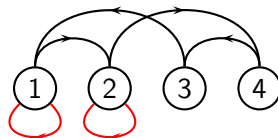
Plugging

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Definition

If $G = (V^G, E^G)$ and $H = (V^H, E^H)$ are two graphs, we define $G \square H$ as the graph $G \uplus H = (V^G \cup V^H, E^G \uplus E^H)$ together with a coloring function δ such that

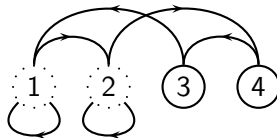
$$\delta(e) = \begin{cases} 0 & \text{if } e \in E^G \\ 1 & \text{if } e \in E^H \end{cases}$$

Execution (Permutations)

We wish to define the execution in the same way it was defined on the first version of Gol with permutations (*Multiplicatives*).

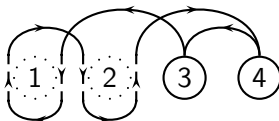
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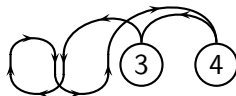
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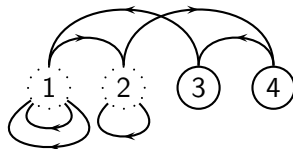


Execution (Graphs)

However, as we are working with graphs, there can be more than one edge from a vertex v to a vertex w .

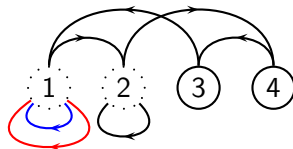
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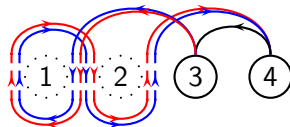
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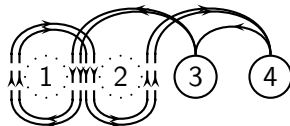
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Definition

We define the execution $F :: G$ between two graphs F and G as

$$F :: G = (V^F \Delta V^G, Path^{alt}(F \square G))$$

where $Path^{alt}(F \square G)$ is the set of alternating paths in $F \square G$.

Remark

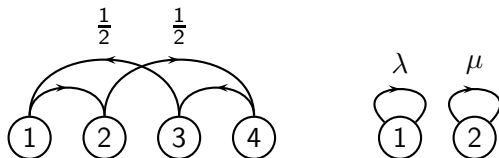
When $V^F \cap V^G = \emptyset$, we will use the notation $F \cup G$ for $F :: G$ (in this particular case, $F :: G = (V^F \cup V^G, E^F \uplus E^G)$).

Weighted graphs

Moreover, our graphs are weighted, so we define the weight of a path as the product of the edges it is composed of.

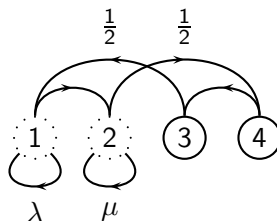
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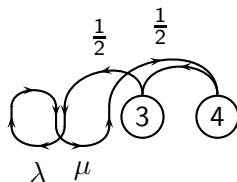
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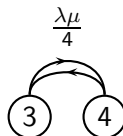
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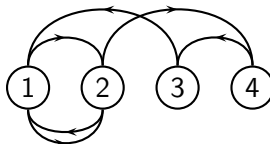


Counting cycles

Sometimes, cycles can appear between two graphs.

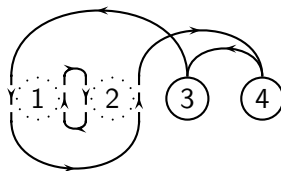
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Definition

We will denote by $\text{cycl}(F, G)$ the set of alternating cycles in $F \square G$, i.e. cycles $\pi = \{e_i\}_{0 \leq i \leq n}$ such that:

- ▶ it is an alternating path: $\delta(e_i) \neq \delta(e_{i+1})$;
- ▶ the concatenation of π with itself is an alternating path: $\delta(e_0) \neq \delta(e_n)$.

Counting cycles

Proposition

Let F, G, H be graphs with $V^F \cap V^G \cap V^H = \emptyset$. Then:

$$\text{cycl}(F, G :: H) \uplus \text{cycl}(G, H) \cong \text{cycl}(F :: G, H) \uplus \text{cycl}(F, G)$$

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Proposition

Let F, G, H be graphs with $V^G \cap V^H = \emptyset$. Then:

$$\text{cycl}(F, G \cup H) \cong \text{cycl}(F, G) \uplus \text{cycl}(F :: G, H)$$

Measuring cycles

Based on this result, we can define a *measurement* of the interaction between two graphs.

Definition

Let $m :]0, 1] \rightarrow \mathbf{R}_{\geq 0} \cup \{\infty\}$. We define, for any two graphs F, G :

$$\llbracket F, G \rrbracket_m = \sum_{\pi \in \text{cycl}(F, G)} m(\omega(\pi))$$

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Proposition

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$$\llbracket F, G \cup H \rrbracket_m + \llbracket G, H \rrbracket_m = \llbracket F :: G, H \rrbracket_m + \llbracket F, G \rrbracket_m$$

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Wagers

In order to deal with the additional term in the adjunction, we will associate a number in $\mathbf{R}_{\geq 0} \cup \{\infty\}$ to our graphs, and extend the execution and the measure to couples (f, F) :

$$\begin{aligned}(f, F) :: (g, G) &= (f + g + \llbracket F, G \rrbracket_m, F :: G) \\ \llbracket (f, F), (g, G) \rrbracket_m &= f + g + \llbracket F, G \rrbracket_m\end{aligned}$$

Then, if F, G, H are three graphs with $V^G \cap V^H = \emptyset$, we get, for any f, g, h :

$$\llbracket (f, F), (g, G) \cup (h, H) \rrbracket_m = \llbracket (f, F) :: (g, G), (h, H) \rrbracket_m$$

Projects

Definition (Projects — Proofs)

A *project* is a couple $\alpha = (a, A)$ where $a \in \mathbf{R}_{\geq 0} \cup \{\infty\}$ is the *wager* and A is a graph. The finite set V_A of vertices of A will be called the *carrier* of α .

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Definition

Let α and $\flat$ be two projects. We define

$$\ll \alpha, \flat \gg_m = a + b + \ll A, B \gg_m$$

Conducts

Definition (Orthogonality)

Two projects $\mathfrak{a}$ and $\mathfrak{b}$ of same carrier are said to be *orthogonal* when $\llbracket \mathfrak{a}, \mathfrak{b} \rrbracket_m \neq 0, \infty$.

Conducts

Definition (Orthogonality)

Two projects a and b of same carrier are said to be *orthogonal* when $\ll a, b \gg_m \neq 0, \infty$.

Definition (Conducts — Formulas)

A conduct is a set of projects (with same carrier) equal to its bi-orthogonal.

Connectives on projects

Definition

Let $\mathfrak{a}, \mathfrak{b}$ be two projects of disjoint carriers, we define

$$\mathfrak{a} \otimes \mathfrak{b} = (\llbracket \mathfrak{a}, \mathfrak{b} \rrbracket_m, A \cup B)$$

Connectives on projects

Definition

Let a, b be two projects of disjoint carriers, we define

$$a \otimes b = (\llbracket a, b \rrbracket_m, A \cup B)$$

Definition (Cut)

Let f and g be two projects. We define the cut between f and g as

$$f \multimap g = (\llbracket f, g \rrbracket_m, F \multimap G)$$

Connectives on Conducts

Definition

Let $\mathbf{A}, \mathbf{B}$ be two conducts of disjoint carriers, we define

$$\mathbf{A} \otimes \mathbf{B} = \{\mathfrak{a} \otimes \mathfrak{b} \mid \mathfrak{a} \in \mathbf{A}, \mathfrak{b} \in \mathbf{B}\}^{\downarrow\downarrow}$$

Theorem

$$(\mathbf{A} \otimes \mathbf{B}^{\downarrow})^{\downarrow} = \mathbf{A} \multimap \mathbf{B}$$

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Definition (Linear Implication)

Let $\mathbf{A}, \mathbf{B}$ be two conducts of disjoint carriers, we define

$$\mathbf{A} \multimap \mathbf{B} = \{f \mid \forall a \in \mathbf{A}, f :: a \in \mathbf{B}\}$$

Theorem

$$(\mathbf{A} \otimes \mathbf{B}^{\downarrow})^{\downarrow} = \mathbf{A} \multimap \mathbf{B}$$

Denotational semantics

Definition

We define the category $\mathcal{Graph}_{MLL}$

$$\begin{aligned}\mathcal{Obj} &= \{\mathbf{A} \mid \mathbf{A} = \mathbf{A}^{\perp\perp} \text{ of carrier } X_{\mathbf{A}} \in \mathcal{P}_f(\mathbb{N})\} \\ \mathcal{Mor}[A, B] &= \{f \in \psi_0(\mathbf{A}) \multimap \psi_1(\mathbf{B})\}\end{aligned}$$

Proposition

We define functors from the connectives $\otimes$, $\multimap$ (with delocations) and the negation. With theses definitions, the category $\mathcal{Graph}_{MLL}$ is a $*$ -autonomous category in which $\otimes \neq \wp$ and $\mathbf{1} \neq \perp$.

Additives

To deal with additives, one needs to replace graphs by *sliced graphs*, which are finite families of graphs.

- ▶ In this more general setting, the adjunction (hence the construction of multiplicatives) is still valid;
- ▶ We can define additive connectives on a subclass of conducts, which we will call behaviours;
- ▶ We show that the class of behaviours is closed under $\otimes, \multimap, (\cdot)^\perp$ (and of course $\&, \oplus$).

Denotational semantics for additives

From these construction, one can define:

- ▶ A $*$ -autonomous category $\mathcal{C}on\mathcal{D}$ (in which objects are conducts) satisfying $\otimes \neq \wp$, $1 \neq \perp$;
- ▶ A full subcategory $\mathcal{B}ehav$ of $\mathcal{C}on\mathcal{D}$ (in which objects are behaviours) where $\&$ defines a categorical product, and satisfying $\& \neq \oplus$, $\top \neq \mathbf{0}$ (and closed under $\otimes, \multimap, (\cdot)^\perp$);
- ▶ The Mix and Weakening rules do not hold in $\mathcal{B}ehav$;
- ▶ However, $1, \perp$ are not in $\mathcal{B}ehav$.

Graph Contraction

Definition (Contracting edges)

We define the operation $\hat{\cdot}$ that associates, to each weighted graph G , a simple weighted graph by replacing the set $E^G(v, w)$ of edges from v to w by a single edge which weight is equal to

$$\sum_{x \in E^G(v, w)} \omega(x).$$

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Theorem

Let $m(x) = -\log(1 - x)$. Then:

$$\llbracket F, G \rrbracket_m = \llbracket F, \hat{G} \rrbracket_m$$

From graphs to operators

- ▶ We restrict ourselves to graphs G such that the matrix of weights $\mathcal{M}(\widehat{G})$ have norm ≤ 1 .

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- ▶ We restrict ourselves to graphs G such that the matrix of weights $\mathcal{M}(\widehat{G})$ have norm ≤ 1 .
- ▶ We define an embedding Ψ associating to a graph F an operator in the hyperfinite factor of type II_∞ (by considering the matrix of weights of $\widehat{F}$), and extend it to an embedding of (graph) projects into Girard's Gol5 projects.

Embedding in Gol5

With $m(x) = -\log(1 - x)$, we have :

Graphs		Gol5
$\llbracket a, b \rrbracket_m$	=	$\llbracket \Psi(a), \Psi(b) \rrbracket$
$a \perp b$	$\Leftrightarrow$	$\Psi(a) \perp \Psi(b)$
$\Psi(a \otimes b)$	=	$\Psi(a) \otimes \Psi(b)$

When $\llbracket a, b \rrbracket_m \neq \infty$:

$$\Psi(a \hat{\vdash} b) = \Psi(a) \vdash \Psi(b)$$

Remark

This embedding can be defined for sliced graphs, and we get in this case the additional property:

$$\Psi(a \& b) = \Psi(a) \& \Psi(b)$$

Nilpotence

One could also take $m(x) = \infty$ for all $x \in]0, 1]$. Then:

- ▶ The weights do not matter anymore, neither does the number of edges between two vertices;
- ▶ Two projects (a, A) and (b, B) are orthogonal if and only if:
 - ▶ $\mathcal{C}(A)\mathcal{C}(B)$ ($\mathcal{C}(\cdot)$ denotes the connectivity matrix) is nilpotent;
 - ▶ $a\mathbf{1}_B + b\mathbf{1}_A \neq 0, \infty$;
- ▶ We therefore get a geometry of interaction for MLL whose orthogonality is based on nilpotency, i.e. "old-style" Gol (with the addition of wagers) giving rise to a *-autonomous category;

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- ▶ We therefore get a geometry of interaction for MLL whose orthogonality is based on nilpotency, i.e. "old-style" Gol (with the addition of wagers) giving rise to a *-autonomous category;
- ▶ If we look at the additive construction, the behaviours are either empty or full (orthogonal of an empty conduct), so we get nothing more than a truth-valued model for MALL.

Conclusion

- ▶ We defined a geometry of interaction for MALL based on graphs by generalizing the constructions of the article *Multiplicatives*;
- ▶ We showed it is possible to construct from it a $*$ -autonomous category, and a full subcategory (closed under $\otimes, \multimap, (\cdot)^\perp$) with products;
- ▶ We obtained a combinatorial version of (a fragment of) Gol5 by letting $m(x) = -\log(1 - x)$;
- ▶ We showed it is possible to define a notion of orthogonality based on nilpotency by letting $m(x) = \infty$.