

INTERACTION GRAPHS: EXPONENTIALS

THOMAS SEILLER

Institut des Hautes Études Scientifiques (IHÉS),
Le Bois-Marie, 35 Route de Chartres, 91440 Bures-sur-Yvette, France
e-mail address: seiller@ihes.fr

ABSTRACT. This paper is the fourth of a series [Sei12a, Sei14a, Sei14c] exposing a systematic combinatorial approach to Girard’s Geometry of Interaction program [Gir89b]. This program aims at obtaining particular realizability models for linear logic that accounts for the dynamics of cut-elimination. This fourth paper tackles the complex issue of defining exponential connectives in this framework. In order to succeed in this, we use the notion of *graphings*, a generalization of graphs which was defined in earlier work [Sei14c]. We explain how we can use this framework to define a GoI for Elementary Linear Logic (ELL) with second-order quantification, a sub-system of linear logic that captures the class of elementary time computable functions.

CONTENTS

1. Introduction	2
2. Interaction Graphs	5
3. Thick Graphs and Contraction	10
4. Construction of an Exponential Connective on the Real Line	21
5. Soundness for Behaviors	34
6. Contraction and Soundness for Polarized Conducts	40
7. Conclusion and Perspectives	49
References	50

Key words and phrases: Interaction Graphs, Geometry of Interaction, Linear Logic, Realizability.
This work was partially supported by the ANR project ANR-10-BLAN-0213 LOGOI.

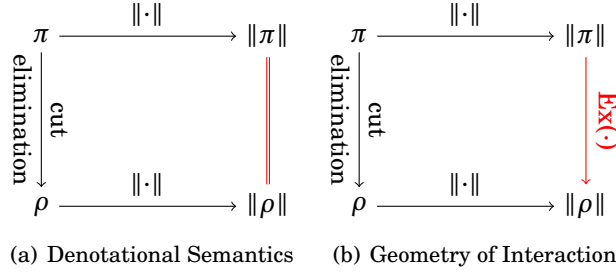


Figure 1: Denotational Semantics vs Geometry of Interaction

1. INTRODUCTION

1.1. Geometry of Interaction. A Geometry of Interaction (GoI) model, i.e. a construction that fulfills the GoI research program [Gir89b], is in a first approximation a representation of linear logic proofs that accounts for the dynamics of cut-elimination. A proof is no longer a morphism from A to B — a function from A into B — but an operator acting on the space $A \oplus B$. As a consequence, the modus ponens is no longer represented by composition. The operation representing cut-elimination, i.e. the obtention of a cut-free proof of B from a cut-free proof of A and a cut-free proof of $A \multimap B$, consists in constructing the solution to an equation called the *feedback equation* (illustrated in Figure 2). A GoI model hence represents both the proofs and the dynamics of their normalization. Contrarily to denotational semantics, a proof π and its normalized form π' are not represented by the same object. However, they remain related since the normalization procedure has a semantical counterpart — the execution formula $\text{Ex}(\cdot)$ — which satisfies $\text{Ex}(\pi) = \pi'$. This essential difference between denotational semantics and GoI is illustrated in Figure 1.

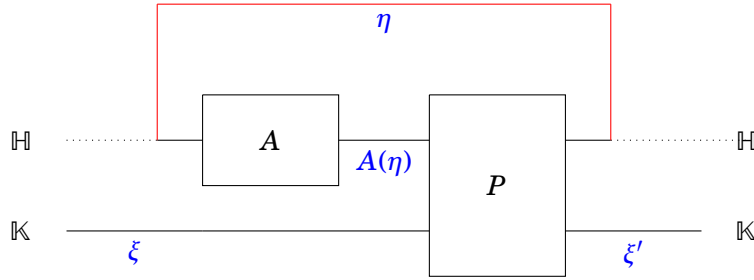
The GoI program has a second aim: define by realizability techniques a reconstruction of logical operations from the dynamical model just exposed. The objects of study in a GoI construction are a generalization of the notion of proof — paraproofs, in the same sense the proof structure where a generalization of the notion of sequent calculus proof. This is reminiscent of game semantics where not all strategies are interpretations of programs, or Krivine’s classical realizability [Kri01, Kri09] where terms containing continuation constants are distinguished from “proof-like terms”. This point of view allows a reconstruction of logic as a description of how paraproofs interact. It is therefore a sort of “discursive syntax” where paraproofs are opposed one to another, where they argue together in a way reminiscent of game semantics, each one trying to prove the other wrong. This argument terminates when one of them gives up. The discussion itself corresponds to the execution formula, which describes the solution to the feedback equation and generalizes the cut-elimination procedure to this generalized notion of proofs. Two paraproofs are then said *orthogonal* — denoted by the symbol $\perp$ — when this argument (takes place and) terminates. A notion of formula is then drawn from this notion of orthogonality: a formula is a set of paraproofs A equal to its bi-orthogonal closure $A^{\perp\perp}$ or, equivalently, a set of paraproofs $A = B^{\perp}$ which is the orthogonal to a given set of paraproofs B .

$P \in \mathcal{L}(\mathbb{H} \oplus \mathbb{K})$ represents ^a a program/proof of implication
 $A \in \mathcal{L}(\mathbb{H})$ represents an argument.
 $R \in \mathcal{L}(\mathbb{K})$ represents the result of the computation if:

$$R(\xi) = \xi' \Leftrightarrow \exists \eta, \eta' \in \mathbb{H}, \begin{cases} P(\eta \oplus \xi) &= \eta' \oplus \xi' \\ A(\eta') &= \eta \end{cases}$$

^aHere, $\mathbb{H}$ and $\mathbb{K}$ are separable infinite-dimensional Hilbert spaces, and $\mathcal{L}(\star)$ denotes the set of operators acting on the Hilbert space $\star$: bounded (or, equivalently, continuous) linear maps from $\star$ to $\star$.

(a) Formal statement



(b) Illustration of the equation

Figure 2: The Feedback Equation

Drawing some intuitions from the Curry-Howard correspondence, one may propose an alternative reading to this construction in terms of programs. Since proofs correspond to well-behaved programs, paraproof is a generalization of those, representing somehow *badly-behaved programs*. If the orthogonality relation represents negation from a logical point of view, it represents a notion of *testing* from a computer science point of view. The notion of formula defined from it corresponds to a notion of type, defined interactively from how (para)programs behave. This point of view is still natural when thinking about programs: a program is of type $\mathbf{nat} \rightarrow \mathbf{nat}$ because it produces a natural number when given a natural number as an argument. On the logical side, this change may be more radical: a proof is a proof of the formula $\mathbf{Nat} \Rightarrow \mathbf{Nat}$ because it produces a proof of $\mathbf{Nat}$ each time it is cut (applied) to a proof of $\mathbf{Nat}$.

Once the notion of type/formula defined, one can reconstruct the connectives: from a "low-level" — between paraproof — definition, one obtains a "high-level" definition — between types. For instance, the connective $\otimes$ is first defined between any two paraproof a, b , and this definition is then extended to types by defining $A \otimes B = \{a \otimes b \mid a \in A, b \in B\}^{\perp\perp}$. As a consequence, the connectives are not defined in an *ad hoc* way, but their definition is a consequence of their computational meaning: the connectives are defined on proofs/programs and their definition at the level of types is just the reflection of the interaction between the execution — the dynamics of proofs — and the low-level definition on paraproof. Logic thus arises as generated by computation, by the normalization of proofs: types/formulas are not there to tame the programs/proofs but only to describe their behavior. This is reminiscent of realizability in the sense that a type is defined as the set of its (para-)proofs. Of course,

the fact that we consider a generalized notion of proofs from the beginning has an effect on the construction: contrarily to usual realizability models (except from classical realizability in the sense of Krivine [Kri01, Kri09]), the types A and $A^\perp$ (the negation of A) are in general both non-empty. This is balanced by the fact that one can define a notion of successful paraproofs, which corresponds in a way to the notion of winning strategy in game semantics. This notion on paraproofs then yields a high-level definition: a formula/type is *true* when it contains a successful paraproof.

1.2. Interaction Graphs and Graphings. Interaction Graphs were first introduced [Sei12a] to define a combinatorial approach to Girard's geometry of interaction in the hyperfinite factor [Gir11]. The main idea was that the execution formula — the counterpart of the cut-elimination procedure — can be computed as the set of alternating paths between graphs, and that the measurement of interaction defined by Girard using the Fuglede-Kadison determinant [FK52] can be computed as a measurement of a set of cycles.

The setting was then extended to deal with additive connectives [Sei14a], showing by the way that the constructions were a combinatorial approach not only to Girard's hyperfinite GoI construction but also to all the earlier constructions [Gir87, Gir89a, Gir88, Gir95a]. This result could be obtained by unveiling a single geometrical property, which we called the *trefoil property*, upon which all the constructions of geometry of interaction introduced by Girard are built.

In a third paper, we explored a wide generalization of the graph framework¹. We introduced the notion of *graphing* which we now informally describe. If $(X, \mathcal{B}, \mu)$ is a measured space and $\mathfrak{m}$ is a monoid of measurable maps² $X \rightarrow X$ (the internal law is composition), then a graphing in $\mathfrak{m}$ is a countable family of restrictions of elements of $\mathfrak{m}$ to measurable subsets. These restrictions of elements of $\mathfrak{m}$ are regarded as edges of a graph *realized* as measurable (partial) maps. We showed that the notions of paths and cycles in a graphing could be defined. As a consequence, one can define the *execution* as the set of alternating paths between graphings, mimicking the corresponding operation of graphs. On the other hand, a more complex argument shows that one can define appropriate measures of cycles in order to insure that the trefoil property holds. As a consequence, we obtained whole hierarchies of models of multiplicative-additive linear logic in this way. The purpose of this paper is to exhibit a family of such models in which one can interpret Elementary Linear Logic [Gir95b, DJ03] with second-order quantification.

1.3. Outline of the paper. In Section 2, we recall some important definitions and properties on directed weighted graphs. This allows us to introduce important notations that will be used later on. We then recall some definitions and properties about the additive construction [Sei14a]. These properties are essential to the understanding of the construction of the multiplicative-additive fragment of linear logic in the setting of interaction graphs.

In Section 3, we define and study the notion of *thick graphs*, and show how it can be used to interpret the contraction $!A \multimap !A \otimes !A$ for some specific formulas A . This motivates the definition of a *perennisation* Ω from which one can define an exponential $A \multimap !_\Omega A$. We also explain why it is necessary to work with a generalization of graphs, namely graphings, in order to define perennisations that are suitably expressive.

¹This generalization, or more precisely a fragment of it, already appeared in the author's PhD thesis [Sei12b].

²For technical reasons, we in fact consider monoids of Borel-preserving non-singular maps [Sei14c].

In Section 4, we give a definition of an exponential connective defined from a suitable notion of perennisation. We show for this a result which allows us to encode any bijection over the natural numbers as a measure-preserving map over the unit interval of the real line. This result is then used to encode some combinatorics as measure-preserving maps and show that functorial promotion can be implemented for the exponential we defined.

We then prove a soundness result for a variant (in Section 5) of Elementary Linear Logic (ELL). This result, though interesting, is not ideal since we restrict to proofs that are in some sense "intuitionnistic". Indeed, for technical reasons explained later on, the introduction of exponentials cannot be performed without being associated to a tensor product. Since the interpretation of elementary time functions in ELL relies heavily on those proofs that are not intuitionnistic in this sense³.

Consequently, we introduce (in Section 6) a notion of polarities which generalize the notion of *perennial*/*co-perennial* formulas defined before. The discussion on polarities leads to a refinement of the sequent calculus considered in the previous section which does not suffer from the drawbacks explained above. We then prove a soundness result for this calculus.

2. INTERACTION GRAPHS

2.1. Basic Definitions. Departing from the realm of infinite-dimensional vector spaces and linear maps between them, we proposed in previous work [Sei12a, Sei14a] a graph-theoretical construction of GoI models. We give here a brief overview of the main definitions and results. The graphs we consider are directed and weighted, where the weights are taken in a *weight monoid* $(\Omega, \cdot)$.

Definition 1. A *directed weighted graph* is a tuple G , where V^G is the set of vertices, E^G is the set of edges, s^G and t^G are two functions from E^G to V^G , the *source* and *target* functions, and ω^G is a function $E^G \rightarrow \Omega$.

The construction is centered around the notion of alternating paths. Given two graphs F and G , an alternating path is a path $e_1 \dots e_n$ such that $e_i \in E^F$ if and only if $e_{i+1} \in E^G$. The set of alternating paths will be used to define the interpretation of cut-elimination in the framework, i.e. the graph $F :: G$ — the *execution of F and G* — is defined as the graph of alternating paths between F and G whose source and target are in the symmetric difference $V^F \Delta V^G$. The weight of a path is naturally defined as the product of the weights of the edges it contains.

Definition 2. Let F, G be directed weighted graphs. The set of alternating paths between F and G is the set of paths $e_0, e_1, \dots, e_n$ such that $e_i \in E^G \Rightarrow e_{i+1} \in E^F$ and $e_i \in E^F \Rightarrow e_{i+1} \in E^G$. We write $\text{Path}(F, G)$ the set of such paths, and $\text{Path}(F, G)_V$ the subset of $\text{Path}(F, G)$ containing the paths whose source and target lie in V .

The execution $F :: G$ of F and G is then defined by:

$$\begin{aligned} V^{F::G} &= V^F \Delta V^G \\ E^{F::G} &= \text{Path}(F, G)_{V^{F::G}} \end{aligned}$$

where the source and target maps are naturally defined, and the weight of a path is the product of the weights of the edges it is composed of.

³This fact was pointed out to the author by Damiano Mazza.

As it is usual in mathematics, this notion of paths cannot be considered without the associated notion of cycle: an *alternating cycle* between two graphs F and G is a cycle which is an alternating path $e_1 e_2 \dots e_n$ such that $e_1 \in V^F$ if and only if $e_n \in V^G$. For technical reasons, we actually consider the related notion of 1-circuit.

Definition 3. A 1-circuit is an alternating cycle $\pi = e_1 \dots e_n$ which is not a proper power of a smaller cycle. In mathematical terms, there do not exist a cycle ρ and an integer k such that $\pi = \rho^k$, where the power represents iterated concatenation.

We denote by $\mathcal{C}(F, G)$ the set of 1-circuits in the following. We show that these notions of paths and cycles satisfy a property we call the *trefoil property* which will turn out to be fundamental. The trefoil property states that there exists weight-preserving bijections:

$$\mathcal{C}(F :: G, H) \cup \mathcal{C}(F, G) \cong \mathcal{C}(G :: H, F) \cup \mathcal{C}(G, H) \cong \mathcal{C}(H :: F, G) \cup \mathcal{C}(H, F)$$

We showed, based only on the trefoil property, how one can define the multiplicative and additive connectives of Linear Logic, obtaining a model fulfilling the GoI research program. This construction is moreover parametrized by a map from the set Ω to $\mathbb{R}_{\geq 0} \cup \{\infty\}$, and therefore yields not only one but a whole family of models. This parameter is introduced to define the notion of orthogonality in our setting, a notion that account for linear negation. Indeed, given a map m and two graphs F, G we define $\llbracket F, G \rrbracket_m$ as the sum $\sum_{\pi \in \mathcal{C}(F, G)} m(\omega(\pi))$, where $\omega(\pi)$ is the weight of the cycle π . The orthogonality is then constructed from this measurement.

We moreover showed how, from any of these constructions, one can obtain a *-autonomous category $\mathcal{G}raph_{MLL}$ with $\wp \not\equiv \otimes$ and $1 \not\equiv \perp$, i.e. a non-degenerate denotational semantics for Multiplicative Linear Logic (MLL). However, as in all the versions of GoI dealing with additive connectives, our construction of additives does not define a categorical product. We solve this issue by introducing a notion of *observational equivalence* within the model. We are then able to define a categorical product from our additive connectives when considering classes of observationally equivalent objects, thus obtaining a denotational semantics for Multiplicative Additive Linear Logic (MALL).

2.2. Models of MALL in a Nutshell. We recall the basic definitions of projects, and behaviors, which will be respectively used to interpret proofs and formulas, as well as the definition of connectives.

- a *project* of carrier V^A is a triple $\mathfrak{a} = (a, V^A, A)$, where a is a real number, $A = \sum_{i \in I^A} \alpha_i^A A_i$ is a finite formal (real-)weighted sum of graphings of carrier included in V^A ;
- two projects $\mathfrak{a}, \mathfrak{b}$ are *orthogonal* when:

$$\llbracket \mathfrak{a}, \mathfrak{b} \rrbracket_m = a \left(\sum_{i \in I^A} \alpha_i^B \right) + b \left(\sum_{i \in I^B} \alpha_i^A \right) + \sum_{i \in I^A} \sum_{j \in I^B} \alpha_i^A \alpha_j^B \llbracket A_i, B_j \rrbracket_m \neq 0, \infty$$

- the *execution* of two projects $\mathfrak{a}, \mathfrak{b}$ is defined as:

$$\mathfrak{a} :: \mathfrak{b} = (\llbracket \mathfrak{a}, \mathfrak{b} \rrbracket_m, V^A \Delta V^B, \sum_{i \in I^A} \sum_{j \in I^B} \alpha_i^A \alpha_j^B A_i :: B_j)$$

- if $\mathfrak{a}$ is a project and V is a measurable set such that $V^A \subset V$, we define the extension $\mathfrak{a}|_V$ as the project (a, V, A) ;
- a *conduct* $\mathbf{A}$ of carrier V^A is a set of projects of carrier V^A which is equal to its bi-orthogonal $\mathbf{A}^{\perp\perp}$;

- a *behavior* $\mathbf{A}$ of carrier V^A is a conduct such that for all $\lambda \in \mathbf{R}$,

$$\begin{aligned} a \in \mathbf{A} &\Rightarrow a + \lambda 0 \in \mathbf{A} \\ b \in \mathbf{A}^\perp &\Rightarrow b + \lambda 0 \in \mathbf{A}^\perp \end{aligned}$$

- we define, for every measurable set the *empty* behavior of carrier V as the empty set $\mathbf{0}_V$, and the *full behavior* of carrier V as its orthogonal $\mathbf{T}_V = \{a \mid a \text{ of support } V\}$;
- if $\mathbf{A}, \mathbf{B}$ are two behaviors of disjoint carriers, we define:

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= \{a :: b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp} \\ \mathbf{A} \multimap \mathbf{B} &= \{f \mid \forall a \in \mathbf{A}, f :: a \in \mathbf{B}\} \\ \mathbf{A} \oplus \mathbf{B} &= (\{a \upharpoonright_{V^A \cup V^B} \mid a \in \mathbf{A}\}^{\perp\perp} \cup \{b \upharpoonright_{V^A \cup V^B} \mid b \in \mathbf{B}\}^{\perp\perp})^{\perp\perp} \\ \mathbf{A} \& \mathbf{B} &= \{a \upharpoonright_{V^A \cup V^B} \mid a \in \mathbf{A}^\perp\}^{\perp\perp} \cap \{b \upharpoonright_{V^A \cup V^B} \mid b \in \mathbf{B}^\perp\}^{\perp\perp} \end{aligned}$$

- two elements a, b of a conduct $\mathbf{A}$ are *observationally equivalent* when:

$$\forall c \in \mathbf{A}^\perp, \ll a, c \gg_m = \ll b, c \gg_m$$

One important point in this work is the fact that all results rely on a single geometric property, namely the previously introduced *trefoil property* which describes how the sets of 1-circuits evolve during an execution. This property insures on its own the four following facts:

- we obtain a $*$ -autonomous category $\mathcal{G}\text{raph}_{MLL}$ whose objects are conducts and morphisms are projects;
- the observational equivalence is a congruence on this category;
- the quotiented category $\mathcal{C}\text{ond}$ inherits the $*$ -autonomous structure;
- the quotiented category $\mathcal{C}\text{ond}$ has a full subcategory $\mathcal{B}\text{ehav}$ with products whose objects are behaviors.

This can be summarized in the following two theorems.

Theorem 4. *For any map $m : \Omega \rightarrow \mathbf{R} \cup \{\infty\}$, the categories $\mathcal{C}\text{ond}$ and $\mathcal{G}\text{raph}_{MLL}$ are non-degenerate categorical models of Multiplicative Linear Logic with multiplicative units.*

Theorem 5. *For any map $m : \Omega \rightarrow \mathbf{R} \cup \{\infty\}$, the full subcategory $\mathcal{B}\text{ehav}$ of $\mathcal{C}\text{ond}$ is a non-degenerate categorical model of Multiplicative-Additive Linear Logic with additive units.*

The categorical model we obtain has two layers (see Figure 3). The first layer consists in a non-degenerate (i.e. $\otimes \neq \wp$ and $\mathbf{1} \neq \perp$) $*$ -autonomous category $\mathcal{C}\text{ond}$, hence a denotational model for MLL with units. The second layer is the full subcategory $\mathcal{B}\text{ehav}$ which does not contain the multiplicative units but is a non-degenerate model (i.e. $\otimes \neq \wp$, $\oplus \neq \&$ and $\mathbf{0} \neq \top$) of MALL with additive units that does not satisfy the mix and weakening rules.

We here recall some technical results obtained in our paper on additives [Sei14a] and that will be useful in the following.

Proposition 6. *If A is a non-empty set of projects of same carrier V^A such that $(a, A) \in A$ implies $a = 0$, then $b \in A^\perp$ implies $b + \lambda 0_{V^A} \in A^\perp$ for all $\lambda \in \mathbf{R}$.*

Proposition 7. *If A is a non-empty set of projects of carrier V such that $a \in A \Rightarrow a + \lambda 0_V \in A$, then any project in $A^\perp$ is wager-free, i.e. if $(a, A) \in A^\perp$ then $a = 0$.*

Lemma 8 (Homothety). *Conducts are closed under homothety: for all $a \in \mathbf{A}$ and all $\lambda \in \mathbf{R}$ with $\lambda \neq 0$, $\lambda a \in \mathbf{A}$.*

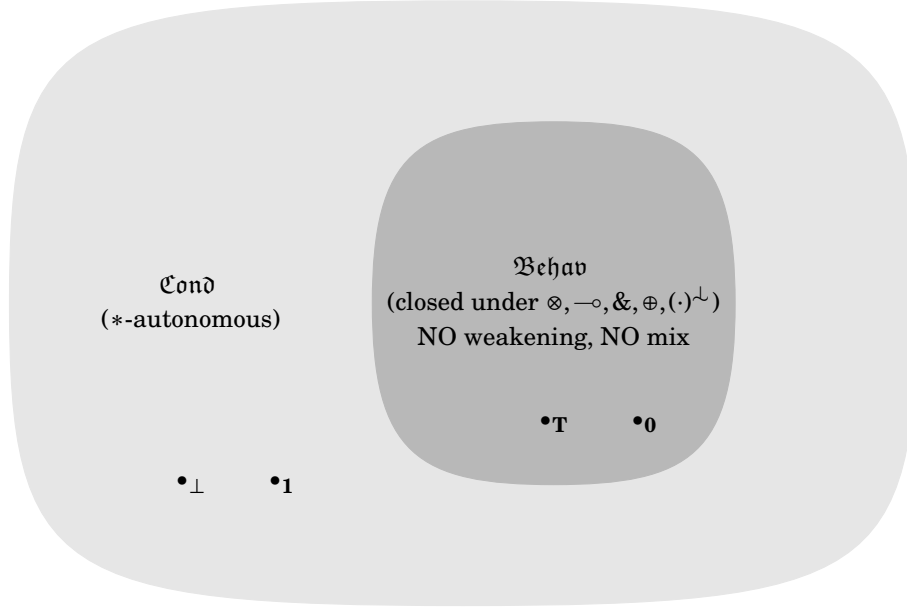


Figure 3: The categorical models

Proposition 9. We denote by $\mathbf{A} \odot \mathbf{B}$ the set $\{a \otimes b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$. Let E, F be non-empty sets of projects of respective carriers V, W with $V \cap W = \emptyset$. Then

$$(E \odot F)^{\perp\perp} = (E^{\perp\perp} \odot F^{\perp\perp})^{\perp\perp}$$

Proposition 10. Let $\mathbf{A}, \mathbf{B}$ be conducts. Then:

$$(\{a \otimes 0_B \mid a \in \mathbf{A}\} \cup \{0_A \otimes b \mid b \in \mathbf{B}\})^{\perp\perp} = \mathbf{A} \oplus \mathbf{B}$$

Proposition 11 (Distributivity). For any behaviors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and delocations ϕ, ψ, θ, ρ of $\mathbf{A}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ respectively, there is a project distr in the behavior

$$((\phi(\mathbf{A}) \multimap \theta(\mathbf{B})) \& (\psi(\mathbf{A}) \multimap \rho(\mathbf{C}))) \multimap (\mathbf{A} \multimap (\mathbf{B} \& \mathbf{C}))$$

2.3. Graphings. In subsequent work [Sei14c], we introduced a generalization of graphs to which the previously described results can be extended. This generalization allows, among other things, for the definition of second order quantification. The main purpose of this generalization is that a vertex can always be cut in an arbitrary (finite) number of sub-vertices, with the idea that these sub-vertices are smaller (hence vertices have a *size*) and form a partition of the initial vertex (where two sub-vertices have the same size). These notions could be introduced and dealt with combinatorially, but we chose to use measure-theoretic notions in order to ease the intuitions and some proofs. In fact, a *graphing* — the notion which is introduced as a generalization of the notion of graph — can be thought of and used as a graph. Another important feature of the construction is the fact that it depends on a *microcosm* — a monoid of non-singular transformations — which somehow describes the computational principles allowed in the model.

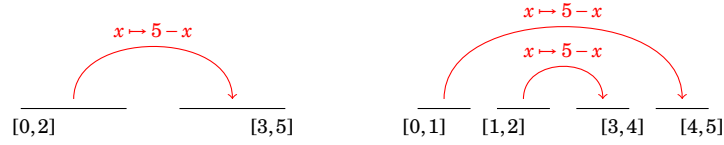


Figure 4: A graphing and one of its refinements

Definition 12. Let $(X, \mathcal{B}, \lambda)$ be a measured space. We denote by $\mathcal{M}(X)$ the set of Borel-preserving non-singular transformations⁴ $X \rightarrow X$. A *microcosm* of the measured space X is a subset $\mathfrak{m}$ of $\mathcal{M}(X)$ which is closed under composition and contains the identity.

As in the graph construction described above, we will consider a notion of graphing depending on a *weight-monoid* Ω , i.e. a monoid $(\Omega, \cdot, 1)$ which contains the possible weights of the edges.

Definition 13 (Graphings). Let $\mathfrak{m}$ be a microcosm of a measured space $(X, \mathcal{B}, \lambda)$ and V^F a measurable subset of X . A Ω -*weighted graphing* in $\mathfrak{m}$ of carrier V^F is a countable family $F = \{(\omega_e^F, \phi_e^F : S_e^F \rightarrow T_e^F)\}_{e \in E^F}$, where, for all $e \in E^F$ (the set of edges):

- ω_e^F is an element of Ω , the *weight* of the edge e ;
- $S_e^F \subset V^F$ is a measurable set, the *source* of the edge e ;
- $T_e^F = \phi_e^F(S_e^F) \subset V^F$ is a measurable set, the *target* of the edge e ;
- ϕ_e^F is the restriction of an element of $\mathfrak{m}$ to S_e^F , the *realization* of the edge e .

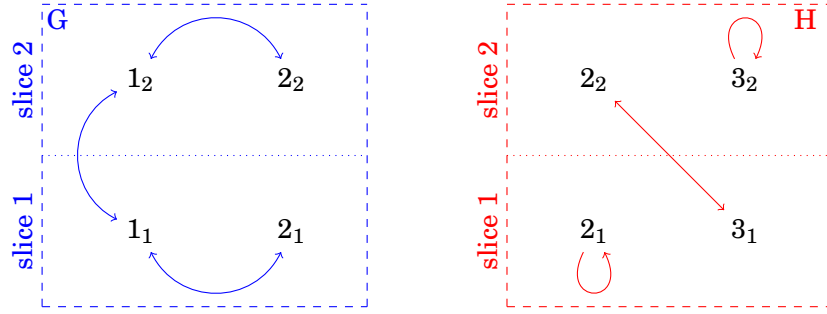
It is natural, as we are working with measure-theoretic notions, to identify two graphings that differ only on a set of null measure. This leads to the definition of an equivalence relation between graphings: that of *almost everywhere equality*. Moreover, since we want vertices to be *decomposable* into any finite number of parts, we want to identify a graphing G with the graphing G' obtained by replacing an edge $e \in E^F$ by a finite family of edges $e_i \in G'$ ($i = 1, \dots, n$) subject to the conditions:

- the family $\{S_{e_i}^{G'}\}_{i=1}^n$ (resp. $\{T_{e_i}^{G'}\}_{i=1}^n$) is a partition of S_e^G (resp. T_e^G);
- for all $i = 1, \dots, n$, $\phi_{e_i}^{G'}$ is the restriction of ϕ_e^G on $S_{e_i}^{G'}$.

Such a graphing G' is an example of a *refinement* of G , and one can easily generalize the previous conditions to define a general notion of refinement of graphings. Figure 4 gives the most simple example of refinement. To be a bit more precise, we define, in order to ease the proofs, a notion of refinement *up to almost everywhere equality*. We then define a second equivalence relation on graphings by saying that two graphings are equivalent if and only if they have a common refinement (up to almost everywhere equality).

The objects under study are thus equivalence classes of graphings modulo this equivalence relation. Most of the technical results on graphings contained in our previous paper [Sei14c] aim at showing that these objects can actually be manipulated as graphs: one can define paths and cycles and these notions are coherent with the quotient by the equivalence relation just mentioned. Indeed, the notions of paths and cycles in a graphings are quite natural, and from two graphings F, G in a microcosm $\mathfrak{m}$ one can define its execution $F :: G$ which

⁴A non-singular transformation $f : X \rightarrow X$ is a measurable map which preserves the sets of null measure, i.e. $\lambda(f(A)) = 0$ if and only if $\lambda(A) = 0$. A Borel-preserving map is a map such that the images of Borel sets are Borel sets.

Figure 5: Two thick graphs G and H , both with dialect $\{1, 2\}$

is again a graphing in $\mathfrak{m}^5$. A more involved argument then shows that the trefoil property holds for a family of measurements $\llbracket \cdot, \cdot \rrbracket_m$, where $m : \Omega \rightarrow \mathbf{R}_{\geq 0} \cup \{\infty\}$ is any measurable map. These results are obtained as a generalization of constructions considered in the author's thesis⁶.

Theorem 14. *Let Ω be a monoid, $\mathfrak{m}$ a microcosm and $m : \Omega \rightarrow \mathbf{R}_{\geq 0} \cup \{\infty\}$ be a measurable map. The set of Ω -weighted graphings in $\mathfrak{m}$ yields a model, denoted by $\mathbb{M}[\Omega, \mathfrak{m}]_m$, of multiplicative-additive linear logic whose orthogonality relation depends on m .*

3. THICK GRAPHS AND CONTRACTION

In this section, we will define the notion of *thick graphs*, and extend the additive construction defined in our earlier paper [Sei14a] to that setting. The introduction of these objects will be motivated in Section 3.3, where we will explain how thick graphs allows for the interpretation of the contraction rule. This contraction rule being satisfied only for a certain kind of conducts — interpretations of formulas, this will justify the definition of the exponentials.

3.1. Thick Graphs.

Definition 15. Let S^G and D^G be finite sets. A directed weighted *thick graph* G of carrier S^G and *dialect* D^G is a directed weighted graph over the set of vertices $S^G \times D^G$.

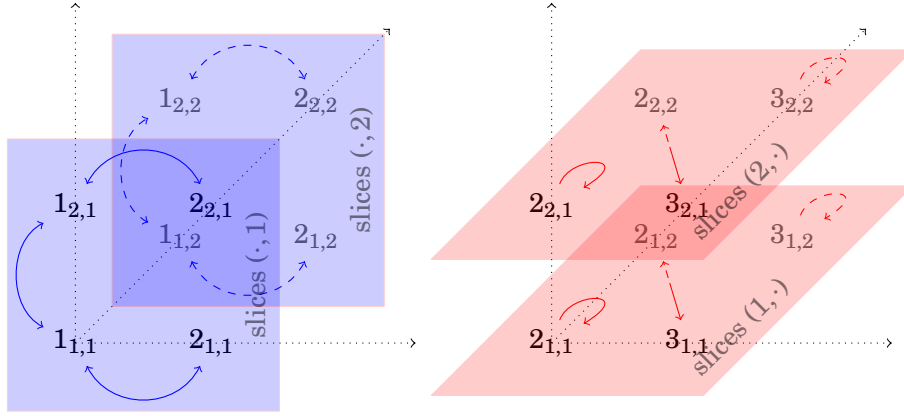
We will call *slices* the set of vertices $S^G \times \{d\}$ for $d \in D^G$.

Figure 5 shows two examples of thick graphs. Thick graphs will be represented following a graphical convention very close to the one we used for sliced graphs:

- Graphs are once again represented with colored edges and delimited by hashed lines;
- Elements of the carrier S^G are represented on a horizontal scale, while elements of the dialect D^G are represented on a vertical scale;
- Inside a given graph, slices are separated by a *dotted* line.

⁵As a consequence, a microcosm is a closed world for the execution of programs.

⁶In the cited work, the results were stated in the particular case of the microcosm of measure-preserving maps on the real line.

Figure 6: The graphs $G^{\dagger_{D^H}}$ and $H^{\dagger_{D^G}}$

Remark 16. If $G = \sum_{i \in I^G} \alpha_i^G G_i$ is a sliced graph such that $\forall i \in I^G, \alpha_i^G = 1$, then G can be identified with a thick graph of dialect I^G . Indeed, one can define the thick graph $\{G\}$ by:

$$\begin{aligned} V^{\{G\}} &= V^G \times I^G \\ E^{\{G\}} &= \uplus_{i \in I^G} E^{G_i} \\ s^{\{G\}} &= e \in E^{G_i} \mapsto (s^{G_i}(e), i) \\ t^{\{G\}} &= e \in E^{G_i} \mapsto (t^{G_i}(e), i) \\ \omega^{\{G\}} &= e \in E^{G_i} \mapsto \omega^{G_i}(e) \end{aligned}$$

Definition 17 (Variants). Let G be a thick graph and $\phi : D^G \rightarrow E$ a bijection. One defines G^ϕ as the graph:

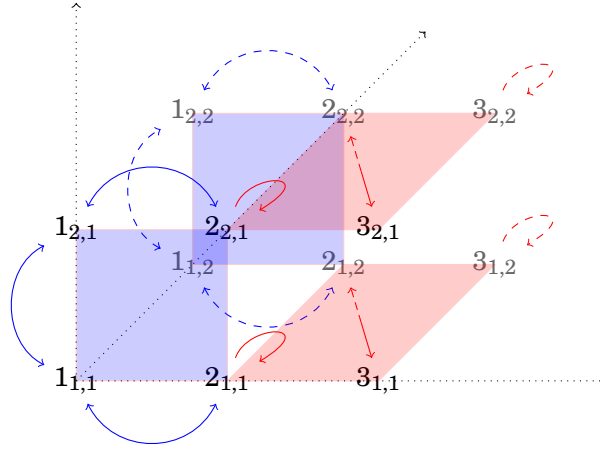
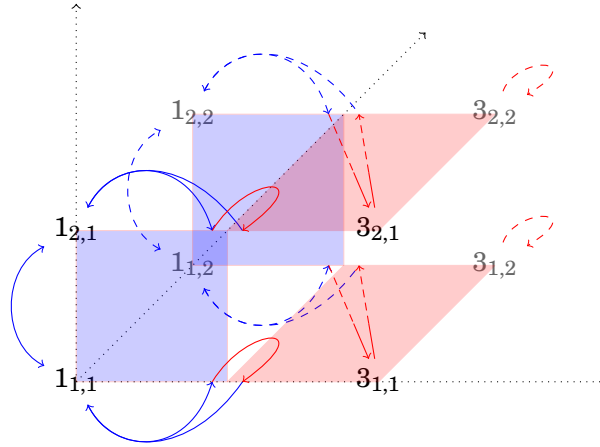
$$\begin{aligned} V^{G^\phi} &= S^G \times E \\ E^{G^\phi} &= E^G \\ s^{G^\phi} &= (Id_{V^G} \times \phi) \circ s^G \\ t^{G^\phi} &= (Id_{V^G} \times \phi) \circ t^G \\ \omega^{G^\phi} &= \omega^G \end{aligned}$$

If G and H are two thick graphs such that $H = G^\phi$ for a bijection ϕ , then H is called a *variant* of G . The relation defined by $G \sim H$ if and only if G is a variant of H can easily be checked to be an equivalence relation.

Definition 18 (Dialectal Interaction). Let G and H be thick graphs.

- (1) We denote by $G^{\dagger_{D^H}}$ the thick graph of dialect $D^G \times D^H$ defined as $\{\sum_{i \in D^H} G_i\}$;
- (2) We denote by $H^{\dagger_{D^G}}$ the thick graph of dialect $D^G \times D^H$ defined as $\{\sum_{i \in D^G} H_i\}^\tau$ where τ is the natural bijection $D^H \times D^G \rightarrow D^G \times D^H, (a, b) \mapsto (b, a)$.

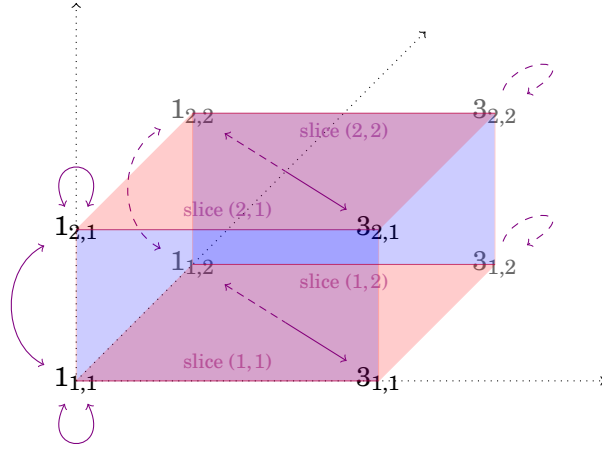
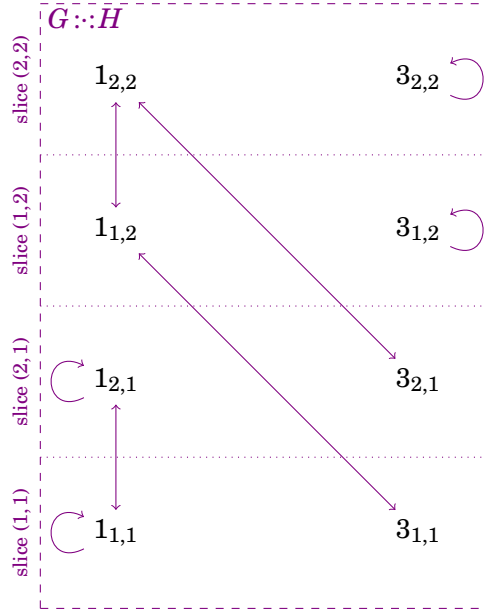
We can then define the plugging $F \square G$ of two thick graphs as the plugging of the graphs $F^{\dagger_{D^G}}$ and $G^{\dagger_{D^F}}$. Figure 7 shows the result of the plugging of G and H , the thick graphs represented in Figure 5.

Figure 7: Plugging of the thick graphs G and H Figure 8: Alternating paths in the plugging of thick graphs G and H

One can then define the execution $G :: H$ of two thick graphs G and H as the execution of the graphs $G^{\dagger}_{DH}$ and $H^{\ddagger}_{DG}$. Figure 8 shows the set of alternating paths in the plugging of the thick graphs G and H introduced in Figure 5. Figure 9 and Figure 10 represent the result of the execution of these two thick graphs, the first is three-dimensional representation which can help make the connection with the set of alternating paths in Figure 8, while the second is a two-dimensional representation of the same graph. In a natural way, the measurement of the interaction between two thick graphs G, H is defined as $\llbracket G^{\dagger}_{DH}, H^{\ddagger}_{DG} \rrbracket_m$.

Definition 19. The execution $F :: G$ of two thick graphs F, G is the thick graph of carrier $S^F \Delta S^G$ and dialect $D^F \times D^G$ defined as $F^{\dagger}_{DG} :: G^{\ddagger}_{DF}$.

Remark 20. Since we only modified the graphs before plugging them together, we can make the following remark. Let F, G, H be thick graphs. Then the thick graph $F :: (G :: H)$ is defined

Figure 9: Result of the execution of the thick graphs G and H Figure 10: The thick graph $G :: H$ represented in two dimensions.

as

$$F^{\dagger}_{D^G \times D^H} :: (G^{\dagger}_{D^H} :: H^{\ddagger}_{D^G})^{\ddagger}_{D^F} = F^{\dagger}_{D^G \times D^H} :: ((G^{\dagger}_{D^H})^{\ddagger}_{D^F} :: H^{\ddagger}_{D^F \times D^G})$$

If one supposes that $S^F \cap S^G \cap S^H = \emptyset$, it is clear that $(S^F \times D) \cap (S^G \times D) \cap (S^H \times D) = \emptyset$. We can deduce from the associativity of execution that

$$F^{\dagger}_{D^G \times D^H} :: ((G^{\dagger}_{D^H})^{\ddagger}_{D^F} :: H^{\ddagger}_{D^F \times D^G}) = (F^{\dagger}_{D^G \times D^H} :: ((G^{\dagger}_{D^H})^{\ddagger}_{D^F})) :: H^{\ddagger}_{D^F \times D^G}$$

But:

$$(F^{\dagger}_{D^G \times D^H} :: ((G^{\dagger}_{D^H})^{\ddagger}_{D^F})) :: H^{\ddagger}_{D^F \times D^G} = ((F^{\dagger}_{D^G} :: G^{\ddagger}_{D^F})^{\dagger}_{D^H}) :: H^{\ddagger}_{D^F \times D^G}$$

The latter is by definition the thick graph $(F :: G) :: H$. This shows that the associativity of $::$ on thick graphs is a simple consequence of the associativity of $::$ on simple graphs.

Proposition 21 (Associativity). *Let F, G, H be thick graphs such that $S^F \cap S^G \cap S^H = \emptyset$. Then:*

$$F :: (G :: H) = (F :: G) :: H$$

Definition 22. Let F and G be two thick graphs. We define $\text{Cy}^e(F, G)$ as the set of circuits in $F^{\dagger_{D^G}} \square G^{\ddagger_{D^F}}$.

We also define, being given a dialect D^H ,

- the set $\text{Cy}^e(F, G)^{\dagger_{D^H}}$ of circuits in the graph $(F^{\dagger_{D^G}})^{\dagger_{D^H}} \square (G^{\ddagger_{D^F}})^{\dagger_{D^H}}$
- the set $\text{Cy}^e(F, G)^{\ddagger_{D^H}}$ of circuits in the graph $(F^{\ddagger_{D^G}})^{\dagger_{D^H}} \square (G^{\ddagger_{D^F}})^{\dagger_{D^H}}$

Proposition 23. *Let F, G, H be thick graphs and $\phi : D^H \rightarrow D$ a bijection. Then:*

$$\begin{aligned} \text{Cy}^e(F, H) &\cong \text{Cy}^e(F, H^\phi) \\ \text{Cy}^e(F, G)^{\dagger_{D^H}} &\cong \text{Cy}^e(F, G)^{\dagger_{\phi(D^H)}} \\ \text{Cy}^e(F, G)^{\ddagger_{D^H}} &\cong \text{Cy}^e(F, G)^{\ddagger_{D^H}} \end{aligned}$$

As in Remark 20, one considers the three thick graphs F, G, H . We are interested in the circuits in $\text{Cy}^e(F, G :: H) \cup (\text{Cy}^e(G, H)^{\ddagger_{D^F}})$. By definition, these are the circuits in one of the following graphs:

$$\begin{aligned} F^{\dagger_{D^G \times D^H}} \square ((G^{\dagger_{D^H}} :: H^{\ddagger_{D^G}})^{\ddagger_{D^F}}) &= F^{\dagger_{D^G \times D^H}} \square ((G^{\dagger_{D^H}})^{\ddagger_{D^F}} :: H^{\ddagger_{D^F \times D^G}}) \\ (G^{\dagger_{D^H}} \square H^{\ddagger_{D^G}})^{\ddagger_{D^F}} &= (G^{\dagger_{D^H}})^{\ddagger_{D^F}} \square H^{\ddagger_{D^F \times D^G}} \end{aligned}$$

We can now use the trefoil property to deduce that these sets of circuits are in bijection with the set of circuits in the following graphs:

$$\begin{aligned} (F^{\dagger_{D^G \times D^H}} :: (G^{\ddagger_{D^F}})^{\dagger_{D^H}}) \square H^{\ddagger_{D^F \times D^G}} &= (F^{\dagger_{D^G}} :: G^{\ddagger_{D^F}})^{\dagger_{D^H}} \square H^{\ddagger_{D^F \times D^G}} \\ (F^{\dagger_{D^G \times D^H}} \square (G^{\ddagger_{D^F}})^{\dagger_{D^H}})^{\ddagger_{D^H}} &= (F^{\dagger_{D^G}} \square G^{\ddagger_{D^F}})^{\dagger_{D^H}} \end{aligned}$$

This shows that the trefoil property holds for thick graphs.

Proposition 24 (Geometric Trefoil Property for Thick Graphs). *If F, G, H are thick graphs such that $S^F \cap S^G \cap S^H = \emptyset$, then:*

$$\text{Cy}^e(F, G :: H) \cup \text{Cy}^e(G, H)^{\ddagger_{D^F}} \cong \text{Cy}^e(F :: G, H) \cup \text{Cy}^e(F, G)^{\dagger_{D^H}}$$

Corollary 25 (Geometric Adjunction for Thick Graphs). *If F, G, H are thick graphs such that $S^G \cap S^H = \emptyset$, we have:*

$$\text{Cy}^e(F, G \cup H) \cong \text{Cy}^e(F :: G, H) \cup \text{Cy}^e(F, G)^{\dagger_{D^H}}$$

Definition 26. Being given a circuit quantifying map m , one can define a measurement of the interaction between thick graphs. For every couple of thick graphs F, G , it is defined as:

$$\llbracket F, G \rrbracket_m = \sum_{\pi \in \text{Cy}^e(F, G)} \frac{1}{\text{Card}(D^F \times D^G)} m(\omega(\pi))$$

Proposition 27 (Numerical Trefoil Property for Thick Graphs). *Let F, G, H be thick graphs such that $S^F \cap S^G \cap S^H = \emptyset$. Then:*

$$\llbracket F, G :: H \rrbracket_m + \llbracket G, H \rrbracket_m = \llbracket H, F :: G \rrbracket_m + \llbracket F, G \rrbracket_m$$

Proof. The proof is a simple calculation using the geometric trefoil property for thick graphs (Proposition 24). We denote by n^F (resp. n^G, n^H) the cardinality of the dialect D^F (resp. D^G, D^H).

$$\begin{aligned}
& \llbracket F, G :: H \rrbracket_m + \llbracket G, H \rrbracket_m \\
&= \sum_{\pi \in \text{Cy}^e(F, G :: H)} \frac{1}{n^F n^G n^H} m(\omega(\pi)) + \sum_{\pi \in \text{Cy}^e(G, H)} \frac{1}{n^G n^H} m(\omega(\pi)) \\
&= \sum_{\pi \in \text{Cy}^e(F, G :: H)} \frac{1}{n^F n^G n^H} m(\omega(\pi)) + \sum_{\pi \in \text{Cy}^e(G, H)^\dagger_{D^F}} \frac{1}{n^F n^G n^H} m(\omega(\pi)) \\
&= \sum_{\pi \in \text{Cy}^e(F, G :: H) \cup \text{Cy}^e(G, H)^\dagger_{D^F}} \frac{1}{n^F n^G n^H} m(\omega(\pi)) \\
&= \sum_{\pi \in \text{Cy}^e(H, F :: G) \cup \text{Cy}^e(F, G)^\dagger_{D^H}} \frac{1}{n^F n^G n^H} m(\omega(\pi)) \\
&= \sum_{\pi \in \text{Cy}^e(H, F :: G)} \frac{1}{n^F n^G n^H} m(\omega(\pi)) + \sum_{\pi \in \text{Cy}^e(F, G)^\dagger_{D^H}} \frac{1}{n^F n^G n^H} m(\omega(\pi)) \\
&= \sum_{\pi \in \text{Cy}^e(H, F :: G)} \frac{1}{n^F n^G n^H} m(\omega(\pi)) + \sum_{\pi \in \text{Cy}^e(F, G)} \frac{1}{n^F n^G} m(\omega(\pi)) \\
&= \llbracket H, F :: G \rrbracket_m + \llbracket F, G \rrbracket_m
\end{aligned}$$

□

Corollary 28 (Numerical Adjunction for Thick Graphs). *Let F, G, H be thick graphs such that $S^G \cap S^H = \emptyset$. Then:*

$$\llbracket F, G :: H \rrbracket_m = \llbracket H, F :: G \rrbracket_m + \llbracket F, G \rrbracket_m$$

Remark 29. ABOUT THE HIDDEN CONVENTION OF THE NUMERICAL MEASURE

The measurement of interaction we defined hides a convention: each slice of a thick graph F is considered as having a "weight" equal to $1/n^F$, so that the total weight of the set of all slices have weight 1. This convention corresponds to the choice of working with a *normalized trace* (such that $\text{tr}(1) = 1$) on the idiom in Girard's hyperfinite geometry of interaction. It would have been possible to consider another convention which would impose that each slice have a weight equal to 1 (this would correspond to working with the usual trace on matrices in Girard's hyperfinite geometry of interaction). In this case, the measurement of the interaction between two thick graphs F, G is defined as:

$$\langle F, G \rangle = \sum_{\pi \in \text{Cy}^e(F, G)} m(\omega(\pi))$$

The numerical trefoil property is then stated differently: for all thick graphs F, G , and H such that $S^F \cap S^G \cap S^H = \emptyset$, we have:

$$\langle F, G :: H \rangle + n^F \langle G, H \rangle = \langle H, F :: G \rangle + n^H \langle F, G \rangle$$

We stress the apparition of the terms n^F and n^H in this equality: their apparition corresponds exactly to the apparition of the terms $\mathbf{1}_F$ and $\mathbf{1}_H$ in the equality stated for the trefoil property for sliced graphs.

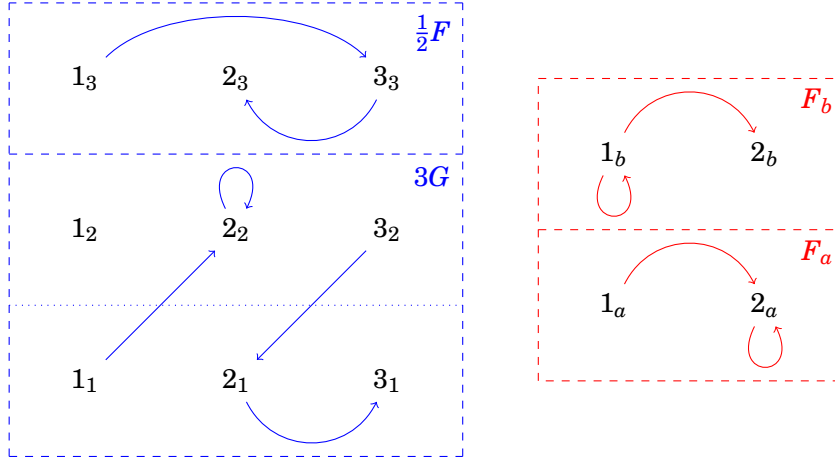


Figure 11: Examples of sliced thick graphs: $\frac{1}{2}F + 3G$ and $F_a + F_b$

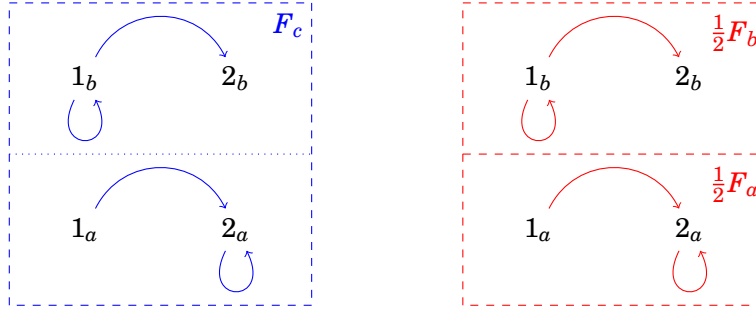
3.2. Sliced Thick Graphs. One can of course apply the additive construction presented in our previous paper [Sei14a] in the case of thick graphs. A *sliced thick graph* G of carrier S^G is a finite family $\sum_{i \in I^G} \alpha_i^G G_i$ where, for all $i \in I^G$, G_i is a thick graph such that $S^{G_i} = S^G$, and $\alpha_i^G \in \mathbf{R}$. We define the *dialect* of G to be the set $\uplus_{i \in I^G} D_{G_i}$. We will abusively call a *slice* a couple (i, d) where $i \in I^G$ and $d \in D_{G_i}$; we will say a graph G is a *one-sliced graph* when $I^G = \{i\}$ and $D_{G_i} = \{d\}$ are both one-element sets.

One can extend the execution and the measurement of the interaction by applying the thick graphs constructions slice by slice:

$$\begin{aligned} \left(\sum_{i \in I^F} \alpha_i^F F_i \right) :: \left(\sum_{i \in I^G} \alpha_i^G G_i \right) &= \sum_{(i,j) \in I^F \times I^G} \alpha_i^F \alpha_j^G F_i :: G_j \\ \llbracket \sum_{i \in I^F} \alpha_i^F F_i, \sum_{i \in I^G} \alpha_i^G G_i \rrbracket_m &= \sum_{(i,j) \in I^F \times I^G} \alpha_i^F \alpha_j^G \llbracket F_i, G_j \rrbracket_m \end{aligned}$$

Figure 11 shows two examples of sliced thick graphs. The graphical convention we will follow for representing sliced and thick graphs corresponds to the graphical convention for sliced graphs, apart from the fact that the graphs contained in the slices are replaced by thick graphs. Thus, two slices are separated by a dashed line, two elements in the dialect of a thick graph (i.e. the graph contained in a slice) are separated by a dotted line.

One should however notice that some sliced thick graphs (for instance the graph $F_a + F_b$ represented in red in Figure 11) can be considered both as a thick graph — hence a sliced thick graph with a single slice — or as a sliced graph with two slices — hence a sliced thick

Figure 12: Les graphes G_1 et G_2

graph with two slices. Indeed, consider the graphs:

F_a	F_b	F_c
$V^{F_a} = \{1, 2\}$	$V^{F_b} = \{1, 2\}$	$V^{F_c} = \{1, 2\} \times \{a, b\}$
$E^{F_a} = \{f, g\}$	$E^{F_b} = \{f, g\}$	$E^{F_c} = \{f_a, f_b, g_a, g_b\}$
$s^{F_a} = \begin{cases} f \mapsto 1 \\ g \mapsto 2 \end{cases}$	$s^{F_b} = \begin{cases} f \mapsto 1 \\ g \mapsto 1 \end{cases}$	$s^{F_c} = \begin{cases} f_i \mapsto s^{F_i}(f) \\ g_i \mapsto s^{F_i}(g) \end{cases}$
$t^{F_a} = \begin{cases} f \mapsto 2 \\ g \mapsto 2 \end{cases}$	$t^{F_b} = \begin{cases} f \mapsto 2 \\ g \mapsto 1 \end{cases}$	$t^{F_c} = \begin{cases} f_i \mapsto t^{F_i}(f) \\ g_i \mapsto t^{F_i}(g) \end{cases}$
$\omega^{F_a} \equiv 1$	$\omega^{F_b} \equiv 1$	$\omega^{F_c} \equiv 1$

One can then define the two sliced thick graphs $G_1 = F_c$ and $G_2 = \frac{1}{2}F_a + \frac{1}{2}F_b$. These two graphs are represented in Figure 12. They are similar in a very precise sense: one can show that if H is any sliced thick graph, and m is any circuit-quantifying map, then $\llbracket G_1, H \rrbracket_m = \llbracket G_2, H \rrbracket_m$. We say they are *universally equivalent*. Notice that this explains in a very formal way the remark about the convention on the measurement of interaction Remark 29.

Definition 30 (Universally equivalent graphs). Let F, G be two graphs. We say that F and G are *universally equivalent* (for the measurement $\llbracket \cdot, \cdot \rrbracket_m$) — which will be denoted by $F \simeq_u G$ — if for all graph H :

$$\llbracket F, H \rrbracket_m = \llbracket G, H \rrbracket_m$$

The next proposition states that if F' is obtained from a graph F by a renaming of edges, then $F \simeq_u F'$.

Proposition 31. Let F, F' be two graphs such that $V^F = V^{F'}$, and ϕ a bijection $E^F \rightarrow E^{F'}$ such that:

$$s^G \circ \phi = s^F, \quad t^G \circ \phi = t^F, \quad \omega^G \circ \phi = \omega^F$$

Then $F \simeq_u F'$.

Proof. Indeed, the bijection ϕ induces, from the hypotheses in the source and target functions, a bijection between the sets of cycles $\text{Cy}(F, H)$ and $\text{Cy}(G, H)$. The condition on the weight map then insures us that this bijection is ω -invariant, from which we deduce the proposition. $\square$

Proposition 32. *Let F, G be sliced graphs. If there exists a bijection $\phi : I^F \rightarrow I^G$ such that $F_i = G_{\phi(i)}$ and $\alpha_i^F = \alpha_{\phi(i)}^G$, then $F \simeq_u G$.*

Proof. By definition:

$$\begin{aligned} \llbracket G, H \rrbracket_m &= \sum_{(i,j) \in I^G \times I^H} \alpha_i^G \alpha_j^H \llbracket G_i, H_j \rrbracket_m \\ &= \sum_{(i,j) \in I^F \times I^H} \alpha_{\phi(i)}^G \alpha_j^H \llbracket G_{\phi(i)}, H_j \rrbracket_m \\ &= \sum_{(i,j) \in I^F \times I^G} \alpha_i^F \alpha_j^G \llbracket F_i, G_j \rrbracket_m \end{aligned}$$

Thus F and G are universally equivalent. $\square$

Proposition 33. *Let F, G be thick graphs. If there exists a bijection $\phi : D^F \rightarrow D^G$ such that $G = F^\phi$, then $F \simeq_u G$.*

Proof. Let F, G be thick graphs such that $G = F^\phi$ for a bijection $\phi : D^G \rightarrow D^F$, and H an arbitrary thick graph. Then the bijection $\phi \times \text{Id} : D^G \times D^H \rightarrow D^F \times D^H$ satisfies that $G^\dagger_{D^H} = (F^\dagger_{D^H})^{\phi \times \text{Id}}$. One can notice that the set of alternating circuits in $F^\dagger \square H^\ddagger$ is the same as the set of alternating circuits in $(F^\dagger)^{\phi \times \text{Id}} \square (H^\dagger)^{\phi \times \text{Id}} = G^\dagger \square H^\ddagger$. Thus:

$$\begin{aligned} \llbracket F, H \rrbracket_m &= \sum_{\pi \in \text{Cy}(F, H)} m(\omega(\pi)) \\ &= \sum_{\pi \in \text{Cy}(G, H)} m(\omega(\pi)) \\ &= \llbracket G, H \rrbracket_m \end{aligned}$$

And finally F and G are universally equivalent. $\square$

Proposition 34. *Let $F = \sum_{i \in I^F} \alpha_i^F F_i$ be a sliced thick graph, and let us define, for all $i \in I^F$, $n^{F_i} = \text{Card}(D^{F_i})$ and $n^F = \sum_{i \in I^F} n^{F_i}$. Suppose that there exists a scalar α such that for all $i \in I^F$, $\alpha_i^F = \alpha \frac{n^{F_i}}{n^F}$. We then define the sliced thick graph with a single slice αG of dialect $\uplus D^{F_i} = \cup_{i \in I^F} D^{F_i} \times \{i\}$ and carrier V^F by:*

$$\begin{aligned} V^G &= V^F \times \uplus D^{F_i} \\ E^G &= \uplus E^{F_i} = \cup_{i \in I^F} E^{F_i} \times \{i\} \\ s^G &= (e, i) \mapsto (s^{F_i}(e), i) \\ t^G &= (e, i) \mapsto (t^{F_i}(e), i) \\ \omega^G &= (e, i) \mapsto \omega^{F_i}(e) \\ \left((e, i) \subset^G (f, j) \right) &\Leftrightarrow (i \neq j) \vee (i = j \wedge e \subset^{F_i} f) \end{aligned}$$

Then F and G are universally equivalent.

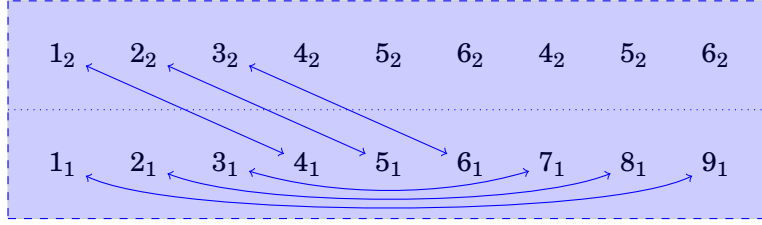


Figure 13: The graph of a contraction project

Proof. Let H be a sliced thick graph. Then:

$$\begin{aligned}
 \llbracket F, H \rrbracket_m &= \sum_{i \in I^H} \sum_{j \in I^F} \alpha_i^H \alpha_j^F \llbracket F_i, H_j \rrbracket_m \\
 &= \sum_{i \in I^H} \sum_{j \in I^F} \alpha_i^H \alpha \frac{n^{F_i}}{n^F} \llbracket F_i, H_j \rrbracket_m \\
 &= \sum_{i \in I^H} \sum_{j \in I^F} \alpha_i^H \alpha \frac{n^{F_i}}{n^F} \frac{1}{n^{F_i} n^{H_j}} \sum_{\pi \in \text{Cy}(F_i, H_j)} m(\omega(\pi)) \\
 &= \sum_{i \in I^H} \sum_{j \in I^F} \alpha_i^H \alpha \frac{1}{n^F n^{H_j}} \sum_{\pi \in \text{Cy}(F_i, H_j)} m(\omega(\pi)) \\
 &= \sum_{i \in I^H} \alpha_i^H \alpha \frac{1}{n^F n^{H_j}} \sum_{j \in I^F} \sum_{\pi \in \text{Cy}(F_i, H_j)} m(\omega(\pi))
 \end{aligned}$$

But one can notice that $\cup_{j \in I^F} \text{Cy}(F_i, H_j) = \text{Cy}(G, H)$. We thus get:

$$\begin{aligned}
 \llbracket F, H \rrbracket_m &= \sum_{i \in I^H} \alpha_i^H \alpha \frac{1}{n^F n^{H_j}} \sum_{j \in I^F} \sum_{\pi \in \text{Cy}(F_i, H_j)} m(\omega(\pi)) \\
 &= \sum_{i \in I^H} \alpha_i^H \alpha \frac{1}{n^F n^{H_j}} \sum_{\pi \in \text{Cy}(F, H_j)} m(\omega(\pi)) \\
 &= \llbracket \alpha G, H \rrbracket_m
 \end{aligned}$$

Finally, we showed that F and αG are universally equivalent. $\square$

One of the consequences of Proposition 31, Proposition 32, and Proposition 33 is that two graphs F, G such that G is obtained from F by a renaming of the sets E^F, I^F, D^F are universally equivalent. We will therefore work from now on with graphs modulo renaming of these sets.

3.3. Thick Graphs and Contraction. In this section, we will explain how the introduction of thick graphs allow the definition of contraction by using the fact that edges can go from a slice to another (contrarily to sliced graphs). In the following, we will be working with sliced thick graphs. The way contraction is dealt with by using slice-changing edges is quite simple, and the graph which will implement this transformation is essentially the same as the graph implementing additive contraction (i.e. the graph implementing distributivity — Proposition 11 — restricted to the location of contexts) modified with a change of slices.

The graph we obtain is then the superimposition of two $\mathfrak{F}\mathfrak{A}\mathfrak{r}$, but where one of them goes from one slice to the other.

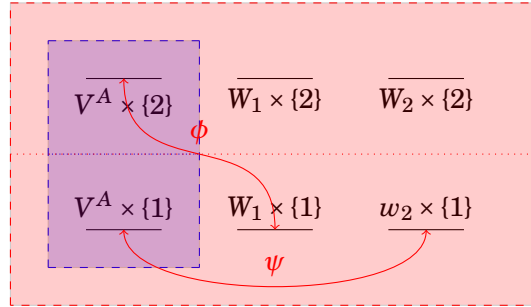
Definition 35 (Contraction). Let $\phi : V^A \rightarrow W_1$ and $\psi : V^A \rightarrow W_2$ be two bijections with $V^A \cap W_1 = V^A \cap W_2 = W_1 \cap W_2 = \emptyset$. We define the project $\mathfrak{Ctr}_\psi^\phi = (0, \text{Ctr}_\psi^\phi)$, where the graph Ctr_ψ^ϕ is defined by:

$$\begin{aligned} V^{\text{Ctr}_\psi^\phi} &= V^A \cup W_1 \cup W_2 \\ D^{\text{Ctr}_\psi^\phi} &= \{1, 2\} \\ E^{\text{Ctr}_\psi^\phi} &= V^A \times \{1, 2\} \times \{i, o\} \\ s^{\text{Ctr}_\psi^\phi} &= \begin{cases} (v, 1, o) \mapsto (\phi(v), 1) \\ (v, 1, i) \mapsto (v, 1) \\ (v, 2, o) \mapsto (\psi(v), 1) \\ (v, 2, i) \mapsto (v, 2) \end{cases} \\ t^{\text{Ctr}_\psi^\phi} &= \begin{cases} (v, 1, o) \mapsto (v, 1) \\ (v, 1, i) \mapsto (\phi(v), 1) \\ (v, 2, o) \mapsto (v, 2) \\ (v, 2, i) \mapsto (\psi(v), 1) \end{cases} \\ \omega^{\text{Ctr}_\psi^\phi} &\equiv 1 \end{aligned}$$

Figure 13 illustrates the graph of the project $\mathfrak{Ctr}_\psi^\phi$, where the functions are defined by $\phi : \{1, 2, 3\} \rightarrow \{4, 5, 6\}, x \mapsto x + 3$ and $\psi : \{1, 2, 3\} \rightarrow \{7, 8, 9\}, x \mapsto 10 - x$.

Proposition 36. Let $\mathfrak{a} = (0, A)$ be a project in a behavior $\mathbf{A}$, such that $D^A \cong \{1\}$. Let ϕ, ψ be two delocations $V^A \rightarrow W_1, V^A \rightarrow W_2$ of disjoint codomains. Then $\mathfrak{Ctr}_\phi^\psi :: \mathfrak{a} \in \phi(\mathbf{A}) \otimes \psi(\mathbf{A})$.

Proof. We will denote by Ctr the graph Ctr_ϕ^ψ to simplify the notations. We first compute $A :: \text{Ctr}$. We get $A^{\ddagger\{1,2\}} = (V^A \times \{1, 2\}, E^A \times \{1, 2\}, s^A \times Id_{\{1,2\}}, t^A \times Id_{\{1,2\}}, \omega^A \circ \pi)$ where π is the projection: $E^A \times \{1, 2\} \rightarrow E^A, (x, i) \mapsto x$. Moreover the graph $\text{Ctr}^{\ddagger D^A}$ is a variant of the graph Ctr since $D^A \cong \{1\}$. Here is what the plugging of $\text{Ctr}^{\ddagger D^A}$ with $A^{\ddagger\{1,2\}}$ looks like:



The result of the execution is therefore a two-sliced graph, i.e. a graph of dialect $D^A \times \{1, 2\} \cong \{1, 2\}$, and which contains the graph $\phi(A) \cup \psi(A)$ in the slice numbered 1 and contains the empty graph in the slice numbered 2.

We deduce from this that $\mathfrak{Ctr}_\phi^\psi :: \mathfrak{a}$ is universally equivalent (Definition 30) to the project $\frac{1}{2}\phi(\mathfrak{a}) \otimes \psi(\mathfrak{a}) + \frac{1}{2}o$ from Proposition 34. Since $\phi(\mathfrak{a}) \otimes \psi(\mathfrak{a}) \in \phi(\mathbf{A}) \otimes \psi(\mathbf{A})$, then the project $\frac{1}{2}(\phi(\mathfrak{a}) \otimes \psi(\mathfrak{a}))$ is an element in $\phi(\mathbf{A}) \otimes \psi(\mathbf{A})$ by the homothety Lemma (Lemma 8). Moreover, $\mathbf{A}$ is a behavior, hence $\phi(\mathbf{A}) \otimes \psi(\mathbf{A})$ is also a behavior and we can deduce that $\frac{1}{2}\phi(\mathfrak{a}) \otimes \psi(\mathfrak{a}) + \frac{1}{2}o$ is an element in $\phi(\mathbf{A}) \otimes \psi(\mathbf{A})$. $\square$

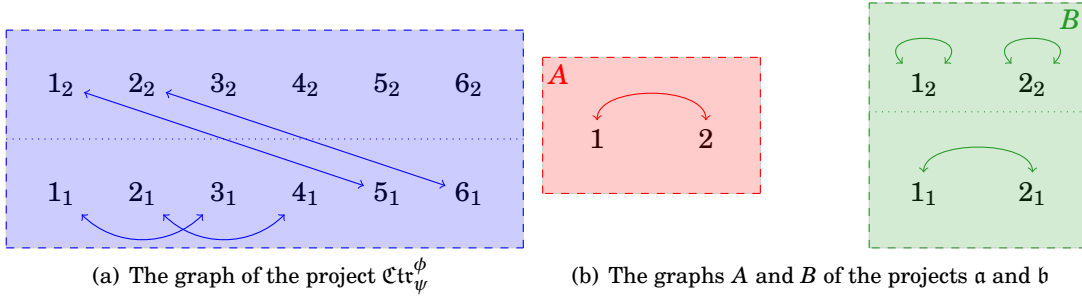


Figure 14: The graphs of the projects Ctr_ψ^ϕ , a and b .

Figure 15, Figure 16 and Figure 17 illustrate the plugging and execution of a contraction with two graphs: the first — A — having a single slice, and the other — B — having two slices (the graphs are shown in Figure 14). One can see that the hypothesis $D^A \equiv \{1\}$ used in the preceding proposition is necessary, and that slice-changing edges allow to implement contraction of graphs with a single slice.

We will use the following direct corollary of Proposition 9.

Proposition 37. *If E is a non-empty set of project sharing the same carrier V^E , $\mathbf{F}$ is a conduct and $\mathfrak{f}$ satisfies that $\forall e \in E, \mathfrak{f} :: e \in \mathbf{F}$, then $\mathfrak{f} \in E^{\perp\perp} \multimap \mathbf{F}$.*

This proposition insures us that if $\mathbf{A}$ is a conduct such that there exists a set E of one-sliced projects with $\mathbf{A} = E^{\perp\perp}$, then the contraction project Ctr_ϕ^ψ belongs to the conduct $\mathbf{A} \multimap \phi(\mathbf{A}) \otimes \psi(\mathbf{A})$.

We find here a geometrical explanation to the introduction of exponential connectives. Indeed, in order to use a contraction, we must be sure we are working with one-sliced graphs. We will therefore define, for all behavior $\mathbf{A}$, a conduct $!\mathbf{A}$ generated by a set of one-sliced graphs.

One should notice that a conduct $!\mathbf{A}$ generated by a set of one-sliced projects cannot be a behavior: the projects $(a, \emptyset)$ necessarily belong to the orthogonal of $!\mathbf{A}$. We will therefore introduce *perennial conducts* as those conducts generated by a set of wager-free one-sliced projects. Dually, we introduce the *co-perennial conducts* as the conducts that are the orthogonal of a perennial conduct.

But first, we will need a way to associate a wager-free one-sliced project to any wager-free project. In order to do so, we will introduce the notion of *thick graphing*.

4. CONSTRUCTION OF AN EXPONENTIAL CONNECTIVE ON THE REAL LINE

We now consider the microcosm mi of measure-inflating maps⁷ on the real line endowed with Lebesgue measure, we fix $\Omega =]0, 1]$ endowed with the usual multiplication and we chose any

⁷A *measure-inflating map* on the real line with Lebesgue measure λ is a non-singular Borel-preserving transformation $\phi : \mathbf{R} \rightarrow \mathbf{R}$ such that there exists a positive real number μ_ϕ with $\lambda(\phi^{-1}(A)) = \mu_\phi \lambda(A)$. In other terms, ϕ transports the measure λ onto $\mu_\phi \lambda$.

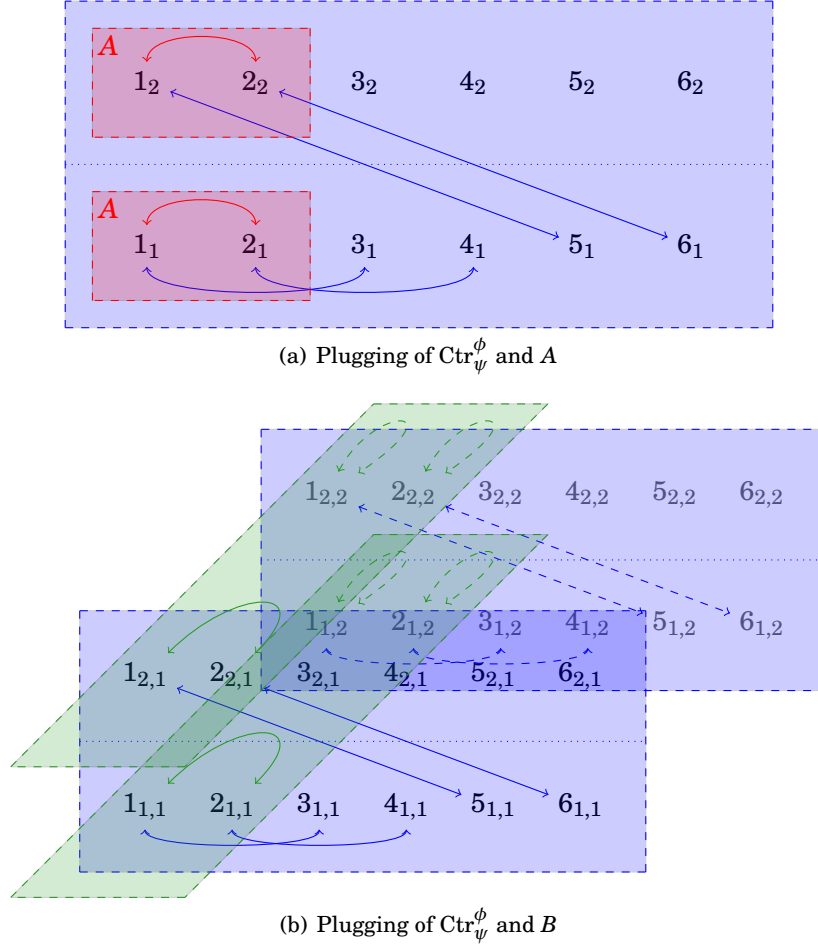
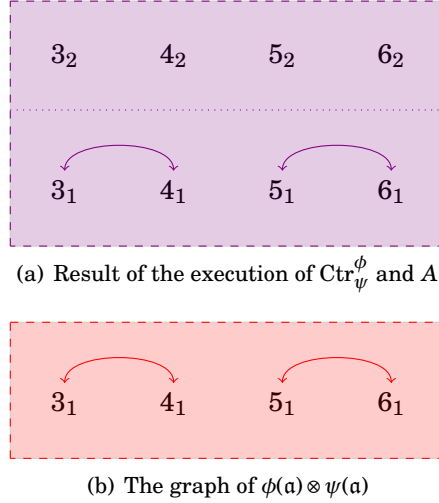


Figure 15: Plugging of Ctr_ψ^ϕ with the two graphs A and B

measurable map $m : \Omega \rightarrow \mathbf{R}_{\geq 0} \cup \{\infty\}$ such that $m(1) = \infty$. We showed in a previous work how this framework can interpret multiplicative-additive linear logic with second order quantification⁸ [Sei14c]. We now show how to interpret elementary linear logic exponential connectives in the model $\mathbb{M}[\Omega, \text{mi}]_m$.

4.1. Sliced Thick Graphings. The sliced graphings are obtained from graphings in the same way we defined sliced thick graphs from directed weighted graphs: we consider formal weighted sums $F = \sum_{i \in I^F} \alpha_i^F F_i$ where the F_i are graphings of carrier V^{F_i} . We define the *carrier of F* as the measurable set $\cup_{i \in I^F} V^{F_i}$. The various constructions are then extended as

⁸We actually showed how one can interpret second-order multiplicative-additive linear logic in the model $\mathbb{M}[\Omega, \text{aff}]_m$ where $\text{aff} \subsetneq \text{mi}$ is the microcosm of affine transformations on the real line. The result is however valid for any super-microcosm $\mathfrak{n} \supset \text{aff}$, hence for mi , since a graphing in aff can be considered as a graphing in $\mathfrak{n}$ in a way that is coherent with execution, orthogonality, sums, etc.

Figure 16: Graphs of the projects $\text{Ctr}_{\psi}^{\phi} :: a$ and $\phi(a) \otimes \psi(a)$

explained above:

$$\begin{aligned} \left(\sum_{i \in I^F} \alpha_i^F F_i \right) :: \left(\sum_{i \in I^G} \alpha_i^G G_i \right) &= \sum_{(i,j) \in I^F \times I^G} \alpha_i^F \alpha_j^G F_i :: G_j \\ \llbracket \sum_{i \in I^F} \alpha_i^F F_i, \sum_{i \in I^G} \alpha_i^G G_i \rrbracket_m &= \sum_{(i,j) \in I^F \times I^G} \alpha_i^F \alpha_j^G \llbracket F_i :: G_j \rrbracket_m \end{aligned}$$

The trefoil property and the adjunction are then easily obtained through the same computations as in the proofs of Proposition 27 and Corollary 28.

We will now consider the most general notion of *thick graphing* one can define. As it was the case in the setting of graphs, a thick graphing is a graphing whose carrier has the form $V \times D$. The main difference between graphings and thick graphings really comes from the way two such objects interact.

Definition 38. Let $(X, \mathcal{B}, \lambda)$ be a measured space and $(D, \mathcal{D}, \mu)$ a probability space (a measured space such that $\mu(D) = 1$). A thick graphing of carrier $V \in \mathcal{B}$ and dialect D is a graphing on $X \times D$ of carrier $V \times D$.

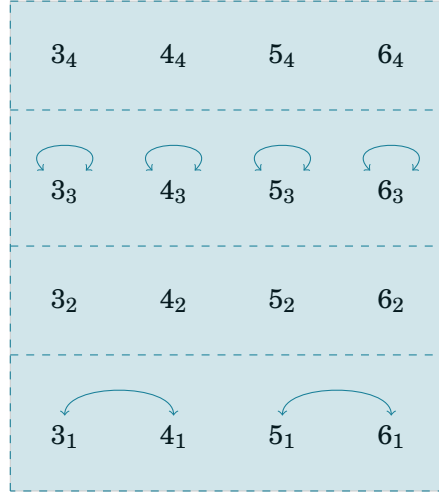
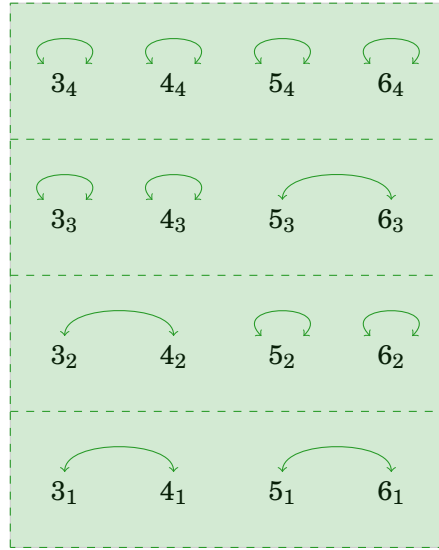
Definition 39 (Dialectal Interaction). Let $(X, \mathcal{B}, \lambda)$ be a measured space and $(D, \mathcal{D}, \mu), (E, \mathcal{E}, \nu)$ two probability spaces. Let F, G be thick graphings of respective carriers $V^F, V^G \in \mathcal{B}$ and respective dialects D, E . We define the graphings $F^{\dagger E}$ and $G^{\ddagger D}$ as the graphings of respective carriers V^F, V^G and dialects $E \times F$:

$$\begin{aligned} F^{\dagger E} &= \{(\omega_e^F, \phi_e^F \times \text{Id}_E : S_e^F \times D \times E \rightarrow T_e^F \times D \times E)\}_{e \in E^F} \\ G^{\ddagger D} &= \{(\omega_e^G, \text{Id}_X \times (\tau \circ (\phi_e^G \times \text{Id}_D) \circ \tau^{-1}) : S_e^G \times D \times E \rightarrow T_e^G \times D \times E)\}_{e \in E^G} \end{aligned}$$

where τ is the natural symmetry: $E \times D \rightarrow D \times E$.

Definition 40 (Plugging). The plugging $F :: G$ of two thick graphings of respective dialects D^F, D^G is defined as $F^{\dagger D^G} \boxdot G^{\ddagger D^F}$.

Definition 41 (Execution). Let F, G be two thick graphings of respective dialects D^F, D^G . Their execution is equal to $F^{\dagger D^G} :: G^{\ddagger D^F}$.

(a) Result of the execution of Ctr_ψ^ϕ and B (b) Graph of the project $\phi(b) \otimes \psi(b)$ Figure 17: Graphs of the projects $\text{Ctr}_\psi^\phi :: \mathfrak{b}$ and $\phi(b) \otimes \psi(b)$

Definition 42 (Measurement). Let F, G be two thick graphings of respective dialects D^F, D^G , and q a circuit-quantifying map. The corresponding measurement of the interaction between F and G is equal to $\llbracket F^{\dagger}_{D^G}, G^{\ddagger}_{D^F} \rrbracket_m$.

As in the setting of graphs, one can show that all the fundamental properties are preserved when we generalize from graphings to thick graphings.

Proposition 43. *Let F, G, H be thick graphings such that $V^F \cap V^G \cap V^H$ is of null measure. Then:*

$$\begin{aligned} F :: (G :: H) &= (F :: G) :: H \\ \llbracket F, G :: H \rrbracket_m + \llbracket G, H \rrbracket_m &= \llbracket G, H :: F \rrbracket_m + \llbracket H, F \rrbracket_m \end{aligned}$$

In a similar way, the extension from thick graphings to sliced thick graphings should now be quite clear. One extends all operations by "linearity" to formal weighted sums of thick graphings, and one obtains, when F, G, H are sliced thick graphings such that $V^F \cap V^G \cap V^H$ is of null measure:

$$\begin{aligned} F :: (G :: H) &= (F :: G) :: H \\ \llbracket F, G :: H \rrbracket_m + \mathbf{1}_F \llbracket G, H \rrbracket_m &= \llbracket G, H :: F \rrbracket_m + \mathbf{1}_G \llbracket H, F \rrbracket_m \end{aligned}$$

4.2. Perennial and Co-perennial conducts. Since we are working with sliced thick graphings, we can follow the constructions of multiplicative and additive connectives as they are studied in the author's second paper on interaction graphs [Sei14a] and which were quickly recalled in Section 2.2.

Definition 44 (Projects). A *project* is a couple $\alpha = (a, A)$ together with a support V^A where:

- $a \in \mathbf{R} \cup \{\infty\}$ is called the *wager*;
- A is a sliced and thick weighted graphing of carrier V^A , of dialect D^A a discrete probability space, and index I^A a finite set.

Remark 45. We made here the choice to stay close to the hyperfinite geometry of interaction defined by Girard [Gir11]. This is why we restrict to discrete probability spaces as dialects, a restriction that corresponds to the restriction to finite von Neumann algebras of type I as idioms in Girard's setting. However, the results of the preceding section about execution and measurement, and the definition of the family of circuit-quantifying maps do not rely on this hypothesis. It should therefore be possible to consider a more general set of project where the dialects may eventually be continuous. It may turn out that this generalization could be used to define more expressive exponential connectives than the one defined in this paper, such as the usual exponentials of linear logic (recall that the exponentials defined here are the exponentials of Elementary Linear Logic).

As we explained at the end of Section 3, we will need to consider a particular kind of conducts which are the kind of conducts obtained from the application of the exponential modality to a conduct and which are unfortunately not behaviors. We now study these types of conducts.

Definition 46 (Perennialisation). A *Perennialisation* is a function that associates a one-sliced weighted graphing to any sliced and thick weighted graphing.

Definition 47 (Exponentials). Let $\mathbf{A}$ be a conduct, and Ω a perennialisation. We define the $!_{\Omega}\mathbf{A}$ as the bi-orthogonal closure of the following set of projects:

$$\sharp_{\Omega}\mathbf{A} = \{\alpha = (0, \Omega(A)) \mid \alpha = (0, A) \in \mathbf{A}, I^A \cong \{1\}\}$$

The dual connective is of course defined as $?_{\Omega}\mathbf{A} = (\sharp_{\Omega}\mathbf{A}^{\perp})^{\perp}$.

Definition 48. A conduct $\mathbf{A}$ is a *perennial conduct* when there exists a set A of projects such that:

- (1) $\mathbf{A} = \mathbf{A}^{\perp\perp}$;
- (2) for all $\mathfrak{a} = (a, A) \in \mathbf{A}$, $a = 0$ and A is a one-sliced graphing.

A *co-perennial* conduct is a conduct $\mathbf{B} = \mathbf{A}^{\perp}$ where $\mathbf{A}$ is a perennial conduct.

Proposition 49. *A co-perennial conduct $\mathbf{B}$ satisfies the inflation property: for all $\lambda \in \mathbf{R}$, $\mathfrak{b} \in \mathbf{B} \Rightarrow \mathfrak{b} + \lambda \mathfrak{b} \in \mathbf{B}$.*

Proof. The conduct $\mathbf{A} = \mathbf{B}^{\perp}$ being perennial, there exists a set A of one-sliced wager-free projects such that $\mathbf{A} = \mathbf{A}^{\perp\perp}$. If A is non-empty, the result is a direct consequence of Proposition 6. If A is empty, then $\mathbf{B} = \mathbf{A}^{\perp} = \mathbf{A}^{\perp}$ is the full behavior $\mathbf{T}_{V^A}$ which obviously satisfies the inflation property. $\square$

Proposition 50. *A co-perennial conduct is non-empty.*

Proof. Suppose that $\mathbf{A}^{\perp}$ is a co-perennial conduct of carrier V^A . Then there exists a set A of one-sliced wager-free projects such that $\mathbf{A} = \mathbf{A}^{\perp\perp}$. If A is empty, then $\mathbf{A}^{\perp} = \mathbf{A}^{\perp}$ is the behavior $\mathbf{T}_{V^A}$. If $\mathbf{A}$ is non-empty, then one can easily check that for all real number $\lambda \neq 0$, the project $\mathfrak{Dai}_{\lambda} = (\lambda, (V^A, \emptyset))$ is an element of $\mathbf{A}^{\perp} = \mathbf{A}^{\perp}$. $\square$

Corollary 51. *Let $\mathbf{A}$ be a perennial conduct. Then $\mathfrak{a} = (a, A) \in \mathbf{A} \Rightarrow a = 0$.*

Proof. Since $\mathbf{A}^{\perp}$ is co-perennial, it is a non-empty set of projects with the same carrier which satisfies the inflation property. The result is then obtained by applying Proposition 6. $\square$

Proposition 52. *If $\mathbf{A}$ is a co-perennial conduct, then for all $a \neq 0$, the project $\mathfrak{Dai}_a = (a, (V^A, \emptyset))$ is an element of $\mathbf{A}$.*

Proof. We write B the set of one-sliced wager-free projects such that $B^{\perp} = \mathbf{A}$. Then for all element $\mathfrak{b} \in \mathbf{B}$, we have that $\mathbf{1}_B = 1$, from which we conclude that $\ll \mathfrak{b}, \mathfrak{Dai}_a \gg_m = a \mathbf{1}_B = a$. Thus $\mathfrak{Dai}_a \in B^{\perp} = \mathbf{A}$ for all $a \neq 0$. $\square$

Proposition 53. *The tensor product of perennial conducts is a perennial conduct.*

Proof. Let $\mathbf{A}, \mathbf{B}$ be perennial conducts. Then there exists two sets of one-sliced wager-free projects E, F such that $\mathbf{A} = E^{\perp\perp}$ and $\mathbf{B} = F^{\perp\perp}$. Using Proposition 9, we know that $\mathbf{A} \otimes \mathbf{B} = (E \odot F)^{\perp\perp}$. But, by definition, $E \odot F$ contains only projects of the form $\mathfrak{e} \otimes \mathfrak{f}$, where $\mathfrak{e}, \mathfrak{f}$ are one-sliced and wager-free. Thus $E \odot F$ contains only one-sliced wager-free projects and $\mathbf{A} \otimes \mathbf{B}$ is therefore a perennial conduct. $\square$

Proposition 54. *If $\mathbf{A}, \mathbf{B}$ are perennial conducts, then $\mathbf{A} \oplus \mathbf{B}$ is a perennial conduct.*

Proof. This is a consequence of Proposition 10. $\square$

Proposition 55. *If $\mathbf{A}$ is a perennial conduct and $\mathbf{B}$ is a co-perennial conduct, then $\mathbf{A} \multimap \mathbf{B}$ is a co-perennial conduct.*

Proof. We recall that $\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp}$. Since $\mathbf{A}$ and $\mathbf{B}^{\perp}$ are perennial conducts, $\mathbf{A} \otimes \mathbf{B}^{\perp}$ is a perennial conduct, and therefore $\mathbf{A} \multimap \mathbf{B}$ is a co-perennial conduct. In particular, $\mathbf{A} \multimap \mathbf{B}$ is non-empty and satisfies the inflation property. $\square$

Proposition 56. *If $\mathbf{A}$ is a perennial conduct and $\mathbf{B}$ is a behavior, then $\mathbf{A} \otimes \mathbf{B}$ is a behavior.*

Proof. If $\mathbf{A} = \mathbf{0}_{V^A}$ with $\mathbf{B} = \mathbf{0}_{V^B}$, then $\mathbf{A} \otimes \mathbf{B} = \mathbf{0}_{V^A \cup V^B}$ which is a behavior.

Let A be the set of one-sliced wager-free projects such that $\mathbf{A} = A^{\perp\perp}$. We have that $\mathbf{A} \otimes \mathbf{B} = (A \odot \mathbf{B})^{\perp\perp}$ by Proposition 9. If $\mathbf{B} \neq \mathbf{0}_{V^B}$ and $A \neq \emptyset$, then $A \odot \mathbf{B}$ is non-empty and contains only wager-free projects. Thus $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ satisfies the inflation property by Proposition 6.

Now suppose there exists $f = (f, F) \in (\mathbf{A} \otimes \mathbf{B})^{\perp}$ such that $f \neq 0$. Then for all $a \in A$ and $b \in \mathbf{B}$, $\ll f, a \otimes b \gg_m \neq 0, \infty$. But, since a is wager-free and $\mathbf{1}_A = 1$, $\ll f, a \otimes b \gg_m = f \mathbf{1}_B + b \mathbf{1}_F + \ll F, A \cup B \gg_m$. We can then define $\mu = (-\ll F, A \cup B \gg_m - b \mathbf{1}_F) / f - \mathbf{1}_B$. Since $\mathbf{B}$ is a behavior, $b + \mu 0 \in \mathbf{B}$. However:

$$\begin{aligned} \ll f, a \otimes (b + \mu 0) \gg_m &= f(\mathbf{1}_B + \mu) + b \mathbf{1}_F + \ll F, A \cup (B + \mu 0) \gg_m \\ &= f(\mathbf{1}_B + \mu) + b \mathbf{1}_F + \ll F, A \cup B \gg_m \\ &= f(\mathbf{1}_B + (-\ll F, A \cup B \gg_m - b \mathbf{1}_F) / f - \mathbf{1}_B) + b \mathbf{1}_F + \ll F, A \cup B \gg_m \\ &= -\ll F, A \cup B \gg_m - b \mathbf{1}_F + b \mathbf{1}_F + \ll F, A \cup B \gg_m \\ &= 0 \end{aligned}$$

But this is a contradiction. Therefore the elements in $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ are wager-free.

If $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ is non-empty, it is a non-empty conduct containing only wager-free projects and satisfying the inflation property: it is therefore a (proper) behavior.

The only case left to treat is when $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ is empty, but then $\mathbf{A} \otimes \mathbf{B} = \mathbf{T}_{V^A \cup V^B}$ is clearly a behavior. $\square$

Corollary 57. *If $\mathbf{A}$ is perennial and $\mathbf{B}$ is a behavior, then $\mathbf{A} \multimap \mathbf{B}$ is a behavior.*

Proof. We recall that $\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp}$. Using the preceding proposition, the conduct $\mathbf{A} \otimes \mathbf{B}^{\perp}$ is a behavior since $\mathbf{A}$ is a perennial conduct and $\mathbf{B}^{\perp}$ is a behavior. Thus $\mathbf{A} \multimap \mathbf{B}$ is a behavior since it is the orthogonal of a behavior. $\square$

Corollary 58. *If $\mathbf{A}$ is a behavior and $\mathbf{B}$ is a co-perennial conduct, then $\mathbf{A} \multimap \mathbf{B}$ is a behavior.*

Proof. One just has to write $\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp}$. Since $\mathbf{A} \otimes \mathbf{B}^{\perp}$ is the tensor product of a behavior with a perennial conduct, it is a behavior. The result then follows from the fact that the orthogonal of a behavior is a behavior. $\square$

Proposition 59. *The weakening (on the left) of perennial conducts holds.*

Proof. Let $\mathbf{A}, \mathbf{B}$ be conducts, and $\mathbf{N}$ be a perennial conduct. Chose $f \in \mathbf{A} \multimap \mathbf{B}$. We will show that $f \otimes 0_{V^N}$ is a project in $\mathbf{A} \otimes \mathbf{N} \multimap \mathbf{B}$. For this, we pick $a \in \mathbf{A}$ and $n \in \mathbf{N}$ (recall that n is necessarily wager-free). Then for all $b' \in \mathbf{B}^{\perp}$,

$$\begin{aligned} \ll (f \otimes 0) :: (a \otimes n), b' \gg_m &= \ll f \otimes 0, (a \otimes n) \otimes b' \gg_m \\ &= \ll f \otimes 0, (a \otimes b') \otimes n \gg_m \\ &= \mathbf{1}_F(\mathbf{1}_A \mathbf{1}_{B'} n + \mathbf{1}_N \mathbf{1}_A b' + \mathbf{1}_N \mathbf{1}_{B'} a) + \mathbf{1}_N \mathbf{1}_A \mathbf{1}_{B'} f + \ll F \cup 0, A \cup B' \cup N \gg_m \\ &= \mathbf{1}_F(\mathbf{1}_N \mathbf{1}_A b' + \mathbf{1}_N \mathbf{1}_{B'} a) + \mathbf{1}_N \mathbf{1}_A \mathbf{1}_{B'} f + \ll F \cup 0, A \cup B' \cup N \gg_m \\ &= \mathbf{1}_N(\mathbf{1}_F(\mathbf{1}_A b' + \mathbf{1}_{B'} a) + \mathbf{1}_A \mathbf{1}_{B'} f) + \mathbf{1}_N \ll F, A \cup B' \gg_m \\ &= \mathbf{1}_N \ll f, a \otimes b' \gg_m \end{aligned}$$

Since $\mathbf{1}_N \neq 0$, $\ll (f \otimes 0) :: (a \otimes n), b' \gg_m \neq 0, \infty$ if and only if $\ll f :: a, b' \gg_m \neq 0, \infty$. Thus for all $a \otimes n \in \mathbf{A} \otimes \mathbf{N}$, $(f \otimes 0) :: (a \otimes n) \in \mathbf{B}$. This shows that $f \otimes 0 \in \mathbf{A} \otimes \mathbf{N} \multimap \mathbf{B}$ by Proposition 37. $\square$

4.3. A Construction of Exponentials. We will begin by showing a technical result that will allow us to define measure preserving transformations from bijections of the set of integers. This result will be used to show that functorial promotion can be implemented for our exponential modality.

Definition 60. Let $\phi : \mathbf{N} \rightarrow \mathbf{N}$ be a bijection and b an integer ≥ 2 . Then ϕ induces a transformation $T_\phi^b : [0, 1] \rightarrow [0, 1]$ defined by $\sum_{i \geq 0} a_i 2^{-k} \mapsto \sum_{i \geq 0} a_{\phi^{-1}(k)} 2^{-k}$.

Remark 61. Suppose that $\sum_{i \geq 0} a_i b^{-i}$ and $\sum_{i \geq 0} a'_i b^{-i}$ are two distinct representations of a real number r . Let us fix i_0 to be the smallest integer such that $a_{i_0} \neq a'_{i_0}$. We first notice that the absolute value of the difference between these digits has to be equal to 1: $|a_{i_0} - a'_{i_0}| = 1$. Indeed, if this was not the case, i.e. if $|a_{i_0} - a'_{i_0}| \geq 2$, the distance between $\sum_{i \geq 0} a_i b^{-i}$ and $\sum_{i \geq 0} a'_i b^{-i}$ would be greater than b^{-i_0} , which contradicts the fact that both sums are equal to r . Let us now suppose, without loss of generality, that $a_{i_0} = a'_{i_0} + 1$. Then $a_j = 0$ for all $j > i_0$ since if there existed an integer $j > i_0$ such that $a_j \neq 0$, the distance between the sums $\sum_{i \geq 0} a_i b^{-i}$ and $\sum_{i \geq 0} a'_i b^{-i}$ would be greater than b^{-j} , which would again be a contradiction. Moreover, $a'_j = b - 1$ for all $j > i_0$: if this was not the case, one could show in a similar way that the difference between the two sums would be strictly greater than 0. We can deduce from this that only the reals with a finite representation in base b have two distinct representations.

Since the set of such elements is of null measure, the transformation T_ϕ associated to a bijection ϕ of $\mathbf{N}$ is well defined as we can define T_ϕ only on the set of reals that have a unique representation. We can however choose to deal with this in another way: choosing between the two possible representations, by excluding for instance the representations as sequences that are almost everywhere equal to zero. Then T_ϕ is defined at all points and bijective. We chose in the following to follow this second approach since it will allow to prove more easily that T_ϕ is measure-preserving. However, this choice is not relevant for the rest of the construction since both transformations are almost everywhere equal.

Lemma 62. Let T be a transformation of $[0, 1]$ such that for all interval $[a, b]$, $\lambda(T([a, b])) = \lambda([a, b])$. Then T is measure-preserving on $[0, 1]$.

Proof. A classical result of measure theory states that if T is a transformation of a measured space $(X, \mathcal{B}, \lambda)$, that $\mathcal{B}$ is generated by $\mathcal{A}$, and that for all $A \in \mathcal{A}$, $\lambda(T(A)) = \lambda(A)$, then T preserves the measure λ on X . Applying this result with $X = [0, 1]$, and $\mathcal{A}$ as the set of intervals $[a, b] \subset [0, 1]$, we obtain the result. $\square$

Lemma 63. Let T be a bijective transformation of $[0, 1]$ that preserves the measure on all interval I of the shape $[\sum_{k=1}^p a_k b^{-k}, \sum_{k=1}^p a_k b^{-k} + b^{-p}]$. Then T is measure-preserving on $[0, 1]$.

Proof. Chose $[a, b] \subset [0, 1]$. One can write $[a, b]$ as a union $\cup_{i=0}^\infty [a_i, a_{i+1}]$, where for all $i \geq 0$, $a_{i+1} = a_i + b^{-k_i}$. We then obtain, using the hypotheses of the statement and the σ -additivity

of the measure λ :

$$\begin{aligned}
\lambda(T([a, b])) &= \lambda(T(\cup_{i=0}^{\infty} [a_i, a_{i+1}[))) \\
&= \lambda(\cup_{i=0}^{\infty} T([a_i, a_{i+1}[))) \\
&= \sum_{i=0}^{\infty} \lambda(T([a_i, a_{i+1}[))) \\
&= \sum_{i=0}^{\infty} \lambda([a_i, a_{i+1}[)) \\
&= \lambda(\cup_{i=0}^{\infty} [a_i, a_{i+1}[)) \\
&= \lambda([a, b])
\end{aligned}$$

We now conclude by using the preceding lemma. $\square$

Theorem 64. *Let $\phi : \mathbf{N} \rightarrow \mathbf{N}$ be a bijection and $b \geq 2$ an integer. Then the transformation T_ϕ^b is measure-preserving.*

Proof. We recall first that the transformation T_ϕ^b is indeed bijective (see Remark 61).

By using the preceding lemma, it suffices to show that T_ϕ^b preserves the measure on intervals of the shape $I = [a, a + b^{-k}]$ with $a = \sum_{i=0}^k a_i b^{-i}$. Let us define $N = \max\{\phi(i) \mid 0 \leq i \leq k\}$. We then write $[0, 1]$ as the union of intervals $A_i = [i \times b^{-N}, (i+1) \times b^{-N}]$ where $0 \leq i \leq b^N - 1$.

Then the image of I by T_ϕ^b is equal to the union of the A_i for $i \times b^{-N} = \sum_{i=0}^N x_i b^{-i}$, where $x_{\phi(j)} = a_j$ for all $0 \leq j \leq k$. The number of such A_j is equal to the number of sequences $\{0, \dots, b-1\}$ of length $N-k$, i.e. b^{N-k} . Since each A_j has a measure equal to b^{-N} , the image of I by T_ϕ^b is of measure $b^{-N} b^{N-k} = b^{-k}$, which is equal to the measure of I since $\lambda(I) = b^{-k}$. $\square$

Remark 65. The preceding theorem can be easily generalized to bijections $\mathbf{N} + \dots + \mathbf{N} \rightarrow \mathbf{N}$ (the domain being the disjoint union of k copies of $\mathbf{N}$, $k \in \mathbf{N}$) which induce measure-preserving bijections from $[0, 1]^k$ onto $[0, 1]$. The particular case $\mathbf{N} + \mathbf{N} \rightarrow \mathbf{N}$, $(n, i) \mapsto 2n + i$ defines the well-known measure-preserving bijection between the unit square and the interval $[0, 1]$:

$$(\sum_{i \geq 0} a_i 2^{-i}, \sum_{i \geq 0} b_i 2^{-i}) \mapsto \sum_{i \geq 0} a_{2i} 2^{-2i} + b_{2i+1} 2^{-2i-1}$$

Let us now define the bijection:

$$\psi : \mathbf{N} + \mathbf{N} + \mathbf{N} \rightarrow \mathbf{N}, \quad (x, i) \mapsto 3x + i$$

We also define the injections ι_i ($i = 0, 1, 2$):

$$\iota_i : \mathbf{N} \rightarrow \mathbf{N} + \mathbf{N} + \mathbf{N}, \quad x \mapsto (x, i)$$

We will denote by ψ_i the composite $\psi \circ \iota_i : \mathbf{N} \rightarrow \mathbf{N}$.

Definition 66. Let $A \subset \mathbf{N} + \mathbf{N} + \mathbf{N}$ be a finite set. We write A as $A_0 + A_1 + A_2$, and define, for $i = 0, 1, 2$, n_i to be the cardinality of A_i if $A_i \neq \emptyset$ and $n_i = 1$ otherwise. We then define a partition of $[0, 1]$, denoted by $\mathcal{P}_A = \{P_A^{i_1, i_2, i_3} \mid \forall k \in \{0, 1, 2\}, 0 \leq i_k \leq n_k - 1\}$, by:

$$P_A^{i_1, i_2, i_3} = \left\{ \sum_{j \geq 1} a_j 2^{-j} \mid \forall k \in \{0, 1, 2\}, \frac{i_k}{n_k} \leq \sum_{j \geq 1} a_{\psi_k(j)} 2^{-j} \leq \frac{i_k + 1}{n_k} \right\}$$

When A_k is empty or of cardinality 1, we will not write the corresponding i_k in the triple (i_1, i_2, i_3) since it is necessarily equal to 0.

Proposition 67. *Let us keep the notations of the preceding proposition and let $X = P_A^{i_1, i_2, i_3}$ and $Y = P_A^{j_1, j_2, j_3}$ be two elements of the partition $\mathcal{P}_A$. For all $x = \sum_{l \geq 1} a_l 2^{-l}$, we define $T_{i_1, i_2, i_3}^{j_1, j_2, j_3}(x) = \sum_{l \geq 1} b_l 2^{-l}$ where the sequence (b_i) is defined by:*

$$\forall k \in \{0, 1, 2\}, \sum_{l \geq 1} b_{\psi_k(l)} 2^{-l} = \sum_{l \geq 1} a_{\psi_k(l)} 2^{-l} + j_k - i_k$$

Then $T_{i_1, i_2, i_3}^{j_1, j_2, j_3} : X \rightarrow Y$ is a measure-preserving bijection.

Proof. For $k = 0, 1, 2$, we will denote by (m_j^k) the sequence such that $j_k - i_k = \sum_{l \geq 1} m_l^k 2^{-l}$. We can define the real number $t = \sum_{l \geq 1} \sum_{k=0,1,2} m_l^k 2^{-3j+k}$. It is then sufficient to check that $T_{i_1, i_2, i_3}^{j_1, j_2, j_3}(x) = x + t$. Since $T_{i_1, i_2, i_3}^{j_1, j_2, j_3}$ is a translation, it is a measure-preserving bijection. $\square$

Definition 68. Let $A \subset \mathbf{N}$ be a finite set endowed with the normalized — i.e. such that A has measure 1 — counting measure, and $X \subset \mathcal{B}(\mathbf{R} \times A)$ be a measurable set. We define the measurable subset $\lceil A \rceil \in \mathbf{R} \times [0, 1]$:

$$\lceil A \rceil = \{(x, y) \mid \exists z \in A, (x, z) \in X, y \in P_A^z\}$$

We will write $\mathcal{P}_A^{-1} : [0, 1] \rightarrow A$ the map that associates to each x the element $z \in A$ such that $x \in P_A^z$.

Proposition 69. *Let $D^A \subset \mathbf{N}$ be a finite set endowed with the normalized counting measure μ (i.e. such that $\mu(A) = 1$), $S, T \in \mathcal{B}(\mathbf{R} \times D^A)$ be measurable sets, and $\phi : S \rightarrow T$ a measure-preserving transformation. We define $\lceil \phi \rceil : \lceil S \rceil \rightarrow \lceil T \rceil$ by:*

$$\lceil \phi \rceil : (x, y) \mapsto (x', y') \quad \phi(x, \mathcal{P}_A^{-1}(y)) = (x', z), \quad y' = T_{\mathcal{P}_A^{-1}(y)}^z$$

Then $\lceil \phi \rceil$ is a measure-preserving bijection.

Proof. For all $(a, b) \in D^A$ we define the set $S_{a,b} = X \cap \mathbf{R} \times \{a\} \cap \phi^{-1}(Y \cap \mathbf{R} \times \{b\})$. The family $(S_{a,b})_{a,b \in D^A}$ is a partition of S , and the family $(\lceil S_{a,b} \rceil)_{a,b \in D^A}$ is a partition of $\lceil A \rceil$. The restriction of $\lceil \phi \rceil$ to $\lceil S_{a,b} \rceil$ can then be defined as the composite $T_a \circ \phi_1$ with:

$$\begin{aligned} \phi_1 &= (\pi_1 \circ \phi) \times \text{Id} \\ T_a &= \text{Id} \times T_a^b \end{aligned}$$

Since the product (resp. the composition) of measure preserving bijections is a measure preserving bijection, the restriction of $\lceil \phi \rceil$ to X_a is a measure preserving bijection. Moreover, it is clear that the image of $\lceil S \rceil$ by $\lceil \phi \rceil$ is equal to $\lceil T \rceil$ and we have finished the proof. $\square$

Definition 70. Let A be a thick graphing, i.e. of support $V^A \subset \mathbf{R} \times D^A$ measurable, where D^A is a finite subset of $\mathbf{N}$ endowed with the normalized counting measure. We define the graphing:

$$\lceil A \rceil = \{(\omega_e^A, \lceil \phi_e^A \rceil : \lceil S_e^A \rceil \rightarrow \lceil T_e^A \rceil)_{e \in E^A}$$

Definition 71. Let A be a thick graphing of dialect D^A , and $\Omega : \mathbf{R} \times [0, 1] \rightarrow \mathbf{R}$ an isomorphism of measured spaces. We define the graphing $!_\Omega A$ by:

$$!_\Omega A = \{(\omega_e^A, \Omega \circ \lceil \phi_e^A \rceil \circ \Omega^{-1} : \Omega(\lceil S_e^A \rceil) \rightarrow \Omega(\lceil T_e^A \rceil))_{e \in E^A}$$

Definition 72. A project α is *balanced* if $\alpha = (0, A)$ where A is a thick graphing, i.e. I^A is a one-element set, for instance $I^A = \{1\}$, and $\alpha_1^A = 1$.

Definition 73. Let α be a balanced project. We define $!_{\Omega}\alpha = (0, !_{\Omega}A)$. If $\mathbf{A}$ is a conduct, we define:

$$!_{\Omega}\mathbf{A} = \{!_{\Omega}\alpha \mid \alpha = (0, A) \in \mathbf{A}, \alpha \text{ balanced}\}^{\perp\perp}$$

We will now show that it is possible to implement the functorial promotion. In order to do this, we define the bijections $\tau, \theta : \mathbf{N} + \mathbf{N} + \mathbf{N} \rightarrow \mathbf{N} + \mathbf{N} + \mathbf{N}$:

$$\begin{aligned} \tau & : \begin{cases} (x, 0) \mapsto (x, 1) \\ (x, 1) \mapsto (x, 0) \\ (x, 2) \mapsto (x, 2) \end{cases} \\ \theta & : \begin{cases} (x, 0) \mapsto (2x, 0) \\ (x, 1) \mapsto (2x+1, 1) \\ (2x, 2) \mapsto (x, 1) \\ (2x+1, 2) \mapsto (x, 2) \end{cases} \end{aligned}$$

These bijections induce bijections of $\mathbf{N}$ onto $\mathbf{N}$ through $\psi : (x, i) \mapsto 3x + i$. We will abusively denote by $T_{\tau} = T_{\psi \circ \tau \circ \psi^{-1}}$ and $T_{\theta} = T_{\psi \circ \theta \circ \psi^{-1}}$ the induced measure-preserving transformations $[0, 1] \rightarrow [0, 1]$.

Pick $\alpha \in \sharp(\phi(\mathbf{A}))$ and $\beta \in \sharp(\mathbf{A} \multimap \mathbf{B})$, where ϕ is a delocation. By definition, $\alpha = (0, \Omega(\ulcorner A \urcorner))$ and $\beta = (0, \Omega(\ulcorner F \urcorner))$ where A, F are graphings of respective dialects D^A, D^F . We define the graphing $T = \{(1, \Omega(\text{Id} \times T_{\tau})), (1, (\Omega(\text{Id} \times T_{\tau}))^{-1})\}$ of carrier $V^{\phi(A)} \cup V^A$, and denote by t, t^* the two edges in E^T . We fix (x, y) an element of V^B and we will try to understand the action of the path $f_0 t \alpha_0 t^* f_1 \dots t \alpha_{k-1} t^* f_k$.

We fix the partition $\mathcal{P}_{D^F + D^A}$ of $[0, 1]$, and denote by (i, j) the integers such that $y \in \mathcal{P}_{D^F + D^A}^{i, j}$. By definition of $\ulcorner F \urcorner$, the map $\ulcorner \phi_{f_0}^F \urcorner$ sends this element to (x_1, y_1) which is an element of $\mathcal{P}_{D^F + D^A}^{i_1, j_1}$ with $j_1 = j$. Then, the function ϕ_t sends this element on (x_2, y_2) , where $x_2 = x_1$ and y_2 is an element of $\mathcal{P}_{D^F + D^A}^{j_1, i_1}$. The function $\ulcorner \phi_{\alpha_0}^A \urcorner$ then produces an element (x_3, y_3) with y_3 in $\mathcal{P}_{D^F + D^A}^{j_2, i_2}$ and $i_2 = i_1$. The element produced by $\phi_{t^*} = \phi_t^{-1}$ is then (x_4, y_4) where y_4 is an element of $\mathcal{P}_{D^F + D^A}^{i_2, j_2}$. One can therefore see how the graphing T simulates the dialectal interaction. The following proposition will show how one can use T to implement functorial promotion.

In order to implement functorial promotion, we will make use of the three bijections we just defined. Though it may seem a complicated, the underlying idea is quite simple. We will be working with three disjoint copies of $\mathbf{N}$, let us say $\mathbf{N}_i$ ($i = 0, 1, 2$). When applying promotion, we will encode the information contained in the dialect on the first copy $\mathbf{N}_0$ (let us stress here that promotion is defined through a non-surjective map, something that will be essential in the following). Suppose now that we have two graphs obtained from two promotions: all the information they contain is located in their first copy $\mathbf{N}_0$. To simulate dialectal information, we need to make these two sets disjoint: this is where the second copy $\mathbf{N}_1$ will be used. Hence, we apply to one of these promoted graphs the bijection τ (in practice we will of course use τ through the induced transformation T_{τ}) which exchanges $\mathbf{N}_0$ and $\mathbf{N}_1$. The information coming from the dialects of the two graphs are now disjoint. We then compute the execution of the two graphs to obtain a graph whose information coming from the dialect is encoded on the two copies $\mathbf{N}_0$ and $\mathbf{N}_1$! In order to be able to see this obtained graph as a graph obtained from a promotion, we need now to move this information so that it is encoded on the first copy $\mathbf{N}_0$ only. This is where we use the third copy $\mathbf{N}_2$: we use the bijection θ (once again, we

use in practice the induced transformation T_θ in order to contract the two copies $\mathbf{N}_0$ and $\mathbf{N}_1$ on the first copy $\mathbf{N}_0$, while deploying the third copy $\mathbf{N}_2$ onto the two copies $\mathbf{N}_1$ and $\mathbf{N}_2$.

Proposition 74. *One can implement functorial promotion: for all delocations ϕ, ψ and conducts $\mathbf{A}, \mathbf{B}$ such that $\phi(\mathbf{A}), \mathbf{A}, \mathbf{B}, \psi(\mathbf{B})$ have pairwise disjoint carriers, there exists a project prom in the conduct*

$$!\phi(\mathbf{A}) \otimes !(\mathbf{A} \multimap \mathbf{B}) \multimap !\psi(\mathbf{B})$$

Proof. Let $f \in \mathbf{A} \multimap \mathbf{B}$ be a balanced project, ϕ, ψ two delocations of $\mathbf{A}$ and $\mathbf{B}$ respectively. We define the graphings $T = \{(1, \Omega(\text{Id} \times T_\tau)), (1, (\Omega(\text{Id} \times T_\tau))^{-1})\}$ of carrier $V^{\phi(\mathbf{A})} \cup V^{\mathbf{A}}$ and $P = \{(1, \Omega(\text{Id} \times T_\theta)), (1, (\Omega(\text{Id} \times T_\theta))^{-1})\}$ of carrier $V^{\mathbf{B}} \cup V^{\psi(\mathbf{B})}$. We define $t = (0, T)$ and $p = (0, P)$, and the project:

$$\text{prom} = (0, T \cup P) = t \otimes p$$

We will now show that prom is an element in $(!\phi(\mathbf{A}) \otimes !(\mathbf{A} \multimap \mathbf{B})) \multimap !\psi(\mathbf{B})$.

We can suppose, up to choosing refinements of A and F , that for all $e \in E^A \cup E^F$, $(S_e)_2$ and $(T_e)_2$ are one-elements sets⁹.

Pick $a \in \sharp\phi(\mathbf{A})$ and $f \in \sharp(\mathbf{A} \multimap \mathbf{B})$. Then, by definition $a = (0, \Omega(\ulcorner A \urcorner))$ and $f = (0, \Omega(\ulcorner F \urcorner))$ where A, F are graphings of dialects D^A, D^F . We get that $a \otimes f :: \text{prom} = ((a :: t) :: f) :: p$ from the associativity and commutativity of $::$ (recall that $a \otimes f = a :: f$).

We show that $\ulcorner A \urcorner :: \ulcorner T \urcorner$ is the graphings composed of the $!^\tau \phi_a$ for $a \in E^A$, where $!^\tau \phi_a$ is defined by:

$$!^\tau \phi_a : (x, y) \mapsto (x', y'), \quad \phi_a(x, \mathcal{P}_{\{0\}+D^A}^{-1}(y)) = (x', z), \quad y' = T_{\mathcal{P}_{\{0\}+D^A}^{-1}(y)}^{(z,1)}(y)$$

This is almost straightforward. An element in $\ulcorner A \urcorner :: \ulcorner T \urcorner$ is a path of the form tat^* . It is therefore the function $\phi_t \circ \ulcorner \phi_a \urcorner \circ \phi_t^{-1}$. By definition,

$$\ulcorner \phi_a \urcorner : (x, y) \mapsto (x', y') \quad n = \mathcal{P}_A^{-1}(y), \quad \phi_a(x, n) = (x', k), \quad y' = T_n^k(y)$$

But $\phi_t : \text{Id} \times T_\tau$ and T_τ is a bijection from $\mathcal{P}_A(y)$ to $\mathcal{P}_{\{0\}+A}(1, y)$.

We now describe the graphing $G = (\ulcorner A \urcorner :: \ulcorner T \urcorner) :: \ulcorner F \urcorner$. It is composed of the paths of the shape $\rho = f_0(ta_0t^*)f_1(ta_1t^*)f_2 \dots f_{n-1}(ta_{n-1}t^*)f_n$. The associated function is therefore:

$$\phi_\rho = \ulcorner \phi_{f_0} \urcorner (\ulcorner !^\tau \phi_{a_0} \urcorner \ulcorner \phi_{f_1} \urcorner \dots \ulcorner \phi_{f_{n-1}} \urcorner (\ulcorner !^\tau \phi_{a_{n-1}} \urcorner \ulcorner \phi_{f_n} \urcorner$$

Let $\pi = f_0a_0f_1 \dots f_{n-1}a_{n-1}f_n$ be the corresponding path in $F :: A$. The function ϕ_π has, by definition, as domain and codomain measurable subsets of $\mathbf{R} \times D^F \times D^A$. We define, for such a function, the function $i\phi_\pi$ by:

$$i\phi_\pi : (x, y) \mapsto (x', y') \\ (n, m) = \mathcal{P}_{D^F+D^A}^{-1}(y), \quad \phi_\pi(x, n, m) = (x', k, l), \quad y' = T_{(n,m)}^{(k,l)}(y)$$

One can then check that $i\phi_\pi = \phi_\rho$.

Finally, $G :: \ulcorner P \urcorner$ is the graphing composed of paths that have the shape $\rho\rho\rho^*$ where ρ is a path in G . But $\phi_p = \text{Id} \times T_\theta$ applies a bijection, for all couple $(k, l) \in D^F \times D^A$, from the set $\mathcal{P}_{D^F+D^A}^{k,l}$ to the set $\mathcal{P}_{\theta(D^F+D^A)}^{\theta(k,l)}$ where:

$$\theta(D^F + D^A) = \{\theta(f, a) \mid f \in D^F, a \in D^A\}$$

⁹The sets S_e and T_e being subsets of a product, we write $(S_e)_2$ (resp. $(T_e)_2$) the result of their projection on the second component.

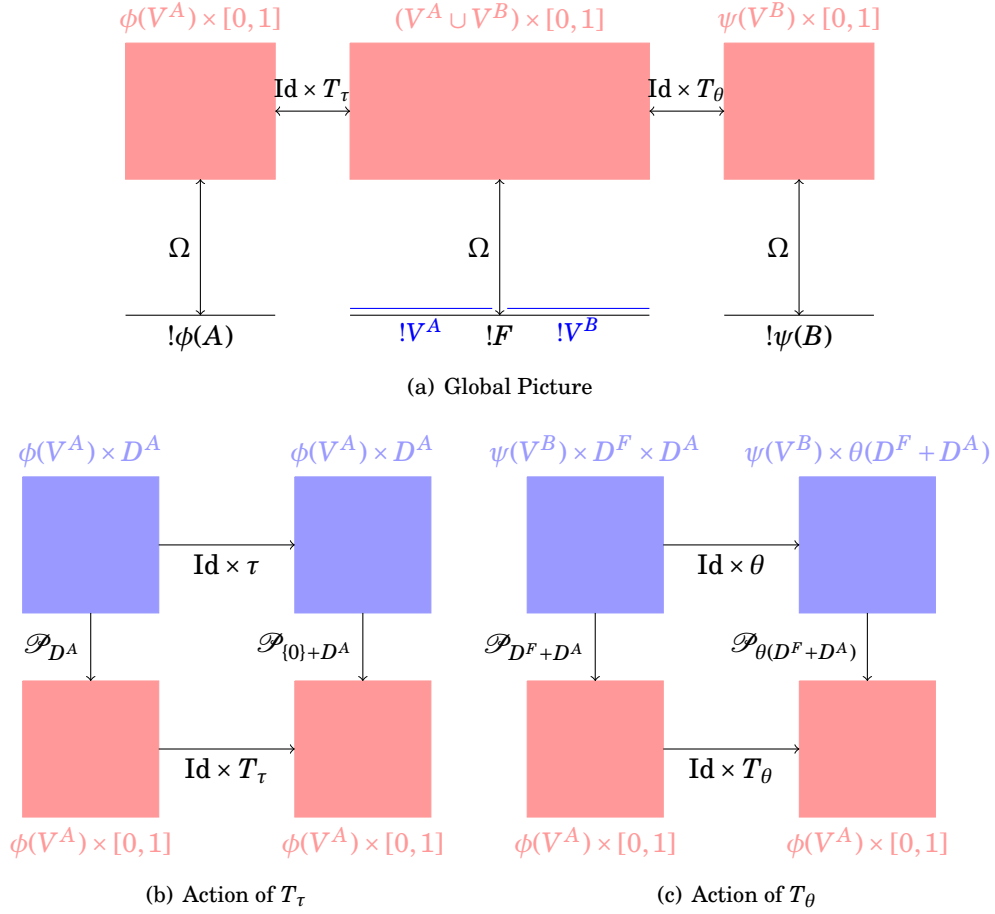


Figure 18: Functorial Promotion

We deduce that:

$$\begin{aligned} \phi_{p\rho p^*} : (x, y) &\mapsto (x', y') \\ n = \theta(k, l) &= \mathcal{P}_{\theta(D^F + D^A)}^{-1}(y) \quad \phi_\pi(x, k, l) = (x', k', l') \quad y' = T_n^{\theta(k', l')}(y) \end{aligned}$$

Modulo the bijection $\mu : D^F \times D^A \rightarrow \theta(D^F + D^A) \subset \mathbf{N}$, we get that $G :: \ulcorner P \urcorner$ is the delocation (along ψ) of the graphing $!(F :: A)$.

Therefore, for all a, f in $\sharp\mathbf{A}, \sharp(\mathbf{A} \multimap \mathbf{B})$ respectively there exists a project b in $\sharp\psi(\mathbf{B})$ such that $\text{prom} :: (a \otimes f) = b$. We showed that for all $g \in \sharp\mathbf{A} \odot \sharp(\mathbf{A} \multimap \mathbf{B})$, one has $\text{prom} :: g \in \sharp\mathbf{B}$, and thus prom is an element in $(\sharp\mathbf{A} \odot \sharp(\mathbf{A} \multimap \mathbf{B}))^{\perp\perp} \multimap \mathbf{B}$ by Proposition 37. But $(\sharp\mathbf{A} \odot \sharp(\mathbf{A} \multimap \mathbf{B}))^{\perp\perp} = \sharp\mathbf{A} \otimes \sharp(\mathbf{A} \multimap \mathbf{B})$ by Proposition 9. $\square$

In the setting of its hyperfinite geometry of interaction [Gir11], Girard shows how one can obtain the exponentials isomorphism as an equality between the conducts $!(\mathbf{A} \& \mathbf{B})$ and $\sharp\mathbf{A} \otimes \sharp\mathbf{B}$. Things are however quite different here. Indeed, if the introduction of behaviors in place of Girard's negative/positive conducts is very interesting when one is interested in the additive connectives, this leads to a (small) complication when dealing with exponentials.

The first thing to notice is that the proof of the implication $!A \otimes !B \multimap !(A \& B)$ in a sequent calculus with functorial promotion and without dereliction and digging rules cannot be written if the weakening rule is restrained to the formulas of the form $?A$:

$$\frac{\frac{\frac{}{\vdash A, A^\perp} \text{ax}}{\vdash A, B^\perp, A^\perp} \text{weak} \quad \frac{\frac{}{\vdash B, B^\perp} \text{ax}}{\vdash B, B^\perp, A^\perp} \text{weak}}{\vdash B^\perp, A^\perp, A \& B} \& \quad \frac{}{\vdash ?B^\perp, ?A^\perp, !(A \& B)} ! \quad \frac{}{\vdash !A \otimes !B \multimap !(A \& B)}$$

In Girard's setting, weakening is available for all positive conducts (the conducts on which one can apply the $?$ modality), something which is coherent with the fact that the inclusion $!A \otimes !B \subset !(A \& B)$ is satisfied. In our setting, however, weakening is never available for behaviors and we think the latter inclusion is therefore not satisfied. This question stays however open.

Concerning the converse inclusion, it does not seem clear at first that it is satisfied in our setting either. This issue comes from the contraction rule. Indeed, since the latter does not seem to be satisfied in full generality (see Definition 5.1), one could think the inclusion $!(A \& B) \subset !A \otimes !B$ is not satisfied either. We will show however in Section 6, through the introduction of alternative "additive connectives", that it does hold (a result that will not be used until the last section).

Proposition 75. *The conduct $\mathbf{1}$ is a perennial conduct, equal to $!\mathbf{T}$.*

Proof. By definition, $\mathbf{1} = \{(0, \emptyset)\}^{\perp\perp}$ is a perennial conduct. Moreover, the balanced projects in $\mathbf{T}$ are the projects of the shape $t_D = (0, \emptyset)$ with dialects $D \subset \mathbf{N}$. Each of these satisfy $!t_D = (0, \emptyset)$. Thus $\#\mathbf{T} = \{(0, \emptyset)\}$ and $!\mathbf{T} = \mathbf{1}$. $\square$

Corollary 76. *The conduct $\perp$ is a co-perennial conduct, equal to $?0$.*

Proof. This is straightforward:

$$\perp = \mathbf{1}^\perp = (!\mathbf{T})^\perp = (\#\mathbf{T})^{\perp\perp\perp} = (\#\mathbf{T})^\perp = (\#0^\perp)^\perp = ?0$$

$\square$

5. SOUNDNESS FOR BEHAVIORS

5.1. Sequent Calculus. To deal with the three kinds of conducts we are working with (behaviors, perennial and co-perennial conducts), we introduce three types of formulas.

Definition 77. We define three types of formulas, (B)ehaviors, (N)egative — perennial, and (P)ositive — co-perennial, inductively defined by the following grammar:

$$\begin{aligned} B &:= \mathbf{T} \mid \mathbf{0} \mid X \mid X^\perp \mid B \otimes B \mid B \wp B \mid B \oplus B \mid B \& B \mid \forall X B \mid \exists X B \mid N \otimes B \mid P \wp B \\ N &:= \mathbf{1} \mid P^\perp \mid !B \mid N \otimes N \mid N \oplus N \\ P &:= \perp \mid N^\perp \mid ?B \mid P \wp P \mid P \& P \end{aligned}$$

We will denote by $\text{FV}(\Gamma)$ the set of free variables in Γ , where Γ is a sequence of formulas (of any type).

Definition 78. We define *pre-sequents* $\Delta \Vdash \Gamma; \Theta$ where Δ, Θ contain negative (perennial) formulas, Θ containing at most one formula, and Γ contains only behaviors.

Definition 35 supposes that we are working with behaviors, and cannot be used to interpret contraction in full generality. It is however possible to show in a similar way that contraction can be interpreted when the sequent contains at least one behavior (this is the next proposition). This restriction of the context is necessary: without behaviors in the sequent one cannot interpret the contraction since the inflation property is essential for showing that $(1/2)\phi(!a) \otimes \psi(!a) + (1/2)o$ is an element of $\phi(!A) \otimes \psi(!A)$.

Proposition 79. Let A be a conduct and ϕ, ψ be disjoint delocations of $!V^A$. Let C be a behavior and θ a delocation disjoint from ϕ and ψ . Then the project $\mathfrak{C}tr_{\phi \cup \theta}^\psi$ is an element of the behavior:

$$(!A \otimes C) \multimap (\phi(!A) \otimes \psi(!A) \otimes \theta(C))$$

Proof. The proof follows the proof of Definition 35. We show in a similar manner that the project $\mathfrak{C}tr_{\phi \cup \theta}^\psi :: (a \otimes c)$ is universally equivalent to:

$$\frac{1}{2}\phi(!a) \otimes \psi(!a) \otimes \theta(C) + \frac{1}{2}o$$

Since $!A$ is a perennial conduct and C is a behavior, $(\phi(!A) \otimes \psi(!A) \otimes \theta(C))$ is a behavior. Thus $\mathfrak{C}tr_{\phi \cup \theta}^\psi :: (a \otimes c)$ is an element in $(\phi(!A) \otimes \psi(!A) \otimes \theta(C))$. Finally we showed that the project $\mathfrak{C}tr_{\phi \cup \theta}^\psi$ is an element of $(!A \otimes C) \multimap (\phi(!A) \otimes \psi(!A) \otimes \theta(C))$, and that the latter is a behavior. $\square$

In a similar way, the proof of distributivity relies on the property that $A + B \subset A \& B$ which is satisfied for behaviors but not in general. It is therefore necessary to restrict to pre-sequents that contain at least one behavior in order to interpret the $\&$ rule. Indeed, we can think of a pre-sequent $\Delta \Vdash \Gamma; \Theta$ as the conduct¹⁰

$$\left(\bigotimes_{N \in \Delta} N^\perp \right) \wp \left(\bigotimes_{B \in \Gamma} B \right) \wp \left(\bigotimes_{N \in \Theta} N \right)$$

Such a conduct is a behavior when the set Γ is non-empty and the set Θ is empty, but it is neither a perennial conduct nor a co-perennial conduct when $\Gamma = \emptyset$. We will therefore restrict to pre-sequents such that $\Gamma \neq \emptyset$ and $\Theta = \emptyset$.

Definition 80 (Sequents). A sequent $\Delta \vdash \Gamma;$ is a pre-sequent $\Delta \Vdash \Gamma; \Theta$ such that Γ is non-empty and Θ is empty.

Definition 81 (The Sequent Calculus ELL_{comp}). A proof in the sequent calculus ELL_{comp} is a derivation tree constructed from the derivation rules shown in Figure 19 page 36.

5.2. Truth. The notion of success is the natural generalization of the corresponding notion on graphs [Sei12a, Sei14a]. The graphing of a successful project will therefore be a disjoint union of "transpositions". Such a graphing can be represented as a graph with a set of vertices that could be infinite, but since we are working with equivalence classes of graphings one can always find a simpler representation: a graphing with exactly two edges.

¹⁰This will actually be the exact definition of its interpretation.

$\frac{}{\vdash C^\perp, C;} \text{ ax} \qquad \frac{\Delta_1 \vdash \Gamma_1, C; \quad \Delta_2 \vdash \Gamma_2, C^\perp;}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2;} \text{ cut}$	
(a) Identity Group	
$\frac{\Delta_1 \vdash \Gamma_1, C_1; \quad \Delta_2 \vdash \Gamma_2, C_2;}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2, C_1 \otimes C_2;} \otimes \qquad \frac{\Delta \vdash \Gamma, C_1, C_2;}{\Delta \vdash \Gamma, C_1 \wp C_2;} \wp$ $\frac{\Delta, N_1, N_2 \vdash \Gamma;}{\Delta, N_1 \otimes N_2 \vdash \Gamma;} \otimes_g^{pol} \qquad \frac{\Delta, P^\perp \vdash \Gamma, C;}{\Delta \vdash \Gamma, P \wp C;} \wp^{mix}$ $\frac{\Delta \vdash \Gamma, C;}{\Delta \vdash \Gamma, C \otimes \mathbf{1};} \mathbf{1}_d \qquad \frac{\Delta \vdash \Gamma;}{\Delta, \mathbf{1} \vdash \Gamma;} \mathbf{1}_g$	
(b) Multiplicative Group	
$\frac{\Delta \vdash \Gamma, C_i;}{\Delta \vdash \Gamma, C_1 \oplus C_2;} \oplus_i \qquad \frac{\Delta \vdash \Gamma, C_1; \quad \Delta \vdash \Gamma, C_2;}{\Delta \vdash \Gamma, C_1 \& C_2;} \&$ $\frac{}{\Delta \vdash \Gamma, \top;} \top \qquad \text{No rules for } 0.$	
(c) Additive Group	
$\frac{\Delta_1 \vdash \Gamma_1, C_1; \quad \Delta_2 \vdash \Gamma_2, C_2;}{\Delta_1, \Delta_2, !\Gamma_1^\perp \vdash \Gamma_2, C_1 \otimes !C_2;} !$ $\frac{\Delta, !A, !A \vdash \Gamma;}{\Delta, !A \vdash \Gamma;} \text{ ctr } (\Gamma \neq \emptyset) \qquad \frac{\Delta \vdash \Gamma;}{\Delta, N \vdash \Gamma;} \text{ weak}$	
(d) Exponential Group	
$\frac{\vdash \Gamma, C; \quad X \notin \text{FV}(\Gamma)}{\vdash \Gamma, \forall X C;} \forall \qquad \frac{\vdash \Gamma, C[A/X];}{\vdash \Gamma, \exists X C;} \exists$	
(e) Quantifier Group	

Figure 19: Rules for the sequent calculus ELL_{comp}

Definition 82. A project $\alpha = (a, A)$ is *successful* when it is balanced, $a = 0$ and A is a disjoint union of transpositions:

- for all $e \in E^A$, $\omega_e^A = 1$;
- for all $e \in E^A$, $\exists e^* \in E^A$ such that $\phi_{e^*}^A = (\phi_e^A)^{-1}$ — in particular $S_e^A = T_{e^*}^A$ and $T_e^A = S_{e^*}^A$;
- for all $e, f \in E^A$ with $f \notin \{e, e^*\}$, $S_e^A \cap S_f^A$ and $T_e^A \cap T_f^A$ are of null measure;

A conduct $\mathbf{A}$ is *true* when it contains a successful project.

The following results were shown in our previous paper [Sei14c]. They ensure that the given definition of truth is coherent.

Proposition 83 (Consistency). *The conducts $\mathbf{A}$ and $\mathbf{A}^\perp$ cannot be simultaneously true.*

Proof. We suppose that $a = (0, A)$ and $b = (0, B)$ are successful project in the conducts $\mathbf{A}$ and $\mathbf{A}^\perp$ respectively. Then:

$$\ll a, b \gg_m = \ll A, B \gg_m$$

If there exists a cycle whose support is of strictly positive measure between A and B , then $\ll A, B \gg_m = \infty$. Otherwise, $\ll A, B \gg_m = 0$. In both cases we obtained a contradiction since a and b cannot be orthogonal. $\square$

Proposition 84 (Compositionnality). *If $\mathbf{A}$ and $\mathbf{A} \multimap \mathbf{B}$ are true, then $\mathbf{B}$ is true.*

Proof. Let $a \in \mathbf{A}$ and $f \in \mathbf{A} \multimap \mathbf{B}$ be successful projects. Then:

- If $\ll a, f \gg_m = \infty$, the conduct $\mathbf{B}$ is equal to $\mathbf{T}_{V_B}$, which is a true conduct since it contains $(0, \emptyset)$;
- Otherwise $\ll a, f \gg_m = 0$ (this is shown in the same manner as in the preceding proof) and it is sufficient to show that $F :: A$ is a disjoint union of transpositions. But this is straightforward: to each path there corresponds an opposite path and the weights of the paths are all equal to 1, the conditions on the source and target sets S_π and T_π are then easily checked.

Finally, if $\mathbf{A}$ and $\mathbf{A} \multimap \mathbf{B}$ are true, then $\mathbf{B}$ is true. $\square$

5.3. Interpretation of proofs. To prove soundness, we will follow the proof technique used in our previous papers [Sei12a, Sei14a, Sei14c]. We will first define a localized sequent calculus and show a result of full soundness for it. The soundness result for the non-localized calculus is then obtained by noticing that one can always *localize* a derivation. We will consider here that the variables are defined with the carrier equal to an interval in $\mathbf{R}$ of the form $[i, i + 1[$.

Definition 85. We fix a set $\mathcal{V} = \{X_i(j)\}_{i,j \in \mathbf{N} \times \mathbf{Z}}$ of *localized variables*. For $i \in \mathbf{N}$, the set $X_i = \{X_i(j)\}_{j \in \mathbf{Z}}$ will be called the *variable name* X_i , and an element of X_i will be called a *variable of name* X_i .

For $i, j \in \mathbf{N} \times \mathbf{Z}$ we define the *location* $\#X_i(j)$ of the variable $X_i(j)$ as the set

$$\{x \in \mathbf{R} \mid 2^i(2j + 1) \leq m < 2^i(2j + 1) + 1\}$$

Definition 86 (Formulas of $\text{locELL}_{\text{comp}}$). We inductively define the formulas of *localized polarized elementary linear logic* $\text{locELL}_{\text{comp}}$ as well as their *locations* as follows:

• **Behaviors:**

- A variable $X_i(j)$ of name X_i is a behavior whose location is defined as $\#X_i(j)$;
- If $X_i(j)$ is a variable of name X_i , then $(X_i(j))^\perp$ is a behavior whose location is $\#X_i(j)$.
- The constants $\mathbf{T}_{\# \Gamma}$ are behaviors whose location is defined as $\# \Gamma$;
- The constants $\mathbf{0}_{\# \Gamma}$ are behaviors whose location is defined as $\# \Gamma$.
- If A, B are behaviors with respective locations X, Y such that $X \cap Y = \emptyset$, then $A \otimes B$ (resp. $A \wp B$, resp. $A \& B$, resp. $A \oplus B$) is a behavior whose location is $X \cup Y$;
- If X_i is a variable name, and $A(X_i)$ is a behavior of location $\#A$, then $\forall X_i A(X_i)$ and $\exists X_i A(X_i)$ are behaviors of location $\#A$.
- If A is a perennial conduct with location X and B is a behavior whose location is Y such that $X \cap Y = \emptyset$, then $A \otimes B$ is a behavior with location $X \cup Y$;
- If A is a co-perennial conduct whose location is X and B is a behavior with location Y such that $X \cap Y = \emptyset$, then $A \wp B$ is a behavior and its location is $X \cup Y$;

- **Perennial conducts:**

- The constant $\mathbf{1}$ is a perennial conduct and its location is $\emptyset$;
- If A is a behavior or a perennial conduct and its location is X , then $!A$ is a perennial conduct and its location is $\Omega(X \times [0, 1])$;
- If A, B are perennial conducts with respective locations X, Y such that $X \cap Y = \emptyset$, then $A \otimes B$ (resp. $A \oplus B$) is a perennial conduct whose location is $X \cup Y$;

- **Co-perennial conducts:**

- The constant $\perp$ is a co-perennial conduct;
- If A is a behavior or a co-perennial conduct and its location is X , then $?A$ is a co-perennial conduct whose location is $\Omega(X \times [0, 1])$;
- If A, B are co-perennial conducts with respective locations X, Y such that $X \cap Y = \emptyset$, then $A \wp B$ (resp. $A \& B$) is a co-perennial conduct whose location is $X \cup Y$;

If A is a formula, we will denote by $\sharp A$ the location of A . A sequent $\Delta \vdash \Gamma$; of $\text{locELL}_{\text{comp}}$ must satisfy the following conditions:

- the formulas of $\Gamma \cup \Delta$ have pairwise disjoint locations;
- the formulas of Δ are all perennial conducts;
- Γ is non-empty and contains only behaviors.

Definition 87 (Interpretations). An *interpretation basis* is a function Φ which associates to each variable name X_i a behavior of carrier $[0, 1]$.

Definition 88 (Interpretation of $\text{locELL}_{\text{comp}}$ formulas). Let Φ be an interpretation basis. We define the interpretation $I_\Phi(F)$ along Φ of a formula F inductively:

- If $F = X_i(j)$, then $I_\Phi(F)$ is the delocation (i.e. a behavior) of $\Phi(X_i)$ defined by the function $x \mapsto 2^i(2j + 1) + x$;
- If $F = (X_i(j))^\perp$, we define the behavior $I_\Phi(F) = (I_\Phi(X_i(j)))^\perp$;
- If $F = \mathbf{T}_{\sharp\Gamma}$ (resp. $F = \mathbf{0}_{\sharp\Gamma}$), we define $I_\Phi(F)$ as the behavior $\mathbf{T}_{\sharp\Gamma}$ (resp. $\mathbf{0}_{\sharp\Gamma}$);
- If $F = \mathbf{1}$ (resp. $F = \perp$), we define $I_\Phi(F)$ as the behavior $\mathbf{1}$ (resp. $\perp$);
- If $F = A \otimes B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \otimes I_\Phi(B)$;
- If $F = A \wp B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \wp I_\Phi(B)$;
- If $F = A \oplus B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \oplus I_\Phi(B)$;
- If $F = A \& B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \& I_\Phi(B)$;
- If $F = \forall X_i A(X_i)$, we define the conduct $I_\Phi(F) = \forall \mathbf{X}_i I_\Phi(A(X_i))$;
- If $F = \exists X_i A(X_i)$, we define the conduct $I_\Phi(F) = \exists \mathbf{X}_i I_\Phi(A(X_i))$;
- If $F = !A$ (resp. $?A$), we define the conduct $I_\Phi(F) = !I_\Phi(A)$ (resp. $?I_\Phi(A)$).

Moreover, a sequent $\Delta \vdash \Gamma$; will be interpreted as the $\wp$ of formulas in Γ and negations of formulas in Δ , which will be written $\wp \Delta^\perp \wp \wp \Gamma$. This formulas can also be written in the equivalent form $\otimes \Delta \multimap (\wp \Gamma)$.

Definition 89 (Interpretation of $\text{locELL}_{\text{comp}}$ proofs). Let Φ be an interpretation basis. We define the interpretation $I_\Phi(\pi)$ — a project — of a proof π inductively:

- if π is a single axiom rule introducing the sequent $\vdash (X_i(j))^\perp, X_i(j')$, we define $I_\Phi(\pi)$ as the project $\wp \text{ax}$ defined by the translation $x \mapsto 2^i(2j' - 2j) + x$;
- if π is composed of a single rule $\mathbf{T}_{\sharp\Gamma}$, we define $I_\Phi(\pi) = \mathbf{0}_{\sharp\Gamma}$;
- if π is obtained from π' by using a $\wp$ rule, a $\wp^{\text{mix}}$ rule, a $\otimes_g^{\text{pol}}$ rule, or a $\mathbf{1}$ rule, then $I_\Phi(\pi) = I_\Phi(\pi')$;
- if π is obtained from π_1 and π_2 by performing a $\otimes$ rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \otimes I_\Phi(\pi_2)$;

- if π is obtained from π' using a weak rule or a $\oplus_i$ rule introducing a formula of location V , we define $I_\Phi(\pi) = I_\Phi(\pi') \otimes \circ_V$;
- if π of conclusion $\vdash \Gamma, A_0 \& A_1$ is obtained from π_0 and π_1 using a $\&$ rule, we define the interpretation of π in the same way it was defined in our previous paper [Sei14a];
- If π is obtained from a $\forall$ rule applied to a derivation π' , we define $I_\Phi(\pi) = I_\Phi(\pi')$;
- If π is obtained from a $\exists$ rule applied to a derivation π' replacing the formula $\mathbf{A}$ by the variable name X_i , we define $I_\Phi(\pi) = I_\Phi(\pi') :: (\otimes[e^{-1}(j) \leftrightarrow X_i(j)])$, using the notations of our previous paper [Sei14c];
- if π is obtained from π_1 and π_2 through the use of a promotion rule $!$, we think of this rule as the following "derivation of pre-sequents":

$$\frac{\frac{\frac{\vdots^{\pi_1}}{\Delta_1 \vdash \Gamma_1, C_1;}}{\Delta_2 \vdash \Gamma_2, C_2;}}{\Delta_2, !\Gamma_2^\perp \Vdash !C_2} ! \quad \frac{\vdots^{\pi_2}}{\Delta_2 \vdash \Gamma_2, C_2;}}{\Delta_2, !\Gamma_2^\perp \Vdash !C_2} ! \quad \frac{}{! \Delta_2, \Delta_1, !\Gamma_2^\perp \vdash \Gamma_1, C_1 \otimes !C_2;}_{\otimes^{mix}}$$

As a consequence, we first define a delocation of $I_\Phi(\pi)$ to which we apply the implementation of the functorial promotion. Indeed, the interpretation of

$$\wp_{\Delta^\perp} \wp_\Gamma$$

can be written as a sequence of implications. The exponential of a well-chosen delocation is then represented as:

$$!(\phi_1(\mathbf{A}_1) \multimap (\phi_2(\mathbf{A}_2) \multimap \dots (\phi_n(\mathbf{A}_n) \multimap \phi_{n+1}(\mathbf{A}_{n+1})) \dots))$$

Applying n instances of the project implementing the functorial promotion to the interpretation of π , we obtain a project p in:

$$!(\phi_1(\mathbf{A}_1)) \multimap !(\phi_2(\mathbf{A}_2)) \multimap \dots !(\phi_n(\mathbf{A}_n)) \multimap !(\phi_{n+1}(\mathbf{A}_{n+1}))$$

which is the same conduct as the one interpreting the conclusion of the promotion "rule" in the "derivation of pre-sequents" we have shown. Now we are left with taking the tensor product of the interpretation of π_2 with the project p to obtain the interpretation of the $!$ rule;

- if π is obtained from π using a contraction rule ctr , we write the conduct interpreting the premise of the rule as $(!\mathbf{A} \otimes !\mathbf{A}) \multimap \mathbf{D}$. We then define a delocation of the latter in order to obtain $(\phi(!\mathbf{A}) \otimes \psi(!\mathbf{A})) \multimap \mathbf{D}$, and take its execution with ctr in $!\mathbf{A} \multimap (!\mathbf{A} \otimes !\mathbf{A})$;
- if π is obtained from π_1 and π_2 by applying a cut rule or a cut^{pol} rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \dot{\cap} I_\Phi(\pi_2)$.

Theorem 90 (locELL_{comp} soundness). *Let Φ be an interpretation basis. Let π be a derivation in locELL_{comp} of conclusion $\Delta \vdash \Gamma$; Then $I_\Phi(\pi)$ is a successful project in $I_\Phi(\Delta \vdash \Gamma)$.*

Proof. The proof is a simple consequence of the proposition and theorems proved before hand. Indeed, the case of the rules of multiplicative additive linear logic was already treated in our previous papers [Sei12a, Sei14a]. The only rules we are left with are the rules dealing with exponential connectives and the rules about the multiplicative units. But the implementation of the functorial promotion (Proposition 74) uses a successful project do not put any restriction on the type of conducts we are working with, and the contraction project (Definition 35 and Proposition 79) is successful. Concerning the multiplicative units, the rules that introduce them do not change the interpretations. $\square$

As it was remarked in our previous papers, one can chose an enumeration of the occurrences of variables in order to "localize" any formula A and any proof π of ELL_{comp} : we then obtain formulas A^e and proofs π^e of $\text{locELL}_{\text{comp}}$. The following theorem is therefore a direct consequence of the preceding one.

Theorem 91 (Full ELL_{comp} Soundness). *Let Φ be an interpretation basis, π an ELL_{comp} proof of conclusion $\Delta \vdash \Gamma$; and e an enumeration of the occurrences of variables in the axioms in π . Then $I_\Phi(\pi^e)$ is a successful project in $I_\Phi(\Delta^e \vdash \Gamma^e)$.*

6. CONTRACTION AND SOUNDNESS FOR POLARIZED CONDUCTS

6.1. Definitions and Properties. In this section, we consider a variation on the definition of additive connectives, which is constructed from the definition of the formal sum $a + b$ of projects. Let us first try to explain the difference between the usual additives $\&$ and $\oplus$ considered until now and the new additives $\tilde{\&}$ and $\tilde{\oplus}$ defined in this section. The conduct $\mathbf{A} \& \mathbf{B}$ contains all the tests that are necessary for the set $\{a' \otimes o \mid a' \in \mathbf{A}^\perp\} \cup \{b' \otimes o \mid b' \in \mathbf{B}^\perp\}$ to generate the conduct $\mathbf{A} \oplus \mathbf{B}$, something for which the set $a + b$ is not sufficient. For the variant of additives considered in this section, it is the contrary that happens: the conduct $\mathbf{A} \& \mathbf{B}$ is generated by the projects of the form $a + b$, but it is therefore necessary to add to the conduct $\mathbf{A} \tilde{\oplus} \mathbf{B}$ all the needed tests.

Definition 92. Let $\mathbf{A}, \mathbf{B}$ be conducts of disjoint carriers. We define $\mathbf{A} \tilde{\&} \mathbf{B} = (\mathbf{A} + \mathbf{B})^{\perp\perp}$. Dually, we define $\mathbf{A} \tilde{\oplus} \mathbf{B} = (\mathbf{A}^\perp \tilde{\&} \mathbf{B}^\perp)^\perp$.

These connectives will be useful for showing that the inclusion $!(\mathbf{A} \tilde{\&} \mathbf{B}) \subset !\mathbf{A} \otimes !\mathbf{B}$ holds when $\mathbf{A}, \mathbf{B}$ are behaviors. We will first dwell on some properties of these connectives before showing this inclusion. Notice that if one of the two conducts $\mathbf{A}, \mathbf{B}$ is empty, then $\mathbf{A} \tilde{\&} \mathbf{B}$ is empty. Therefore, the behavior $\mathbf{0}_\emptyset$ is a kind of absorbing element for $\tilde{\&}$. But the latter connective also has a neutral element, namely the neutral element $\mathbf{1}$ of the tensor product! Notice that the fact that $\tilde{\&}$ and $\otimes$ share the same unit appeared in Girard's construction¹¹ of geometry of interaction in the hyperfinite factor [Gir11].

Notice that at the level of denotational semantics, this connective is almost the same as the usual $\&$ (apart from units). The differences between them are erased in the quotient operation.

Proposition 93. *Distributivity for $\tilde{\&}$ and $\tilde{\oplus}$ is satisfied for behaviors.*

Proof. Using the same project than in the proof of Proposition 11, the proof consists in a simple computation. $\square$

¹¹Our construction [Sei14a] differs slightly from Girard's, which explains why our additives don't share the same unit as the multiplicatives.

Proposition 94. *Let $\mathbf{A}, \mathbf{B}$ be behaviors. Then*

$$\{a \otimes o_{VB} \mid a \in \mathbf{A}\} \cup \{b \otimes o_{VA} \mid b \in \mathbf{B}\} \subset \mathbf{A} \tilde{\otimes} \mathbf{B}$$

Proof. We will show only one of the inclusions, the other one can be obtained by symmetry. Chose $f + g \in \mathbf{A}^\perp + \mathbf{B}^\perp$ and $a \in \mathbf{A}$. Then:

$$\begin{aligned} \ll f + g, a \otimes o \gg_m &= \ll f, a \otimes o \gg_m + \ll g, a \otimes o \gg_m \\ &= \ll f, a \gg_m \end{aligned}$$

Using the fact that g and a have null wagers. □

Recall (this notion is defined and studied in our second paper [Sei14a]) that a behavior $\mathbf{A}$ is *proper* if both $\mathbf{A}$ and its orthogonal $\mathbf{A}^\perp$ are non-empty. Proper behavior can be characterized as those conducts $\mathbf{A}$ such that:

- $(a, A) \in \mathbf{A}$ implies that $a = 0$;
- for all $a \in \mathbf{A}$ and $\lambda \in \mathbf{R}$, the project $a + \lambda o \in \mathbf{A}$;
- $\mathbf{A}$ is non-empty.

Proposition 95. *Let $\mathbf{A}, \mathbf{B}$ be proper behaviors. Then every element in $\mathbf{A} \tilde{\otimes} \mathbf{B}$ is observationally equivalent to an element in $\{a \otimes o_{VB} \mid a \in \mathbf{A}\} \cup \{b \otimes o_{VA} \mid b \in \mathbf{B}\} \subset \mathbf{A} \tilde{\otimes} \mathbf{B}$.*

Proof. Let $c \in \mathbf{A} \tilde{\otimes} \mathbf{B}$. Since $(\mathbf{A}^\perp + \mathbf{B}^\perp)^\perp = \mathbf{A} \tilde{\otimes} \mathbf{B}$, we know that $c \perp a + b$ for all $a + b \in \mathbf{A}^\perp + \mathbf{B}^\perp$. By the homothety lemma (Lemma 8), we obtain, for all λ, μ non-zero real numbers 0:

$$\ll c, \lambda a + \mu b \gg_m = \lambda \ll c, a \gg_m + \mu \ll c, b \gg_m \neq 0, \infty$$

We deduce that one expression among $\ll c, a \gg_m$ and $\ll c, b \gg_m$ is equal to 0. Suppose, without loss of generality, that it is $\ll c, a \gg_m$. Then $\ll c, a' \gg_m = 0$ for all $a' \in \mathbf{A}^\perp$. Thus $\ll b, c \gg_m \neq 0, \infty$ for all $b \in \mathbf{B}^\perp$. But $\ll b \otimes o, c \gg_m = \ll b, c :: o \gg_m$. We finally have that $c :: o \in \mathbf{B}^\perp$ and $c :: o \cong_{\mathbf{A} \tilde{\otimes} \mathbf{B}} c$. □

Proposition 96. *Let $\mathbf{A}, \mathbf{B}$ be proper behaviors. Then $\mathbf{A} \tilde{\otimes} \mathbf{B}$ is a proper behavior.*

Proof. By definition, $\mathbf{A} \tilde{\otimes} \mathbf{B} = (\mathbf{A} + \mathbf{B})^{\perp\perp}$. But $\mathbf{A}, \mathbf{B}$ are non empty contain only one-sliced wager-free projects. Thus $\mathbf{A} + \mathbf{B}$ is non empty and contains only one-sliced wager-free projects. Thus $(\mathbf{A} + \mathbf{B})^\perp$ satisfies the inflation property. Moreover, if $a + b \in \mathbf{A} + \mathbf{B}$, we have that $a + b + \lambda o = (a + \lambda o) + b$. Since $\mathbf{A}$ has the inflation property, $\mathbf{A} + \mathbf{B}$ has the inflation property. Thus $(\mathbf{A} + \mathbf{B})^\perp$ contains only wager-free projects. Moreover, $(\mathbf{A} + \mathbf{B})^\perp = \mathbf{A}^\perp \tilde{\otimes} \mathbf{B}^\perp$ and it is therefore non-empty by the preceding proposition (because $\mathbf{A}^\perp, \mathbf{B}^\perp$ are non empty). Then $(\mathbf{A} + \mathbf{B})^\perp$ is a proper behavior, which allows us to conclude. □

Proposition 97. *Let $\mathbf{A}, \mathbf{B}$ be behaviors. Then $!(\mathbf{A} \tilde{\otimes} \mathbf{B}) \subset !\mathbf{A} \otimes !\mathbf{B}$.*

Proof. If one of the behaviors among $\mathbf{A}, \mathbf{B}$ is empty, $!(\mathbf{A} \tilde{\otimes} \mathbf{B}) = \mathbf{0} = !\mathbf{A} \otimes !\mathbf{B}$. We will now suppose that $\mathbf{A}, \mathbf{B}$ are both non empty.

Chose $f = (0, F)$ a one-sliced wager-free project. We have that $f' = n_F / (n_F + n_G) f \in \mathbf{A}$ if and only if $f \in \mathbf{A}$ from the homothety lemma (Lemma 8). Moreover, since $\mathbf{A}$ is a behavior, $f' \in \mathbf{A}$ is equivalent¹² to $f'' = f' + \sum_{i=1}^{n_G} (1 / (n_F + n_G)) o \in \mathbf{A}$. Since the weighted thick and sliced graphing $\frac{n_F}{n_F + n_G} F + \sum_{i=1}^{n_G} \frac{1}{n_F + n_G} \emptyset$ is universally equivalent to (Definition 30) a one-sliced weighted thick and sliced graphing F' , we obtain finally that the project $(0, F')$ is an element of $\mathbf{A}$ if and only

¹²The implication $a \in \mathbf{A} \Rightarrow a + \lambda o \in \mathbf{A}$ comes from the definition of behaviors, its reciprocal is shown by noticing that $a + \lambda o - \lambda o$ is equivalent to a .

if $f \in \mathbf{A}$. We define in a similar way, being given a project g , a weighted graphing with a single slice G' such that $(0, G') \in \mathbf{B}$ if and only if $g \in \mathbf{B}$.

We are now left to show that $!(0, F') \otimes !(0, G') = !(f + g)$. By definition, the graphing of $!(0, F') \otimes !(0, G')$ is equal to $!_{\Omega} F' \uplus !_{\Omega} G'$. By definition again, the graphing of $!(f + g)$ is equal to $!_{\Omega}(F \uplus G) = !_{\Omega} F^{\iota_1} \uplus !_{\Omega} G^{\iota_2}$, where ι_1 (resp. ι_2) denotes the injection of D^F (resp. D^G) into $D^F \uplus D^G$. We now are left to notice that $!_{\Omega} F^{\iota_1} = !_{\Omega} F'$ since F^{ι_1} and F' are variants one of the other. Similarly, $!_{\Omega} G^{\iota_2} = !_{\Omega} G'$. Finally, we have that $\sharp(\mathbf{A} + \mathbf{B}) \subset \sharp\mathbf{A} \odot \sharp\mathbf{B}$ which is enough to conclude. $\square$

Lemma 98. *Let $\mathbf{A}$ be a conduct, and ϕ, ψ disjoint delocations. There exists a successful project in the conduct*

$$\mathbf{A} \multimap \phi(\mathbf{A}) \tilde{\&} \psi(\mathbf{A})$$

Proof. We define $c = \mathfrak{F}ar_{\phi} \otimes o_{\psi(V^A)} + \mathfrak{F}ar_{\psi} \otimes o_{\phi(V^A)}$. Then for all $a \in \mathbf{A}$:

$$c :: a = \phi(a) \otimes o_{\psi(V^A)} + \psi(a) \otimes o_{\phi(V^A)}$$

Thus $c \in \mathbf{A} \multimap \phi(\mathbf{A}) \tilde{\&} \psi(\mathbf{A})$. Moreover, c is obviously successful. $\square$

Proposition 99. *Let $\mathbf{A}$ be a behavior, and ϕ, ψ be disjoint delocations. There exists a successful project in the conduct*

$$?\phi(\mathbf{A}) \wp ?\psi(\mathbf{A}) \multimap ?\mathbf{A}$$

Proof. If $f \in ?\phi(\mathbf{A}) \wp ?\psi(\mathbf{A})$, then we have $f \in ?(\phi(\mathbf{A}) \tilde{\&} \psi(\mathbf{A}))$ by Proposition 97. Moreover, we have a successful project c in $\mathbf{A}^{\perp} \multimap \phi(\mathbf{A}^{\perp}) \tilde{\&} \psi(\mathbf{A}^{\perp})$ using the preceding lemma. Using the successful project implementing functorial promotion we obtain a successful project $c' \in !\mathbf{A}^{\perp} \multimap !(\phi(\mathbf{A}^{\perp}) \tilde{\&} \psi(\mathbf{A}^{\perp}))$. Thus c' is a successful project in $?\phi(\mathbf{A}) \tilde{\&} \psi(\mathbf{A}) \multimap ?\mathbf{A}$. Finally, we obtain, by composition, that $f :: c'$ is a successful project in $?\mathbf{A}$. $\square$

Corollary 100. *Let $\mathbf{A}, \mathbf{B}$ be behaviors, and ϕ, ψ be respective delocations of $\mathbf{A}$ and $\mathbf{B}$. There exists a successful project in the conduct*

$$!(\mathbf{A} \& \mathbf{B}) \multimap !\phi(\mathbf{A}) \otimes !\psi(\mathbf{B})$$

Proof. It is obtained as the interpretation of the following derivation (well formed in the sequent calculus we define later on):

$$\frac{\frac{\frac{}{\Vdash A, A^{\perp};} \text{ax}}{\Vdash A^{\perp} \oplus B^{\perp}, A} \oplus_{d,2}}{\frac{}{\Vdash A^{\perp} \oplus B^{\perp}, B} \oplus_{d,1}} \frac{\frac{}{\Vdash A, A^{\perp};} \text{ax}}{\Vdash A^{\perp} \oplus B^{\perp}, A} \oplus_{d,2} \frac{\frac{}{\Vdash B, B^{\perp};} \text{ax}}{\Vdash A^{\perp} \oplus B^{\perp}, B} \oplus_{d,1}}{\frac{}{!(A \& B) \Vdash; !A} !} \frac{\frac{}{!(A \& B) \Vdash; !B} !}{} \frac{}{!(A \& B), !(A \& B) \Vdash; !A \otimes !B} \otimes^{pol} \frac{}{!(A \& B) \Vdash; !A \otimes !B} \text{ctr}$$

The fact that it is successful is a consequence of the soundness theorem (Theorem 122). $\square$

6.2. Polarized conducts. The notions of perennial and co-perennial conducts are not completely satisfactory. In particular, we are not able to show that an implication $\mathbf{A} \multimap \mathbf{B}$ is either perennial or co-perennial when $\mathbf{A}$ is a perennial conduct (resp. co-perennial) and $\mathbf{B}$ is a co-perennial conduct (resp. perennial). This is an important issue when one considers the sequent calculus: the promotion rule has to be associated with a rule involving behaviors in order to in the setting of behaviors (using Proposition 56). Indeed, a sequent $\vdash ?\Gamma, !A$ would be interpreted by a conduct which is neither perennial nor co-perennial in general. The sequents considered are for this reason restricted to pre-sequent containing behaviors.

We will define now the notions of negative and positive conducts. The idea is to relax the notion of perennial conduct in order to obtain a notion *negative conduct*. The main interest of this approach is that positive/negative conducts will share the important properties of perennial/co-perennial conducts while interacting in a better way with connectives. In particular, we will be able to interpret the usual functorial promotion (not associated to a $\otimes$ rule), and we will be able to use the contraction rule without all the restrictions we had in the previous section.

Definition 101 (Polarized Conducts). A positive conduct $\mathbf{P}$ is a conduct satisfying the inflation property and containing all daemons:

- $p \in \mathbf{P} \Rightarrow p + \lambda o \in \mathbf{P}$;
- $\forall \lambda \in \mathbf{R} - \{0\}, \mathcal{D}ai_\lambda = (\lambda, (V^P, \emptyset)) \in \mathbf{P}$.

A conduct $\mathbf{N}$ is negative when its orthogonal $\mathbf{N}^\perp$ is a positive conduct.

Proposition 102. *A perennial conduct is negative. A co-perennial conduct is positive.*

Proof. We already showed that the perennial conducts satisfy the inflation property (Proposition 49) and contain daemons (Proposition 52). $\square$

Proposition 103. *A conduct $\mathbf{A}$ is negative if and only if:*

- $\mathbf{A}$ contains only wager-free projects;
- $a \in \mathbf{A} \Rightarrow \mathbf{1}_A \neq 0$.

Proof. If $\mathbf{A}^\perp$ is a positive conduct, then it is non-empty and satisfies the inflation property, thus $\mathbf{A}$ contains only wager-free projects by Proposition 7. As a consequence, if $a \in \mathbf{A}$, we have that $\ll a, \mathcal{D}ai_\lambda \gg_m = \lambda \mathbf{1}_A$ thus the condition $\ll a, \mathcal{D}ai \gg_m \neq 0$ implies that $\mathbf{1}_A \neq 0$.

Conversely, if $\mathbf{A}$ satisfies that stated properties, we distinguish two cases. If $\mathbf{A}$ is empty, then it is clear that $\mathbf{A}^\perp$ is a positive conduct. Otherwise, $\mathbf{A}$ is a non-empty conduct containing only wager-free projects, thus $\mathbf{A}^\perp$ satisfies the inflation property (Proposition 6). Moreover, $\ll a, \mathcal{D}ai \gg_m = \mathbf{1}_A \lambda \neq 0$ as a consequence of the second condition and therefore $\mathcal{D}ai \in \mathbf{A}^\perp$. Finally, $\mathbf{A}^\perp$ is a positive conduct, which implies that $\mathbf{A}$ is a negative conduct. $\square$

The polarized conducts do not interact very well with the connectives $\tilde{\&}$ and $\tilde{\oplus}$. Indeed, if $\mathbf{A}, \mathbf{B}$ are negative conducts, the conduct $\mathbf{A} \tilde{\&} \mathbf{B}$ is generated by a set of wager-free projects, but it does not satisfy the second property needed to be a negative conduct. Similarly, if $\mathbf{A}, \mathbf{B}$ are positive conducts, then $\mathbf{A} \tilde{\&} \mathbf{B}$ will obviously have the inflation property, but it will contain the project $\mathcal{D}ai_0$ (which implies that any element c in its orthogonal is such that $\mathbf{1}_C = 0$). We are also not able to characterize in any way the conduct $\mathbf{A} \tilde{\&} \mathbf{B}$ when $\mathbf{A}$ is a positive conduct and $\mathbf{B}$ is a negative conduct, except that it has the inflation property. However, the notions of positive and negative conducts interacts in a nice way with the connectives $\otimes, \&, \wp, \oplus$.

Proposition 104. *The tensor product of negative conducts is a negative conduct. The $\&$ of negative conducts is a negative conduct. The $\oplus$ of negative conducts is a negative conduct.*

Proof. We know that $\mathbf{A} \otimes \mathbf{B} = \emptyset$ if one of the two conducts $\mathbf{A}$ and $\mathbf{B}$ is empty, which leaves us to treat the non-empty case. In this case, $\mathbf{A} \otimes \mathbf{B} = (\mathbf{A} \odot \mathbf{B})^{\perp\perp}$ is the bi-orthogonal of a non-empty set of wager-free projects. Thus $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ satisfies the inflation property. Moreover $\ll a \otimes b, \mathcal{D}ai \gg_m = \mathbf{1}_B \mathbf{1}_A \lambda$ which is different from zero since $\mathbf{1}_A, \mathbf{1}_B$ both are different from zero. Thus $\mathcal{D}ai \in (\mathbf{A} \otimes \mathbf{B})^{\perp}$, which shows that $\mathbf{A} \otimes \mathbf{B}$ is a negative conduct since $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ is a positive conduct.

The set $\mathbf{A}^{\perp} \upharpoonright_B$ contains all daemons $\mathcal{D}ai_{\lambda} \otimes o = \mathcal{D}ai_{\lambda}$, and $\mathcal{D}ai \in \mathbf{A}^{\perp}$. It has the inflation property since $(b + \lambda o) \otimes o = b \otimes o + \lambda o$. Thus $((\mathbf{A}^{\perp}) \upharpoonright_B)^{\perp}$ is a negative conduct. Similarly, $((\mathbf{B}^{\perp}) \upharpoonright_A)^{\perp}$ is a negative conduct, and their intersection is a negative conduct since the properties defining negative conducts are preserved by intersection. As a consequence, $\mathbf{A} \& \mathbf{B}$ is a negative conduct.

In the case of $\oplus$, we will use the fact that $\mathbf{A} \oplus \mathbf{B} = (\mathbf{A} \upharpoonright_B \cup \mathbf{B} \upharpoonright_A)^{\perp}$. If $a \in \mathbf{A}$, $a \otimes o = b$ has a null wager and $\mathbf{1}_B = \mathbf{1}_A \neq 0$. If $\mathbf{A}$ is empty, $(\mathbf{A} \upharpoonright_B)^{\perp}$ is a positive conduct. If $\mathbf{A}$ is non-empty, then Proposition 6 allows us to state that $(\mathbf{A} \upharpoonright_B)^{\perp}$ has the inflation property. Moreover, the fact that all elements in $a \otimes o = b$ satisfy $\mathbf{1}_B \neq 0$ implies that $\mathcal{D}ai_{\lambda} \in (\mathbf{A} \upharpoonright_B)^{\perp}$ for all $\lambda \neq 0$. Therefore, $(\mathbf{A} \upharpoonright_B)^{\perp}$ is a positive conduct. As a consequence, $\mathbf{A} \upharpoonright_B$ is a negative conduct. We show in a similar way that $\mathbf{B} \upharpoonright_A$ is a negative conduct. We can deduce from this that $\mathbf{A} \upharpoonright_B \cup \mathbf{B} \upharpoonright_A$ contains only projects c with zero wager and such that $\mathbf{1}_C \neq 0$. Finally, we showed that $\mathbf{A} \oplus \mathbf{B}$ is a negative conduct. $\square$

Corollary 105. *The $\mathfrak{V}$ of positive conducts is a positive conduct, the $\&$ of positive conducts is a positive conduct, and the $\oplus$ of positive conducts is a positive conduct.*

Proposition 106. *Let $\mathbf{A}$ be a positive conduct and $\mathbf{B}$ be a negative conduct. Then $\mathbf{A} \otimes \mathbf{B}$ is a positive conduct.*

Proof. Pick $f \in (\mathbf{A} \otimes \mathbf{B})^{\perp} = \mathbf{B} \multimap \mathbf{A}^{\perp}$. Then for all $b \in \mathbf{B}$, $f :: b = (\mathbf{1}_B f + \mathbf{1}_F b, F :: B)$ is an element of $\mathbf{A}^{\perp}$. Since $\mathbf{A}^{\perp}$ is a negative conduct, we have that $\mathbf{1}_F \mathbf{1}_B \neq 0$ and $\mathbf{1}_B f + \mathbf{1}_F b = 0$. Thus $\mathbf{1}_F \neq 0$. Moreover, $\mathbf{B}$ is a negative conduct, therefore $\mathbf{1}_B \neq 0$ and $b = 0$. The condition $\mathbf{1}_B f + \mathbf{1}_F b = 0$ then becomes $\mathbf{1}_B f = 0$, i.e. $f = 0$.

Thus $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ is a negative conduct, which implies that $\mathbf{A} \otimes \mathbf{B}$ is a positive conduct. $\square$

Corollary 107. *If $\mathbf{A}$ is a positive conduct and $\mathbf{B}$ is a positive conduct, $\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp}$ is a positive conduct.*

Corollary 108. *If $\mathbf{A}, \mathbf{B}$ are negative conducts, then $\mathbf{A} \multimap \mathbf{B}$ is a negative conduct.*

Proof. We know that $\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp}$. We also just showed that $\mathbf{A} \otimes \mathbf{B}^{\perp}$ is a positive conduct, thus $\mathbf{A} \multimap \mathbf{B}$ is a negative conduct. $\square$

Proposition 109. *The tensor product of a negative conduct and a behavior is a behavior.*

Proof. Let $\mathbf{A}$ be a negative conduct and $\mathbf{B}$ be a behavior. If either $\mathbf{A}$ or $\mathbf{B}$ is empty (or both), $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ equals $\mathbf{T}_{V^A \cup V^B}$ and we are done. We now suppose that $\mathbf{A}$ and $\mathbf{B}$ are both non empty.

Since $\mathbf{A}, \mathbf{B}$ contain only wager-free projects, the set $\{a \otimes b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$ contains only wager-free projects. Thus $(\mathbf{A} \otimes \mathbf{B})^{\perp}$ has the inflation property: this is a consequence of Proposition 6. Suppose now that there exists $f \in (\mathbf{A} \otimes \mathbf{B})^{\perp}$ such that $f \neq 0$. Chose $a \in \mathbf{A}$ and $b \in \mathbf{B}$. Then $\ll f, a \otimes b \gg_m = f \mathbf{1}_B \mathbf{1}_A + \ll F, A :: B \gg_m$. Since $\mathbf{1}_A \neq 0$, we can define $\mu = -\ll F, A \cup B \gg_m / (\mathbf{1}_A f)$,

$\frac{\otimes \parallel N \mid P}{N \parallel N \mid P}$	$\frac{\wp \parallel N \mid P}{N \parallel ? \mid N}$	$\frac{\& \parallel N \mid P}{N \parallel N \mid ?}$	$\frac{\oplus \parallel N \mid P}{N \parallel N \mid ?}$
$\frac{P \parallel P \mid ?}{P \parallel P \mid ?}$	$\frac{P \parallel N \mid P}{P \parallel N \mid P}$	$\frac{P \parallel ? \mid P}{P \parallel ? \mid P}$	$\frac{P \parallel ? \mid P}{P \parallel ? \mid P}$
(a) Tenseur	(b) Parr	(c) Avec(1)	(d) Plus(1)

Figure 20: Connectives and Polarization

and $b + \mu o \in \mathbf{B}$ since $\mathbf{B}$ has the inflation property. We then have:

$$\begin{aligned}
\ll f, a \otimes (b + \mu o) \gg_m &= f \mathbf{1}_A (\mathbf{1}_B + \mu) + \ll F, A :: (B + \mu o) \gg_m \\
&= f \mathbf{1}_A \frac{-\ll F, A \cup B \gg_m}{\mathbf{1}_A f} + \ll F, A :: B \gg_m \\
&= 0
\end{aligned}$$

This is a contradiction, since $f \in (\mathbf{A} \otimes \mathbf{B})^\perp$. Thus $f = 0$.

Finally, we have shown that $(\mathbf{A} \otimes \mathbf{B})^\perp$ has the inflation property and contains only wager-free projects. $\square$

Corollary 110. *If $\mathbf{A}$ is a negative conduct and $\mathbf{B}$ is a behavior, $\mathbf{A} \multimap \mathbf{B}$ is a behavior.*

Proposition 111. *The weakening (on the left) of negative conducts holds.*

Proof. Let $\mathbf{A}, \mathbf{B}$ be conducts, $\mathbf{N}$ be a negative conduct, and pick $f \in \mathbf{A} \multimap \mathbf{B}$. We will show that $f \otimes o_{V^N}$ is an element of $\mathbf{A} \otimes \mathbf{N} \multimap \mathbf{B}$. For this, we pick $a \in \mathbf{A}$ and $n \in \mathbf{N}$. Then for all $b' \in \mathbf{B}^\perp$,

$$\begin{aligned}
\ll (f \otimes o) :: (a \otimes n), b' \gg_m &= \ll f \otimes o, (a \otimes n) \otimes b' \gg_m \\
&= \ll f \otimes o, (a \otimes b') \otimes n \gg_m \\
&= \mathbf{1}_F (\mathbf{1}_A \mathbf{1}_{B'} n + \mathbf{1}_N \mathbf{1}_A b' + \mathbf{1}_N \mathbf{1}_{B'} a) + \mathbf{1}_N \mathbf{1}_A \mathbf{1}_{B'} f + \ll F \cup 0, A \cup B' \cup N \gg_m \\
&= \mathbf{1}_F (\mathbf{1}_N \mathbf{1}_A b' + \mathbf{1}_N \mathbf{1}_{B'} a) + \mathbf{1}_N \mathbf{1}_A \mathbf{1}_{B'} f + \ll F \cup 0, A \cup B' \cup N \gg_m \\
&= \mathbf{1}_N (\mathbf{1}_F (\mathbf{1}_A b' + \mathbf{1}_{B'} a) + \mathbf{1}_A \mathbf{1}_{B'} f) + \mathbf{1}_N \ll F, A \cup B' \gg_m \\
&= \mathbf{1}_N \ll f, a \otimes b' \gg_m
\end{aligned}$$

Since $\mathbf{1}_N \neq 0$, $\ll (f \otimes o) :: (a \otimes n), b' \gg_m \neq 0, \infty$ if and only if $\ll f :: a, b' \gg_m \neq 0, \infty$. Therefore, for all $a \otimes n \in \mathbf{A} \otimes \mathbf{N}$, $(f \otimes o) :: (a \otimes n) \in \mathbf{B}$. This shows that $f \otimes o$ is an element of $\mathbf{A} \otimes \mathbf{N} \multimap \mathbf{B}$ by Proposition 37. $\square$

6.3. Sequent Calculus and Soundness. We now describe a sequent calculus which is much closer to the usual sequent calculus for Elementary Linear Logic. We introduce once again three types of formulas: (B)ehaviors, (P)ositive, (N)egative. The sequents we will be working with will be the equivalent to the notion of pre-sequent introduced earlier.

Definition 112. We once again define three types of formulas — (B)ehavior, (P)ositive, (N)egative — by the following grammar:

$$\begin{aligned}
B &:= X \mid X^\perp \mid \mathbf{0} \mid \mathbf{T} \mid B \otimes B \mid B \wp B \mid B \oplus B \mid B \& B \mid \forall X B \mid \exists X B \mid B \otimes N \mid B \wp P \\
N &:= \mathbf{1} \mid !B \mid !N \mid N \otimes N \mid N \& N \mid N \oplus N \mid N \wp P \\
P &:= \perp \mid ?B \mid ?P \mid P \wp P \mid P \& P \mid P \oplus P \mid N \otimes P
\end{aligned}$$

Definition 113. A sequent $\Delta \Vdash \Gamma; \Theta$ is such that Δ, Θ contain only negative formulas, Θ containing at most one formula and Γ containing only behaviors.

Definition 114 (The System ELL_{pol}). A proof in the system ELL_{pol} is a derivation tree constructed from the derivation rules shown in Figure 21.

Remark 115. Even though one can consider the conduct $\mathbf{A} \& \mathbf{B}$ when $\mathbf{A}, \mathbf{B}$ are negative conducts, no rule of the sequent calculus ELL_{pol} allows one to construct such a formula. The reason for that is simple: since in this case the set $\mathbf{A} + \mathbf{B}$ is not necessarily included in the conduct $\mathbf{A} \& \mathbf{B}$, one cannot interpret the rule in general (since distributivity does not necessarily holds). The latter can be interpreted when the context contains at least one behavior, but imposing such a condition on the rule could lead to difficulties when considering the cut-elimination procedure (in case of commutations). We therefore choose to work with a system in which one introduces additive connectives only between behaviors. Notice however that a formula built with an additive connective between negative sub-formulas can still be introduced by a weakening rule.

The following proposition is obtained easily by standard proof techniques.

Proposition 116. *The system ELL_{pol} possesses a cut-elimination procedure.*

We now define the interpretation of the formulas and proofs of the localized sequent calculus in the model $\mathbb{M}[\Omega, \text{mi}]_m$.

Definition 117. We fix $\mathcal{V} = \{X_i(j)\}_{i,j \in \mathbf{N} \times \mathbf{Z}}$ a set of *localized variables*. For $i \in \mathbf{N}$, the set $X_i = \{X_i(j)\}_{j \in \mathbf{Z}}$ will be referred to as *the name of the variable X_i* , and an element of X_i will be referred to as a *variable of name X_i* .

For $i, j \in \mathbf{N} \times \mathbf{Z}$ we define the *location* $\#X_i(j)$ of the variable $X_i(j)$ as the set

$$\{x \in \mathbf{R} \mid 2^i(2j+1) \leq m < 2^i(2j+1)+1\}$$

Definition 118 (Formulas of $\text{locELL}_{\text{pol}}$). We inductively define the formulas of $\text{locELL}_{\text{pol}}$ together with their *locations* as follows:

• **Behaviors:**

- A variable $X_i(j)$ of name X_i is a behavior whose location is defined as $\#X_i(j)$;
- If $X_i(j)$ is a variable of name X_i , then $(X_i(j))^\perp$ is a behavior of location $\#X_i(j)$.
- The constants $\mathbf{T}_{\# \Gamma}$ are behaviors of location $\# \Gamma$;
- The constants $\mathbf{0}_{\# \Gamma}$ are behaviors of location $\# \Gamma$.
- If A, B are behaviors of respective locations X, Y such that $X \cap Y = \emptyset$, then $A \otimes B$ (resp. $A \wp B$, resp. $A \& B$, resp. $A \oplus B$) is a behavior of location $X \cup Y$;
- If X_i is a variable name, and $A(X_i)$ is a behavior of location $\#A$, then $\forall X_i A(X_i)$ and $\exists X_i A(X_i)$ are behaviors of location $\#A$.
- If A is a negative conduct of location X and B is a behavior of location Y such that $X \cap Y = \emptyset$, then $A \otimes B$ is a behavior of location $X \cup Y$;
- If A is a positive conduct of location X and B is a behavior of location Y such that $X \cap Y = \emptyset$, then $A \wp B$ is a behavior of location $X \cup Y$;

• **Negative Conducts:**

- The constant $\mathbf{1}$ is a negative conduct;
- If A is a behavior or a negative conduct of location X , then $!A$ is a negative conduct of location $\Omega(X \times [0, 1])$;

$$\overline{\Vdash B^\perp, B;}^{\text{ax}}$$

$$\frac{\Delta_1 \Vdash \Gamma_1; N \quad \Delta_2, N \Vdash \Gamma_2; \Theta}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2; \Theta} \text{cut}^{pol} \quad \frac{\Delta_1 \Vdash \Gamma_1, B; \Theta \quad \Delta_2 \Vdash \Gamma_2, B^\perp; \Theta}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2; \Theta} \text{cut}$$

(a) Identity Group

$$\frac{\Delta_1 \Vdash \Gamma_1, B_1; \Theta \quad \Delta_2 \Vdash \Gamma_2, B_2; \Theta}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2, B_1 \otimes B_2; \Theta} \otimes \quad \frac{\Delta \Vdash \Gamma, B_1, B_2; \Theta}{\Delta \Vdash \Gamma, B_1 \wp B_2; \Theta} \wp$$

$$\frac{\Delta, N_1, N_2 \Vdash \Gamma; \Theta}{\Delta, N_1 \otimes N_2 \Vdash \Gamma; \Theta} \otimes_g^{pol} \quad \frac{\Delta_1 \Vdash \Gamma_1; N_1 \quad \Delta_2 \Vdash \Gamma_2; N_2}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2; N_1 \otimes N_2} \otimes_d^{pol}$$

$$\frac{\Delta, P_1^\perp \Vdash \Gamma; N_2}{\Delta \Vdash \Gamma; P_1 \wp N_2} \wp_d^{pol} \quad \frac{\Delta_1 \Vdash \Gamma_1; P_1^\perp \quad \Delta_2, N_2 \Vdash \Gamma_2; \Theta}{\Delta_1, \Delta_2, P_1 \wp N_2 \Vdash \Gamma_1, \Gamma_2; \Theta} \wp_g^{pol}$$

$$\frac{\Delta, P^\perp \Vdash \Gamma, B; \Theta}{\Delta \Vdash \Gamma, P \wp B; \Theta} \wp^{mix} \quad \frac{\Delta_1 \Vdash \Gamma_1; N \quad \Delta_2 \Vdash \Gamma_2, B; \Theta}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2, N \otimes B; \Theta} \otimes^{mix}$$

$$\overline{\Vdash; \mathbf{1}}^{\mathbf{1}_d} \quad \frac{\Delta \Vdash \Gamma; \Theta}{\Delta, \mathbf{1} \Vdash \Gamma; \Theta} \mathbf{1}_g$$

(b) Multiplicative Group

$$\frac{\Delta \Vdash \Gamma, B_i; \Theta}{\Delta \Vdash \Gamma, B_1 \oplus B_2; \Theta} \oplus_i \quad \frac{\Delta \Vdash \Gamma, B_1; \Theta \quad \Delta \Vdash \Gamma, B_2; \Theta}{\Delta \Vdash \Gamma, B_1 \& B_2; \Theta} \&$$

$$\overline{\Delta \Vdash \Gamma, \top; \Theta}^\top \quad \text{No rules for } 0.$$

(c) Additive Group

$$\frac{\Delta \Vdash \Gamma; N}{!\Delta, !\Gamma^\perp \Vdash; !N} !^{pol} \quad \frac{\Delta \Vdash \Gamma, B; \Theta}{!\Delta, !\Gamma^\perp \Vdash; !B} !$$

$$\frac{\Delta, !B, !B \Vdash \Gamma; \Theta}{\Delta, !B \Vdash \Gamma; \Theta} \text{ctr (B Behavior)} \quad \frac{\Delta \Vdash \Gamma; \Theta}{\Delta, N \Vdash \Gamma; \Theta} \text{weak}$$

(d) Exponential Group

$$\frac{\Delta \Vdash \Gamma, C; \Theta \quad X \notin \text{FV}(\Gamma, \Delta, \Theta)}{\Delta \Vdash \Gamma, \forall X C; \Theta} \forall \quad \frac{\Delta \Vdash \Gamma, C[A/X]; \Theta}{\Delta \Vdash \Gamma, \exists X C; \Theta} \exists$$

(e) Quantifier Group

Figure 21: Sequent Calculus ELL_{pol}

- If A, B are negative conducts of locations X, Y such that $X \cap Y = \emptyset$, then $A \otimes B$ (resp. $A \oplus B$, resp. $A \& B$) is a negative conduct of location $X \cup Y$;
- If A is a negative conduct of location X and B is a positive conduct of location Y , $A \wp B$ is a negative conduct of location $X \cup Y$.
- **Positive Conducts:**
 - The constant $\perp$ is a positive conduct;
 - If A is a behavior or a positive conduct of location X , then $?A$ is a positive conduct of location $\Omega(X \times [0, 1])$;
 - If A, B are positive conducts of locations X, Y such that $X \cap Y = \emptyset$, then $A \wp B$ (resp. $A \& B$, resp. $A \oplus B$) is a positive conduct of location $X \cup Y$;
 - If A is a negative conduct of location X and B is a positive conduct of location Y , $A \otimes B$ is a positive conduct of location $X \cup Y$.

If A is a formula, we will denote by $\sharp A$ its location. We also define sequents $\Delta \Vdash \Gamma; \Theta$ of $\text{locELL}_{\text{pol}}$ when:

- formulas in $\Gamma \cup \Delta \cup \Theta$ have pairwise disjoint locations;
- formulas in Δ and Θ are negative conducts;
- there is at most one formula in Θ ;
- Γ contains only behaviors.

Definition 119 (Interpretations). We define an *interpretation basis* as a function Φ which maps every variable name X_i to a behavior of carrier $[0, 1[$.

Definition 120 (Interpretation of $\text{locELL}_{\text{pol}}$ formulas). Let Φ be an interpretation basis. We define the interpretation $I_\Phi(F)$ along Φ of a formula F inductively:

- If $F = X_i(j)$, then $I_\Phi(F)$ is the delocation (i.e. a behavior) of $\Phi(X_i)$ along the function $x \mapsto 2^i(2j+1) + x$;
- If $F = (X_i(j))^\perp$, we define the behavior $I_\Phi(F) = (I_\Phi(X_i(j)))^\perp$;
- If $F = \mathbf{T}_{\sharp\Gamma}$ (resp. $F = \mathbf{0}_{\sharp\Gamma}$), we define $I_\Phi(F)$ as the behavior $\mathbf{T}_{\sharp\Gamma}$ (resp. $\mathbf{0}_{\sharp\Gamma}$);
- If $F = \mathbf{1}$ (resp. $F = \perp$), we define $I_\Phi(F)$ as the behavior $\mathbf{1}$ (resp. $\perp$);
- If $F = A \otimes B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \otimes I_\Phi(B)$;
- If $F = A \wp B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \wp I_\Phi(B)$;
- If $F = A \oplus B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \oplus I_\Phi(B)$;
- If $F = A \& B$, we define the conduct $I_\Phi(F) = I_\Phi(A) \& I_\Phi(B)$;
- If $F = \forall X_i A(X_i)$, we define the conduct $I_\Phi(F) = \forall \mathbf{X}_i I_\Phi(A(X_i))$;
- If $F = \exists X_i A(X_i)$, we define the conduct $I_\Phi(F) = \exists \mathbf{X}_i I_\Phi(A(X_i))$;
- If $F = !A$ (resp. $?A$), we define the conduct $I_\Phi(F) = !I_\Phi(A)$ (resp. $?I_\Phi(A)$).

Moreover a sequent $\Delta \vdash \Gamma; \Theta$ will be interpreted as the $\wp$ of the formulas in Γ and Θ and the negations of formulas in Δ , which we will write $\wp \Delta^\perp \wp \wp \Gamma \wp \wp \Theta$. We will also represent this formula by the equivalent formula $\otimes \Delta \multimap (\wp \Gamma \wp \wp \Theta)$.

Definition 121 (Interpretation of $\text{locELL}_{\text{pol}}$ proofs). Let Φ be an interpretation basis. We define the interpretation $I_\Phi(\pi)$ — a project — of a proof π inductively:

- if π consists in an axiom rule introducing $\vdash (X_i(j))^\perp, X_i(j')$, we define $I_\Phi(\pi)$ as the project $\wp \wp$ defined by the translation $x \mapsto 2^i(2j' - 2j) + x$;
- if π consists solely in a $\mathbf{T}_{\sharp\Gamma}$ rule, we define $I_\Phi(\pi) = \mathbf{0}_{\sharp\Gamma}$;
- if π consists solely in a $\mathbf{1}_d$ rule, we define $I_\Phi(\pi) = \mathbf{0}_\emptyset$;
- if π is obtained from π' by a $\wp$ rule, a $\otimes_g^{\text{pol}}$ rule, a $\wp_d^{\text{pol}}$ rule, a $\wp^{\text{mix}}$ rule, or a $\mathbf{1}_g$ rule, then $I_\Phi(\pi) = I_\Phi(\pi')$;

- if π is obtained from π_1 and π_2 by applying a $\otimes$ rule, a $\otimes_d^{pol}$ rule, a $\wp_g^{pol}$ rule or a $\otimes^{mix}$ rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \otimes I_\Phi(\pi_2)$;
- if π is obtained from π' by a weak rule or a $\oplus_i$ rule introducing a formula of location V , we define $I_\Phi(\pi) = I_\Phi(\pi') \otimes \circ_V$;
- if π of conclusion $\vdash \Gamma, A_0 \& A_1$ is obtained from π_0 and π_1 by applying a $\&$ rule, we define the interpretation of π as it was done in our earlier paper [Sei14a];
- If π is obtained from a $\forall$ rule applied to a derivation π' , we define $I_\Phi(\pi) = I_\Phi(\pi')$;
- If π is obtained from a $\exists$ rule applied to a derivation π' replacing the formula $\mathbf{A}$ by the variable name X_i , we define $I_\Phi(\pi) = I_\Phi(\pi') :: (\otimes[e^{-1}(j) \leftrightarrow X_i(j)])$, using the notations of our previous paper [Sei14c] for the *measure-inflating faces* $[e^{-1}(j) \leftrightarrow X_i(j)]$ where e is an enumeration of the occurrences of $\mathbf{A}$ in π' ;
- if π is obtained from π' by applying a promotion rule $!$ or $!^{pol}$, we apply the implementation of the functorial promotion rule to the project $!I_\Phi(\pi')$ $n - 1$ times, where n is the number of formulas in the sequent;
- if π is obtained from π' by applying a contraction rule ctr , we define the interpretation of π as the execution between the interpretation of π' and the project implementing contraction described in Proposition 79;
- if π is obtained from π_1 and π_2 by applying a cut rule or a cut^{pol} rule, we define $I_\Phi(\pi) = I_\Phi(\pi_1) \dot{\cap} I_\Phi(\pi_2)$.

Once again, one can choose an enumeration e of the occurrences of variables in order to "localize" any formula A and any proof π of ELL_{pol} : and define formulas A^e and proofs π^e of $\text{locELL}_{\text{pol}}$. One easily shows a soundness result for the localized calculus $\text{locELL}_{\text{pol}}$ which implies the following result.

Theorem 122. *Let Φ be an interpretation basis, π a proof of ELL_{pol} of conclusion $\Delta \Vdash \Gamma; \Theta$, and an enumeration of the occurrences of variables in the axioms of π . Then $I_\Phi(\pi^e)$ is a successful project in $I_\Phi(\Delta^e \vdash \Gamma^e; \Theta^e)$.*

7. CONCLUSION AND PERSPECTIVES

In this paper, we extended the setting of Interaction Graphs in order to deal with all connectives of linear logic. We showed how one can obtain a soundness result for two versions of Elementary Linear Logic. The first system, which is conceived so that the interpretation of sequents are behaviors, seems to lack expressivity and it may appear that elementary functions cannot be typed in this system. The second system, however, is very closed to usual ELL sequent calculus, and, even though one should prove it, the proofs of type $!nat \multimap nat$ to itself seem to correspond to elementary functions from natural numbers to natural numbers, as it is the case with traditional Elementary Linear Logic [DJ03].

Though the generalization from graphs to graphings may seem a big effort, we believe the resulting framework to be extremely interesting. We should stress that with little work on the definition of exponentials, one should be able to show that interpretations of proofs can be described by finite means. Indeed, the only operation that seems to turn an interval into an infinite number of intervals is the promotion rule. One should however be able to show that, up to a suitable delocation, the promotion of a project defined on a finite number of rational intervals is defined on a finite number of rational intervals.

Another interesting perspective would consist in considering continuous dialects in addition to discrete ones. All the definitions and properties of thick and sliced graphings obviously

hold in this setting and one can obtain all the results described in this paper, although no finite description of projects could be expected in this case. The question of whether we would gain some expressivity by extending the framework in this way is still open. We believe that it may be a way to obtain more expressive exponentials, such as the usual exponentials of linear logic.

More generally, now that this framework has been defined and that we have shown its interest by providing a construction for *elementary exponentials*, we believe the definition and study of other exponential connectives may be a work of great interest. First, these new exponentials would co-exist with each other, making it possible to study their interactions. Secondly, even if the definition of exponentials for full linear logic may be a complicated task, the definition of low-complexity exponentials may be of great interest.

Finally, we explained in our previous paper how the systematic construction of models of linear logic based on graphings [Sei14c] give rise to a hierarchy of models mirroring subtle distinctions concerning computational principles. In particular, it gives rise to a hierarchy of models characterizing complexity classes [Sei14b] by adapting results obtained using operator theory [AS12, AS13]. The present work will lead to characterizations of larger complexity classes such as **Ptime** or **Exptime** predicates and/or functions, following the work of Baillot [Bai11].

REFERENCES

- [AS12] Clément Aubert and Thomas Seiller. Characterizing co-nl by a group action. *CoRR*, abs/1209.3422, 2012.
- [AS13] Clément Aubert and Thomas Seiller. Logarithmic space and permutations. *CoRR*, abs/1301.3189, 2013.
- [Bai11] Patrick Baillot. Elementary linear logic revisited for polynomial time and an exponential time hierarchy. In Hongseok Yang, editor, *APLAS*, volume 7078 of *Lecture Notes in Computer Science*, pages 337–352. Springer, 2011.
- [DJ03] Vincent Danos and Jean-Baptiste Joinet. Linear logic & elementary time. *Information and Computation*, 183(1):123–137, 2003.
- [FK52] Bent Fuglede and Richard V. Kadison. Determinant theory in finite factors. *Annals of Mathematics*, 56(2), 1952.
- [Gir87] Jean-Yves Girard. Multiplicatives. In Lolli, editor, *Logic and Computer Science : New Trends and Applications*, pages 11–34, Torino, 1987. Università di Torino. Rendiconti del seminario matematico dell’università e politecnico di Torino, special issue 1987.
- [Gir88] Jean-Yves Girard. Geometry of interaction II: Deadlock-free algorithms. In *Proc. of COLOG, 1988, LNCS 417*, pages 76–93. Springer, 1988.
- [Gir89a] Jean-Yves Girard. Geometry of interaction I: Interpretation of system F. In *In Proc. Logic Colloquium 88*, 1989.
- [Gir89b] Jean-Yves Girard. Towards a geometry of interaction. In *Proceedings of the AMS Conference on Categories, Logic and Computer Science*, 1989.
- [Gir95a] Jean-Yves Girard. Geometry of interaction III: Accommodating the additives. In *Advances in Linear Logic*, number 222 in Lecture Notes Series, pages 329–389. Cambridge University Press, 1995.
- [Gir95b] Jean-Yves Girard. Light linear logic. In *Selected Papers from the International Workshop on Logical and Computational Complexity, LCC ’94*, pages 145–176, London, UK, UK, 1995. Springer-Verlag.
- [Gir11] Jean-Yves Girard. Geometry of interaction V: Logic in the hyperfinite factor. *Theoretical Computer Science*, 412:1860–1883, 2011.
- [Kri01] Jean-Louis Krivine. Typed lambda-calculus in classical zermelo-fraenkel set theory. *Archive for Mathematical Logic*, 40(3):189–205, 2001.
- [Kri09] Jean-Louis Krivine. Realizability in classical logic. *Panoramas et synthèses*, 27:197–229, 2009.
- [Sei12a] Thomas Seiller. Interaction graphs: Multiplicatives. *Annals of Pure and Applied Logic*, 163:1808–1837, December 2012.
- [Sei12b] Thomas Seiller. *Logique dans le facteur hyperfini : géométrie de l’interaction et complexité*. PhD thesis, Université de la Méditerranée, 2012.

- [Sei14a] Thomas Seiller. Interaction graphs: Additives. *Accepted for publication in Annals of Pure and Applied Logic*, 2014.
- [Sei14b] Thomas Seiller. Interaction graphs and complexity. Extended Abstract, 2014.
- [Sei14c] Thomas Seiller. Interaction graphs: Graphings. *CoRR*, abs/1405.6331, 2014.