

Geometry of Interaction I

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Proof nets

- ▶ Recall that a *proof structure* is a directed multigraph with edges labelled by formulas and vertices labelled by elements in the set $\{\text{ax}, \text{cut}, \text{c}, \otimes, \wp\}$. A *proof net* is a proof structure which is a translation of a MLL sequent calculus proof, or, equivalently, a proof structure that satisfies any correctness criterion.
- ▶ In the following we will only consider η -expanded proof nets, i.e. proof nets where the outgoing edges of axiom vertices are labelled by atoms.

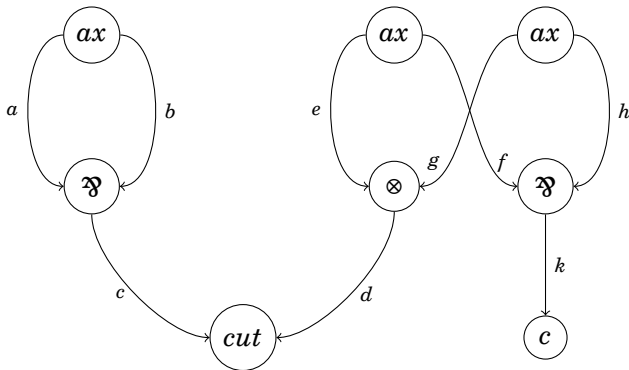
Paths in proof nets

- ▶ We will be interested in paths in proof nets. For this, we will consider *reverted edges*: if e of source s and target t is an edge in a graph, we define the reverted edge e^* of source t and target s . The label of a reverted edge s^* is the same as the label of the edge e . The edge $(e^*)^*$ is defined to be equal to e .

Definition

A path $(e_i)_{i=0}^n$ in a proof net $\mathcal{R}$ is a sequence of edges and reverted edges of $\mathcal{R}$ such that the source of e_i is equal to the target of e_{i+1} .

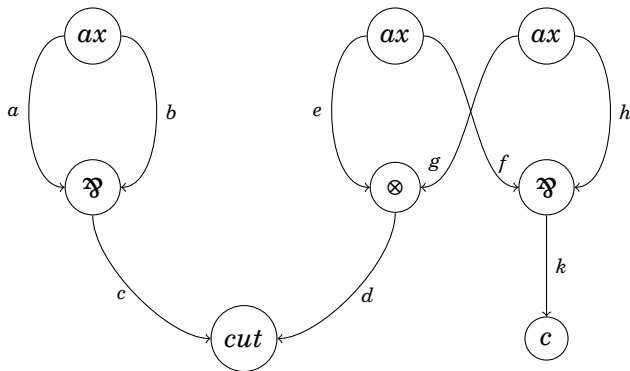
Paths in proof nets



- Path (bouncing):

$$d^* def^*$$

Paths in proof nets



- Path (twisting):

$$hg^*ef^*k^*$$

Straight paths

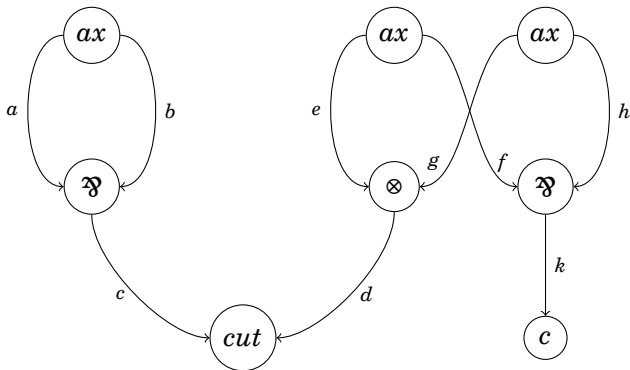
Definition

A path $(e_i)_{i=0}^n$ in $\mathcal{R}$ is *straight* if it is:

- ▶ *non-bouncing*: if for all $0 \leq i \leq n-1$, $e_{i+1} \neq e_i^*$;
- ▶ *non-twisting*: if it does not contain $\alpha^* \beta$ where α, β are distinct premises of a $\otimes$ or $\wp$ vertex;

We define $\mathcal{P}(\mathcal{R})$ to be the set of conclusion-to-conclusion straight paths in $\mathcal{R}$ to which we add a formal 0 element.

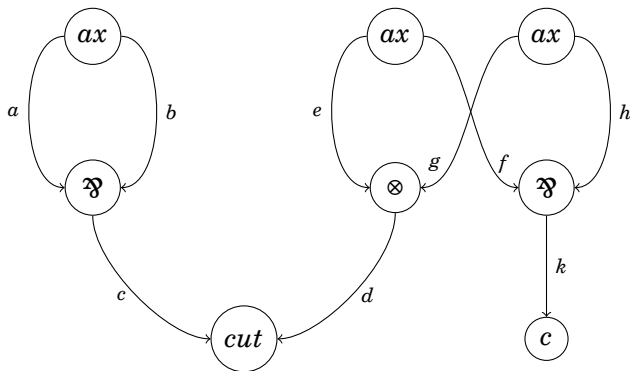
Straight Paths



- Path (straight):

$$e^* d^* c b a^*$$

Straight Paths



- Path (straight, conclusion-to-conclusion):

$$kfg^*d^*cba^*c^*deh^*k^*$$

Cut-Elimination

We first recall the cut-elimination steps on proof nets in order to introduce notations that will be used in the next paragraph. In our setting of η -expanded MLL proof nets, there are only two cut-eliminations steps, the $ax - ax$ -step and the $\otimes - \wp$ -step.

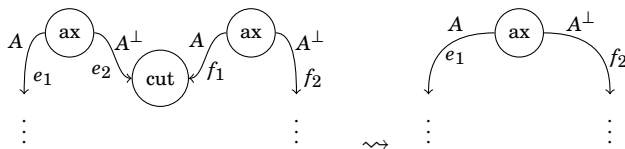


Figure : Cut-elimination: the $ax - ax$ -step

Cut-Elimination

We first recall the cut-elimination steps on proof nets in order to introduce notations that will be used in the next paragraph. In our setting of η -expanded MLL proof nets, there are only two cut-eliminations steps, the $ax - ax$ -step and the $\otimes - \wp$ -step.

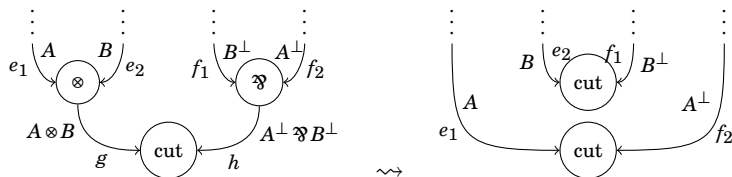
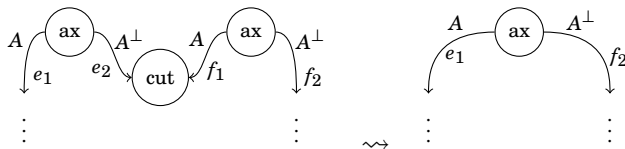


Figure : Cut-elimination: the $\otimes - \wp$ -step

Cut-Elimination II

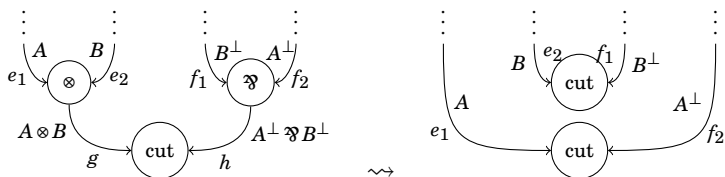
For a cut-elimination step $\rho : \mathcal{R} \rightarrow \mathcal{R}'$, we can define a map φ_ρ from $\mathcal{P}(\mathcal{R})$ to $\mathcal{P}(\mathcal{R}')$ as follows:



- Ax-Ax steps.** If the path p contains the sequence $e_1^* e_2 f_1^* f_2$ (resp. $f_2^* f_1 e_2^* e_1$), $\varphi_\rho(p)$ is the path p where this sequence is replaced by $e_1^* f_2$ (resp. $f_2^* e_1$). If p does not contain any of e_1, e_2, f_1, f_2 or their reverse, then $\varphi_\rho(p) = p$.

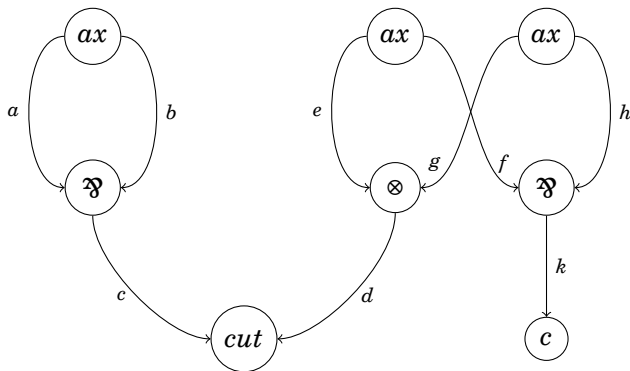
Cut-Elimination II

For a cut-elimination step $\rho : \mathcal{R} \rightarrow \mathcal{R}'$, we can define a map φ_ρ from $\mathcal{P}(\mathcal{R})$ to $\mathcal{P}(\mathcal{R}')$ as follows:



- **$\otimes - \wp$ steps.** If the path p contains the sequence $e_1 g h^* f_1^*$ (resp. $e_2 g h^* f_2^*$, resp. $f_1 h g^* e_1^*$, resp. $f_2 h g^* e_2^*$), $\varphi_\rho(p)$ is the path p where this sequence is replaced by $e_1 f_1^*$ (resp. $e_2 f_2^*$, resp. $f_1 e_1^*$, resp. $f_2 e_2^*$). In all other cases, $\varphi_\rho(p) = 0$.

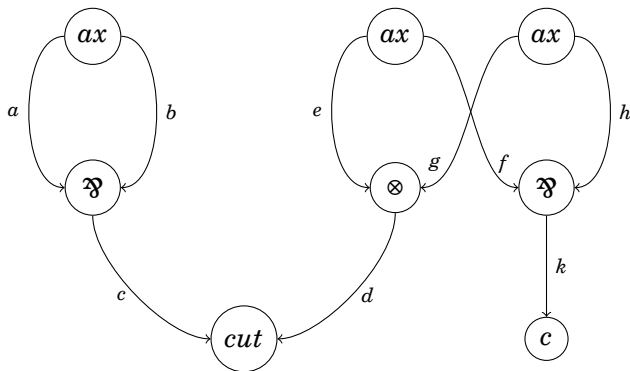
Cut-Elimination



- Path (straight, conclusion-to-conclusion):

$$kfe^*d^*cab^*c^*dgh^*k^*$$

Cut-Elimination



- Path (straight, conclusion-to-conclusion):

$$kfe^*d^*cba^*c^*dgh^*k^*$$

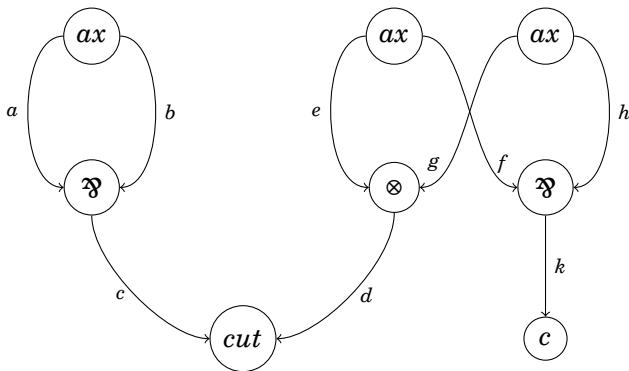
Persistent Paths

- ▶ This defines a map φ from $\mathcal{P}(\mathcal{R})$ to $\mathcal{P}(\mathcal{R}')$ ($\mathcal{R}'$ is the normal form of $\mathcal{R}$).

Definition

A conclusion-to-conclusion straight path p is called *persistent* if its image by φ is different from 0.

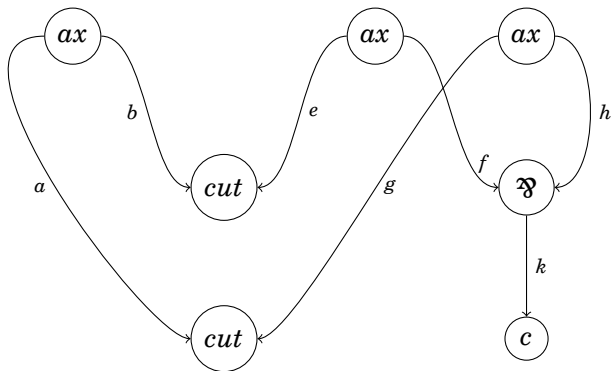
Persistent Paths



$$kfe^*d^*\underbrace{cba^*c^*d}_{\text{cut}}gh^*k^* \rightarrow kfe^*d^*cba^*gh^*k^*$$

$$kfe^*\underbrace{d^*cba^*}_{\text{cut}}gh^*k^* \rightarrow kfe^*ba^*gh^*k^*$$

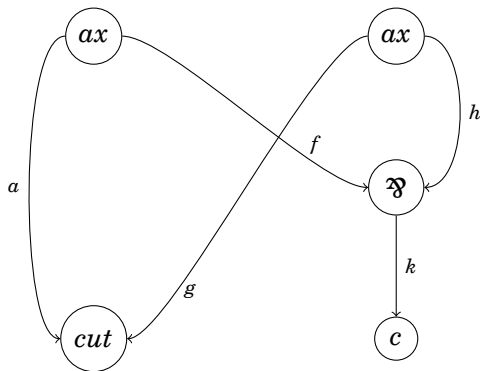
Persistent Paths



$$kfe^*d^*cba^*c^*dgh^*k^* \rightarrow kfe^*ba^*gh^*k^*$$

$$\underbrace{kfe^*ba^*}_{\text{persistent path}}gh^*k^* \rightarrow kfa^*gh^*k^*$$

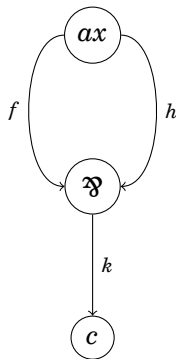
Persistent Paths



$$kfe^*d^*cba^*c^*dgh^*k^* \rightarrow kfa^*gh^*k^*$$

$$\underbrace{kfa^*gh^*k^*}_{\text{persistent path}} \rightarrow kfh^*k^*$$

Persistent Paths



$$kfe^*d^*cba^*c^*dgh^*k^* \rightarrow kfh^*k^*$$

The L^* involutive monoid

Definition

Define the free involutive monoid M^* generated by the elements r, l . This is the set of words over the alphabet $\{l, r, l^*, r^*\}$ endowed with the concatenation product and satisfying the equations $(w^*)^* = w$ and $(vw)^* = w^*v^*$.

Definition

We define L^* to be:

- ▶ an involutive monoid with two generators r, l (free as a monoid);
- ▶ with an absorbing element 0 , i.e. $w \cdot 0 = 0 \cdot w = 0$;
- ▶ satisfying the equations $l^*r = 0$ and $l^*l = r^*r = 1$.

Weight of Paths

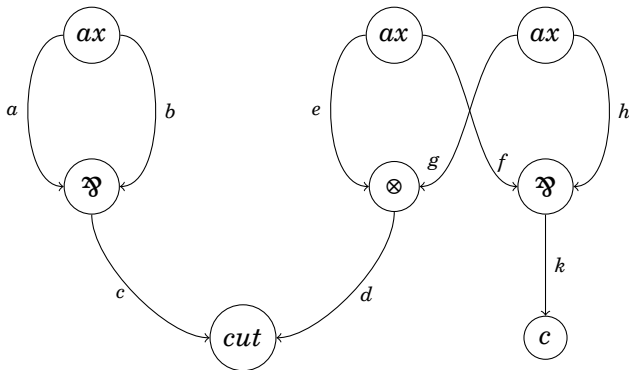
Definition

We define a weight function ω on the set of edges of a proof net $\mathcal{R}$:

- ▶ $\omega(e) = 1$ if e is a A -labelled edge of target $a \otimes$ (resp. $\wp$) vertex whose conclusion is labelled $A \otimes B$ (resp. $A \wp B$);
- ▶ $\omega(e) = r$ if e is a B -labelled edge of target $a \otimes$ (resp. $\wp$) vertex whose conclusion is labelled $A \otimes B$ (resp. $A \wp B$);
- ▶ $\omega(e) = 1$ otherwise.

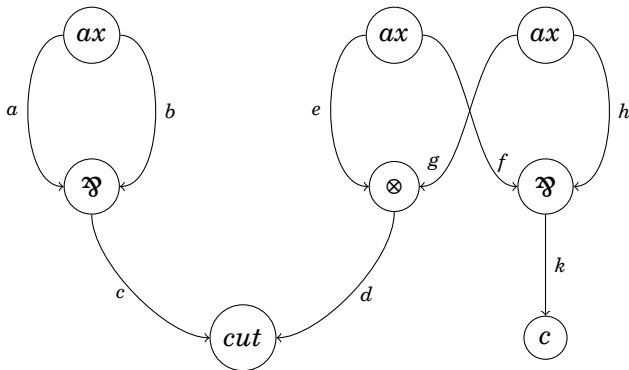
This weight function extends to reversed edges by defining $\omega(e^*) = \omega(e)^*$. It can then be extended to paths by letting $\omega(vw) = \omega(v)\omega(w)$.

Weight of paths



$$\omega(khg^*d^*cba^*c^*dgh^*k^*) = rr^*lr^*rr^* = 0$$

Weight of paths



$$\omega(kfe^*d^*cba^*c^*dgh^*k^*) = 1l^*1r^*rr^* = 1l^*1r^* = 1r^*$$

Regular paths

Definition

A path p is said *regular* when $\omega(p) \neq 0$.

Theorem

A straight path p is persistent if and only if it is regular. Moreover, if p_1, p_2 are persistent paths with same source and target and such that neither is a subpath of the other, then $\omega(p_2^*p_1) = 0$.

Main Theorems

Definition (The Execution)

Let $\mathcal{R}$ be a proof net. The execution of $\mathcal{R}$ is defined as:

$$\text{Ex}(\mathcal{R}) = \sum_{p \in \mathcal{P}(\mathcal{R})} \omega(p)$$

Theorem

Let $\mathcal{R}$ be a proof net and $\mathcal{R}'$ its normal form. Then:

$$\text{Ex}(\mathcal{R}) = \text{Ex}(\mathcal{R}')$$

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\text{Ex}(\cdot)} & \text{Ex}(\mathcal{R}) \\ \text{CE} \downarrow & & \parallel \\ \mathcal{R}' & \xrightarrow{\text{Ex}(\cdot)} & \text{Ex}(\mathcal{R}') \end{array}$$

Regular Paths, Axioms and Cuts

- ▶ A regular (therefore straight) path bounces up and down the net, changing direction only in axioms and cuts;
- ▶ A regular path can therefore be represented as an alternated sequence of axioms and cuts;
- ▶ The GoI interpretation will consist in representing the axioms on one hand and the cuts on another;
- ▶ The regular paths will then be recovered as an algebraic computation: the execution formula.

Definition

An *address* in a proof net $\mathcal{R}$ is a sequence $(e_i)_{i=0}^n$ such that $e_0 = p_A$, where A is a conclusion of $\mathcal{R}$, and for all $1 \leq i \leq n$ the element e_i is either equal to r or to l .

Flows I

Definition

A *flow* is an ordered pair of addresses (a, a') written as $a \leftarrow a'$.

Definition

Let $a \leftarrow a'$ and $b \leftarrow b'$ be two flows. We define their composition:

$$(a \leftarrow a').(b \leftarrow b') = \begin{cases} a \leftarrow b' & \text{if } a' = b \\ 0 & \text{otherwise} \end{cases}$$

Flows II

- ▶ An *extension* of an address $a = (e_i)_{i=0}^n$ is an address $a.w = (f_i)_{i=0}^{n+m}$ such that $(f_i)_{i=0}^n = a$ and $w = (f_i)_{i=n+1}^{n+m}$ is a word in the free monoid generated by $\{l, r\}$.
- ▶ A *bundle of addresses* is a sequence $(e_i)_{i=0}^n$ such that $a = (e_i)_{i=0}^{n-1}$ is an address and $e_n = \star$. A bundle of addresses represents the union of all extensions of a and will be written as $a.\star$.
- ▶ A *bundle of flows* is an ordered pair $a \leftarrow a'$ where a and a' are bundles of addresses. It represents the union of all the flows $b \leftarrow b'$ where $b = a.w$ and $b' = a'.w'$ are extensions of a and a' such that $w = w'$.

Flows III

This definition can be extended to compose a flow with a sum of flows (and thus to a composition between two sums of flows):

$$(\sum_{i=0}^n a_i \leftarrow b_i).(c \leftarrow c') = \sum_{i=0}^n (a_i \leftarrow b_i).(c \leftarrow c')$$

It is also easily extended to a composition of a flow with a bundle of flows:

$$(a.\star \leftarrow a'.\star).(c \leftarrow c') = \sum_w (a.w \leftarrow a'.w).(c \leftarrow c')$$

Definition

Let $a = (e_i)_{i=0}^n$ and b be addresses, and let $w = (e_i)_{i=1}^n$. We define the substitution:

$$a\{p_A \mapsto b\} = \begin{cases} b.w & \text{if } e_0 = p_A \\ a & \text{otherwise} \end{cases}$$

Interpretation of proofs I

We represent proofs by triples

$$(\{A_1, \dots, A_n\}, \{(C_1, C_1^\perp), \dots, (C_k, C_k^\perp)\}, P)$$

where:

- ▶ $A_1, \dots, A_n$ are the conclusions of the proof;
- ▶ $\{(C_1, C_1^\perp), \dots, (C_k, C_k^\perp)\}$ are the cut formulas of the proof;
- ▶ P is a sum of flows on extensions of the addresses p_{A_i} ($1 \leq i \leq n$) and p_{C_i} ($1 \leq i \leq k$).

Interpretation of proofs II

Axioms. A proof π consisting in a single axiom rule introducing the atom X and its orthogonal $X^\perp$ is represented by the triple

$$\|\pi\| = (\{X, X^\perp\}, \emptyset, p_X \leftarrow p_{X^\perp} + p_{X^\perp} \leftarrow p_X)$$

Interpretation of proofs III

Tensor. Let π_1 and π_2 be proofs of conclusions $A_1, \dots, A_n, A$ and $B_1, \dots, B_m, B$ respectively. Let π be the proof obtained from π_1 and π_2 by a tensor rule on A and B . Define

$$\|\pi_1\| = (\{A_1, \dots, A_n, A\}, C_1, a_1)$$

$$\|\pi_2\| = (\{B_1, \dots, B_m, B\}, C_2, a_2)$$

and

$$a_\pi = a_{\pi_1} \{p_A \mapsto p_{A \otimes B} \mathbf{l}\} + a_{\pi_2} \{p_B \mapsto p_{A \otimes B} \mathbf{r}\}$$

The interpretation $\|\pi\|$ is then defined as

$$(\{A_1, \dots, A_n, B_1, \dots, B_m, A \otimes B\}, C_1 \cup C_2, a_\pi)$$

Interpretation of proofs IV

Parr. Let π' be a proof of conclusions $A_1, \dots, A_n, A, B$ and π the proof obtained from π' by applying a $\wp$ rule on A and B . We denote by $(\{A_1, \dots, A_n, A, B\}, C, a_{\pi'})$ the interpretation $\|\pi'\|$ of π' . The interpretation of π is then defined as:

$$\|\pi\| = (\{A_1, \dots, A_n, A \wp B\}, C, a_{\pi'}\{p_A \mapsto p_{A \wp B}^l; p_B \mapsto p_{A \wp B}^r\})$$

Interpretation of proofs V

Cut. Let π_1 and π_2 be proofs of conclusions $A_1, \dots, A_n, A$ and $B_1, \dots, B_m, A^\perp$ respectively. Let π be the proof obtained from π_1 and π_2 by a cut rule on A and $A^\perp$. Define:

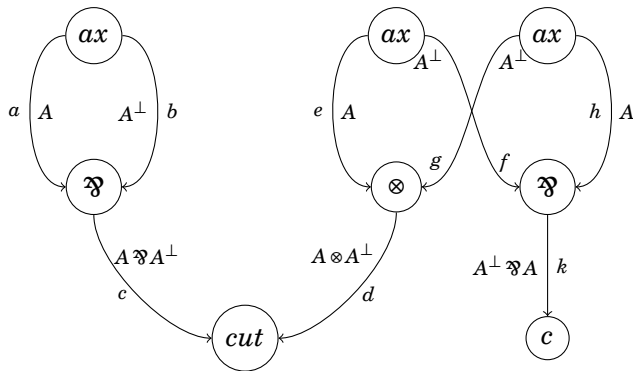
$$\|\pi_1\| = (\{A_1, \dots, A_n, A\}, C_1, P_1)$$

$$\|\pi_2\| = (\{B_1, \dots, B_m, A^\perp\}, C_2, P_2)$$

Then:

$$\|\pi\| = (\{A_1, \dots, A_n, B_1, \dots, B_m\}, C_1 \cup C_2 \cup \{(A, A^\perp)\}, P_1 + P_2)$$

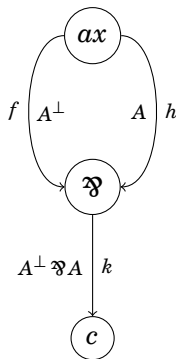
Interpretation of Proofs



$$(\{A^\perp \wp A\}, \{(A \wp A^\perp, A \otimes A^\perp)\}, P)$$

$$P = p_{A \wp A^\perp} \cdot r \Leftrightarrow p_{A \wp A^\perp} \cdot 1 + p_{A \otimes A^\perp} \cdot 1 \Leftrightarrow p_{A^\perp \wp A} \cdot 1 + p_{A \otimes A^\perp} \cdot r \Leftrightarrow p_{A^\perp \wp A} \cdot r$$

Interpretation of Proofs



$$(\{A^\perp \text{⌘} A\}, \emptyset, P')$$

$$P' = p_{A^\perp \text{⌘} A} \cdot r \Longleftrightarrow p_{A^\perp \text{⌘} A} \cdot 1$$

Execution I

- ▶ Let (A, C, P) be the interpretation of a proof, and $C' \subset C$ a set of cut formulas. Let us denote by R the set of cut formulas in $C - C'$. We define the sum of bundles of flows $1 - C'$ as:

$$1 - C' = \sum_{A \cup R} p_{A \cdot \star} \leftarrow p_{A \cdot \star}$$

This expression plays the role of a partial identity.

- ▶ Moreover, the set $C' \subset C$ defines a sum of bundles of flows we will abusively denote by C' :

$$S_{C'} = \sum_{(A, A^\perp) \in C'} (p_{A \cdot \star} \leftarrow p_{A^\perp \cdot \star} + p_{A^\perp \cdot \star} \leftarrow p_{A \cdot \star})$$

Execution II

- ▶ Let $\|\pi\| = (A, C, P)$ be the interpretation of a proof, and $C' \subset C$ a set of cut formulas. We define the *execution of (A, C, P) along C'* by:

$$\text{Ex}(\|\pi\|, C') = (A, C - C', (1 - C').P(\sum_{i \geq 0} (S_{C'} P)^i).(1 - C'))$$

- ▶ If $C' = C$, we write $\text{Ex}(\|\pi\|)$ the expression $\text{Ex}(\|\pi\|, C)$.

Theorem

Let $\|\pi\| = (A, C, P)$ be the interpretation of a proof, and $C_1, C_2 \subset C$ two disjoint sets of cut formulas. Then:

$$\text{Ex}(\text{Ex}(\|\pi\|, C_1), C_2) = \text{Ex}(\|\pi\|, C_1 \cup C_2) = \text{Ex}(\text{Ex}(\|\pi\|, C_2), C_1)$$

Main Theorems

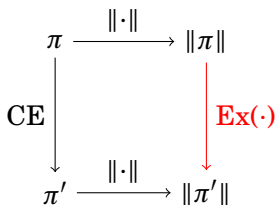
Theorem (Nilpotency)

Let $\mathcal{R}$ be a proof net, $\|\mathcal{R}\| = (A, C, P)$ its interpretation and $C' \subset C$. Then $S_{C'}P$ is nilpotent, i.e. the expression $\text{Ex}(\|\mathcal{R}\|, C')$ is finite.

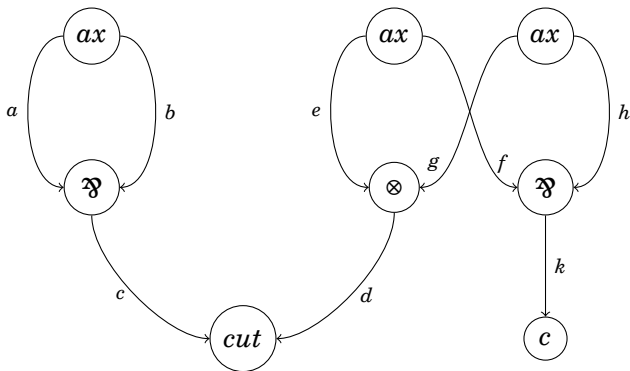
- GoI is not a denotational semantics.

Theorem (Execution and Cut-Elimination)

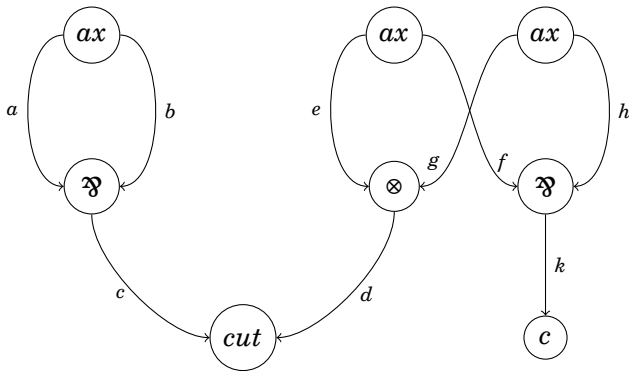
If π is a proof and ρ its normal form, then $\text{Ex}(\|\pi\|) = \|\rho\|$.



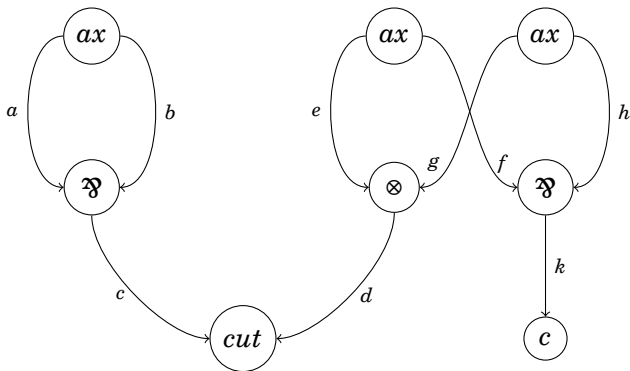
Exercises



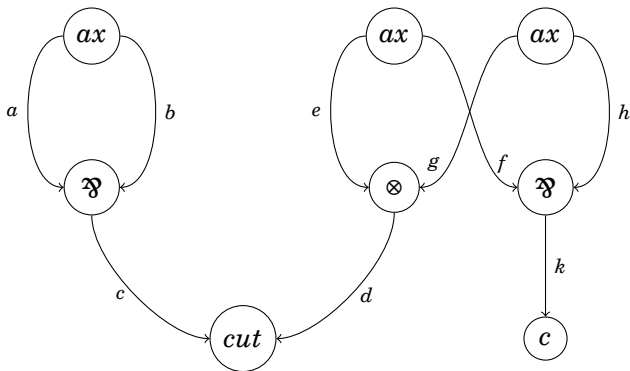
- Compute the set of persistent paths;
- Compute the execution $\text{Ex}(\mathcal{R})$.



- ▶ Compute the GoI interpretation of this net;
- ▶ Compute the GoI interpretation of its normal form:
 - ▶ By eliminating cuts and then computing the GoI interpretation;
 - ▶ Using the execution formula.



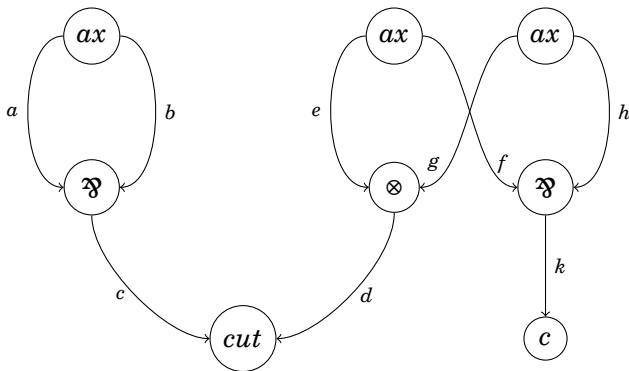
- Compute the set of persistent paths: $kfe^*d^*cab^*c^*dgh^*k^*$
and $khg^*d^*cba^*c^*def^*k^*$



- Compute the execution: $\text{Ex}(\mathcal{R}) = \text{lr}^* + \text{rl}^*$:

$$\omega(kfe^*d^*cab^*c^*dgh^*k^*) = \text{ll}^*\text{lr}^*\text{rr}^* = \text{lr}^*$$

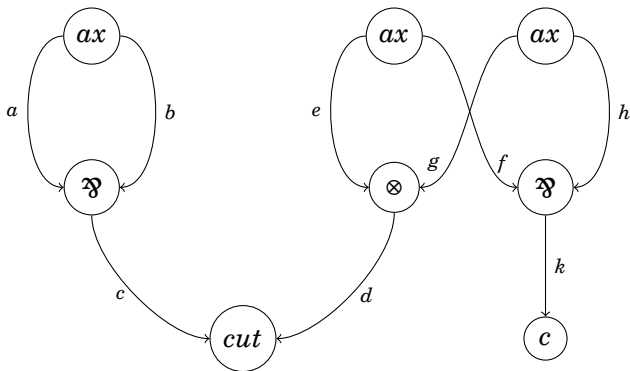
$$\omega(khg^*d^*cba^*c^*def^*k^*) = \text{rr}^*\text{rl}^*\text{ll}^* = \text{rl}^*$$



- Compute the GoI interpretation of this net:

$$(\{A^\perp \wp A\}, \{(A \wp A^\perp, A \otimes A^\perp)\}, P)$$

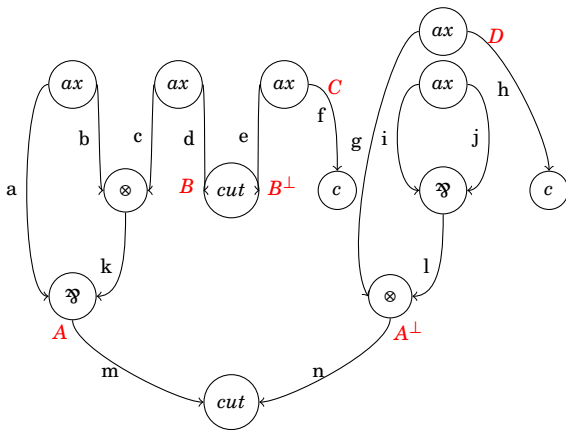
$$P = p_{A \wp A^\perp} \cdot r \Leftrightarrow p_{A \wp A^\perp} \cdot l + p_{A \otimes A^\perp} \cdot l \Leftrightarrow p_{A^\perp \wp A} \cdot l + p_{A \otimes A^\perp} \cdot r \Leftrightarrow p_{A^\perp \wp A} \cdot r$$



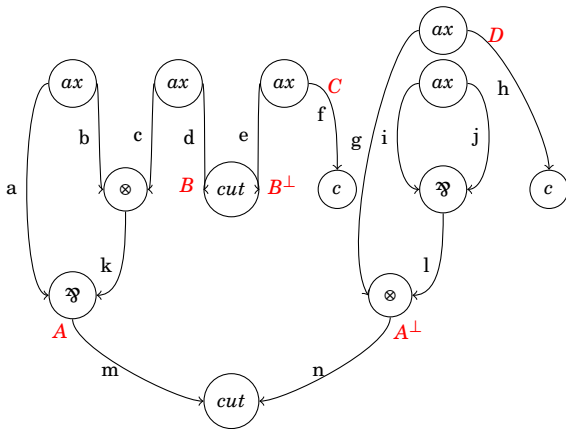
- Compute the GoI interpretation of its normal form:

$$(\{A^\perp \wp A\}, \emptyset, P')$$

$$P' = p_{A^\perp \wp A} \cdot r \leq p_{A^\perp \wp A} \cdot 1$$



- Compute the set of persistent paths;
- Compute the execution $\text{Ex}(\mathcal{R})$.



- Compute the GoI interpretation of this net;
- Compute the GoI interpretation of its normal form:
 - By eliminating some cut(s) and then computing the GoI interpretation;
 - Using the execution formula.

- ▶ Try to prove the theorems in these slides.