

Interaction Graphs: Exponentials

Thomas Seiller



Réunion LOGOI
February 1st, 2013

Thick Graphs

Graphings

Exponentials

Thick Graphs

Definition

Let S^G and D^G be finite sets. A *thick graph* (weighted and directed) G of carrier S^G and *dialect* D^G is a directed and weighted graph whose set of vertices is $S^G \times D^G$.

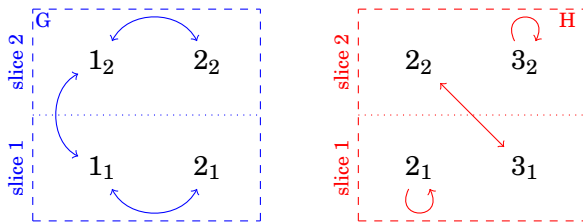


Figure : Two thick graphs G and H , of dialect $\{1,2\}$

Thick Graphs

Definition (Dialectal Interaction)

Let G and H be thick graphs.

1. We write $G^{\dagger}_{D^H}$ the thick graph of dialect $D^G \times D^H$ defined as $\{\sum_{i \in D^H} G\}$;
2. We write $H^{\ddagger}_{D^G}$ the thick graph of dialect $D^G \times D^H$ defined as $\{\sum_{i \in D^G} H\}^{\tau}$ where τ is the bijection $D^H \times D^G \rightarrow D^G \times D^H, (a, b) \mapsto (b, a)$.

Definition

The execution $F :: G$ of two thick graphs F, G is the thick graph of carrier $S^F \Delta S^G$ and dialect $D^F \times D^G$ defined as $F^{\dagger}_{D^G} :: G^{\ddagger}_{D^F}$.

Thick Graphs

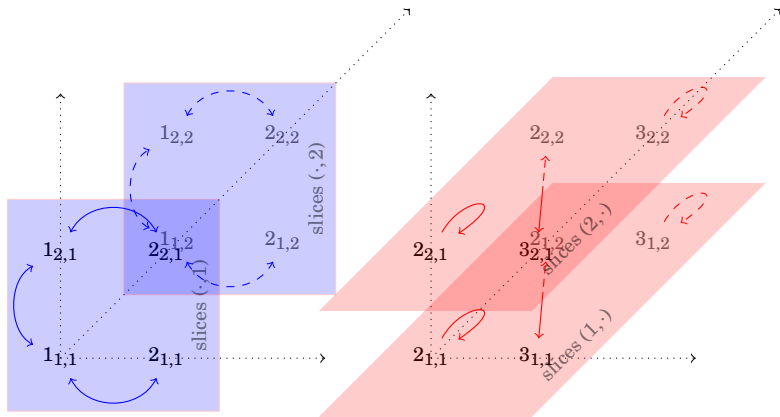


Figure : The graphs $G^{\dagger_{D^H}}$ and $H^{\ddagger_{D^G}}$

Thick Graphs

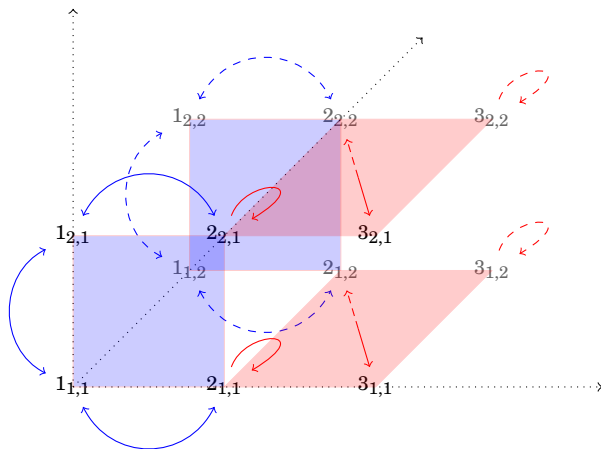


Figure : Plugging of $G^{\dagger D^H}$ and $H^{\dagger D^G}$

Thick Graphs

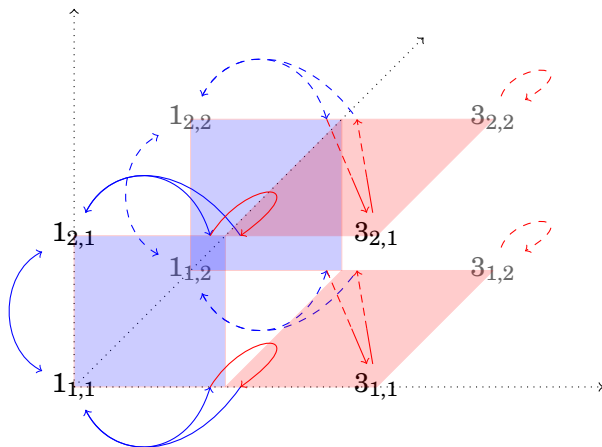


Figure : Alternating paths in the plugging of $G^{\dagger}_{D^H}$ and $H^{\dagger}_{D^G}$

Thick Graphs

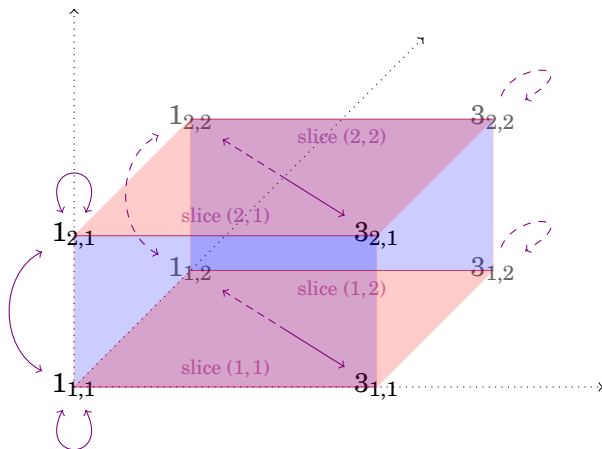


Figure : The thick graph $G :: H$

Thick Graphs

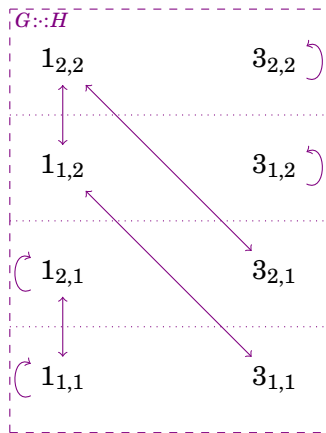


Figure : The thick graph $G :: H$

Thick Graphs

Proposition (Associativity)

Let F, G, H be thick graphs such that $S^F \cap S^G \cap S^H = \emptyset$. Then:

$$F :: (G :: H) = (F :: G) :: H$$

Definition

If F and G are thick graphs, one defines $\text{Cy}^e(F, G)$ as the set of alternating cycles in $F^{\dagger}_{D^G} \square G^{\ddagger}_{D^F}$.

Proposition (Geometric Cyclic Property)

If F, G, H are thick graphs such that $S^F \cap S^G \cap S^H = \emptyset$, then:

$$\text{Cy}^e(F, G :: H) \cup \text{Cy}^e(G, H)^{\dagger}_{D^F} \cong \text{Cy}^e(F :: G, H) \cup \text{Cy}^e(F, G)^{\dagger}_{D^H}$$

Thick Graphs

Definition

Let $m : \Omega \rightarrow \mathbf{R}_{\geq 0} \cup \{\infty\}$. For every couple (F, G) of thick graphs, one defines:

$$\llbracket F, G \rrbracket = \sum_{\pi \in \text{Cy}^e(F, G)} \frac{1}{\text{Card}(D^F \times D^G)} m(\omega(\pi))$$

Proposition (Scalar cyclic property)

If F, G, H are thick graphs such that $S^F \cap S^G \cap S^H = \emptyset$, then:

$$\llbracket F, G :: H \rrbracket + \llbracket G, H \rrbracket = \llbracket H, F :: G \rrbracket + \llbracket F, G \rrbracket$$

Slice-Changes and Contraction

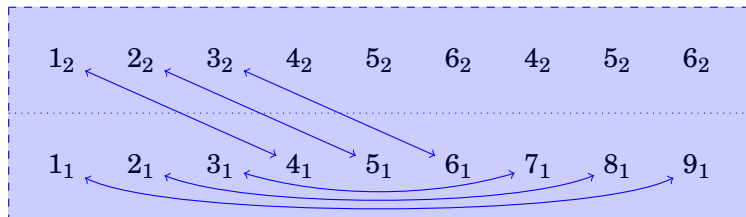
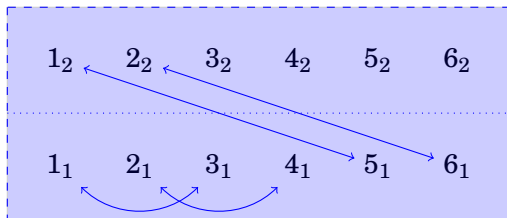
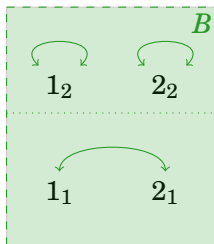
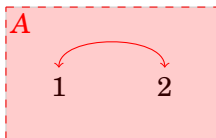


Figure : A graph implementing contraction

Slice-Changes and Contraction

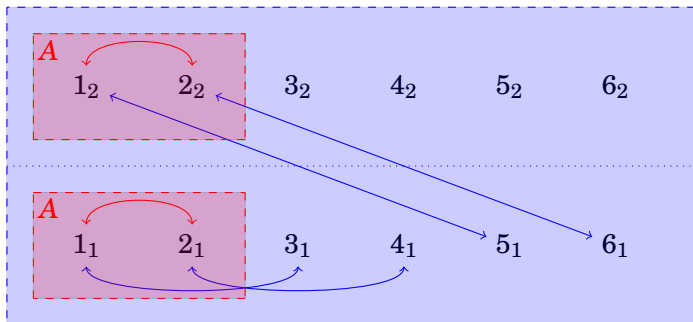


(a) Graph Ctr_{ψ}^{ϕ}



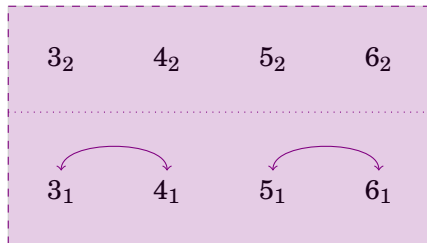
(b) Graphs A and B

Slice-Changes and Contraction

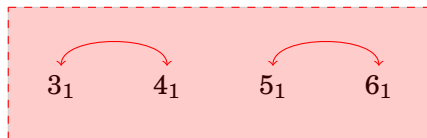


(c) Plugging of Ctr_ψ^ϕ and A

Slice-Changes and Contraction

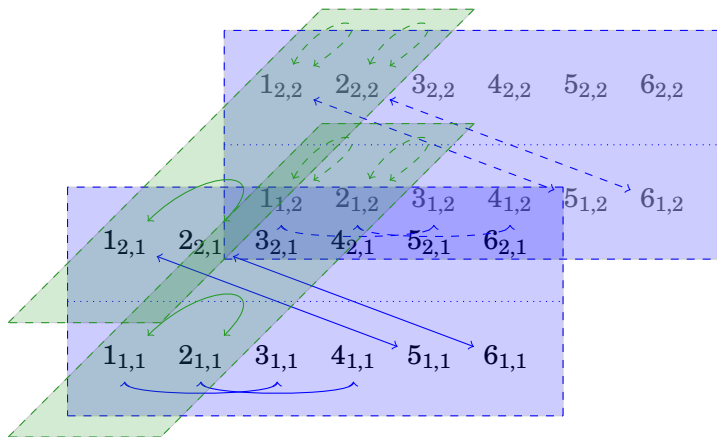


(d) $\text{Ctr}_{\psi}^{\phi} :: A$



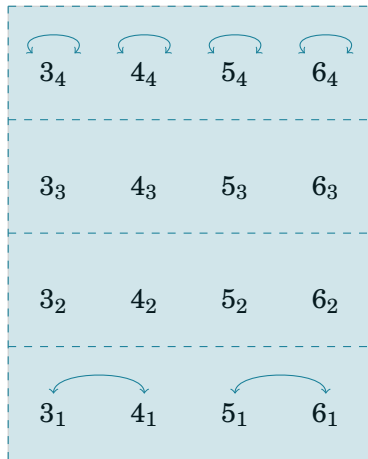
(e) $\phi(A) \otimes \psi(A)$

Slice-Changes and Contraction

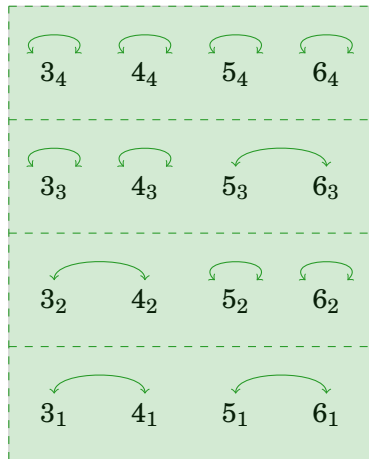


(f) Plugging of Ctr_ψ^ϕ and B

Slice-Changes and Contraction



(g) $\text{Ctr}_\psi^\phi :: B$



(h) $\phi(B) \otimes \psi(B)$

Slice-Changes and Contraction

Proposition

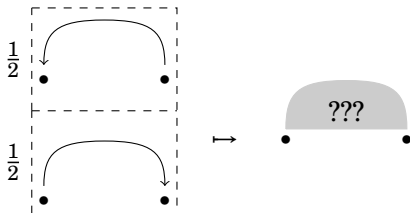
*Let $\mathfrak{a} = (0, A)$ be a project in the behavior $\mathbf{A}$, such that $D^A \cong \{1\}$.
Then $\mathfrak{C}\mathfrak{t}\mathfrak{r}_{\phi}^{\psi} :: \mathfrak{a} \in \phi(\mathbf{A}) \otimes \psi(\mathbf{A})$.*

Proposition

Suppose E is a non-empty set of projects of carrier V^E , $\mathbf{F}$ is a non-empty conduct and $\mathfrak{f}$ satisfies $\forall \mathfrak{e} \in E, \mathfrak{f} :: \mathfrak{e} \in \mathbf{F}$. Then $\mathfrak{f} \in E^{\perp\perp} \multimap \mathbf{F}$.

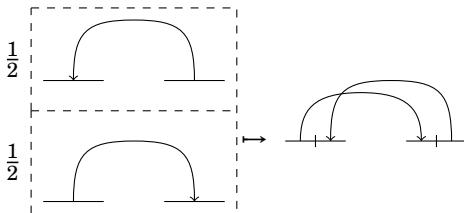
From graphs to graphings

- Perennisation: a map from thick graphs to graphs (i.e. thick graphs with a unique slice).



From graphs to graphings

- ▶ Perennisation: a map from thick graphs to graphs (i.e. thick graphs with a unique slice).
- ▶ We will replace graphs by *graphings*: families of measure-preserving maps (m.p.m.) between measurable sets.



From graphs to graphings

We fix $X = (X, \mathcal{B}, \lambda)$ a measured space.

Definition (Weighted Graphings)

Let $V^F \in \mathcal{B}$ be a set of finite measure. A *graphing* on X of carrier V^F is a (countable) family $F = \{(\omega_e^F, \phi_e^F : S_e^F \rightarrow T_e^F)\}_{e \in E^F}$, where:

- ▶ $\forall e \in E^F$, ω_e^F is in Ω , the set of weights (think of $[0, 1]$);
- ▶ $\forall e \in E^F$, ϕ_e^F is a m.p.m. between the measurable sets $S_e^F \subset V^F$ and $T_e^F \subset V^F$.



Figure : A graphing

From graphs to graphings

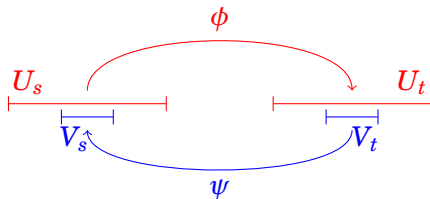


Figure : Example of a plugging

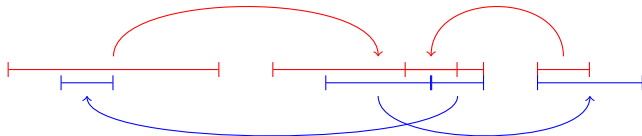


Figure : Example of a plugging

From graphs to graphings

Definition (Paths)

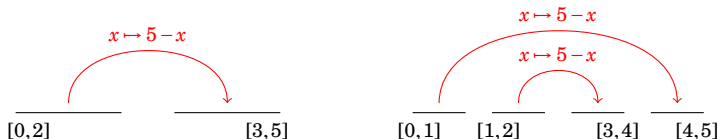
A path in the graphing F is a finite sequence $\{e_i\}_{i=0}^n$ of elements of E^F such that for all $0 \leq i \leq n-1$, $T_{e_i}^F \cap S_{e_{i+1}}^F$ is of non-null measure.

Proposition (Associativity of Execution)

Let F, G, H be graphings such that $\lambda(V^F \cap V^G \cap V^H) = 0$. Then:

$$F:m:(G:m:H) = (F:m:G):m:H$$

From graphs to graphings



(a) Refinements

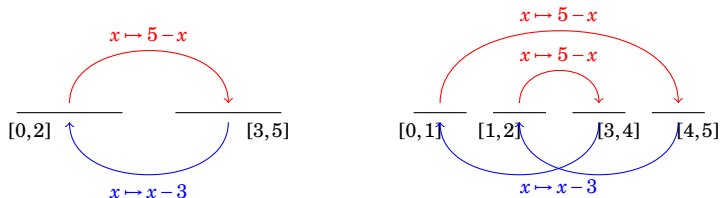
Proposition

The relation $\sim_{\leq}$ between graphings defined as:

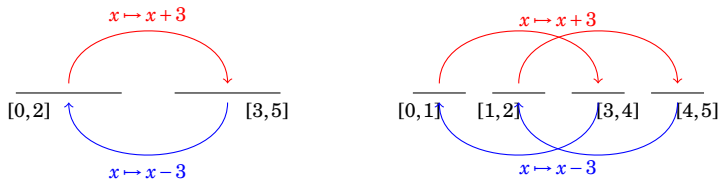
$$F \sim_{\leq} G \Leftrightarrow \exists H, (H \leq F) \wedge (H \leq G)$$

is an equivalence relation compatible with execution.

From graphs to graphings



(b) A cycle can become a longer cycle



(c) A cycle can become two (or more) cycles

Figure : Evolution of cycles through refinement

From graphs to graphings

Definition (Circuit-quantifying maps)

A map q from the set of circuits to $\mathbf{R}_{\geq 0}$ is a *circuit-quantifying map* when:

1. for all representants π_1, π_2 of a circuit π , $q(\pi_1) = q(\pi_2)$;
2. q is invariant by refinement.

Definition (Measure)

Let q be a circuit-quantifying map. One defines the measure associated to q by:

$$\llbracket F, G \rrbracket = \sum_{\pi \in \text{Circ}^m(F, G)} q(\pi)$$

Where $\text{Circ}^m(F, G)$ is a set of representants of circuits.

From graphs to graphings

Theorem

Let F, G be graphings and $F' \leq F$. Then:

$$\llbracket F, G \rrbracket = \llbracket F', G \rrbracket$$

Proposition

If F, G, H are graphings such that $\lambda(V^F \cap V^G \cap V^H) = 0$, then:

$$\llbracket F, G :_m H \rrbracket + \llbracket G, H \rrbracket = \llbracket H :_m F, G \rrbracket + \llbracket H, F \rrbracket$$

Circuit-Quantifying maps

We will now work with the measure space $(\mathbf{R}, \mathcal{B}, \lambda)$.

Theorem

Let $\phi : X \rightarrow X$ be a m.p.m., where $X \subset \mathbf{R}$ is a set of finite measure.
Then the following map is measurable:

$$\rho_\phi : \begin{cases} X & \rightarrow \mathbf{N} \cup \{\infty\} \\ x & \mapsto \inf\{n \in \mathbf{N} \mid \phi^n(x) = x\} & \text{if } \{n \in \mathbf{N} \mid \phi^n(x) = x\} \neq \emptyset \\ x & \mapsto \infty & \text{otherwise} \end{cases}$$

Definition

Let m be a measurable map $\Omega \rightarrow \mathbf{R} \cup \{\infty\}$. One defines:

$$q_m : \pi \mapsto \int_{\rho_\pi^{-1}(\mathbf{N})} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda(x)$$

Circuit-Quantifying maps

We will now work with the measure space $(\mathbf{R}, \mathcal{B}, \lambda)$.

Definition

Let m be a measurable map $\Omega \rightarrow \mathbf{R} \cup \{\infty\}$. One defines:

$$q_m : \pi \mapsto \int_{\rho_{\phi_\pi}^{-1}(\mathbf{N})} \frac{m(\omega(\pi)^{\rho_{\phi_\pi}(x)})}{\rho_{\phi_\pi}(x)} d\lambda(x)$$

Theorem

For all measurable map $m : \Omega \rightarrow \mathbf{R} \cup \{\infty\}$, the map q_m is a circuit-quantifying map.

A perennisation

- ▶ Let F be a graphing on $V^F \times D^F$, where $D^F = \{0, 1, \dots, k\}$ with normalized counting measure.
- ▶ The trick: we use three copies of $\mathbf{N}$, and consider $\iota_0(D^F)$, a subset of $\mathbf{N} + \mathbf{N} + \mathbf{N}$ located in the first copy of $\mathbf{N}$.
- ▶ Through the bijection $\psi : \mathbf{N} + \mathbf{N} + \mathbf{N} \rightarrow \mathbf{N}$ defined as $(x, i) \mapsto 3x + i$ one gets a graphing (a *variant* of F) on $V^F \times \psi(\iota_0(D^F))$.
- ▶ One defines $\lceil F \rceil$, a graphing on $V^F \times [0, 1]$, in a natural way.

Definition

Let A be a thick graphing, and $\Omega : \mathbf{R} \times [0, 1] \rightarrow \mathbf{R}$ an isomorphism of measure spaces. One defines $!_{\Omega}A = \Omega(\lceil A \rceil)$.

Functorial Promotion

Definition

Let $\phi : \mathbf{N} \rightarrow \mathbf{N}$ be a bijection, b an integer ≥ 2 . Then ϕ induces a transformation $T_\phi^b : [0, 1] \rightarrow [0, 1]$ defined by

$$\sum_{i \geq 0} a_i 2^{-i} \mapsto \sum_{i \geq 0} a_{\phi^{-1}(i)} 2^{-i}.$$

Theorem

Let $\phi : \mathbf{N} \rightarrow \mathbf{N}$ be a bijection, $b \geq 2$ an integer. The transformation T_ϕ^b is measure-preserving.

Functorial Promotion

- ▶ F, G are graphings of dialects D^F, D^G .
- ▶ $\lceil F \rceil, \lceil G \rceil$ are obtained through $\lceil \cdot \rceil$ as images of the variants of dialects $\psi(\iota_0(D^F))$ and $\psi(\iota_0(D^G))$.
- ▶ Define $\sigma : (x, 0) \mapsto (x, 1), (x, 1) \mapsto (x, 0), (x, 2) \mapsto (x, 2)$
- ▶ Using the preceding theorem, the bijection $\psi \circ \sigma \circ \psi^{-1}$ induces a m.p.m. T_σ of $[0, 1]$.
- ▶ The graphing $\lceil G \rceil :: T_\sigma$ is then the graphing obtained through $\lceil \cdot \rceil$ as the image of the variant of G of dialect $\psi(\iota_1(D^G))$.
- ▶ The execution of $\lceil G \rceil :: T_\sigma$ and $\lceil F \rceil$ is then a variant of $\lceil F^{\dagger}_{D^G} :: G^{\ddagger}_{D^F} \rceil$.

Behaviors' sequent calculus

$$\frac{}{\vdash C^\perp, C;}^{\text{ax}} \quad \frac{\Delta_1 \vdash \Gamma_1, C; \quad \Delta_2 \vdash \Gamma_2, C^\perp;}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2;}^{\text{cut}}$$

(a) Identity group

$$\frac{\Delta_1 \vdash \Gamma_1, C_1; \quad \Delta_2 \vdash \Gamma_2, C_2;}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2, C_1 \otimes C_2;}^{\otimes} \quad \frac{\Delta \vdash \Gamma, C_1, C_2;}{\Delta \vdash \Gamma, C_1 \wp C_2;}^{\wp}$$

$$\frac{\Delta, N_1, N_2 \vdash \Gamma;}{\Delta, N_1 \otimes N_2 \vdash \Gamma;}^{\otimes_g^{pol}} \quad \frac{\Delta, P^\perp \vdash \Gamma, C;}{\Delta \vdash \Gamma, P \wp C;}^{\wp^{mix}}$$

$$\frac{\Delta \vdash \Gamma, C;}{\Delta \vdash \Gamma, C \otimes \mathbf{1};}^{\mathbf{1}_d} \quad \frac{\Delta \vdash \Gamma;}{\Delta, \mathbf{1} \vdash \Gamma;}^{\mathbf{1}_g}$$

(b) Multiplicative group

Figure : Sequent Calculus ELL_{comp}

Behaviors' sequent calculus

$$\begin{array}{c}
 \frac{\Delta \vdash \Gamma, C_i;}{\Delta \vdash \Gamma, C_1 \oplus C_2;} \oplus_i \qquad \frac{\Delta \vdash \Gamma, C_1; \quad \Delta \vdash \Gamma, C_2;}{\Delta \vdash \Gamma, C_1 \& C_2;} \& \\
 \\
 \frac{}{\Delta \vdash \Gamma, \top;} \top \qquad \text{No rules for 0.}
 \end{array}$$

(a) Additive group

$$\begin{array}{c}
 \frac{\Delta_1 \vdash \Gamma_1, C_1; \quad \Delta_2 \vdash \Gamma_2, C_2;}{! \Delta_1, \Delta_2, ! \Gamma_1^\perp \vdash \Gamma_2, ! C_1 \otimes C_2;} ! \\
 \\
 \frac{\Delta, !A, !A \vdash \Gamma;}{\Delta, !A \vdash \Gamma;} \text{ctr}(\Gamma) \qquad \frac{\Delta \vdash \Gamma;}{\Delta, N \vdash \Gamma;} \text{weak}
 \end{array}$$

(b) Exponential group

Figure : Sequent Calculus ELL_{comp}

Negative/Positive Conducts

Definition (Polarized conducts)

A positive conduct $\mathbf{P}$ is a conduct satisfying:

- ▶ $p \in \mathbf{P} \Rightarrow p + \lambda o \in \mathbf{P}$;
- ▶ $\forall \lambda \in \mathbf{R} - \{0\}, \mathcal{D} \text{ai}_\lambda = (\lambda, (V^P, \emptyset)) \in \mathbf{P}$.

Proposition

A conduct $\mathbf{A}$ is negative iff:

- ▶ $\alpha = (a, A) \in \mathbf{A} \Rightarrow a = 0$;
- ▶ $\alpha \in \mathbf{A} \Rightarrow \mathbf{I}_A \neq 0$.

Negative/Positive Conducts

Proposition

A perennial conduct is negative.

Proposition

If $\mathbf{A}, \mathbf{B}$ are negative conducts, $\mathbf{A} \multimap \mathbf{B}$ is a negative conduct.

Proposition

Let $\mathbf{A}$ be a behavior or a positive conduct. There exists a successful project in

$$?\phi(\mathbf{A}) \wp ?\psi(\mathbf{A}) \multimap ?\mathbf{A}$$

Polarized Conducts' sequent calculus

$$\begin{array}{c}
 \frac{}{\Vdash C^\perp, C;}^{\text{ax}} \\
 \\
 \frac{\Delta_1 \Vdash \Gamma_1; N \quad \Delta_2, N \Vdash \Gamma_2; \Theta}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2; \Theta}^{\text{cut}^{pol}} \\
 \\
 \frac{\Delta_1 \Vdash \Gamma_1, C; \Theta \quad \Delta_2 \Vdash \Gamma_2, C^\perp;}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2; \Theta}^{\text{cut}}
 \end{array}$$

(a) Identity group

Figure : Sequent calculus ELL_{pol}

Polarized Conducts' sequent calculus

$$\begin{array}{c}
 \frac{\Delta_1 \Vdash \Gamma_1, C_1; \Theta \quad \Delta_2 \Vdash \Gamma_2, C_2; \Theta}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2, C_1 \otimes C_2; \Theta} \otimes \\
 \\
 \frac{}{\Vdash; \mathbf{1}} \mathbf{1}_d \quad \frac{\Delta \Vdash \Gamma; \Theta}{\Delta, \mathbf{1} \Vdash \Gamma; \Theta} \mathbf{1}_g \quad \frac{\Delta \Vdash \Gamma, C_1, C_2; \Theta}{\Delta \Vdash \Gamma, C_1 \wp C_2; \Theta} \wp \\
 \\
 \frac{\Delta, N_1, N_2 \Vdash \Gamma; \Theta}{\Delta, N_1 \otimes N_2 \Vdash \Gamma; \Theta} \otimes_g^{pol} \quad \frac{\Delta_1 \Vdash \Gamma_1; N_1 \quad \Delta_2 \Vdash \Gamma_2; N_2}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2; N_1 \otimes N_2} \otimes_d^{pol} \\
 \\
 \frac{\Delta, P_1^\perp \Vdash \Gamma; N_2}{\Delta \Vdash \Gamma; P_1 \wp N_2} \wp_d^{pol} \quad \frac{\Delta_1 \Vdash \Gamma_1; P_1^\perp \quad \Delta_2, N_2 \Vdash \Gamma_2; \Theta}{\Delta_1, \Delta_2, P_1 \wp N_2 \Vdash \Gamma_1, \Gamma_2; \Theta} \wp_g^{pol} \\
 \\
 \frac{\Delta, P^\perp \Vdash \Gamma, C; \Theta}{\Delta \Vdash \Gamma, P \wp C; \Theta} \wp^{mix} \quad \frac{\Delta_1 \Vdash \Gamma_1; N \quad \Delta_2 \Vdash \Gamma_2, C; \Theta}{\Delta_1, \Delta_2 \Vdash \Gamma_1, \Gamma_2, N \otimes C; \Theta} \otimes^{mix}
 \end{array}$$

(a) Multiplicative group

Polarized Conducts' sequent calculus

$$\boxed{
 \begin{array}{c}
 \frac{\Delta \Vdash \Gamma, C_i; \Theta}{\Delta \Vdash \Gamma, C_1 \oplus C_2; \Theta} \oplus_i \quad \frac{\Delta \Vdash \Gamma, C_1; \Theta \quad \Delta \Vdash \Gamma, C_2; \Theta}{\Delta \Vdash \Gamma, C_1 \& C_2; \Theta} \& \\
 \\
 \frac{}{\Delta \Vdash \Gamma, \top; \Theta} \top \quad \text{No rules for } 0.
 \end{array}
 }$$

(a) Additive group

$$\boxed{
 \begin{array}{c}
 \frac{\Delta \Vdash \Gamma; N}{!\Delta, !\Gamma^\perp \Vdash; !N} !^{pol} \quad \frac{\Delta \Vdash \Gamma, C;}{!\Delta, !\Gamma^\perp \Vdash; !C} ! \\
 \\
 \frac{\Delta, !C, !C \Vdash \Gamma; \Theta}{\Delta, !C \Vdash \Gamma; \Theta} \text{ctr} \quad \frac{\Delta \Vdash \Gamma; \Theta}{\Delta, N \Vdash \Gamma; \Theta} \text{aff}
 \end{array}
 }$$

(b) Exponential group

Figure : Sequent calculus ELL_{pol}



The End