Non-commutative logic I : the multiplicative fragment

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We introduce proof nets and sequent calculus for the multiplicative fragment of non-commutative logic, which is an extension of both linear logic and cyclic linear logic. The two main technical novelties are a third switching position for the non-commutative disjunction, and the structure of order variety.

1 INTRODUCTION

Unrestricted exchange rules of Girard's linear logic [8] force the commutativity of the multiplicative connectives $\otimes$ (*times*, conjunction) and $\parr$ (*par*, disjunction), and henceforth the commutativity of all logic. This *a priori* commutativity is not always desirable — it is quite problematic in applications like linguistics or computer science —, and actually the desire of a non-commutative logic goes back to the very beginning of LL [9].

Previous works on non-commutativity deal essentially with *non-commutative fragments* of LL, obtained by removing the exchange rule at all.
At that point, a simple remark on the status of exchange in the sequent calculus is necessary to be clear: there are two presentations of exchange in commutative LL, either sequents are finite sets of occurrences of formulas and exchange is obviously *implicit*, or sequents are finite sequences of formulas and the (unrestricted) exchange rule is *explicit*:

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$$\frac{\vdash A_1, \dots, A_n}{\vdash A_{\sigma(1)}, \dots, A_{\sigma(n)}} \; \sigma \text{ any permutation of } \{1, \dots, n\}.$$

Now, removing the exchange rule in LL is possible because, in the second style calculus, the cut elimination procedure of LL preserves crucially the absence of exchanges [2].
The resulting non-commutative fragment enjoys an important and rather unexpected property [3]: provability is closed under the rule of *cyclic exchange*:

$$\frac{\vdash A_1, \dots, A_n}{\vdash A_{\sigma(1)}, \dots, A_{\sigma(n)}} \; \sigma \text{ cyclic permutation of } \{1, \dots, n\}.$$

So the right name for the non-commutative fragment of LL is *cyclic linear logic*, cyLL. CyLL has been proposed by Girard [9] and expounded by Yetter [20], but presented with cyclic exchange as a rule of the sequent calculus — and the first wrong impression has been that cyLL is not really non-commutative!
The above result on the provability in cyLL leads naturally to a nice formulation of cyLL obtained by defining sequents as finite cycles of occurrences of formulas. Cyclic proof structures can also be defined, and a correctness criterion is obtained very easily from [2]: cyclic proof nets are usual proof nets of LL satisfying a certain additional condition.
It is also possible to consider two negations instead of one [1], but this introduces complications, both in the sequent calculus, in proof nets and in the phase semantics (for associativity, as noticed by Girard in [10] Appendix F), not to speak about the "semantics" of proofs. In both cases, the intuitionistic version is the extension of Lambek's syntactic calculus (introduced thirty years before LL [12] for linguistic needs: categorial grammars) with additives and exponentials. Remark indeed that the multiplicative fragment of cyLL is a conservative extension of Lambek's calculus [3].

However purely non-commutative fragments of LL are too limited in practice. We must find a non-commutative logic that is *more general* than commutative logic. Retoré shows in [16] that LL enlarged with the Mix rule contains a self-dual non-commutative connective which is intermediate between $\otimes$ and $\parr$: the connective $\lhd$ (*before*); he gives proof nets and a coherent semantics, the drawback being the complicated sequent calculus and (up to now) the absence of a sequentialization theorem. There have also been attempts to add modalities in order to recover commutativity in a non-commutative framework (e.g., [14]), but there are too many possibilities and these modalities introduce many complications.

A simple solution arised recently through the interaction of two independent works:

[2] A nice topological study of proof nets with explicit exchange rule has been carried out by Fleury [7].

— The first author found a direct characterization of proof nets of CyLL as graphs satisfying a geometrical property which implies (but does not presuppose) that cyclic proof nets are proof nets of LL. Let $\odot$ (*next*) denote the non-commutative conjunction and $\triangledown$ (*sequential*) the non-commutative disjunction [3]. The idea is to consider only one switching position, say the right one, for $\odot$-links and to introduce a new switching position for $\triangledown$-links. Then there is a simple definition of proof nets by a trip condition, which can be generalized in presence of commutative connectives.
— The second author introduced a mixed non-commutative / commutative sequent calculus enjoying cut elimination and a corresponding phase semantics [17], starting from the intuitionistic version of De Groote [6] and questions arising in the theory of concurrency [19]. The main technical ingredient is the structure of *order varieties*, which enable to express symmetry constraints in a sequent. An order variety is a structure which, provided a point of view (an element x in the base set), can be seen as a partial order on the complement of $\{x\}$. Order varieties can therefore be presented in different ways by changing the viewpoint, and are invariant under the change of presentation. In the sequent calculus, this idea of presentation corresponds to the ability of focusing on any formula to apply a rule. A good analogy is with cyclic permutations in cyLL, which enable to move the desired formula to the position where the rule is applicable, typically avoiding the problems of the 2-negations fragment.

Some important results are the following:
1. To any (strict) order ω we associate an order variety $V(\omega) = \overline{\omega}$ (definition 3.5 and proposition 3.8 (i)). If one considers that ω *presents* the order variety $\overline{\omega}$, one can raise the question of existence and unicity of the presentation.
2. *Existence, no unicity.* An order variety can always be presented by a partial order, indeed by focusing on one point of the support (theorem 3.10 and corollary 3.11). However the presentation is not unique since [4] $\overline{\omega_1 \parallel \omega_2} = \overline{\omega_1 < \omega_2} = \overline{\omega_1 > \omega_2}$ (proposition 3.8 (ii)).
3. *Unicity, no existence.* If the support of the variety $\boldsymbol{\alpha}$ is partitionned into two non-empty sets X_1 and X_2, there is at most one pair of partial orders ω_1, ω_2 on X_1, X_2 respectively, such that $\boldsymbol{\alpha} = \overline{\omega_1 \parallel \omega_2} = \overline{\omega_1 < \omega_2}$ (proposition 3.9). For an arbitrary partition, such a presentation does not necessarily exist. The existence of a presentation holds, however, when X_2 is a singleton $\{x\}$, and this corresponds to focusing on one point as above.

[3] Another possible — maybe preferable — notation for the sequential connective is the Weierstraß symbol $\wp$. However, $\wp$ has already been used in many papers as an alternative to $\parr$ for the commutative par, so we prefer to make use of a different symbol for sequential, namely $\triangledown$, whose design also has the advantage of recalling both $\parr$ and disjunction.
[4] $\omega_1 \parallel \omega_2$ and $\omega_1 < \omega_2$ are respectively the parallel and serial sums of ω_1 and ω_2. See definition 3.7.

4. Sequents are order varieties of (occurrences of) formulas. If a sequent — i.e. an order variety $\boldsymbol{\alpha}$ on a set Γ of formulas — is provable, then Γ, the support of $\boldsymbol{\alpha}$, is the set of conclusions of a proof-net π (adequacy: theorem 4.3) and $\boldsymbol{\alpha}$ is induced by the proof net π via definition 3.23.
5. In practice, in the sequent calculus, one may prefer to work with presentations, i.e. partial orders, rather than order varieties. It is possible to define a sequent calculus on presentations: this is done in [17,18]. In that case, one needs to add structural rules, *seesaw* and *entropy*:

$$\frac{\vdash \Gamma \parallel \Delta}{\vdash \Gamma < \Delta}\,\text{seesaw} \qquad \frac{\vdash \Gamma[\Delta < \Sigma]}{\vdash \Gamma[\Delta \parallel \Sigma]}\,\text{entropy}$$

which enable to change the presentation. In the present paper, we simply consider the sequent calculus on order varieties, which is completely free from structural rules. In particular, there is no rule of entropy, in the same way as there is no need for an entropy link in proof nets. In the sequent calculus on order varieties, entropy is ingeniously dissimulated in the absence of condition on the introduction rule for $\parr$: it is always possible to focus on 2 formulas A and B in a sequent (order variety), but glueing the two formulas (to form $A \parr B$) might result in a weakening of the order variety. In proof nets, entropy is given by the fact that every switching for $\parr$ is a switching for $\triangledown$.
6. A third alternative sequent calculus would be a kind of focalized calculus in the style of Andreoli, on order varieties, *with* entropy. Order varieties could be written $\Gamma \mid \Delta$, the symbol $\mid$ standing for $\parallel$ or $<$ indifferently. Then $\Gamma \mid A$ for instance would stand for an order variety where we have focused on A. Nevertheless, we would have to add structural rules to change the focus, like $\vdash \Gamma < \Delta \mid \Sigma$ into $\vdash \Gamma \mid \Delta < \Sigma$ ...

Still a difficulty: the name for the resulting logic? "Mixed non-commutative / commutative linear logic" is too long. On the other hand non-commutativity practically implies linearity and it includes commutativity as a particular case, so we have chosen to call it simply *non-commutative logic*, NL.

The present paper introduces the multiplicative fragment MNL of non-commutative logic, which extends both linear logic and cyclic linear logic: proof nets and cut elimination (section 2), order varieties (section 3), sequent calculus and sequentialization (section 4 and appendix A).

2 PROOF NETS AND CUT ELIMINATION

2.1 Language

Definition 2.1 (Formulas of MNL) *The* formulas (of MNL) *are built from atoms $p, q, \ldots, p^\perp, q^\perp, \ldots$ and the following multiplicative connectives:*
— *the non-commutative conjunction $\odot$,* next*,*
— *the non-commutative disjunction $\triangledown$,* sequential*,*
— *the commutative conjunction $\otimes$,* times*,*
— *the commutative disjunction $\parr$,* par*.*

Definition 2.2 (Negation) Negation *is defined by De Morgan rules:*

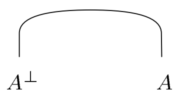
$$
\begin{array}{ll}
(p)^\perp = p^\perp & (p^\perp)^\perp = p \\
(A \odot B)^\perp = B^\perp \triangledown A^\perp & (A \triangledown B)^\perp = B^\perp \odot A^\perp \\
(A \otimes B)^\perp = B^\perp \parr A^\perp & (A \parr B)^\perp = B^\perp \otimes A^\perp
\end{array}
$$

Negation is then an involution: for any formula A, $A^{\perp\perp} = A$.

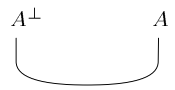
Definition 2.3 (Formulas of MLL, McyLL) — *The* formulas of MLL (resp. McyLL) *are built from atoms $p, q, \ldots, p^\perp, q^\perp, \ldots$ and the connectives $\otimes$ and $\parr$ (resp. $\odot$ and $\triangledown$).*
— *For every formula A of MNL, we define the formula A^* of MLL, called the* commutative translation of *A, by induction: $A^* = A$ if A is atomic, $(A \otimes B)^* = A^* \otimes B^*$, $(A \parr B)^* = A^* \parr B^*$, $(A \odot B)^* = A^* \otimes B^*$, $(A \triangledown B)^* = A^* \parr B^*$.*

2.2 Proof nets

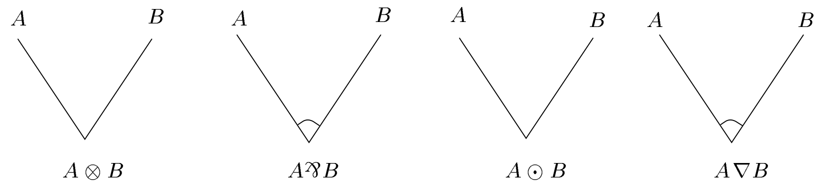
Definition 2.4 (Links) Links of MNL *are the following graphs where the vertices are labeled by formulas of MNL:*
— identity *links:*



with two conclusions *$A^\perp$ and A and no* premisse*;*
— cut *links:*



with two premisses $A^\perp$ *and* A *and no* conclusion*;*
— $\otimes$,⅋,$\odot$,$\triangledown$*-links:*



where the formula A is the first premisse*, the formula B is the* second premisse *and the third formula is the* conclusion *of the link.*

Definition 2.5 (Proof structures) — *A* proof structure (of MNL) *is a graph built from links of MNL such that every occurrence of formula is the conclusion of exactly one link of MNL and the premisse of at most one link.*
— *If π is a proof structure of MNL, the* conclusions *of π are the occurrences of formulas in π which are not premisses of a link.*
— *A* proof structure of MLL (resp. McyLL) *is a proof structure labeled with only MLL (resp. McyLL) formulas.*

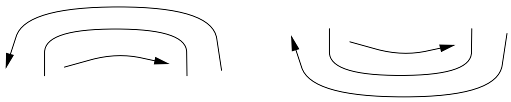
Definition 2.6 (π^*) *If π is a proof structure of MNL, then its* commutative translation π^* *is the proof structure of MLL obtained by replacing every occurrence of formula A by A^*, every $\odot$-link by a $\otimes$-link and every $\triangledown$-link by a* ⅋*-link.*

We consider as [8] formulas with *decorations*: $\uparrow$ (question) or $\downarrow$ (answer). A *decorated formula* is of the form $A^\uparrow$ or $A^\downarrow$, where A is a formula of MNL. Define $\bar{\uparrow} = \downarrow$, $\bar{\downarrow} = \uparrow$. For each link l of MNL, we can consider two sets of decorated formulas:

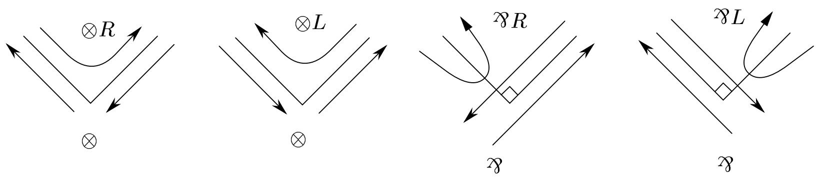
— l^{in} is the set of all decorated formulas A^x, where A is a premisse of l and x is $\downarrow$, or A is a conclusion of l and x is $\uparrow$;
— l^{out} is the set of all A^x, where A is a premisse of l and x is $\uparrow$, or A is a conclusion of l and x is $\downarrow$.

Definition 2.7 (Switchings) *For each link l of MNL we define a set $S(l)$ of (partial) functions from l^{in} to l^{out}, called the* switching positions *of l, as follows:*
— *if l is an identitylink $\overline{A^\perp \;\; A}$, then $S(l) = \{id\}$ where*
$id : (A^\perp)^\uparrow \mapsto A^\downarrow, A^\uparrow \mapsto (A^\perp)^\downarrow$*;*

— *if l is a cut link $\underline{A^\perp \quad A}$, then $S(l) = \{cut\}$ where* $cut : (A^\perp)^\downarrow \mapsto A^\uparrow, A^\downarrow \mapsto (A^\perp)^\uparrow$;

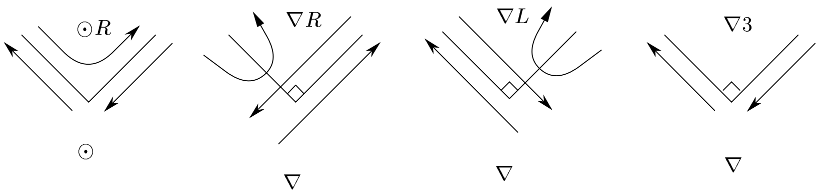


— *if l is a $\otimes$-link $\frac{A \quad B}{A \otimes B}$, then $S(l) = \{\otimes R, \otimes L\}$ where*
$\otimes R : (A \otimes B)^\uparrow \mapsto A^\uparrow, A^\downarrow \mapsto B^\uparrow, B^\downarrow \mapsto (A \otimes B)^\downarrow$ *and*
$\otimes L : (A \otimes B)^\uparrow \mapsto B^\uparrow, A^\downarrow \mapsto (A \otimes B)^\downarrow, B^\downarrow \mapsto A^\uparrow$;
— *if l is a $\parr$-link $\frac{A \quad B}{A \parr B}$, then $S(l) = \{\parr R, \parr L\}$ where*
$\parr R : (A \parr B)^\uparrow \mapsto B^\uparrow, A^\downarrow \mapsto A^\uparrow, B^\downarrow \mapsto (A \parr B)^\downarrow$ *and*
$\parr L : (A \parr B)^\uparrow \mapsto A^\uparrow, A^\downarrow \mapsto (A \parr B)^\downarrow, B^\downarrow \mapsto B^\uparrow$;



— *if l is a $\odot$-link $\frac{A \quad B}{A \odot B}$, then $S(l) = \{\odot R\}$ where*
$\odot R : (A \odot B)^\uparrow \mapsto A^\uparrow, A^\downarrow \mapsto B^\uparrow, B^\downarrow \mapsto (A \odot B)^\downarrow$;

— *if l is a $\triangledown$-link $\frac{A \quad B}{A \triangledown B}$, then $S(l) = \{\triangledown R, \triangledown L, \triangledown 3\}$ where*
$\triangledown R : (A \triangledown B)^\uparrow \mapsto B^\uparrow, A^\downarrow \mapsto A^\uparrow, B^\downarrow \mapsto (A \triangledown B)^\downarrow$,
$\triangledown L : (A \triangledown B)^\uparrow \mapsto A^\uparrow, A^\downarrow \mapsto (A \triangledown B)^\downarrow, B^\downarrow \mapsto B^\uparrow$ *and*
$\triangledown 3 : (A \triangledown B)^\uparrow \mapsto A^\uparrow, B^\downarrow \mapsto (A \triangledown B)^\downarrow$.



Given a proof structure π, a switching *for π is a function s such that for every link l of π, $s(l) \in S(l)$. A switching s for π is* $\triangledown$3-free *if for every $\triangledown$-link l, $s(l) \neq \triangledown 3$.*

Definition 2.8 (Trips) — *Let π be a proof structure and s a switching for π. The* switched proof structure *$s(\pi)$ is the oriented graph with vertices the decorated formulas labeling π, and with an oriented edge from A^x to B^y iff either $B^y = s(l)(A^x)$ for some link l in π, or $A^x = C^\downarrow$ and $B^y = C^\uparrow$ for some conclusion C of π.*

— *A* trip *in $s(\pi)$ is a cycle or a maximal path in $s(\pi)$.*

Remark. Let π be a proof structure of MNL and s a switching for π. If v is a trip in $s(\pi)$ and not a cycle, v begins with $B^{\uparrow}$ where B is the second premisse of a $\triangledown$-link l with $s(l) = \triangledown 3$, and ends with $A^{\downarrow}$ where A is the first premisse of a $\triangledown$-link l' with $s(l') = \triangledown 3$.

Facts 2.9 *(i) If v and v' are different trips in $s(\pi)$, then v and v' are disjoint. (ii) If s is $\triangledown 3$-free, then every trip in $s(\pi)$ is a cycle.*

We can now define proof nets for MNL, a class containing all the usual proof nets of MLL and McyLL.

Definition 2.10 (Long trips and bilateral trips) *Let π be a proof structure of MNL and s a switching for π. — A trip v in $s(\pi)$ is a* long trip *if v is a cycle and in v every occurrence of formula A in π occurs twice, once as $A^{\uparrow}$ once as $A^{\downarrow}$.*
— A cycle v in $s(\pi)$ is bilateral *(see [5]) if v is not of the form $A^{x}, \dots, B^{y}, \dots, A^{\bar{x}}, \dots, B^{\bar{y}}, \dots, A^{x}$ where A and B are occurrences of formulas in π.*

Definition 2.11 (Proof nets) *π is a* proof net (of MNL) *iff π is a proof structure of MNL and for every switching s for π:*
(1) there is exactly one cycle σ in $s(\pi)$,
(2) σ contains all the conclusions,
(3) σ is bilateral.

Facts 2.12 *(i) If π is a proof net of MNL, and s is a $\triangledown 3$-free switching for π, then the unique cycle σ in $s(\pi)$ is a long trip.*
(ii) If π is a proof net of MNL and s a switching for π, then the oriented graph with vertices the conclusions of π and an oriented edge from a conclusion A to a conclusion B iff there is no conclusion between $B^{\uparrow}$ and $A^{\downarrow}$ in the unique cycle in $s(\pi)$, is an oriented cycle, called the cycle of the conclusions in $s(\pi)$.

Definition 2.13 (Proof nets of MLL (McyLL)) *A* proof net of MLL (resp. McyLL) *is a proof structure of MLL (resp. McyLL) that is a proof net of MNL.*

Proposition 2.14 *(i) π is a proof net of MLL iff for every switching s for π there is a long trip in $s(\pi)$.*
(ii) If π is a proof net of MCyLL and s and s' are switchings for π, then the cycle of the conclusions in $s(\pi)$ is equal to the cycle of the conclusions in $s'(\pi)$.

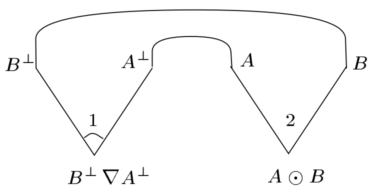
Proof. **(i)** If π is a proof structure of MLL, and s is a switching for π, all the trips in $s(\pi)$ are cycles (facts 2.9); but since π is a proof net, there is exactly one cycle σ, whence σ is a long trip. Conversely, assume that there is a long trip in $s(\pi)$ for every switching s for π: if s is a switching for π, the long trip

σ in $s(\pi)$ is the unique cycle in $s(\pi)$ and satisfies (2) (obvious) and (3) (see Danos-Régnier [5]).

(ii) If π has no $\triangledown$-link, the result is obvious. If π has $\triangledown$-links, and l is a $\triangledown$-link in π with conclusion $A\triangledown B$, then for every switching s, no conclusion occurs in the unique cycle σ in $s(\pi)$ between $B^{\uparrow}$ and $A^{\downarrow}$: indeed, otherwise, by taking the switching s' such that $s'(l) = \triangledown 3$ and $s'(l') = s(l)$ for every link $l' \neq l$, we get a contradiction with the fact that π is a proof net. This gives what is stated in the lemma. ■

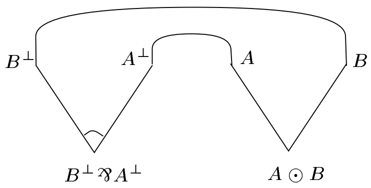
Examples.

- $\psi_1 =$

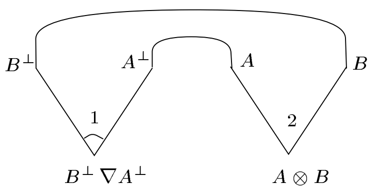


is a proof net (in fact a proof net of McyLL). The two trips for $s(l_1) = \triangledown 3$ are: $v_1 = (B^{\perp}\triangledown A^{\perp})^{\uparrow}B^{\perp\uparrow}B^{\downarrow}(A \odot B)^{\downarrow}(A \odot B)^{\uparrow}A^{\uparrow}A^{\perp\downarrow}(B^{\perp}\triangledown A^{\perp})^{\downarrow}(B^{\perp}\triangledown A^{\perp})^{\uparrow}$, a cycle containing both conclusions and bilateral, and
$v_2 = A^{\perp\uparrow}A^{\downarrow}B^{\uparrow}B^{\perp\downarrow}$.

- $\psi_2 =$

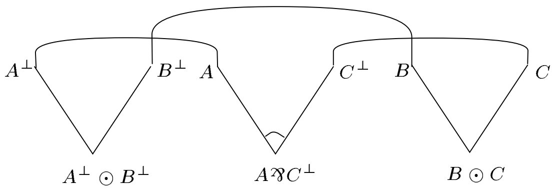


is a proof net, but



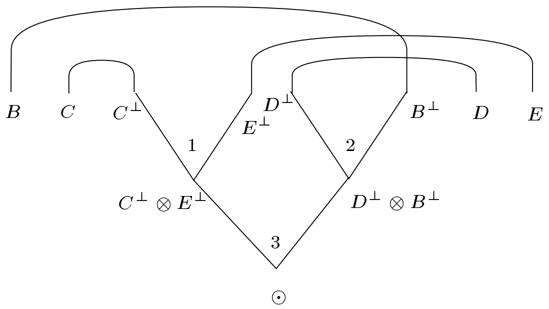
is not a proof net: with $s(l_1) = \triangledown 3$ and $s(l_2) = \otimes L$, the trips are
$A^{\perp\uparrow}A^{\downarrow}(A \otimes B)^{\downarrow}(A \otimes B)^{\uparrow}B^{\uparrow}B^{\perp\downarrow}$ and
$(B^{\perp}\triangledown A^{\perp})^{\uparrow}B^{\perp\uparrow}B^{\downarrow}A^{\uparrow}A^{\perp\downarrow}(B^{\perp}\triangledown A^{\perp})^{\downarrow}(B^{\perp}\triangledown A^{\perp})^{\uparrow}$, a cycle which does not contain the conclusion $A \otimes B$, contradicting condition (2).

- $\psi_3 =$



is a proof net. For every switching s, the cycle of the conclusions in $s(\psi_3)$ is $B \odot C \to A ⅋ C^\perp \to A^\perp \odot B^\perp \to B \odot C$. But the proof structure obtained by replacing ⅋ by $\triangledown$ is not a proof net (even though, of course, its commutative translation is a proof net of LL): to see why, take $\triangledown 3$ for the $\triangledown$-link.

- $\psi_4 =$



is a proof net. Call A the conclusion $(C^\perp \otimes E^\perp) \odot (D^\perp \otimes B^\perp)$. The cycle of the conclusions in $s(\psi_4)$ is
$A \to B \to D \to E \to C \to A$ if $s(l_1) = s(l_2) = \otimes R$,
$A \to D \to B \to E \to C \to A$ if $s(l_1) = \otimes R$ and $s(l_2) = \otimes L$,
$A \to B \to D \to C \to E \to A$ if $s(l_1) = \otimes L$ and $s(l_2) = \otimes R$,
$A \to D \to B \to C \to E \to A$ if $s(l_1) = s(l_2) = \otimes L$.

2.3 An equivalent definition of proof nets

Theorem 2.20 tells that the correctness criterion in definition 2.11 is equivalent to the correctness in the commutative sense plus some conditions on the *inner* parts of $\triangledown$ links. To prove it, we need a few definitions.

Definition 2.15 *Let π be a proof structure of MNL.*
— *Let s be a $\triangledown 3$-free switching for π. Define a switching s^* for π^* by: for every link l of π,*

$$s^*(l^*) = (s(l))^*$$

where for $x = R$ or $x = L$, $(\otimes x)^* = \otimes x$, $(⅋ x)^* = ⅋ x$, $(\triangledown x)^* = ⅋ x$ and $(\odot R)^* = \otimes R$.
For every trip v in $s(\pi)$, v^* is obtained from v by replacing each decorated occurrence of formula A^x in v by $(A^*)^x$.

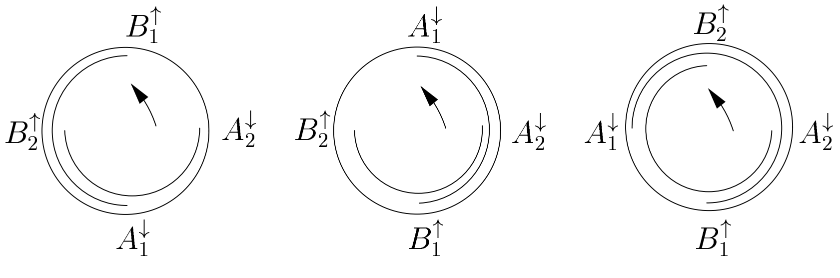
— *Let s be a switching for π^* such that for all the $\odot$-links l in π, $s(l^*) \neq \otimes L$. Define a $\triangledown 3$-free switching $s^\bullet$ for π by: for every $\odot$-link or $\triangledown$-link l of π,*

$$s^\bullet(l) = (s(l^*))^\bullet \text{ if } l \text{ is a } \odot\text{-link or a } \triangledown\text{-link,}$$
$$s^\bullet(l) = s(l^*) \text{ otherwise,}$$

where $(\otimes R)^\bullet = \odot R$, $(⅋R)^\bullet = \triangledown R$ and $(⅋L)^\bullet = \triangledown L$.
For every trip v in $s(\pi^*)$, $v^\bullet$ is obtained from v by replacing each decorated occurrence of formula $(A^*)^x$ in v by A^x.

Facts 2.16 *Let π be a proof structure of MNL.*
(i) Let s be a $\triangledown 3$-free switching for π.
— $s^{*\bullet} = s$.
— *If v is a trip in $s(\pi)$, then v^* is a trip in $s^*(\pi^*)$.*
— *If v is a cycle (resp. a long trip, a bilateral trip, a trip containing all the conclusions), then so is v^*.*
(ii) Let s be a switching for π^*, such that for all the $\odot$-links l in π, $s(l^*) \neq \otimes L$.
— $s^{\bullet *} = s$
— *If v is a trip in $s(\pi^*)$, then $v^\bullet$ is a trip in $s^\bullet(\pi)$.*
— *If v is a cycle (a long trip, a bilateral trip, a trip containing all the conclusions), then so is $v^\bullet$.*

Definition 2.17 (Inner, outer, inf parts of $\triangledown$, ⅋-links. Overlapping) *Let π be a proof structure of MNL, s a switching for π, v a trip in $s(\pi)$.*
— *Let l be a ⅋-link or a $\triangledown$-link of π, with first premisse A, second premisse B and conclusion C. When it exists, the part of v from $B^\uparrow$ to $A^\downarrow$ (resp. the part of v from $A^\uparrow$ to $B^\downarrow$, the part of v from $C^\downarrow$ to $C^\uparrow$) is called the* inner *(resp.* outer, inf*) part of l in v. The* sup *part of l in v is the union of the inner and outer parts of l in v.*
— *Let $l_1 = \frac{A_1 \quad B_1}{A_1 \triangledown B_1}$ and $l_2 = \frac{A_2 \quad B_2}{A_2 \triangledown B_2}$ be two $\triangledown$-links of π. The inner parts of l_1 and l_2 in v* overlap *if v is of one of the following forms:*



in other terms they do not overlap *if either one is included into the other, or they are disjoint.*

Definition 2.18 (Deletion and insertion of inner parts) *Let π be a proof structure of MNL, and $l = \frac{A \quad B}{A \triangledown B}$ be a $\triangledown$-link of π.*
(i) If s is a switching for π, then s^{l3} is the switching for π defined as follows:

$$s^{l3}(l') = s(l') \text{ if } l' \neq l,$$
$$s^{l3}(l) = \triangledown 3.$$

If v a trip in $s(\pi)$ containing the inner part of l, v^{l3} is obtained from v by deleting the inner part of the $\triangledown$-link l.
(ii) If s is a switching for π such that $s(l) = \triangledown 3$, then s^{lR} is the switching for π defined as follows:

$$s^{lR}(l') = s(l') \text{ if } l' \neq l,$$
$$s^{lR}(l) = \triangledown R.$$

If v a trip in $s(\pi)$ and there is a trip v' in $s(\pi)$ containing the inner part of l, then v^{lR} is obtained from v by inserting between $(A \triangledown B)^{\uparrow}$ and $A^{\uparrow}$ the inner part of the $\triangledown$-link l (contained in v'). (If $(A \triangledown B)^{\uparrow}$ and $A^{\uparrow}$ are not in v, then $v^{lR} = v$.)

Facts 2.19 *Let π be a proof structure of MNL, and $l = \frac{A \quad B}{A \triangledown B}$ be a $\triangledown$-link of π.*
(i) Let s be a switching for π, and v a trip in $s(\pi)$ containing the inner part of l.
— If $s(l) = \triangledown R$, then $(s^{l3})^{lR} = s$.
— v^{l3} is a trip in $s^{l3}(\pi)$.
— If v is a cycle (resp. a bilateral trip), then so is v^{l3}.
— If v contains all the conclusions, and no conclusion is in the inner part of l, then v^{l3} contains all the conclusions.
(ii) Let s be a switching for π such that $s(l) = \triangledown 3$, and v a trip in $s(\pi)$, and assume there is a trip v' in $s(\pi)$ containing the inner part of l.
— $(s^{lR})^{l3} = s$.
— v^{lR} is a trip in $s^{lR}(\pi)$.
— If v is a cycle (resp. a cycle containing all the conclusions), then so is v^{lR}.

Theorem 2.20 *π is a proof net of MNL iff π is a proof structure of MNL such that*
(i) π^* is a proof net of MLL,
(ii) for every $\triangledown 3$-free switching s for π, the inner parts of $\triangledown$-links in the unique cycle σ in $s(\pi)$ contain no conclusion and do not overlap.

Proof. ($\Rightarrow$) Assume π is a proof net of MNL.
(i) We prove that π^* is a proof net of MLL. Let s be a switching for π^*, and n be the number of $\odot$-links l in π such that $s(l^*) = \otimes L$. We prove, by induction on n, that there is a bilateral long trip in $s(\pi^*)$.

If $n = 0$, then $s^\bullet$ is a $\triangledown 3$-free switching for π; since π is a proof net of MNL, there is a unique cycle σ in $s^\bullet(\pi)$ which is a bilateral long trip; but then σ^* is a bilateral long trip in $s(\pi^*)$ by facts 2.19.

If $n > 0$, then let $l = \frac{A \quad B}{A \odot B}$ be a $\odot$-link in π such that $s(l^*) = \otimes L$: change s into s' by taking $s'(l^*) = \otimes R$ and $s'(l'^*) = s(l'^*)$ for all the links $l' \neq l$. By induction hypothesis, there is a unique bilateral long trip σ in $s'(\pi^*)$: since σ is a long trip, in σ we find the following adjacent decorated occurrences of formulas:

$(A^* \otimes B^*)^\uparrow, A^{*\uparrow}$
$A^{*\downarrow}, B^{*\uparrow}$
$B^{*\downarrow}, (A^* \otimes B^*)^\downarrow$

and since σ is bilateral the cycle σ must have the following form:

$$(A^* \otimes B^*)^\uparrow, (A^*)^\uparrow, \overset{1}{\cdots}, (A^*)^\downarrow, (B^*)^\uparrow, \overset{2}{\cdots}, (B^*)^\downarrow, (A^* \otimes B^*)^\downarrow, \overset{3}{\cdots}, (A^* \otimes B^*)^\uparrow.$$

Therefore, by taking the switching s, we get the following long trip in $s(\pi^*)$:

$$(A^* \otimes B^*)^\uparrow, (B^*)^\uparrow, \overset{2}{\cdots}, (B^*)^\downarrow, (A^*)^\uparrow, \overset{1}{\cdots}, (A^*)^\downarrow, (A^* \otimes B^*)^\downarrow, \overset{3}{\cdots}, (A^* \otimes B^*)^\uparrow$$

which is a bilateral trip as well.

(ii) We prove that for every $\triangledown 3$-free switching s for π the inner parts of $\triangledown$-links in the unique cycle σ in $s(\pi)$ contain no conclusion: this is immediate. Indeed, by absurdum, let s be a $\triangledown 3$-free switching for π and assume the inner part of a $\triangledown$-link l contains a conclusion: by facts 2.12, σ is a long trip in $s(\pi)$, so the inner part of l is included in σ; but then σ^{l3} is a cycle in $s^{l3}(\pi)$, whence the unique cycle in $s^{l3}(\pi)$ (since π is a proof net of MNL), and it does not contain all the conclusions: contradiction.

Finally, we prove that for every $\triangledown 3$-free switching s for π the inner parts of $\triangledown$-links in the long trip σ in $s(\pi)$ do not overlap. By absurdum, assume s is a $\triangledown 3$-free switching for π, and $l_1 = \frac{A_1 \quad B_1}{A_1 \triangledown B_1}$ and $l_2 = \frac{A_2 \quad B_2}{A_2 \triangledown B_2}$, are two $\triangledown$-links in π such that, in the long trip, the inner parts of l_1 and l_2 overlap. So, $s(l_1)$ and $s(l_2)$ are not $\triangledown 3$ and we have two possible cases:
a) in the inner part of l_1 there is ${B_2}^\uparrow$ but not ${A_2}^\downarrow$,
b) in the inner part of l_2 there is ${B_1}^\uparrow$ but not ${A_1}^\downarrow$,
c) the union of the two inner parts is σ.

In case a) we have four possible subcases, depending on $s(l_1)$ and $s(l_2)$:
a1) the unique cycle σ is
$(A_1 \triangledown B_1)^\uparrow, {B_1}^\uparrow, \overset{1}{\cdots}, (A_2 \triangledown B_2)^\uparrow, {B_2}^\uparrow, \overset{2}{\cdots}, {A_1}^\downarrow, {A_1}^\uparrow, \overset{3}{\cdots}, {A_2}^\downarrow, {A_2}^\uparrow, \overset{4}{\cdots}, (A_1 \triangledown B_1)^\uparrow,$
a2) the unique cycle σ is

$(A_1 \triangledown B_1)^\uparrow, B_1{}^\uparrow, \overset{1}{\cdots}, B_2{}^\downarrow, B_2{}^\uparrow, \overset{2}{\cdots}, A_1{}^\downarrow, A_1{}^\uparrow, \overset{3}{\cdots}, A_2{}^\downarrow, (A_2 \triangledown B_2)^\downarrow, \overset{4}{\cdots}, (A_1 \triangledown B_1)^\uparrow$,
a3) the unique cycle σ is
$(B_1)^\downarrow, B_1{}^\uparrow, \overset{1}{\cdots}, (A_2 \triangledown B_2)^\uparrow, B_2{}^\uparrow, \overset{2}{\cdots}, A_1{}^\downarrow, (A_1 \triangledown B_1)^\downarrow, \overset{3}{\cdots}, A_2{}^\downarrow, A_2{}^\uparrow, \overset{4}{\cdots}, B_1{}^\downarrow$,
a4) the unique cycle σ is
$(B_1)^\downarrow, B_1{}^\uparrow, \overset{1}{\cdots}, B_2{}^\downarrow, B_2{}^\uparrow, \overset{2}{\cdots}, A_1{}^\downarrow, (A_1 \triangledown B_1)^\downarrow, \overset{3}{\cdots}, A_2{}^\downarrow, (A_2 \triangledown B_2)^\downarrow, \overset{4}{\cdots}, B_1{}^\downarrow$.

Now, consider $(s^{l_1 3})^{l_2 3}$: it is easy to see that there is no cycle in $(s^{l_1 3})^{l_2 3}(\pi)$; for instance, in the case a1), there are only the following two trips (none of them being a cycle):
$(B_1)^\uparrow, \overset{1}{\cdots}, (A_2 \triangledown B_2)^\uparrow, (A_2)^\uparrow, \overset{4}{\cdots}, (A_1 \triangledown B_1)^\uparrow, (A_1)^\uparrow, \overset{3}{\cdots}, (A_2)^\downarrow$
$(B_2)^\uparrow, \overset{2}{\cdots}, (A_1)^\downarrow$.
But this contradicts the hypothesis that π is a proof net of MNL.
The case b) is very similar.
The case c) is impossible here because inner parts do not contain conclusions.

($\Leftarrow$) Assume π is a proof structure of MNL such that:
(i) π^* is a proof net of MLL, and
(ii) for every $\triangledown 3$-free switching s for π, the inner parts of $\triangledown$-links in the unique cycle σ of $s(\pi)$ contain no conclusion and do not overlap.

Properties (i) and (ii) imply:
(ii') for every switching s for π, the inner parts of $\triangledown$-links in the unique cycle σ of $s(\pi)$ contain no conclusion and do not overlap.
Indeed, let s be a switching for π, and assume for a contradiction that there are inner parts of $\triangledown$-links containing conclusions or overlappings. Consider the $\triangledown$-free switching $s^{l_1 R, \cdots, l_n R}$ where $l_1, \ldots, l_n$ are the $\triangledown$-links such that $s(l_i) = \triangledown 3$. Then $(s^{l_1 R, \cdots, l_n R})^*$ is a switching for the proof net π^* of MLL, so there is a long trip σ^* in $(s^{l_1 R, \cdots, l_n R})^*(\pi^*)$, and therefore by facts 2.19, σ is a long trip in $s^{l_1 R, \cdots, l_n R}(\pi)$ where there are inner parts of $\triangledown$-links containing conclusions or overlappings: contradiction.

Now let s be a switching for π. By induction on the number n of the $\triangledown$-links l such that $s(l) = \triangledown 3$, we show that:
a) in $s(\pi)$ there is exactly one cycle σ,
b) σ contains all the conclusions,
c) σ is bilateral,
d) for every $\triangledown$-link l, there is a trip in $s(\pi)$ containing the inner part of l.

If $n = 0$, then s is $\triangledown 3$-free, so s^* is a switching for π^*. Since π^* is a proof net of MLL, there is a unique bilateral long trip σ^* in $s^*(\pi^*)$, and therefore σ is the unique bilateral long trip in $s(\pi)$, thus that a),b),c),d) are satisfied.

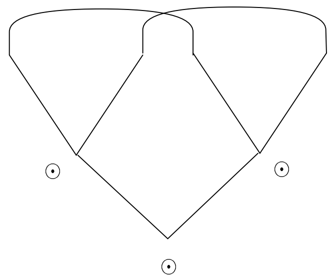
If $n > 0$, take a $\triangledown$-link l such that $s(l) = \triangledown 3$, and consider the switching s^{lR} for π. By induction hypothesis, there is exactly one cycle σ in $s^{lR}(\pi)$, σ is bilateral and contains all the conclusions, and d) is satisfied. Since $s^{lR}(\pi)$ satisfies d), either the inner part of l is in σ or the inner part of l is outside σ. If the inner part of l is in σ, then σ^{l3} is the unique cycle in $s(\pi)$, it contains all the conclusions (since by hypothesis the inner part of l contains no conclusion) and it is bilateral; moreover $s(\pi)$ satisfies d) because the inner parts of $\triangledown$-links do not overlap. If the inner part of l is outside σ, then the inner part of l is contained in some trip v, since $s^{lR}(\pi)$ satisfies d); therefore $\sigma^{l3} = \sigma$ is the unique cycle in $s(\pi)$, and it contains all the conclusions and is bilateral, and $s(\pi)$ satisfies d) again, because the inner parts of $\triangledown$-links do not overlap. ■

Remarks.

• Let π be a proof net of MLL, with atomic identity links, satisfying the following property: there exist $\otimes$-links $l_1, \ldots, l_n$, and $⅋$-links $l'_1, \ldots, l'_m$, such that for every switching s with $s(l_i) = \otimes R$, the inner parts of $l'_1, \cdots, l'_m$ contain no conclusion and do not overlap. Then by theorem 2.20, if we replace $l_1, \ldots, l_n$ by $\odot$-links and $l'_1, \ldots, l'_m$ by $\triangledown$-links, and the occurrences of formulas in a corresponding way, then we get a proof net π' of MNL.

The reader may check for instance that $\psi_1 \ldots \psi_4$ come from proof nets of MLL satisfying conditions stated in theorem 2.20.

• Theorem 2.20 tells in particular that condition (3) is necessary. Without it a proof net can even not be correct in the commutative sense. For instance, the following structure satisfies (1) and (2) for every switching, but is obviously not correct:

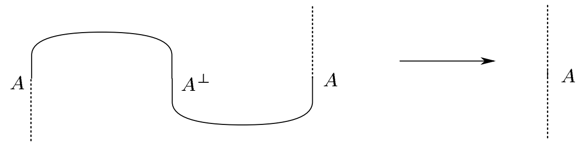


Condition (3) is in fact equivalent, modulo the other conditions, to the usual constraint on the order of passage through $\odot$ links (see lemma 2.9.7 of [8] where this constraint is given for commutative times links).

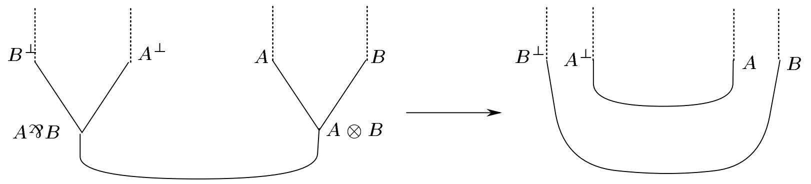
2.4 Reduction

Definition 2.21 Reduction rules *of proof structures are the following local rules:*

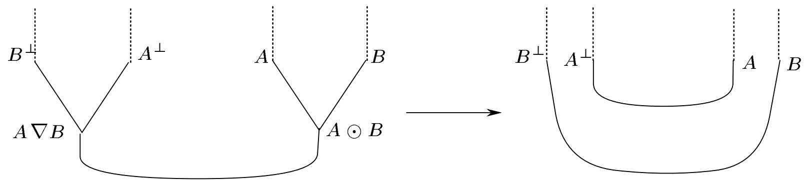
— *identity link:*



— *commutative links:*



— *non-commutative links:*



Theorem 2.22 (Stability of proof nets by reduction) *If π is a proof net and $\pi \to \pi'$, then π' is a proof net.*

Proof. Let π be a proof net and π' the structure obtained by eliminating the cut bearing on the pair of links $\frac{A \quad B}{AtB}$ and $\frac{B^\perp \quad A^\perp}{B^\perp t^\perp A^\perp}$ ($t \in \{\otimes, \odot\}$ and $\otimes^\perp = \parr$, $\odot^\perp = \triangledown$). We show the stability of conditions (1), (2) and (3) using the equivalent formulation given by theorem 2.20.

1. *Correctness of π'^*.* It has already been proved: see [8].

2. *Absence of conclusions in the inner parts of $\triangledown$-links.* Let s be a $\triangledown$3-free switching for π, $\frac{C \quad D}{C\nabla D}$ be a link in π', and s' be the restriction of s to π.

— $t = \odot$. Depending on the position of the link $\frac{B^\perp \quad A^\perp}{B^\perp \nabla A^\perp}$, the (long) trip v in $s(\pi)$ goes either like:

$A^{\perp\uparrow} \stackrel{1}{\ldots} B^{\perp\downarrow} B^{\perp\uparrow} \stackrel{2}{\ldots} A^{\perp\downarrow}(B^\perp \triangledown A^\perp)^\downarrow (A\odot B)^\uparrow A^\uparrow \stackrel{3}{\ldots} A^\downarrow B^\uparrow \stackrel{4}{\ldots} B^\downarrow (A\odot B)^\downarrow (B^\perp\triangledown A^\perp)^\uparrow \ldots A^{\perp\uparrow}$
or like

$B^{\perp\uparrow} \stackrel{2}{\ldots} A^{\perp\downarrow} A^{\perp\uparrow} \stackrel{1}{\ldots} B^{\perp\downarrow}(B^\perp \triangledown A^\perp)^\downarrow (A\odot B)^\uparrow A^\uparrow \stackrel{3}{\ldots} A^\downarrow B^\uparrow \stackrel{4}{\ldots} B^\downarrow (A\odot B)^\downarrow (B^\perp\triangledown A^\perp)^\uparrow \ldots B^{\perp\uparrow}$.

After reduction, the long trip v' in $s'(\pi')$ is:

$$B^{\downarrow}B^{\perp\uparrow} \overset{2}{\ldots} A^{\perp\downarrow}A^{\uparrow} \overset{3}{\ldots} A^{\downarrow}A^{\perp\uparrow} \overset{1}{\ldots} B^{\perp\downarrow}B^{\uparrow} \overset{4}{\ldots} B^{\downarrow}.$$

By hypothesis 1 contains no conclusion. The condition of non-overlapping of inner parts of $\triangledown$-links in π implies that $D^{\uparrow}$ is in 1 iff $C^{\downarrow}$ is in 1 iff $D^{\uparrow}\ldots C^{\downarrow}$ is in 1, and in this case the inner part $D^{\uparrow}\ldots C^{\downarrow}$ contains no conclusion. Otherwise $D^{\uparrow}$ and $C^{\downarrow}$ are in $2,3,4$: now the segments $2,3$ and 4 are in this order in the long trip, before and after reduction, so $D^{\uparrow}\ldots C^{\downarrow}$ contains a conclusion in v' iff it contains a conclusion in v, and this is false by hypothesis, qed.

— $t = \otimes$. With the same notations as above for the 4 segments of the cycle, the trip v in π is $1,2,3,4$ or $2,1,3,4$ or $1,2,4,3$ or $2,1,4,3$ up to cyclic permutation, and the trip v' in π' is always $1,4,2,3$ up to cyclic permutation[5]. Assume $D^{\uparrow}\ldots C^{\downarrow}$ contains a conclusion ϕ in π'. The position of these three formulas in $1,4,2,3$ is a triple $(D^{\uparrow},\phi,C^{\downarrow}) \in \{1,2,3,4\}^3$. But for any triple $(x,y,z) \in \{1,2,3,4\}^3$ such that (x,y,z) is in the order variety (1423), (x,y,z) is in at least one of the order varieties (1234) or (2134) or (1243) or (2143): if two positions among x,y,z are equal, it is obvious; besides $(1,4,2) \in (2134)$, $(1,4,3) \in (1243)$, $(1,2,3) \in (1234)$ and $(4,2,3) \in (2134)$. So some trip in π contains a conclusion ϕ: contradiction.

3. *Non-overlapping of the inner parts of $\triangledown$-links.* Let s be a $\triangledown$3-free switching. Let $\frac{C_1 \quad C_2}{C_1 \triangledown C_2}$ and $\frac{D_1 \quad D_2}{D_1 \triangledown D_2}$ be two others links.

— $t = \odot$. We use the above notations. In π', $D_2^{\uparrow}$ is in 1 iff $D_1^{\downarrow}$ is in 1 iff $D_2^{\uparrow}\ldots D_1^{\downarrow}$ is in 1, it is therefore also the case in π, and then for every position of $C_2^{\uparrow}$ and $C_1^{\downarrow}$, an overlapping in π' comes from an overlapping in at least one of the trips $1,2,3,4$ and $2,1,3,4$ in π.

The case when $C_2^{\uparrow}\ldots C_1^{\downarrow}$ is in 1 is similar.

Otherwise all of $D_2^{\uparrow}$, $D_1^{\downarrow}$, $C_2^{\uparrow}$ and $C_1^{\downarrow}$ are in $2,3,4$, and the overlapping is preserved, as the order of the segments $2,3$ and 4 is preserved.

— $t = \otimes$. Let $C_2^{\uparrow}, D_2^{\uparrow}, C_1^{\downarrow}, D_1^{\downarrow}$ be an overlapping in π' (the other case is absolutely similar), let us show that it comes from an overlapping in π. If the 4 formulas are spread over 1, 2 or 3 segments (among the segments $1,2,3$ and 4), then it is clear by the same argument as above. Besides, they cannot be spread over the 4 segments. Indeed, if for instance $C_2^{\uparrow}, D_2^{\uparrow}, C_1^{\downarrow}, D_1^{\downarrow}$ are respectively in $1,4,2,3$, then the trip of the form $2,1,4,3$ in π forbids on one hand

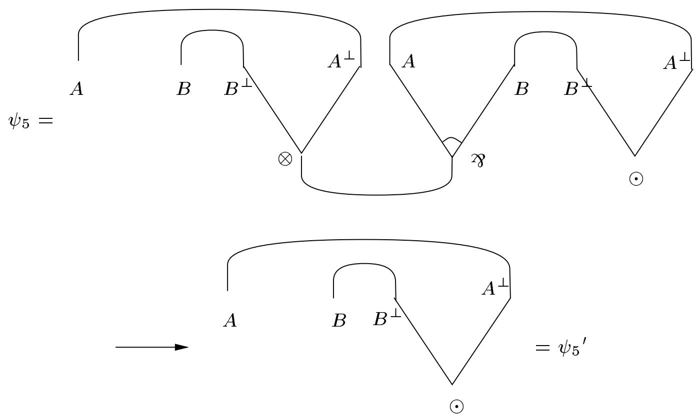
[5] We shall refer to "$1,4,2,3$ up to cyclic permutation" as "the order variety (1423)". Although it should be intuitive enough in this case, precise definitions are in section 3.

any conclusion in 3 and 4 (no conclusion in the inner part $C_2^{\uparrow} \dots C_1^{\downarrow}$ in π), and on the other hand any conclusion in 1 and 2 (no conclusion in $D_2^{\uparrow} \dots D_1^{\downarrow}$), what is impossible, because π^* is a proof net of MLL and must therefore have at least one conclusion. The others cyclic permutations of $C_2^{\uparrow}, D_2^{\uparrow}, C_1^{\downarrow}, D_1^{\downarrow}$ in $1, 4, 2, 3$ give rise to an analogous argument. ■

Theorem 2.23 *Reduction of proof nets is strongly normalizing and confluent.*

Proof. Obviously $\pi \to \pi'$ iff $\pi^* \to \pi'^*$, therefore see [8]. ■

Example.



The cycle of the conclusions in $s(\psi_5)$ is: $A \to B \to B^\perp \odot A^\perp \to A$ or $B \to A \to B^\perp \odot A^\perp \to B$, depending on the switching s.
In $s(\psi_5')$, the cycle of the conclusions is: $A \to B \to B^\perp \odot A^\perp \to A$ for every swiching s.

3 ORDER VARIETIES

3.1 Order varieties and partial orders

Order varieties are structures that can be presented as *partial* orders in several ways, the idea being that of the oriented circle which becomes a total order as soon as an origin is fixed.

Definition 3.1 (Order varieties) *Let E be a set. An* order variety *on E is a ternary relation $\boldsymbol{\alpha}$ which is*
— cyclic*: $\forall x, y, z \in E, \boldsymbol{\alpha}(x, y, z) \Rightarrow \boldsymbol{\alpha}(y, z, x)$,*
— anti-reflexive*: $\forall x, y \in E, \neg\boldsymbol{\alpha}(x, x, y)$,*
— transitive*: $\forall x, y, z, t \in E,\ \boldsymbol{\alpha}(x, y, z) \wedge \boldsymbol{\alpha}(z, t, x) \Rightarrow \boldsymbol{\alpha}(y, z, t)$,*
— spreading*: $\forall x, y, z, t \in E,\ \boldsymbol{\alpha}(x, y, z) \Rightarrow \boldsymbol{\alpha}(t, y, z) \vee \boldsymbol{\alpha}(x, t, z) \vee \boldsymbol{\alpha}(x, y, t)$.*

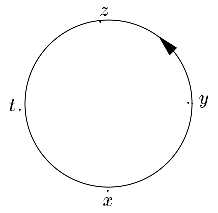
An order variety $\boldsymbol{\alpha}$ *on* E *is said* total *when* $\forall x, y, z \in E$, $x \neq y \neq z \neq x \Rightarrow \boldsymbol{\alpha}(x, y, z) \vee \boldsymbol{\alpha}(z, y, x)$.

Ternary relations satisfying the first three axioms have been studied by Novák [15] and called *cyclic orders*, but only the spreading condition gives a satisfactory connection between order varieties and their presentations by orders (theorem 3.10).

A few elementary properties and examples:

Lemma 3.2 *(i) Transitivity of an order variety* $\boldsymbol{\alpha}$ *implies* $\boldsymbol{\alpha}(x, y, z) \wedge \boldsymbol{\alpha}(z, t, x) \Rightarrow \boldsymbol{\alpha}(t, x, y)$ *as well.*
(ii) An order variety $\boldsymbol{\alpha}$ *on a set* E *is* anti-symetrical*:* $\forall x, y, z \in E$, $\neg\boldsymbol{\alpha}(x, y, z) \vee \neg\boldsymbol{\alpha}(z, y, x)$.

Proof. **(i)**



(ii) Let $\boldsymbol{\alpha}$ be an order variety. $\boldsymbol{\alpha}(x, y, z) \wedge \boldsymbol{\alpha}(z, y, x)$ implies $\boldsymbol{\alpha}(y, x, y)$ (transitivity) i.e. $\boldsymbol{\alpha}(y, y, x)$ (cyclicity), and this does not hold. ■

Examples.
• As expected, if $\boldsymbol{\alpha}$ is a total order variety, $\boldsymbol{\alpha}(x, y, z)$ can be read as "y is between x and z".
• The empty ternary relation on any set E is an order variety on E, called the *empty order variety* on E, denoted by $\emptyset_E$, or simply $\emptyset$ if there is no ambiguity.
• The cyclic closure of $\{(a, b, c)\}$ satifies the first three axioms, but it is not an order variety on $\{a, b, c, d\}$ (only on $\{a, b, c\}$).

Definition 3.3 ($\rightarrow_{\boldsymbol{\alpha}}$) *Any order variety* $\boldsymbol{\alpha}$ *on* E *induces an oriented graph* $\rightarrow_{\boldsymbol{\alpha}}$ *on* E *with an oriented edge between* x *and* $y \in E$ *iff* $\forall z \in E, z \neq x \wedge z \neq y \Rightarrow \boldsymbol{\alpha}(x, y, z)$.
Any oriented cycle G *induces a ternary relation* $r(G)$ *on* $|G|$ *by:* $r(G)(x, y, z)$ *iff* y *is between* x *and* z *in* G.

Facts 3.4 *(i) If* $\boldsymbol{\alpha}$ *is a total order variety on* E*, then* $\rightarrow_{\boldsymbol{\alpha}}$ *is an oriented cycle.*
(ii) If G *is an oriented cycle, then* $r(G)$ *is a total order variety.*
(iii) The set of finite oriented cycles is isomorphic to the set of finite total order varieties.

Notation. The finite total order variety corresponding to the oriented cycle $a_1 \to \cdots \to a_n \to a_1$ will be simply denoted $(a_1 \ldots a_n)$.

Definition 3.5 *(i) Let $\boldsymbol{\alpha}$ be an order variety on E and $x \in E$. Define the binary relation $\boldsymbol{\alpha}_x$ on $E \setminus \{x\}$ by: $\boldsymbol{\alpha}_x(y,z)$ iff $\boldsymbol{\alpha}(x,y,z)$.*
(ii) Let (E,ω) be a strict order and $z \in E$. Define the binary relation $\overset{z}{<}_\omega$ by:

— *$x \overset{z}{<}_\omega y$ iff $x <_\omega y$ and z is comparable with neither x nor y,*

and the ternary relation ϖ on E by:

$$\text{— } \varpi(x,y,z) \quad \textit{iff} \quad \begin{array}{lll} x <_\omega y <_\omega z \quad \vee & y <_\omega z <_\omega x \quad \vee & z <_\omega x <_\omega y \quad \vee \\ x \overset{z}{<}_\omega y \quad \vee & y \overset{x}{<}_\omega z \quad \vee & z \overset{y}{<}_\omega x. \end{array}$$

Proposition 3.6 *If $\boldsymbol{\alpha}$ is an order variety on E and $x \in E$, then $\boldsymbol{\alpha}_x$ is a strict order on $E \setminus \{x\}$. It is called the order* induced by $\boldsymbol{\alpha}$ and x.

Proof. If $\boldsymbol{\alpha}$ is an order variety , then $\boldsymbol{\alpha}_x$ indeed is
— anti-reflexive: $\boldsymbol{\alpha}_x(y,z)$ iff $\boldsymbol{\alpha}(x,y,z)$, and this implies $y \neq z$ (anti-reflexivity of $\boldsymbol{\alpha}$),
— anti-symmetrical: $\boldsymbol{\alpha}_x(y,z) \wedge \boldsymbol{\alpha}_x(z,y)$ iff $\boldsymbol{\alpha}(x,y,z) \wedge \boldsymbol{\alpha}(x,z,y)$, and this does not hold (lemma 3.2),
— transitive: $\boldsymbol{\alpha}_x(y,z) \wedge \boldsymbol{\alpha}_x(z,t)$ iff $\boldsymbol{\alpha}(x,y,z) \wedge \boldsymbol{\alpha}(x,z,t)$, iff $\boldsymbol{\alpha}(x,y,z) \wedge \boldsymbol{\alpha}(z,t,x)$ (cyclicity of $\boldsymbol{\alpha}$), and this implies $\boldsymbol{\alpha}(t,x,y)$ (transitivity of $\boldsymbol{\alpha}$), whence $\boldsymbol{\alpha}(x,y,t)$ i.e. $\boldsymbol{\alpha}_x(y,t)$. ■

We shall make use of the familiar notions of serial and parallel sums of orders:

Definition 3.7 *Let ω_1 and ω_2 be orders on disjoint sets E and F respectively; define two orders on $E \cup F$, their* serial *and* parallel *sums, $\omega_1 < \omega_2$ and $\omega_1 \parallel \omega_2$ respectively, by:*
- *$(\omega_1 < \omega_2)(x,y)$ iff $x <_{\omega_1} y$ or $x <_{\omega_2} y$ or ($x \in E$ and $y \in F$),*
- *$(\omega_1 \parallel \omega_2)(x,y)$ iff $x <_{\omega_1} y$ or $x <_{\omega_2} y$.*

Proposition 3.8 *(i) If (E,ω) is a strictly ordered set, then ϖ is an order variety on E.*
(ii) If (E_1,ω_1) and (E_2,ω_2) are two strict orders on disjoints sets E_1 and E_2, then $\overline{\omega_1 < \omega_2} = \overline{\omega_1 \parallel \omega_2} = \overline{\omega_2 < \omega_1}$.

Proof. **(i)** If (E,ω) is strictly ordered, then ϖ is indeed:

— cyclic: it is clear;

— anti-reflexive: as ω is a strict order, $\neg x <_\omega x$ and $\neg x \overset{x}{<}_\omega y$, thus $\neg\varpi(x,x,z)$;

— transitive: $\overline{\omega}(x,y,z)$ and $\overline{\omega}(z,t,x)$ iff $(x <_\omega y <_\omega z \vee y <_\omega z <_\omega x \vee z <_\omega x <_\omega y \vee x \overset{z}{<}_\omega y \vee y \overset{x}{<}_\omega z \vee z \overset{y}{<}_\omega x) \wedge (z <_\omega t <_\omega x \vee t <_\omega x <_\omega z \vee x <_\omega z <_\omega t \vee z \overset{x}{<}_\omega t \vee t \overset{z}{<}_\omega x \vee x \overset{t}{<}_\omega z)$. If $x <_\omega y <_\omega z$, then $t <_\omega x <_\omega y <_\omega z$ or $x <_\omega y <_\omega z <_\omega t$ or t incomparable with x, y and z, and in all cases $\overline{\omega}(t,x,y)$. The cases when $y <_\omega z <_\omega x$ or $z <_\omega x <_\omega y$ are similar (cyclic permutation).
If $x \overset{z}{<}_\omega y$, then $z \overset{x}{<}_\omega t \vee t \overset{z}{<}_\omega x$. In the first case, $x <_\omega y$ and t is incomparable with x and y ($t <_\omega y \Rightarrow z <_\omega y$ contradicts $x \overset{z}{<}_\omega y$, and $y <_\omega t \Rightarrow x <_\omega t$ contradicts $z \overset{x}{<}_\omega t$), whence $x \overset{t}{<}_\omega y$. In the second case, $t <_\omega x <_\omega y$, thus once again $\overline{\omega}(t,x,y)$. The cases when $y \overset{x}{<}_\omega z$ or $z \overset{y}{<}_\omega x$ are similar (cyclic permutation).

— spreading: assume $\overline{\omega}(x,y,z)$ and let $t \in E$. If $x <_\omega y <_\omega z$, then either $t <_\omega y$ (so $t <_\omega y <_\omega z$, whence $\overline{\omega}(y,z,t)$), or $y <_\omega t$ (so $x <_\omega y <_\omega t$, whence $\overline{\omega}(x,y,t)$), or y and t are comparable (and in that case either t is incomparable with x, so $\overline{\omega}(x,y,t)$, or t is incomparable with z, so $\overline{\omega}(y,z,t)$, or $x <_\omega t <_\omega z$, so $\overline{\omega}(z,x,t)$).
If $x \overset{z}{<}_\omega y$, then either $t <_\omega x <_\omega y$ (whence $\overline{\omega}(x,y,t)$), or $x <_\omega t <_\omega y$ (whence $x \overset{z}{<}_\omega t$ and also $t \overset{z}{<}_\omega y$), or $x <_\omega y <_\omega t$ (whence $\overline{\omega}(x,y,t)$), or $x <_\omega t$ and t and y are incomparable (whence $\overline{\omega}(z,x,t)$ if $\neg z <_\omega t$, or $\overline{\omega}(y,z,t)$ if $z <_\omega t$), or t and x are incomparable and $t <_\omega y$ (and apply the same argument as above), or t is incomparable with x and y (so $\overline{\omega}(x,y,t)$).

(ii) If $x, y \in E_1$, then $x(\omega_1 < \omega_2)y$ iff $x <_{\omega_1} y$, thus if $x,y,z \in E_1$, $\overline{\omega_1 < \omega_2}(x,y,z)$ iff $\overline{\omega_1}(x,y,z)$ iff $\overline{\omega_1 \,\|\, \omega_2}(x,y,z)$.

Similarly for $x,y,z \in E_2$.

If $x, y \in E_1$ and $z \in E_2$, then $\overline{\omega_1 \,\|\, \omega_2}(x,y,z)$ iff $x <_{\omega_1} y$ iff x $(\omega_1 < \omega_2)$ y $(\omega_1 < \omega_2)$ z iff $\overline{\omega_1 < \omega_2}(x,y,z)$. ■

Propositions 3.6 and 3.8 are essential: they express the possibility to focus on an arbitrary element x in an order variety ($\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}_x \,\|\, x$) to perform operations (the usual operations on binary orders) and then come back to an order variety ($\omega \mapsto \overline{\omega}$). They are at the core of the operations on order varieties, see section 3.2.

Given an order variety $\boldsymbol{\alpha}$ on E and a non-trivial bipartition $E = X \cup Y$ ($X, Y \neq \emptyset, X \cap Y = \emptyset$), one may ask whether there exist relations (necessarily orders) ω_X and ω_Y respectively on X and Y such that $\boldsymbol{\alpha}$ can be presented by $(\omega_X \,\|\, \omega_Y)$ (equivalently by $(\omega_X < \omega_Y)$ or $(\omega_{F^c} < \omega_F)$, cf. proposition 3.8), i.e. $\boldsymbol{\alpha} = \overline{\omega_X \,\|\, \omega_Y}$. Theorem 3.10 gives the existence of a presentation when F is a singleton. For an arbitrary partition, there is no such presentation in general; however, when they exist, the relations ω_X and ω_Y are easily seen to be unique:

Proposition 3.9 (Partitions) *Let $\boldsymbol{\alpha}$ be an order variety on $E = X \cup Y$, with $X, Y \neq \emptyset$ and $X \cap Y = \emptyset$. Then there exists at most one pair of partial orders ω_X, ω_Y respectively on X and Y such that $\boldsymbol{\alpha} = \overline{\omega_X \parallel \omega_Y} (= \overline{\omega_X < \omega_Y} = \overline{\omega_X > \omega_Y})$.*

Proof. Let ω_X, ω_Y and ω'_X, ω'_Y be two such pairs. Assume for a contradiction that $\omega_X \neq \omega'_X$. Then necessarily X has cardinality at least 2, and there exist $x, x' \in X$ such that for instance $x < x'$ in ω and not in ω' (or vice-versa). Take $y \in Y$: we have $\overline{\omega_X \parallel \omega_Y}(y, x, x')$ but not $\overline{\omega'_X \parallel \omega'_Y}(y, x, x')$, contradiction. Therfore $\omega_X = \omega'_X$. Similarly $\omega_Y = \omega'_Y$, qed. ∎

Theorem 3.10 *Let $\boldsymbol{\alpha}$ be an order variety on a set E, $a \in E$, and ω be one of the following three strict orders on E: $(\boldsymbol{\alpha}_a \parallel a)$, $(\boldsymbol{\alpha}_a < a)$ or $(a < \boldsymbol{\alpha}_a)$. Then $\overline{\omega} = \boldsymbol{\alpha}$.*

Proof. According to proposition 3.8 (ii), the three choices for ω give the same order variety $\overline{\omega}$. Let us therefore just consider the case of $(a < \boldsymbol{\alpha}_a)$.

If $x = a \ \vee \ y = a \ \vee \ z = a$, then $\boldsymbol{\alpha}(x, y, z)$ iff $\overline{\omega}(x, y, z)$ by definition of $\boldsymbol{\alpha}_a$.

Let then $x, y, z \in E$ be all different from a and such that $\boldsymbol{\alpha}(x, y, z)$. As $\boldsymbol{\alpha}$ is spreading, $\boldsymbol{\alpha}(x, y, a) \vee \boldsymbol{\alpha}(y, z, a) \vee \boldsymbol{\alpha}(z, x, a)$, i.e. $x \ \boldsymbol{\alpha}_a \ y \vee y \ \boldsymbol{\alpha}_a \ z \vee z \ \boldsymbol{\alpha}_a \ x$, whence $a <_\omega x <_\omega y \vee a <_\omega y <_\omega z \vee a <_\omega z <_\omega x$. For instance $a <_\omega x <_\omega y$ (the other two cases are similar). If $y <_\omega z \vee z <_\omega x$, then obviously $\overline{\omega}(x, y, z)$. On the other hand, if $x <_\omega z$ then $\boldsymbol{\alpha}(a, x, z)$, and as $\boldsymbol{\alpha}(x, y, z)$, then $\boldsymbol{\alpha}(a, y, z)$ by transitivity, thus $y <_\omega z$. Similarly if $z <_\omega y$, then $z <_\omega x$. Now the only remaining possible case is: z incomparable with x and y, and again $\overline{\omega}(x, y, z)$.

Conversely, let $x, y, z \in E$ be all different from a and such that $\overline{\omega}(x, y, z)$, i.e. $x <_\omega y <_\omega z \ \vee \ y <_\omega z <_\omega x \ \vee \ z <_\omega x <_\omega y \ \vee \ x \overset{z}{<}_\omega y \ \vee \ y \overset{x}{<}_\omega z \ \vee \ z \overset{y}{<}_\omega x$. In the first case, $\boldsymbol{\alpha}_a(x, y) \ \wedge \ \boldsymbol{\alpha}_a(y, z)$ (by definition of ω), whence $\boldsymbol{\alpha}(a, x, y) \wedge \boldsymbol{\alpha}(a, y, z)$ (by definition of $\boldsymbol{\alpha}_a$), thus $\boldsymbol{\alpha}(x, y, z)$ since $\boldsymbol{\alpha}$ is transitive. Similarly for the other two cases.

If $x \overset{z}{<}_\omega y$, then in particular $\boldsymbol{\alpha}(a, x, y)$. As $\boldsymbol{\alpha}$ is spreading, this forces $\boldsymbol{\alpha}(a, x, z) \vee \boldsymbol{\alpha}(x, y, z) \ \vee \ \boldsymbol{\alpha}(y, a, z)$. Besides $x \overset{z}{<}_\omega y$ implies among others: $\neg\boldsymbol{\alpha}(a, z, y) \ \wedge \ \neg\boldsymbol{\alpha}(a, x, z)$. Therefore $\boldsymbol{\alpha}(x, y, z)$, qed. The cases $y \overset{x}{<}_\omega z$ and $z \overset{y}{<}_\omega x$ are identical. ∎

Remark. For the above theorem, the spreading condition is necessary: for example, as already mentioned, the cyclic closure of $\{(a, b, c)\}$ is not an order variety on $\{a, b, c, d\}$, and actually it does not come from any order on $\{a, b, c, d\}$.

Corollary 3.11 *Any order variety $\boldsymbol{\alpha}$ on a set E can be presented by a partial order ω on E, i.e. there exists an order ω on E such that $\boldsymbol{\alpha} = \overline{\omega}$.*

It turns out that theorem 3.10 can be very simply formulated in terms of species of structures (a branch of enumerative combinatorics introduced by Joyal [11]); this is not essential in the present paper, but we mention it since it might be exploitable in the future. Recall that a *species of structures* is a functor from the category $\mathbf{B}$ of finite sets and bijections as morphisms to the category **FinSet** of finite sets and functions. Two species F and G are said *isomorphic*, in symbols $F \simeq G$, when there is a natural isomorphism between F and G. If F is a species, its *derivative* F' is the species defined by: $F'(x) = F(x \cup \{*\})$ where $* \notin x$ and $F'(\sigma) = F(\sigma + *)$ for a bijection $\sigma : x \to y$.

For instance the functor O_t that maps a set x to the set of total orders on x, the functor O that maps a set x to the set of all orders on x, and the functor C that maps a set to the set of its cyclic permutations, are species of structures. O_t and C are related by: $C' \simeq O_t$. Order varieties are an integral of (partial) binary orders:

Theorem 3.12 *The functor V that maps a set x to the set of all order varieties on x has derivative the species O of orders:*

$$V' \simeq O.$$

Proof. The transformations $\theta : V' \to O$ defined by $\theta_x(\boldsymbol{\alpha}) = \boldsymbol{\alpha}_*$ for any order variety $\boldsymbol{\alpha}$ on $x \cup \{*\}$, and $\eta : O \to V'$ defined by $\eta_x(\omega) = \overline{\omega \parallel *}$ for any order ω on x, are clearly natural in x. Besides they are inverse of each other because $(\overline{\omega \parallel *})_* = \omega$ (obvious) and $\overline{\boldsymbol{\alpha}_* \parallel *} = \boldsymbol{\alpha}$ (theorem 3.10), qed. ∎

3.2 Compositions

We will use the following constructions of order varieties.

Definition 3.13 *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on the sets E and F respectively, with $E \cap F = \{x\}$. Define:*

$$\boxed{\boldsymbol{\alpha} \odot_x \boldsymbol{\beta} \;=\; \overline{\boldsymbol{\alpha}_x < x < \boldsymbol{\beta}_x}} \quad \textit{and} \quad \boxed{\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta} \;=\; \overline{\boldsymbol{\alpha}_x \parallel x \parallel \boldsymbol{\beta}_x}}\,.$$

Proposition 3.14 *If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are order varieties on the sets E and F respectively, with $E \cap F = \{x\}$, then $\boldsymbol{\alpha} \odot_x \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta}$ are order varieties on $E \cup F$.*

Proof. According to proposition 3.6, $\boldsymbol{\alpha}_x$ and $\boldsymbol{\beta}_x$ are strict orders on $E \setminus \{x\}$ and $F \setminus \{x\}$, so by proposition 3.8 (i), $\boldsymbol{\alpha} \odot_x \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta}$ are order varieties on $E \cup F$. ■

Example. If $E \cap F = \{x\}$, $\emptyset_E \otimes_x \emptyset_F = \emptyset_{E\cup F}$, but $\emptyset_E \odot_x \emptyset_F \neq \emptyset_{E\cup F}$.

The following is a straightforward calculation:

Proposition 3.15 *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on the sets E and F respectively, with $E \cap F = \{x\}$, and let $y \in E \setminus \{x\}$, $z \in F \setminus \{x\}$.*

$$\begin{array}{rclrcl}
(\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_x & = & \boldsymbol{\beta}_x < \boldsymbol{\alpha}_x & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_x & = & \boldsymbol{\beta}_x \parallel \boldsymbol{\alpha}_x \\
(\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_y & = & \boldsymbol{\alpha}_y[(x < \boldsymbol{\beta}_x)/x] & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_y & = & \boldsymbol{\alpha}_y[(x \parallel \boldsymbol{\beta}_x)/x] \\
(\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_z & = & \boldsymbol{\beta}_z[(\boldsymbol{\alpha}_x < x)/x] & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_z & = & \boldsymbol{\beta}_z[(\boldsymbol{\alpha}_x \parallel x)/x]
\end{array}$$

3.3 Restriction

Proposition 3.16 *(i) If $\boldsymbol{\alpha}$ is an order variety on a set E and $F \subseteq E$, then the restriction $\boldsymbol{\alpha}\restriction_F$ of $\boldsymbol{\alpha}$ to F (as a set of triples) is an order variety on F.*
(ii) Let ω be an order on E and $F \subseteq E$. Then $(\overline{\omega})\restriction_F = \overline{\omega\restriction_F}$.
(iii) If $\boldsymbol{\alpha}$ is an order variety on $E \cup \{x\}$ with $x \notin E$, then $\boldsymbol{\alpha}\restriction_E = \overline{\boldsymbol{\alpha}_x}$.

Proof. **(i)** $\boldsymbol{\alpha}\restriction_F$ is cyclic ($\boldsymbol{\alpha}\restriction_F (x,y,z) \Rightarrow x,y,z \in F \ \wedge\ \boldsymbol{\alpha}(x,y,z)$ whence $\boldsymbol{\alpha}(y,z,x)$, so $\boldsymbol{\alpha}\restriction_F (y,z,x)$), anti-reflexive ($\boldsymbol{\alpha}\restriction_F (x,y,z) \Rightarrow \boldsymbol{\alpha}(x,y,z)$ whence $x \neq y \wedge y \neq z \wedge z \neq x$), transitive ($\boldsymbol{\alpha}\restriction_F (x,y,z) \wedge \boldsymbol{\alpha}\restriction_F (z,t,x) \Rightarrow x,y,z,t \in F \wedge \boldsymbol{\alpha}(x,y,z) \ \wedge\ \boldsymbol{\alpha}(z,t,x) \Rightarrow \boldsymbol{\alpha}\restriction_F (t,x,y)$), and spreading ($\forall x,y,z,t \in F \subseteq E$, $\boldsymbol{\alpha}(x,y,z) \Rightarrow \boldsymbol{\alpha}(x,y,t) \vee \boldsymbol{\alpha}(y,z,t) \vee \boldsymbol{\alpha}(z,x,t)$).

(ii) Obvious.

(iii) Let $\boldsymbol{\alpha}$ be an order variety on $E \cup \{x\}$ with $x \notin E$. $\boldsymbol{\alpha}\restriction_E$ and $\overline{\boldsymbol{\alpha}_x}$ are both order varieties on E, and by theorem 3.10 $\boldsymbol{\alpha}\restriction_E = (\overline{\boldsymbol{\alpha}_x \parallel x})\restriction_E = \overline{\boldsymbol{\alpha}_x}$. ■

3.4 Intersections and interior

An intersection of order varieties is obviously cyclic, anti-reflexive and transitive — hence a cyclic order —, but not necessarily an order variety: for instance $(abcd) \cap (abdc) \cap (acbd) = (abd)$ is not an order variety on $\{a,b,c,d\}$.

In section 4 we will be dealing with order varieties associated to proof nets (see also section 3.6), but on the other hand we shall also take intersections of order varieties. So we need a way to transform an intersection of order varieties, or more generally a cyclic order, into an order variety. This is the purpose of the following:

Definition 3.17 *Let $\boldsymbol{\alpha}$ be a cyclic order on E. Define its* interior $\natural\boldsymbol{\alpha}$ *by:*

$$\boxed{\natural\boldsymbol{\alpha} = \bigcap_{x\in E} \overline{\boldsymbol{\alpha}_x \,\|\, x}}.$$

Example. If $\boldsymbol{\alpha} = (xyzt) \cup (xyu)$, then $\natural\boldsymbol{\alpha} = (xyz) \cup (xyt) \cup (xyu)$.

Proposition 3.18 *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be cyclic orders on E.*
(i) $\natural\boldsymbol{\alpha}$ is an order variety on E.
(ii) $\natural\boldsymbol{\alpha} \subseteq \boldsymbol{\alpha}$.
(iii) $\natural\natural\boldsymbol{\alpha} = \natural\boldsymbol{\alpha}$.
(iv) $\natural\boldsymbol{\alpha}$ is the largest order variety included in $\boldsymbol{\alpha}$.
(v) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta} \Rightarrow \natural\boldsymbol{\alpha} \subseteq \natural\boldsymbol{\beta}$.
(vi) If $F \subseteq E$ then $(\natural\boldsymbol{\alpha})\restriction_F \subseteq \natural(\boldsymbol{\alpha}\restriction_F)$.
(vii) $\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \subseteq \natural\boldsymbol{\alpha} \cap \natural\boldsymbol{\beta}$.
(viii) $\natural(\natural\boldsymbol{\alpha} \cap \natural\boldsymbol{\beta}) = \natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$.

Proof. **(i)** As an intersection of order varieties, $\natural\boldsymbol{\alpha}$ is a cyclic order. It is spreading because if $(\natural\boldsymbol{\alpha})(x, y, z)$ and $t \in E \setminus \{x, y, z\}$ then $\overline{\boldsymbol{\alpha}_t}(x, y, z)$, so at least one of the pairs $(x, y), (y, z), (z, x)$ is in $\boldsymbol{\alpha}_t$, whence either $\boldsymbol{\alpha}(x, y, t)$ or $\boldsymbol{\alpha}(y, z, t)$ or $\boldsymbol{\alpha}(z, x, t)$, qed.

(ii) $(\natural\boldsymbol{\alpha})(x, y, z) \Rightarrow \overline{\boldsymbol{\alpha}_x \,\|\, x}(x, y, z) \Leftrightarrow (y, z) \in \boldsymbol{\alpha}_x \Leftrightarrow \boldsymbol{\alpha}(x, y, z)$.

(iii) $\natural\boldsymbol{\alpha}$ is an order variety so for any $x \in E$, $\overline{(\natural\boldsymbol{\alpha})_x \,\|\, x} = \natural\boldsymbol{\alpha}$ by theorem 3.10.

(iv) Let $\boldsymbol{\beta}$ be an order variety on E included in $\boldsymbol{\alpha}$. If $\boldsymbol{\beta}(x, y, z)$, then for any $t \in E\setminus\{x, y, z\}$, $\overline{\boldsymbol{\alpha}_t}(x, y, z)$: indeed $\boldsymbol{\alpha}(x, y, z)$, and $\boldsymbol{\beta}$ is spreading so either $\boldsymbol{\beta}(t, y, z)$ or $\boldsymbol{\beta}(x, t, z)$ or $\boldsymbol{\beta}(x, y, t)$, say for instance $\boldsymbol{\beta}(t, y, z)$ (the two other cases being similar), so $\boldsymbol{\alpha}(t, y, z)$ whence $y <_{\boldsymbol{\alpha}_t} z$; besides $x <_{\boldsymbol{\alpha}_t} z \Rightarrow x <_{\boldsymbol{\alpha}_t} y$ and $y <_{\boldsymbol{\alpha}_t} x \Rightarrow z <_{\boldsymbol{\alpha}_t} x$ by transitivity of $\boldsymbol{\alpha}$; this implies $\overline{(\boldsymbol{\alpha}_t \,\|\, t)}(x, y, z)$ as well. Furthermore, if $\boldsymbol{\beta}(x, y, z)$, then obviously for any $t \in \{x, y, z\}$, $\overline{(\boldsymbol{\alpha}_t \,\|\, t)}(x, y, z)$. Therefore $\boldsymbol{\beta} \subseteq \natural\boldsymbol{\alpha}$.

(v) Immediate consequence of (i), (ii) and (iv).

(vi) If $F \subseteq E$ then $(\natural\boldsymbol{\alpha})\restriction_F = (\bigcap_{x\in E} \overline{\boldsymbol{\alpha}_x \,\|\, x})\restriction_F \subseteq (\bigcap_{x\in F} \overline{\boldsymbol{\alpha}_x \,\|\, x})\restriction_F = \bigcap_{x\in F} \overline{(\boldsymbol{\alpha}_x \,\|\, x)\restriction_F} = \bigcap_{x\in F} \overline{(\boldsymbol{\alpha}\restriction_F)_x \,\|\, x} = \natural(\boldsymbol{\alpha}\restriction_F)$.

(vii) $\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$ is an order variety (by (i)) included in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (by (ii)), so by (iv) $\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \subseteq \natural\boldsymbol{\alpha}$ and $\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \subseteq \natural\boldsymbol{\beta}$, qed.

(viii) $\natural(\natural\boldsymbol{\alpha} \cap \natural\boldsymbol{\beta}) \subseteq \natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$ is a consequence of (ii) and (v). Conversely,

$\natural(\boldsymbol{\alpha}\cap\boldsymbol{\beta}) \subseteq \natural\boldsymbol{\alpha}$ and $\natural(\boldsymbol{\alpha}\cap\boldsymbol{\beta}) \subseteq \natural\boldsymbol{\beta}$, so $\natural(\boldsymbol{\alpha}\cap\boldsymbol{\beta}) \subseteq \natural\boldsymbol{\beta}\cap\natural\boldsymbol{\beta}$; but $\natural(\boldsymbol{\alpha}\cap\boldsymbol{\beta})$ is an order variety, so by (iv), $\natural(\boldsymbol{\alpha}\cap\boldsymbol{\beta}) \subseteq \natural(\natural\boldsymbol{\alpha}\cap\natural\boldsymbol{\beta})$, qed. ■

In the sequent calculus, we shall make extensive use of the relation $\rightarrow_{\boldsymbol{\alpha}}$ of definition 3.3. An essential property of $\natural\boldsymbol{\alpha}$ is that it is basically a simplification of $\boldsymbol{\alpha}$ relative to $\rightarrow_{\_}$: it is an order variety, and useless information has been removed, as shown by the following proposition.

Proposition 3.19 *Let $\boldsymbol{\alpha}$ be a cyclic order. Then $\rightarrow_{\natural\boldsymbol{\alpha}} = \rightarrow_{\boldsymbol{\alpha}}$.*

Proof. Clearly $\rightarrow_{\natural\boldsymbol{\alpha}} \subseteq \rightarrow_{\boldsymbol{\alpha}}$. Now if $a \rightarrow_{\boldsymbol{\alpha}} b$, then the cyclic closure $R_{a,b}$ of the relation $\{(a,b,x) \mid x \neq a,b\}$ is included into $\boldsymbol{\alpha}$ (by definition of $\rightarrow_{\boldsymbol{\alpha}}$) and it is cyclic, anti-reflexive, transitive (trivial) and spreading, hence an order variety, and therefore $R_{a,b} \subseteq \natural\boldsymbol{\alpha}$ by proposition 3.18 (iv), so $a \rightarrow_{\natural\boldsymbol{\alpha}} b$. ■

3.5 Pasting

Definition 3.20 *Let $\boldsymbol{\alpha}$ be an order variety on a set $E\cup\{x,y\}$, with $x,y \notin E$, $x \neq y$, and let $z \notin E$. Define the* pasting $\boldsymbol{\alpha}[z/x,y]$ *of x and y along z in $\boldsymbol{\alpha}$ by:*

$$\boxed{\boldsymbol{\alpha}[z/x,y] \;=\; \natural(\boldsymbol{\alpha}\restriction_{E\cup\{x\}}[z/x]\cap\boldsymbol{\alpha}\restriction_{E\cup\{y\}}[z/y])}\,.$$

The following proposition is obvious:

Proposition 3.21 *(i) If $\boldsymbol{\alpha}$ is an order variety, so is $\boldsymbol{\alpha}[z/x,y]$.*
(ii) If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are order varieties on $E\cup\{x,y\}$, with $x,y \notin E$, $x \neq y$, and $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$, then $\boldsymbol{\alpha}[z/x,y] \subseteq \boldsymbol{\beta}[z/x,y]$.

Given a cyclic order $\boldsymbol{\alpha}$, we shall need to compute $(\natural\boldsymbol{\alpha})[z/x,y]$:

Lemma 3.22 *Let $\boldsymbol{\alpha}$ be a cyclic order on $E\cup\{x,y\}$, with x,y distinct and not in E, and let $z \notin E\cup\{x,y\}$.*
Then $(\natural\boldsymbol{\alpha})[z/x,y] = \natural(\boldsymbol{\alpha}\restriction_{E\cup\{x\}}[z/x]\cap\boldsymbol{\alpha}\restriction_{E\cup\{y\}}[z/y])$.

Proof. $(\natural\boldsymbol{\alpha})[z/x,y] = \natural((\natural\boldsymbol{\alpha})\restriction_{E\cup\{x\}}[z/x]\cap(\natural\boldsymbol{\alpha})\restriction_{E\cup\{y\}}[z/y])$. Now:

$$\begin{aligned}
(\natural\boldsymbol{\alpha})\restriction_{E\cup\{x\}} &= (\textstyle\bigcap_{a\in E}\overline{\boldsymbol{\alpha}_a \parallel a}\cap\overline{\boldsymbol{\alpha}_x \parallel x}\cap\overline{\boldsymbol{\alpha}_y \parallel y})\restriction_{E\cup\{x\}}\\
&= \textstyle\bigcap_{a\in E}\overline{(\boldsymbol{\alpha}\restriction_{E\cup\{x\}})_a \parallel a}\cap\overline{(\boldsymbol{\alpha}\restriction_{E\cup\{x\}})_x \parallel x}\cap\overline{\boldsymbol{\alpha}_y}\\
&= \natural(\boldsymbol{\alpha}\restriction_{E\cup\{x\}})\cap\overline{\boldsymbol{\alpha}_y}\\
&= \natural(\boldsymbol{\alpha}\restriction_{E\cup\{x\}})\cap\boldsymbol{\alpha}\restriction_{E\cup\{y\}},
\end{aligned}$$

and similarly $(\natural\boldsymbol{\alpha})\restriction_{E\cup\{y\}} = \natural(\boldsymbol{\alpha}\restriction_{E\cup\{y\}})\cap\boldsymbol{\alpha}\restriction_{E\cup\{x\}}$. Hence:

$$
\begin{aligned}
(\natural\boldsymbol{\alpha})[z/x,y] &= \natural(\natural(\boldsymbol{\alpha}\restriction_{E\cup\{x\}})[z/x] \cap \natural(\boldsymbol{\alpha}\restriction_{E\cup\{y\}})[z/y] \\
&\quad \cap \boldsymbol{\alpha}\restriction_{E\cup\{y\}}[z/y] \cap \boldsymbol{\alpha}\restriction_{E\cup\{x\}}[z/x]) \\
&= \natural(\natural(\boldsymbol{\alpha}\restriction_{E\cup\{x\}}[z/x]) \cap \natural(\boldsymbol{\alpha}\restriction_{E\cup\{y\}}[z/y]) \\
&\quad \cap \boldsymbol{\alpha}\restriction_{E\cup\{y\}}[z/y] \cap \boldsymbol{\alpha}\restriction_{E\cup\{x\}}[z/x]) \\
&= \natural(\natural(\boldsymbol{\alpha}\restriction_{E\cup\{x\}}[z/x]) \cap \natural(\boldsymbol{\alpha}\restriction_{E\cup\{y\}}[z/y])) \\
&= \natural(\boldsymbol{\alpha}\restriction_{E\cup\{x\}}[z/x] \cap \boldsymbol{\alpha}\restriction_{E\cup\{y\}}[z/y])
\end{aligned}
$$

by proposition 3.18 (viii), qed. ■

3.6 Order varieties and proof nets

Definition 3.23 **($\alpha_{\pi,s}, \alpha_\pi$)** *Let π be a proof net of MNL with conclusion Γ.*
(i) If s is a switching for π, $\alpha_{\pi,s}$ is the total order variety on Γ corresponding to the cycle of the conclusions in $s(\pi)$.
(ii) $\alpha_\pi = \natural(\bigcap_s \alpha_{\pi,s})$.

By definition, α_π is always an order variety.

Proposition 3.24 *(i) If π is a proof net of MLL, then $\alpha_\pi = \emptyset$.*
(ii) If π is a proof net of McyLL, then α_π is a total order variety.

Proof. **(i)** Assume $\alpha_{\pi,s}(A,B,C)$ for some conclusions A,B,C of π and some switching s. Deleting for each $\parr$-link l the left edge of l if $s(l)=R$ or the right edge of l if $s(l)=L$, produces a graph which is a tree (Danos-Régnier criterion [5]), and therefore determines a ternary link l_0 (which has to be a $\otimes$-link) where the three paths between the three leaves A,B,C meet. Let s' be the switching obtained from s by changing the position of l_0: then $\alpha_{\pi,s'}(C,B,A)$.
This proves that $\bigcap_s \alpha_{\pi,s} = \emptyset$, so $\alpha_\pi = \emptyset$.

(ii) Corollary of proposition 2.14. ■

The following lemma is an immediate consequence of theorem 2.20.

Lemma 3.25 *Let π be a proof net, s a $\triangledown$3-free switching for π and s' a switching for π such that $s'(l) = s(l)$ for any link $l \neq \triangledown$. Then $\alpha_{\pi,s} = \alpha_{\pi,s'}$.*

Proposition 3.26 *If π is a proof net and $\pi \to \pi'$, then $\alpha_\pi \subseteq \alpha_{\pi'}$.*

Proof. We shall prove that $\bigcap_s \alpha_{\pi,s} \subseteq \bigcap_s \alpha_{\pi',s}$, from which the result follows by proposition 3.18 (v).

Assume that $(\bigcap_s \alpha_{\pi,s})(C,D,E)$ for some conclusions C,D,E of π. The proof is very similar to the proof of theorem 2.22, and we shall use the notations of this proof.

If the reduction bears on an identity link, then the result is trivial.

Assume the reduction bears on a pair of $\triangledown/\odot$-links, and let s' be a $\triangledown$3-free switching for π'; considering the two positions L and R (not 3) for the cut $\triangledown$-link, we get two long trips in π: $1,2,3,4$ and $2,1,3,4$, the long trip in π' being $2,3,1,4$. As the inner part of a $\triangledown$-link, 1 contains no conclusion, so C, D, E are in $2,3,4$. Now $(2,3,4)$ belongs to the three total order varieties (1234), (2134) and (2314), hence $\alpha_{\pi',s'}(C,D,E)$ still holds after reduction, and by lemma 3.25, $\alpha_{\pi',s''}(C,D,E)$ for any switching s'' of π', qed.

Assume the reduction bears on a pair of $⅋/\otimes$-links, and let s' be a $\triangledown$3-free switching for π'; choosing positions for the cut links, we get a switching s for π and the long trip in π is $1,2,3,4$ or $2,1,3,4$ or $1,2,4,3$ or $2,1,4,3$, the long trip in π' being $1,4,2,3$. As $(\bigcap_s \alpha_{\pi,s})(C,D,E)$, we have $(C,D,E) \in (1234) \cap (2134) \cap (1243) \cap (2143)$, hence at least two formulas among C, D, E have to be in the same segment ($1,2,3$ ou 4), and therefore $(C,D,E) \in (1423) = \alpha_{\pi',s'}$. Again by lemma 3.25, this extends to non-$\triangledown$3-free switchings. ■

Examples. $\alpha_{\psi_1} = \emptyset$ and $\alpha_{\psi_2} = \emptyset$, since $\emptyset$ is the order variety corresponding to the cycle of two elements (and is by the way the only order variety on a two-elements set).

$\alpha_{\psi_3} = (B \odot C, A⅋C^\perp, A^\perp \odot B)$.

$(\bigcap_s \alpha_{\psi_4,s}) = (C,A,D) \cup (C,A,B) \cup (E,A,D) \cup (E,A,B)$ is already an order variety, so $\alpha_{\psi_4} = (C,A,D) \cup (C,A,B) \cup (E,A,D) \cup (E,A,B)$.

$(\bigcap_s \alpha_{\psi_5,s}) = (A,B,B^\perp \odot A^\perp) \cap (B,A,B^\perp \odot A^\perp) = \emptyset$, so $\alpha_{\psi_5} = \emptyset$. But $\psi_5 \to \psi'_5$ and $\alpha_{\psi'_5} = (A,B,B^\perp \odot A^\perp)$.

4 SEQUENT CALCULUS AND SEQUENTIALIZATION

4.1 Sequent calculus

Definition 4.1 (Sequents) *(i) A* sequent (of MNL) $\vdash \Gamma\ \langle \boldsymbol{\alpha} \rangle$ *is a set* Γ *of occurrences of formulas together with an order variety* $\boldsymbol{\alpha}$ *on* Γ.
(ii) A sequent of MLL *is a sequent* $\vdash \Gamma\ \langle \emptyset \rangle$ *where* Γ *is a set of occurrences of formulas of MLL. So* $\vdash \Gamma\ \langle \emptyset \rangle$ *can be denoted* $\vdash \Gamma$.
(iii) A sequent of cyLL *is a sequent* $\vdash \Gamma\ \langle \boldsymbol{\alpha} \rangle$ *where* Γ *is a set of occurrences*

Identity - Cut

$$\vdash A^{\perp}, A \ \langle \emptyset \rangle \qquad \frac{\vdash \Gamma, A \ \langle \boldsymbol{\alpha} \rangle \qquad \vdash A^{\perp}, \Delta \ \langle \boldsymbol{\beta} \rangle}{\vdash \Gamma, \Delta \ \langle \boldsymbol{\alpha} \asymp_{A,A^{\perp}} \boldsymbol{\beta} \rangle}\,\mathrm{cut}$$

Non-commutatives

$$\frac{\vdash \Gamma, A \ \langle \boldsymbol{\alpha} \rangle \qquad \vdash B, \Delta \ \langle \boldsymbol{\beta} \rangle}{\vdash \Gamma, A \odot B, \Delta \ \langle \boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta} \rangle}\,\odot \qquad \frac{\vdash \Gamma, A, B \ \langle \boldsymbol{\alpha} \rangle}{\vdash \Gamma, A \triangledown B \ \langle \boldsymbol{\alpha}[A \triangledown B/A, B] \rangle}\,\triangledown, \text{ if } A \to_{\boldsymbol{\alpha}} B$$

Commutatives

$$\frac{\vdash \Gamma, A \ \langle \boldsymbol{\alpha} \rangle \qquad \vdash B, \Delta \ \langle \boldsymbol{\beta} \rangle}{\vdash \Gamma, A \otimes B, \Delta \ \langle \boldsymbol{\alpha} \otimes_{A,B} \boldsymbol{\beta} \rangle}\,\otimes \qquad \frac{\vdash \Gamma, A, B \ \langle \boldsymbol{\alpha} \rangle}{\vdash \Gamma, A \parr B \ \langle \boldsymbol{\alpha}[A \parr B/A, B] \rangle}\,\parr$$

Table 1
Sequent calculus of MNL.

of formulas of McyLL and $\boldsymbol{\alpha}$ *is total. So* $\vdash A_1, \dots, A_n \ \langle (A_1, \dots, A_n) \rangle$ *can be denoted by* $\vdash (A_1, \dots, A_n)$.

Notation. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on disjoint sets of formula occurrences $\Gamma \cup \{A\}$ and $\Delta \cup \{B\}$ respectively. Define:

$\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta} = \overline{\boldsymbol{\alpha}_A < A \odot B < \boldsymbol{\beta}_B} = \boldsymbol{\alpha}[A \odot B/A] \odot_{A \odot B} \boldsymbol{\beta}[A \odot B/B]$ and
$\boldsymbol{\alpha} \otimes_{A,B} \boldsymbol{\beta} = \overline{\boldsymbol{\alpha}_A \parallel A \otimes B \parallel \boldsymbol{\beta}_B} = \boldsymbol{\alpha}[A \otimes B/A] \otimes_{A \otimes B} \boldsymbol{\beta}[A \otimes B/B]$,
two order varieties on $\Gamma \cup \Delta \cup \{A \odot B\}$ and $\Gamma \cup \Delta \cup \{A \otimes B\}$ respectively, and:

$\boldsymbol{\alpha} \asymp_{A,B} \boldsymbol{\beta} \ = \ (\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta}) \restriction_{\Gamma \cup \Delta} = \ (\boldsymbol{\alpha} \otimes_{A,B} \boldsymbol{\beta}) \restriction_{\Gamma \cup \Delta} = \ \overline{\boldsymbol{\alpha}_A \parallel \boldsymbol{\beta}_B}$,
an order variety on $\Gamma \cup \Delta$.

The *rules* of the multiplicative sequent calculus are given in table 1.

Recall from definition 3.3 that $A \to_{\boldsymbol{\alpha}} B$ iff $\boldsymbol{\alpha}(A, B, C)$ for any $C \in \Gamma$.

Examples.
- A sequent calculus proof corresponding to ψ_3 is

$$\dfrac{\dfrac{\vdash A^\perp, A\ \langle\emptyset\rangle \qquad \vdash B^\perp, B\ \langle\emptyset\rangle}{\vdash A^\perp \odot B^\perp, A, B\ \langle(A, A^\perp \odot B^\perp, B)\rangle}\odot \qquad \vdash C^\perp, C\ \langle\emptyset\rangle}{\dfrac{\vdash A^\perp \odot B^\perp, B \odot C, A, C^\perp\ \langle(A, A^\perp \odot B^\perp, B \odot C, C^\perp)\rangle}{\vdash A^\perp \odot B^\perp, B \odot C, A \parr C^\perp\ \langle(A^\perp \odot B^\perp, B \odot C, A \parr C^\perp)\rangle}\parr}\odot$$

Note that the last rule could also be an introduction of $C^\perp \triangledown A$ since $C^\perp \to A$.

- A sequent calculus proof corresponding to ψ_5 is

$$\dfrac{\dfrac{\vdash A^\perp, A\ \langle\emptyset\rangle \qquad \vdash B^\perp, B\ \langle\emptyset\rangle}{\vdash A, B, B^\perp \otimes A^\perp\ \langle\emptyset\rangle}\otimes \qquad \dfrac{\dfrac{\vdash A^\perp, A\ \langle\emptyset\rangle \qquad \vdash B^\perp, B\ \langle\emptyset\rangle}{\vdash A, B, B^\perp \odot A^\perp\ \langle(A, B^\perp \odot A^\perp, B)\rangle}\odot}{\vdash A \parr B, B^\perp \odot A^\perp\ \langle\emptyset\rangle}\parr}{\vdash A, B, B^\perp \odot A^\perp\ \langle\emptyset\rangle}\text{cut}$$

4.2 Sequentialization theorem

Definition 4.2 (D^-) *To a proof D of conclusion $\vdash \Gamma\ \langle\boldsymbol{\alpha}\rangle$ in sequent calculus, is associated in the obvious way a proof structure D^- with conclusion Γ.*

Theorem 4.3 (Adequacy) *If D is a sequential proof with conclusion $\vdash \Gamma\ \langle\boldsymbol{\alpha}\rangle$, then D^- is a proof net and $\boldsymbol{\alpha} = \alpha_{D^-}$.*

Proof. We prove by induction on D that D^- satisfies the axioms of theorem 2.20 (π^* is a proof net of MLL and for any switching, the inner parts of $\triangledown$-links contain no conclusion and do not overlap) and $\boldsymbol{\alpha} = \natural(\bigcap_s \alpha_{D^-,s})$. For the axioms of theorem 2.20, it is straightforward. For the order varieties:

- D is an identity: there is only one switching s and $\alpha_{D^-,s} = \emptyset_{\{A^\perp, A\}} = \boldsymbol{\alpha}$.

- D is

$$\dfrac{\begin{matrix} {\scriptstyle D_1}\vdots \\ \vdash \Gamma, A\ \langle\boldsymbol{\alpha}\rangle\end{matrix} \qquad \begin{matrix} \vdots{\scriptstyle D_2} \\ \vdash B, \Delta\ \langle\boldsymbol{\beta}\rangle\end{matrix}}{\vdash \Gamma, A \odot B, \Delta\ \langle\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta}\rangle}$$

A switching in D^- is a pair (s,t) where s is a switching of ${D_1}^-$ and t is a switching of ${D_2}^-$, and $\alpha_{D^-,(s,t)}$ is then $\overline{(\alpha_{{D_1}^-,s})_A < A \odot B < (\alpha_{{D_2}^-,t})_B} = (\alpha_{{D_1}^-,s}) \odot_{A,B} (\alpha_{{D_2}^-,t})$.

By induction hypothesis, $\boldsymbol{\alpha} = \natural \bigcap_s \alpha_{{D_1}^-,s}$ and $\boldsymbol{\beta} = \natural \bigcap_t \alpha_{{D_2}^-,t}$, therefore we have to show: $\natural \bigcap_{s,t} (\alpha_{{D_1}^-,s}) \odot_{A,B} (\alpha_{{D_2}^-,t}) = (\natural \bigcap_s \alpha_{{D_1}^-,s}) \odot_{A,B} (\natural \bigcap_t \alpha_{{D_2}^-,t})$.

— Let $x, y \in \Gamma \cup \Delta$: $(\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta})(A \odot B, x, y)$ iff $x \in \Delta$ and $y \in \Gamma$. Besides

$$
\begin{aligned}
& (\natural \textstyle\bigcap_{s,t} \alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(A \odot B, x, y) \\
\Rightarrow\ & \textstyle\bigcap_{s,t} (\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(A \odot B, x, y) \\
\Leftrightarrow\ & x \in \Delta \wedge y \in \Gamma,
\end{aligned}
$$

but $\overline{A \odot B < \emptyset_\Delta < \emptyset_\Gamma}$ is an order variety included in $\bigcap_{s,t}(\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})$, so $\overline{A \odot B < \emptyset_\Delta < \emptyset_\Gamma} \subseteq \natural \bigcap_{s,t}(\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})$. Therefore $(\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta})(A \odot B, x, y)$ iff $x \in \Delta$ and $y \in \Gamma$ iff $\natural \bigcap_{s,t}(\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(A \odot B, x, y)$.

— Let $x, y, z \in \Gamma$: $(\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta})(x, y, z)$ iff $\overline{\boldsymbol{\alpha}_A}(x, y, z)$ iff $\overline{\boldsymbol{\alpha}_A \parallel A}(x, y, z)$ iff $\boldsymbol{\alpha}(x, y, z)$ as $\boldsymbol{\alpha}$ is an order variety. Besides

$$
\begin{aligned}
& (\natural \textstyle\bigcap_{s,t} \alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(x, y, z) \\
\Leftrightarrow\ & (\natural \textstyle\bigcap_{s,t} \alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t}) \restriction_{\Gamma,A} (x, y, z) \\
\Rightarrow\ & \natural (\textstyle\bigcap_{s,t} (\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t}) \restriction_{\Gamma,A})(x, y, z) \quad \text{by proposition 3.18 (vi)} \\
\Leftrightarrow\ & \boldsymbol{\alpha}(x, y, z).
\end{aligned}
$$

because $\bigcap_{s,t}(\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t}) \restriction_{\Gamma,A} = \bigcap_s \alpha_{D_1^-,s}$. But

$$
\begin{aligned}
& \boldsymbol{\alpha}(x, y, z) \\
\Leftrightarrow\ & (\natural \textstyle\bigcap_s \alpha_{D_1^-,s})[\emptyset_\Delta, A/A](x, y, z) \\
\Rightarrow\ & \natural (\textstyle\bigcap_s \alpha_{D_1^-,s}[\emptyset_\Delta, A/A])(x, y, z) \qquad \text{by proposition 3.18 (vi)} \\
\Rightarrow\ & (\natural \textstyle\bigcap_{s,t} \alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(x, y, z) \quad \text{by proposition 3.18 (v)}
\end{aligned}
$$

because $(\bigcap_s \alpha_{D_1^-,s}[\emptyset_\Delta, A/A]) \subseteq \bigcap_{s,t}(\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})$. Therefore $(\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta})(x, y, z)$ iff $\natural \bigcap_{s,t}(\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(x, y, z)$.

— For $x, y, z \in \Delta$ apply the above argument.

— Let $x, y \in \Gamma$ and $z \in \Delta$: $(\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta})(x, y, z)$ iff $\boldsymbol{\alpha}(x, y, A)$. Besides

$$
\begin{aligned}
& (\natural \textstyle\bigcap_{s,t} \alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(x, y, z) \\
\Leftrightarrow\ & (\natural \textstyle\bigcap_{s,t} \alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t}) \restriction_{\Gamma,z} (x, y, z) \\
\Rightarrow\ & \natural (\textstyle\bigcap_{s,t} (\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t}) \restriction_{\Gamma,z})(x, y, z) \quad \text{by proposition 3.18 (vi)}
\end{aligned}
$$

but $\natural(\bigcap_{s,t}(\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t}) \restriction_{\Gamma,z}) = \natural \bigcap_{s,t} \overline{(\alpha_{D_1^-,s})_A \parallel z} = \boldsymbol{\alpha}[z/A]$, so $(\natural \bigcap_{s,t} \alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(x, y, z) \Rightarrow \boldsymbol{\alpha}(x, y, A)$. Now

$$
\begin{array}{ll}
& \boldsymbol{\alpha}(x,y,A) \\
\Leftrightarrow & (\natural \bigcap_s \alpha_{D_1^-,s})[\emptyset_\Delta, A/A](x,y,A) \\
\Leftrightarrow & \natural(\bigcap_s \alpha_{D_1^-,s}[\emptyset_\Delta, A/A])(x,y,A) \quad \text{by proposition 3.18 (vi)} \\
\Leftrightarrow & \natural(\bigcap_s \alpha_{D_1^-,s}[\emptyset_\Delta, A/A])(x,y,z) \\
& \\
\Rightarrow & (\natural \bigcap_{s,t} \alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(x,y,z).
\end{array}
$$

Therefore $(\boldsymbol{\alpha} \odot_{A,B} \boldsymbol{\beta})(x,y,z)$ iff $\natural \bigcap_{s,t}(\alpha_{D_1^-,s} \odot_{A,B} \alpha_{D_2^-,t})(x,y,z)$.
— For $x \in \Gamma$ and $y, z \in \Delta$ apply the above argument.

- D is

$$
\frac{\overset{D_1 \vdots}{\vdash \Gamma, A \ \langle \boldsymbol{\alpha} \rangle} \qquad \overset{\vdots D_2}{\vdash B, \Delta \ \langle \boldsymbol{\beta} \rangle}}{\vdash \Gamma, A \otimes B, \Delta \ \langle \boldsymbol{\alpha} \otimes_{A,B} \boldsymbol{\beta} \rangle} \quad \text{or} \quad \frac{\overset{D_1 \vdots}{\vdash \Gamma, A \ \langle \boldsymbol{\alpha} \rangle} \qquad \overset{\vdots D_2}{\vdash A^\perp, \Delta \ \langle \boldsymbol{\beta} \rangle}}{\vdash \Gamma, \Delta \ \langle \boldsymbol{\alpha} \asymp_{A,A^\perp} \boldsymbol{\beta} \rangle}
$$

Similar argument.

- D is

$$
\frac{\overset{D_1 \vdots}{\vdash \Gamma, A, B \ \langle \boldsymbol{\alpha} \rangle}}{\vdash \Gamma, A \triangledown B \ \langle \boldsymbol{\alpha}[A \triangledown B/A, B] \rangle} \ A \to_{\boldsymbol{\alpha}} B \quad \text{or} \quad \frac{\overset{D_1 \vdots}{\vdash \Gamma, A, B \ \langle \boldsymbol{\alpha} \rangle}}{\vdash \Gamma, A \parr B \ \langle \boldsymbol{\alpha}[A \parr B/A, B] \rangle}
$$

A switching s' in D^- is a switching s of $D_1{}^-$ together with a position of for the link $\frac{A \quad B}{ApB}$, $p \in \{\triangledown, \parr\}$. Let us consider the introduction of $\parr$: $\bigcap_{s'} \alpha_{D^-,s'} = \bigcap_s(\alpha_{D_1^-,s} \restriction_{\Gamma,A} [A \parr B/A] \cap \alpha_{D_1^-,s} \restriction_{\Gamma,B} [A \parr B/B])$.

Now by induction hypothesis $\boldsymbol{\alpha} = \natural(\bigcap_s \alpha_{D_1^-,s})$, so

$$
\begin{array}{ll}
& \boldsymbol{\alpha}[A \parr B/A, B] \\
= & (\natural(\bigcap_s \alpha_{D_1^-,s}))[A \parr B/A, B] \\
= & \natural((\bigcap_s \alpha_{D_1^-,s}) \restriction_{\Gamma,A} [A \parr B/A] \cap (\bigcap_s \alpha_{D_1^-,s}) \restriction_{\Gamma,B} [A \parr B/B]) \\
& \text{by lemma 3.22} \\
= & \natural \bigcap_s(\alpha_{D_1^-,s} \restriction_{\Gamma,A} [A \parr B/A] \cap \alpha_{D_1^-,s} \restriction_{\Gamma,B} [A \parr B/B]) \\
= & \bigcap_{s'} \alpha_{D^-,s'}.
\end{array}
$$

For the introduction of $\triangledown$, the proof is the same because D^- is a proof net, so the internal part of $\frac{A \quad B}{A \triangledown B}$ contains no conclusion and we can concentrate on positions R and L. ■

Theorem 4.4 (Sequentialization) *Let π be a cut-free proof net with conclusion Γ. There exists a sequent calculus proof D, with conclusion $\vdash \Gamma\ \langle \alpha_\pi \rangle$, such that $\pi = D^-$.*

Proof. We proceed as in [8], with terminal $\triangledown$ or $⅋$-links and splitting $\odot$ or $\otimes$-links.
In the case of tensor links, remark that the absence of cut link ensures correctness of both proof structures. In the case of a terminal $\triangledown$-links, remark that that position 3 implies the required condition $A \to_{\alpha_\pi} B$. ■

Appendix A deals with sequentialization of proof nets with cuts. Let us first give an application of theorem 4.4 to cut elimination in the sequent calculus.

4.3 Cut elimination

Theorem 4.5 *If D is a sequent calculus proof with conclusion $\vdash \Gamma\ \langle \boldsymbol{\alpha} \rangle$, then $\vdash \Gamma\ \langle \boldsymbol{\alpha}' \rangle$ is provable in the sequent calculus without the cut rule, with $\boldsymbol{\alpha} \subseteq \boldsymbol{\alpha}'$.*

Proof. By theorem 4.3, D^- is a proof net, and by theorem 2.23 and proposition 3.26, $D^- \to \pi$ with π a cut-free proof net such that $\alpha_{D^-} \subseteq \alpha_\pi$. By theorem 4.4 there exists a sequent calculus proof D' such that $\pi = D'^-$, and by construction D' is cut-free.

$$\begin{tikzcd} D \arrow[r, "\text{adequacy}"] \arrow[d, dotted] & D^- \arrow[d, "\text{cut-elimination}"] \\ D' & \pi \arrow[l, "\text{sequentialization}"'] \end{tikzcd}$$

The inclusion of order varieties is a consequence of proposition 3.26. ■

Corollary 4.6 *MNL is a conservative extension of both MLL and McyLL:*
(i) if $\Gamma\ \langle \boldsymbol{\alpha} \rangle$ is provable in the sequent calculus, and the formulas of Γ are formulas of MLL, then $\boldsymbol{\alpha} = \emptyset$;
(ii) if $\Gamma\ \langle \boldsymbol{\alpha} \rangle$ is provable in the sequent calculus, and the formulas of Γ are formulas of McyLL, then $\boldsymbol{\alpha}$ is total.

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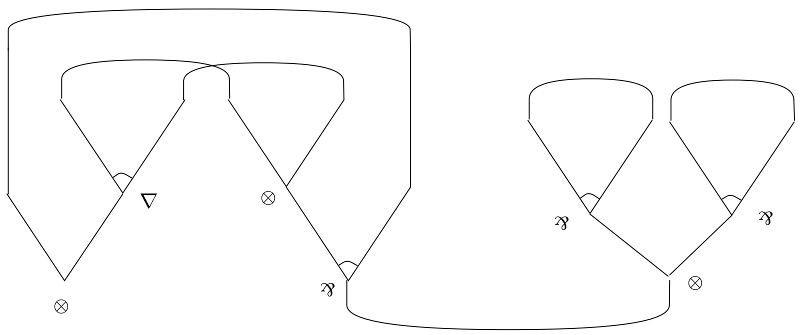
A APPENDIX: ON THE SEQUENTIALIZATION OF PROOF NETS WITH CUTS

So far, the theorems of section 4 say that given a sequent calculus proof D with cuts, D^- is a proof net and can therefore be reduced to its normal form, a cut-free proof net π which in turn comes from a sequent calculus proof. We might also wonder what happens during intermediate steps of reduction: are proof nets still sequentializable during cut elimination $D^- \to^* \psi \to^* \pi$? More generally we may ask whether theorem 4.4 can be extended to proof nets with cuts, as in commutative LL.

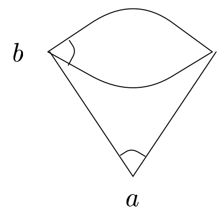
Sequentialization of proof nets with cuts fails in general, with our definition of proof nets and sequent calculus; for instance the following structure χ is a proof net but does not come from the sequent calculus:



However this is not a serious problem, indeed there are at least three possible solutions:

1) One approach is to add a condition to the definition of proof nets, that is preserved during reduction and implies sequentialization for all proof nets. In the above counter-example, the point is that the inner part of the $\triangledown$-link goes through a $\parr$-link which is "below" it, so that the two disjunction links on the left block each other. It is straightforward to formalize this notion of "being above": we define two orders $<_\pi$ and $\Vdash_\pi$ between the disjunction links of a proof net [6].

Definition A.1 **($<_\pi, \Vdash_\pi$)** *Let π be a proof net, and a and b be two arbitrary disjunction links of π.*
Let $s = s_{a,b}$ be a switching à la Danos-Regnier (cut one of the two branches of a link) for all the disjunction links of π but a and b. The graph obtained contains exactly two independent cycles, and some pending edges. We note $\pi^s_{a,b}$ the graph obtained by erasing these pending edges.
Define the relation $<_\pi$ by : $a <_\pi b$ iff for some $\triangledown 3$-free switching of π, both the sup part and the inf part of b go through a (i.e. they contain some premisse of a), in other words iff for some switching $s = s_{a,b}$ à la Danos-Regnier for all disjunctions but a and b, $\pi^s_{a,b}$ is the graph:



Define the relation $\Vdash_\pi$ by : $a \Vdash_\pi b$ iff a is a $\triangledown$-link and for some switching for π, the inner part of a goes through b (i.e. it contains a premisse of b).

Lemma A.2 *Let π be a proof net. Then $<_\pi$ and $\Vdash_\pi$ are orders on disjunction links.*

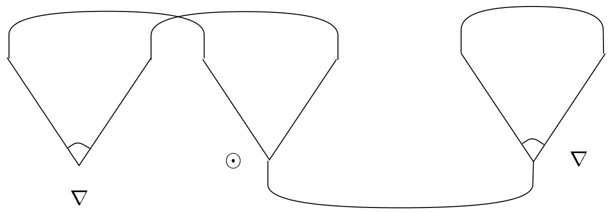
Proof. The result is very easy for $<_\pi$. For $\Vdash_\pi$, use the facts that the inner parts of $\triangledown$-links do not overlap (theorem 2.20) and that the inf parts of disjunction links do not cross (i.e., if one inf part contains the beginning or the end of the other one, then it contains both ends: this holds for commutative proof nets as well). ∎

One can verify easily the following lemma:

[6] $<_\pi$ can also be defined very simply via Métayer's homological criterion [13] for proof nets of MLL.

Lemma A.3 *Let π be a proof net and a a disjunction link of π. a is minimal for $<_\pi$ iff it is splitting, i.e. the graph obtained by erasing both edges of a has two connected components.*

Of course, the point here is that the two connected components may not both be proof nets in general, for instance in the following proof net (which is sequentializable!) the cut and the right $\triangledown$-link are splitting but then the left component is no more a proof net:



Now it is natural to prove sequentialization with splitting disjunctions ($\triangledown$ and $\parr$), in a way similar to Danos [4]. We consider proof structures with non-logical axioms, and one can easily imagine the obvious necessary adaptations.

Definition A.4 (Condition 4) *A proof net π will be said to satisfy* condition 4 *if the two relations $<_\pi$ and $\Vdash_\pi$ are orthogonal (i.e. $<_\pi \cup \Vdash_\pi$ does not contain a cycle).*

Lemma A.5 *(i) If D is a sequential proof with conclusion $\vdash \Gamma \langle \boldsymbol{\alpha} \rangle$, then D^- satisfies condition 4.*
(ii) If π is a proof net and $\pi \to \pi'$, then $<_\pi \supseteq <_{\pi'}$ and $\Vdash_\pi \supseteq \Vdash_{\pi'}$.
(iii) If π is a proof net satisfying condition 4, and $\pi \to \pi'$, then π' is a proof net satisfying condition 4.

Proof. **(i)** Obvious induction on D.
(ii) This is proved by pulling back configurations of 3 or 4 points before reduction, as for theorem 2.22.
(iii) Follows immediately from (ii). ∎

Note that in the above counter-example, the cycle has lenght 2, but there are bigger counter-examples, and condition 4 cannot be reduced to the absence of a simple configuration.

Proposition A.6 *Let π be any proof net with conclusion Γ satisfying condition 4. There exists a sequent calculus proof D, with conclusion $\vdash \Gamma \langle \alpha_\pi \rangle$, such that $\pi = D^-$.*

Proof. Proceed by induction on the number n of disjunction links. If $n = 0$, π is a tree: clear. If $n > 0$, then by lemma A.2 there is a link a which is minimal for $<_\pi \cup \Vdash_\pi$; by definition of condition 4, a is minimal for $<_\pi$ so it is splitting (lemma A.3), and it is minimal for $\Vdash_\pi$ so it is not in the inner part of any $\triangledown$-link. Let π_1 and π_2 be the two components of π obtained by erasing the two edges of a: by [4], ${\pi_1}^*$ and ${\pi_2}^*$ are proof nets; we have chosen a so that in particular no inner part of a $\triangledown$-link of π_1 or π_2 goes through a, thus the inner parts of $\triangledown$-links of π_1 and π_2 contain no conclusion; non-overlapping and condition 4 for π_1 and π_2 are immediate. Therefore π_1 and π_2 are proof nets satisfying condition 4, and we can apply the induction hypothesis, qed. ■

2) Another approach is to keep the correctness criterion for proof nets, and try and slightly modify the syntax. An obvious idea is to consider cuts as ternary links (with a conclusion) both in the sequent calculus and in the proof structures: then of course theorem 4.4 holds for all proof nets, and there is an interesting phenomenon, namely there are two kinds of cuts (the "parallel cuts" with conclusion $A \otimes A^\perp$, and the "sequential cuts" with conclusion $A \odot A^\perp$) and they are no more innocuous (adding cuts may for instance destroy the planarity).

Another possibility is to authorize some kind of revision in the sequent calculus (introduce a $\invamp$ *a priori*, then replace it by $\triangledown$ if it is *a posteriori* possible), the idea being that there is essentially one disjunction and one conjunction, but different ways to view them geometrically. One could add for instance a "purgatory" in sequents: a sequent then consists in an order variety $\boldsymbol{\alpha}$ on Γ plus a set Δ of formula occurrences (with no structure), and the introduction of $\triangledown$ is not subject to a condition any more; on the contrary it can be performed freely, but the problematic formulas (those formulas $C \in \Gamma$ such that $\neg\boldsymbol{\alpha}(A, B, C)$) are send to the purgatory. Formulas in the purgatory can be removed by cutting with proofs whose conclusion sequent has only one conclusion:

$$\frac{\vdash \Gamma\ \langle\boldsymbol{\alpha}\rangle; \Delta, A \qquad \vdash A^\perp\ \langle\emptyset\rangle}{\vdash \Gamma\ \langle\boldsymbol{\alpha}\rangle; \Delta}\,\text{cut},$$

and the usual sequents are those sequents which have empty purgatory.

We leave the details to a further paper.