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Asynchronous Boolean networks and hereditarily bijective maps

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Abstract The study of relationships between structure and dynamics of asynchronous Boolean networks has recently led to the introduction of hereditarily bijective maps and even or odd self-dual networks. We show here that these two notions can be simply characterized geometrically: through orthogonality between certain affine subspaces. We also use this characterization to study operations preserving hereditary bijectiveness, and to provide effective methods for constructing hereditarily bijective maps and even or odd self-dual networks.

1 Introduction

Introduced by von Neumann [22], Boolean networks represent the dynamic interaction of components which can take two values, 0 and 1. They have been extensively used to model various biological networks, especially genetic regulatory networks, since the early works of S. Kauffman and R. Thomas [7,8,18]. In the context of genetic networks, they are a discrete alternative to differential equations models, in which sufficently precise quantitative data often lack to accurately define the parameters. Regulatory interactions also exhibit strong thresholds effects (sigmoids), so that differential models are often conveniently approximated by piecewise linear equations [17], or one step further discretized into Boolean (or more generally multivalued) networks. See [3,1] for recent surveys.

An increasingly active field of research is the study of asynchronous Boolean networks [20]. An asynchronous Boolean network with n components may be presented by its phase space, which is a partial orientation of the lattice $\{0,1\}^n$, *i.e.* a directed graph whose vertex set is $\{0,1\}^n$ and whose edges only relate vertices which are 1-distant from each other for the Hamming distance. Asynchronous networks are thus nondeterministic dynamical systems, in which the value of only one component is updated at a time.

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Although other update schemes of Boolean networks are studied as models of biological networks (in particular random [7] and synchronous networks [8,4], as well as comparisons between update schemes [6]), asynchrony provides a simple mathematical framework in which all possible trajectories are considered. In the context of genetic networks, one may observe that trajectories of piecewise linear models almost surely (in the sense of measure theory) cross only one threshold hyperplane at a time, so that only the value of the corresponding component is updated in the discretized Boolean network, which thus follows the asynchronous update scheme.

The asymptotic dynamical properties of asynchronous Boolean networks (nature and number of attractors, e.g., existence and unicity of fixed points or attractive cycles) depend on their structure (the directed graph of interactions between components), but precise relationships between dynamics and structure are very difficult to characterize in general. In [19,21], R. Thomas conjectures rules relating positive or negative cycles in the interaction graphs to non-unicity of fixed points (related to cellular differentiation) or sustained oscillations (related to homeostasis). It is possible to give a precise mathematical status to these rules in the framework of asynchronous networks, by identifying sustained oscillations with attractive cycles, or more generally cyclic attractors, and by defining local interaction graphs in a way similar to Jacobian matrices (Section 2 recalls the useful definitions). In this framework, while the positive rule is well understood [9,11], the rule relating (local) negative cycles to the existence of an attractive cycle is unproved in general. In [13], the special case of and-or nets is partly solved, and in [14], the special case of antipodal attractive cycles for and-or nets is proved.

In the course of better understanding these relationships, two opposite notions have been independently introduced recently: *even or odd self-dual* networks in [12], and *hereditarily bijective* maps in [14]. They seem particularly relevant to the study of asynchronous Boolean networks which have a non trivial dynamics: indeed, the dynamics of a hereditarily bijective map is weakly terminating to a unique fixed point, hence particularly "simple", while even or odd self-dual subnetworks generalize the chordless cycles considered in [10] and are necessary for the emergence of a "complex" dynamical behaviour.

In this article, we develop the theory of these classes of networks. We show that hereditary bijectiveness, and hence even or odd self-duality, can be simply characterized geometrically: through orthogonality between certain affine subspaces (Section 4). We then use this characterization:

- to study operations preserving hereditary bijectiveness, in particular to prove that this property is preserved by inverse (Corollary 4),
- and to provide constructions of the classes of hereditarily bijective maps and even or odd self-dual networks (Section 5).

We also study the relationship between the invertibility of a Boolean map and of its Jacobian matrices: we show in Section 3.3 that if a map from $\{0,1\}^n$ to itself has all its Jacobian matrices hereditarily invertible, then it is hereditarily bijective (while the same statement without heredity is known to be false [14]).

This article is an extended version of [15]. In addition to the results already presented in [15], it contains several new results: Lemma 3 and a short proof of Theorem 3 (1) in Section 3.2, Corollary 2 and Lemma 7 in Section 4. Section 5 has

$$f_1(x) = (x_2 + 1)x_3$$
$$f_2(x) = (x_3 + 1)x_1$$
$$f_3(x) = (x_1 + 1)x_2$$

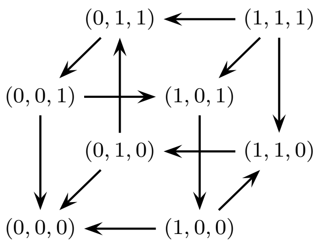

Fig. 1 A Boolean map $f : \mathbb{F}_2^3 \to \mathbb{F}_2^3$ and the asynchronous dynamics $\Gamma(f)$ associated to it. For instance, the point $x = (1, 0, 0)$ in $\Gamma(f)$ has two outgoing edges to $x + e_1 = (0, 0, 0)$ and $x + e_2 = (1, 1, 0)$ because $f(x) = (0, 1, 0) = x + e_1 + e_2$.

been particularly expanded: the details of statements mentioned in [15] have been explained (Theorems 7 and 8, Proposition 2), and some open questions discussed.

2 Asynchronous Boolean networks

We need some preliminary definitions and notations. $\mathbb{F}_2$ denotes the two-element field, equipped with the Boolean sum $(+)$ and product $(\cdot)$ operations.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of the vector space $\mathbb{F}_2^n$, and for each $I \subseteq \{1, \ldots, n\}$, $e_I = \sum_{i \in I} e_i$. For $x, y \in \mathbb{F}_2^n$, $v(x, y)$ denotes the subset $I \subseteq \{1, \ldots, n\}$ such that $x+y = e_I$, and the Hamming distance $d(x, y)$ is defined as the cardinality of $v(x, y)$.

Asynchronous Boolean networks can be equivalently presented in terms of directed graphs or in terms of Boolean maps. An *asynchronous Boolean network* can be defined:

1. either as a directed graph whose vertex set is $\mathbb{F}_2^n$ and whose edges only relate vertices which are 1-distant from each other (for any edge from x to y, $d(x, y) = 1$);
2. or as a map from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$.

The two presentations indeed carry the same information (see Figure 1):

1. To a directed graph γ as above, we may associate a map $\Phi(\gamma) : \mathbb{F}_2^n \to \mathbb{F}_2^n$ by $\Phi(\gamma)(x) = x + e_I$, where $\{(x, x + e_i), i \in I\}$ is the set of edges going from x in γ.
2. Conversely, given a map $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ we may define a directed graph $\Gamma(f)$ with vertex set $\mathbb{F}_2^n$ and an edge from x to y when for some i, $y = x + e_i$ and $f_i(x) \neq x_i$. Here, f_i is defined by $f_i(x) = f(x)_i$. In that case, $d(x, y) = 1$, and clearly, Γ and Φ are inverses of each other.

We call $\Gamma(f)$ the *asynchronous dynamics* associated to f. As we shall consider asynchronous Boolean networks as dynamical systems, the coordinates i such that $f_i(x) \neq x_i$ may naturally be considered as the *degrees of freedom* of x.

An asynchronous Boolean network is therefore a kind of asynchronous *non-uniform* cellular automaton. More specifically, a Boolean network $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ may be viewed as a cellular automaton with global rule $F : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ arising from local rules $F(x)_i = h_i(x)$ defined for $x \in \mathbb{F}_2^{\mathbb{Z}}$ and $i \in \mathbb{Z}$ by:

$$h_i(x) = \begin{cases} f_i(x_1, \ldots, x_n) & \text{if } 1 \leqslant i \leqslant n \\ x_i & \text{otherwise.} \end{cases}$$

The local rule h_i has radius $< n$ for $1 \leqslant i \leqslant n$ and 0 otherwise. Such a cellular automaton is said to be non-uniform because the different local rules prevent the global rule from commuting with the shift map. Under the above identification, a Boolean network is actually a non-uniform automaton with *default* rule (same local rule for large enough $|i|$), *i.e.* a $d\nu$-CA as defined in [5].

2.1 Dynamical properties

We shall be interested in the following asymptotic dynamical properties of asynchronous Boolean networks.

Let $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$. A *trajectory* is a path in $\Gamma(f)$, and an *attractor* is a terminal strongly connected component of $\Gamma(f)$. An attractor which is not a singleton (*i.e.* which does not consist in a fixed point) is called a *cyclic attractor*. *Attractive cycles*, *i.e.* cyclic trajectories θ such that for each point $x \in \theta$, $d(x, f(x)) = 1$, are examples of cyclic attractors. Observe that attractive cycles are deterministic, since any point in θ has a unique degree of freedom.

Recall that f is said to be *weakly terminating* when for any $x \in \mathbb{F}_2^n$, some trajectory leaving x leads to a fixed point. Therefore, f has a cyclic attractor if and only if it is not weakly terminating. A stronger form of weak termination may be defined as follows. Given $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$, a path from $x \in \mathbb{F}_2^n$ to $y \in \mathbb{F}_2^n$ in $\Gamma(f)$ is called a *direct trajectory* when its length is minimal, *i.e.* equals $d(x, y)$. And $\Gamma(f)$ is said to be *directly terminating* when for any point $x \in \mathbb{F}_2^n$ there exists a direct trajectory from x to some fixed point.

For instance, the network f defined in Figure 1 has a (non-attractive) cyclic trajectory $(1,0,0) \to (1,1,0) \to (0,1,0) \to (0,1,1) \to (0,0,1) \to (1,0,1) \to (1,0,0)$, but it is directly terminating to a unique fixed point. We shall see in Section 3 that this property of direct termination is actually a consequence of the fact that $f + \mathrm{id}$ is hereditarily bijective (Theorem 2).

2.2 Subcubes and subnetworks

Given $x \in \mathbb{F}_2^n$ and $I \subseteq \{1, \ldots, n\}$, the subset $x[I]$ consists of all points y such that $y_i = x_i$ for each $i \notin I$; subsets of the form $x[I]$ are called *I-subcubes*, or simply *subcubes* of $\mathbb{F}_2^n$. If $y = x + e_I$, the subcube $x[I]$ is also denoted by $[x, y]$.

Subcubes of $\mathbb{F}_2^n$ are affine subspaces: indeed, the vector space $0[I] = \{e_J | J \subseteq I\}$ acts faithfully and transitively on $x[I]$. However, not every affine subspace is a subcube: the subset $\{(0,0), (1,1)\}$ is an affine subspace because $(1,1) + (1,1) = (0,0)$, but it is clearly not a subcube.

For any subcube κ, let $\pi_\kappa : \mathbb{F}_2^n \to \kappa$ be the projection onto κ, defined as follows: if $\kappa = x[I]$,

$$(\pi_\kappa(y))_i = \begin{cases} y_i & \text{if } i \in I \\ x_i & \text{otherwise.} \end{cases}$$

By observing that an I-subcube and a $(\{1,\dots,n\} \setminus I)$-subcube have a unique common point, it is easy to show that

$$\pi_\kappa(y) = \kappa \cap y[\{1,\dots,n\} \setminus I].$$

Let now $\iota_\kappa : \kappa \to \mathbb{F}_2^n$ be the inclusion map. It is immediate that $\pi_\kappa \circ \iota_\kappa$ is the identity. For any $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$, let

$$f\restriction_\kappa = \pi_\kappa \circ f \circ \iota_\kappa : \kappa \to \kappa.$$

A *subnetwork* of f is a map $f\restriction_\kappa$ for some subcube κ. The asynchronous dynamics $\Gamma(f\restriction_\kappa)$ is easily shown to be the subgraph of $\Gamma(f)$ induced by vertices in κ, a characterization which may be taken as an alternative, more intuitive, definition of subnetworks.

We shall need the fact that projections are affine maps, together with some simple consequences explained in the following lemmas.

Lemma 1 *The projection onto any I-subcube is an affine map, with associated linear transformation the projection $\pi_{0[I]}$ from $\mathbb{F}_2^n$ onto the linear subspace $0[I]$.*

Proof If κ is an I-subcube, then:

$$\pi_\kappa(x) = \pi_\kappa(0) + \pi_{0[I]}(x). \tag{1}$$

Indeed, for any $i \notin I$, $(\pi_\kappa(x))_i = (\pi_\kappa(0))_i$ and $(\pi_{0[I]}(x))_i = 0$. And for any $i \in I$, $(\pi_\kappa(x))_i = (\pi_{0[I]}(x))_i = x_i$ and $(\pi_\kappa(0))_i = 0$. □

Lemma 2 *Let $\kappa = x[I]$ and $\lambda = y[J]$ be any two subcubes of $\mathbb{F}_2^n$. The image of λ under π_κ is the subcube $\pi_\kappa(y)[I \cap J]$. In other terms, $\pi_\kappa(y[J]) = \pi_\kappa(y)[J] \cap \kappa$.*

Proof A point of λ is of the form $y + e_K$ for some $K \subseteq J$. By Equation (1) in the proof of Lemma 1, for any $K \subseteq J$:

$$\pi_\kappa(y + e_K) = \pi_\kappa(y) + \pi_{0[I]}(e_K) = \pi_\kappa(y) + e_{I\cap K}.$$

When K varies among all subsets of J, $I \cap K$ varies among all subsets of $I \cap J$. Therefore the image of λ under π_κ equals $\pi_\kappa(y)[I \cap J]$.

Now, observe that $z[I \cap J] = z[I] \cap z[J]$ for any z. Since $\pi_\kappa(y)[I] = x[I] = \kappa$, we conclude that $\pi_\kappa(y)[I \cap J] = \pi_\kappa(y)[J] \cap \kappa$. □

Corollary 1 *If $x, y \in \mathbb{F}_2^n$ and κ is any subcube, then $\pi_\kappa([x, y]) = [\pi_\kappa(x), \pi_\kappa(y)]$.*

3 Hereditarily bijective maps and even or odd self-dual networks

3.1 Hereditarily bijective maps

A map $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is said to be *hereditarily bijective* (resp. *hereditarily ufp*) [14] when for any subcube κ, $f\restriction_\kappa$ is bijective (resp. has a unique fixed point). A pair $(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$ is called a *mirror pair* of $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ when $x \neq y$ and $(f + \mathrm{id})\restriction_{[x,y]}(x) = (f + \mathrm{id})\restriction_{[x,y]}(y)$, *i.e.* when x and y have the same degrees of freedom for the projected map $f\restriction_{[x,y]}$.

For any $x \in \mathbb{F}_2^n$, the translation t_x maps $y \in \mathbb{F}_2^n$ to $x + y$. The following proposition establishes some immediate preservation properties of hereditary bijectiveness.

Proposition 1 *hereditary bijectiveness is preserved by:*

1. *composition with translations: if* $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ *is hereditarily bijective and* $x, y \in \mathbb{F}_2^n$, *then so is* $t_x \circ f \circ t_y$;
2. *permutation of coordinates: if* $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ *is hereditarily bijective and* $\sigma \in \mathfrak{S}_n$, *then so is* $f^\sigma = \sigma \circ f \circ \sigma^{-1}$, *where* σ *acts on* $\mathbb{F}_2^n$ *by permuting coordinates.*

Proof The first property follows from the fact that translations are (hereditarily) bijective and stable under projection on subcubes. The second property follows from the fact that $f^\sigma \upharpoonright_\kappa = (f \upharpoonright_{\sigma(\kappa)})^\sigma$. □

We shall use the results of Section 4 to give another preservation property in Corollary 4. Let now id denote the identity map from $\mathbb{F}_2^n$ to itself. The following theorem relates the above three definitions.

Theorem 1 (Ruet [14]) *For any* $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$, *the following are equivalent:*

1. $f + \mathrm{id}$ *is hereditarily bijective;*
2. f *is hereditarily ufp;*
3. f *has no mirror pair.*

The asynchronous dynamics of hereditarily bijective maps can then be characterized as follows.

Theorem 2 (Richard [12], Ruet [14]) *If* $f + \mathrm{id}$ *is hereditarily bijective, then* $\Gamma(f)$ *has a unique attractor, this attractor is a fixed point and* $\Gamma(f)$ *is directly terminating (in particular, it is weakly terminating).*

3.2 Even or odd self-dual networks

On the other hand, the definition of even or odd self-dual networks is introduced in [12] and recalled here.

A point $x \in \mathbb{F}_2^n$ is said to be *even* (resp. *odd*) when $\sum_{i=1}^n x_i = 0$ (resp. 1). The sum here is again addition in the field $\mathbb{F}_2$. A map $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is *even* (resp. *odd*) when the image $\mathrm{Im}(f + \mathrm{id})$ of $f + \mathrm{id}$ is the set of even (resp. odd) points of $\mathbb{F}_2^n$. Letting $\overline{x}$ denote the *antipode* $x + e_1 + \cdots + e_n$ of $x \in \mathbb{F}_2^n$, a map f is said to be *self-dual* when for any $x \in \mathbb{F}_2^n$, $f(\overline{x}) = \overline{f(x)}$, *i.e.* $(x, \overline{x})$ is a mirror pair.

Point 2 of the following theorem essentially asserts that if $f + \mathrm{id}$ is not hereditarily bijective, not only has f a mirror pair, but it has an even or odd self-dual subnetwork.

Theorem 3 (Richard [12]) *Let* $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$.

1. *If for any* $i \in \{1, \ldots, n\}$ *and* $\{1, \ldots, n\} \setminus \{i\}$*-subcube* κ, $(f + \mathrm{id})\upharpoonright_\kappa$ *is bijective, and if* $f + \mathrm{id}$ *is not bijective, then* f *is even or odd, and self-dual.*
2. $f + \mathrm{id}$ *is hereditarily bijective if and only if* f *has no even or odd self-dual subnetwork.*

As we have already observed, for the network f of Figure 1, $f + \mathrm{id}$ is hereditarily bijective and indeed, f is directly terminating to a unique fixed point. On the other hand, flipping the arrow from $(0, 1, 0)$ to $(0, 0, 0)$ in f gives rise to the network g of Figure 2 without fixed point: $g + \mathrm{id}$ is not bijective (hence not hereditarily bijective)

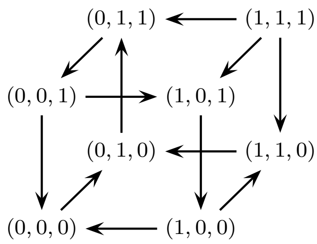

Fig. 2 A network $g : \mathbb{F}_2^3 \to \mathbb{F}_2^3$ such that $g + \mathrm{id}$ is not (hereditarily) bijective.

as it never takes value $(0,0,0)$, and for $\kappa = (0,0,0)[2,3] = [(0,0,0),(0,1,1)]$, $g\!\restriction_\kappa$ is an odd self-dual subnetwork.

The proof of point 1 of Theorem 3 given in [12] relies on an interesting property of subsets of $\mathbb{F}_2^n$. We propose here an alternative, shorter proof, which relies on a single observation:

Lemma 3 *Assume $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ satisfies the hypothesis of point 1 of Theorem 3, and let $F = f + \mathrm{id}$. For any $a \in \mathbb{F}_2^n$ and $i \in \{1,\ldots,n\}$, $F^{-1}(\{a, a+e_i\})$ has cardinality* 2.

Proof If κ denotes the $\{1,\ldots,n\} \setminus \{i\}$-subcube containing a and λ the one containing $a + e_i$, it suffices to note that

$$F^{-1}(\{a, a+e_i\}) = F\!\restriction_\kappa^{-1}(a) \cup F\!\restriction_\lambda^{-1}(a+e_i)$$

is the union of two disjoint singletons, because $F\!\restriction_\kappa$ and $F\!\restriction_\lambda$ are bijective. □

Now, point 1 of Theorem 3 can be proved as follows. If for some a, $F^{-1}(a)$ were a singleton, then by Lemma 3, so would be $F^{-1}(a+e_i)$ for any i, and by "capillarity", so would be all preimages $F^{-1}(b)$ for any b: this would contradict the hypothesis that F is not a bijection. Therefore, for any a and i, $F^{-1}(a)$ has cardinality 2 if and only if $F^{-1}(a+e_i) = \varnothing$. In particular, the distance between any two points in $\mathrm{Im}(F)$ has to be even, so that points in $\mathrm{Im}(F)$ are either all even or all odd, and f is even or odd. It remains to show that f is self-dual. To this end, let $a \in \mathrm{Im}(F)$ and $F^{-1}(a) = \{x, y\}$: x and y are antipodes because otherwise, they would both belong to the same $\{1,\ldots,n\} \setminus \{i\}$-subcube κ for some i, contradicting the bijectiveness of $F\!\restriction_\kappa$.

3.3 Dynamics and structure

The above definitions are used in [12,14] to understand the relationships between the dynamics of an asynchronous Boolean network f and the structure of its Jacobian matrices $\mathscr{J}(f)(x)$ and interaction graphs $\mathscr{G}(f)(x)$. Before turning to the main topic of this paper (the link between hereditary bijectiveness and orthogonality in Section 4), we now recall the definitions of $\mathscr{J}(f)(x)$ and $\mathscr{G}(f)(x)$ and the main results of [12,14], and we mention another interesting property of hereditary bijectiveness (Theorem 5).

Given $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2$ and $i \in \{1, \ldots, n\}$, the *discrete i^{th} partial derivative* $\partial\varphi/\partial x_i = \partial_i \varphi : \mathbb{F}_2^n \to \mathbb{F}_2$ maps each $x \in \mathbb{F}_2^n$ to

$$\partial_i \varphi(x) = \varphi(x) + \varphi(x + e_i).$$

The $+$ here is the addition of the field $\mathbb{F}_2$, therefore $\partial_i \varphi(x) = 1$ if and only if $\varphi(x) \neq \varphi(x + e_i)$. Now, given $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ and $x \in \mathbb{F}_2^n$, the *discrete Jacobian matrix* $\mathscr{J}(f)(x)$ is the $n \times n$ matrix with entries

$$\mathscr{J}(f)(x)_{i,j} = \partial_j f_i(x).$$

And $\mathscr{G}(f)(x)$, the *interaction graph of f at x*, is defined [16] to be the directed graph with vertex set $\{1, \ldots, n\}$ and an edge from j to i when $\mathscr{J}(f)(x)_{i,j} = 1$. The Jacobian matrix $\mathscr{J}(f)(x)$ is therefore simply the transpose of the adjacency matrix of $\mathscr{G}(f)(x)$, and $\mathscr{G}(f)(x)$ is the graph underlying the signed interaction graph defined in [9,11].

In [12,14], a theorem of Shih and Dong on unicity of fixed points is improved as follows.

Theorem 4 *Let $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$.*

1. *If for any $x \in \mathbb{F}_2^n$, $\mathscr{G}(f)(x)$ has no cycle, then f has a unique fixed point (Shih and Dong [16]).*
2. *If $f + \mathrm{id}$ is not hereditarily bijective, then there exist two points $x, y \in \mathbb{F}_2^n$ such that $\mathscr{G}(f)(x)$ and $\mathscr{G}(f)(y)$ have a cycle (Ruet [14]).*
3. *If $f + \mathrm{id}$ is not bijective, then for some $k \in \{1, \ldots, n\}$, there exist 2^k points $x \in \mathbb{F}_2^n$ such that $\mathscr{G}(f)(x)$ has a cycle of length at most k (Richard [12]).*

It is also observed in [16] that a graph is acyclic if and only if all its eigenvalues (under a suitable definition) are 0, so that point 1 of Theorem 4 may be viewed as a Boolean counterpart of the conjecture of Cima, Gasull and Mañosas [2]. In view of the fact that this conjecture on complex polynomials is equivalent to the Jacobian conjecture, it was observed in [14] that, by contrast, the invertibility of all Jacobian matrices $\mathscr{J}(f)(x)$ does not entail invertibility of $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$. We prove here that the stronger condition of hereditary invertibility does.

Definition 1 (Hereditary invertibility, nilpotence) An $n \times n$ matrix $M = (M_{i,j})_{i,j \in \{1,\ldots,n\}}$ with entries in $\mathbb{F}_2$ is said to be *hereditarily invertible* (resp. *hereditarily nilpotent*) when so are all square submatrices $M_I = (M_{i,j})_{i,j \in I}$, for $I \subseteq \{1, \ldots, n\}$.

Lemma 4 *Let M be an $n \times n$ matrix with entries in $\mathbb{F}_2$. The following are equivalent:*

1. *the graph whose adjacency matrix is M has no cycle;*
2. *M is hereditarily nilpotent;*
3. *$\mathscr{I} + M$ is hereditarily invertible, where $\mathscr{I}$ denotes the identity matrix.*

Proof It is proved in [14] that (1) implies M is nilpotent, hence hereditarily nilpotent because induced subgraphs of M have no cycle either. Therefore, (1) implies (2). Clearly, (2) implies (3).

It remains to prove that (3) implies (1). If $\mathscr{I} + M$ is hereditarily invertible, its diagonal is necessarily $(1, \ldots, 1)$. Assume for a contradiction that the graph whose adjacency matrix is M has a cycle: let C be a minimal cycle, *i.e.* one without chord, and $|C|$ be its vertex set. Then the submatrix $(\mathscr{I}+M)_{|C|}$ is the sum of $\mathscr{I}_{|C|}$ with the matrix of a cyclic permutation: it is therefore not invertible, contradicting the hypothesis. $\square$

Theorem 5 *Let $f:\mathbb{F}_2^n\to\mathbb{F}_2^n$. If $\mathscr{J}(f)(x)$ is hereditarily invertible for each $x\in\mathbb{F}_2^n$, then f is hereditarily bijective.*

Proof By Lemma 4, the condition of the theorem implies that, for any $x\in\mathbb{F}_2^n$, $\mathscr{I}+\mathscr{J}(f)(x)=\mathscr{J}(f+\mathrm{id})(x)$ has no cycle. By Theorem 4, f is then hereditarily bijective. $\square$

4 Hereditary bijectiveness and orthogonality

A symmetric and nondegenerate bilinear form $\langle\cdot,\cdot\rangle:\mathbb{F}_2^n\times\mathbb{F}_2^n\to\mathbb{F}_2$ is defined on the vector space $\mathbb{F}_2^n$ by $\langle x,y\rangle=\sum_{i=1}^n x_iy_i$, with sum and product in $\mathbb{F}_2$. As usual, two vectors x,y are orthogonal when $\langle x,y\rangle=0$, a symmetric relation denoted by $x\perp y$. Let A and B be two affine subspaces of $\mathbb{F}_2^n$, with underlying vector spaces V and W respectively: A and B are said to be orthogonal (denoted by $A\perp B$) when for any two vectors $v\in V$ and $w\in W$, $v\perp w$.

Although the quadratic form is isotropic (for instance $\langle(1,1),(1,1)\rangle=0$), orthogonality of subcubes characterizes hereditary bijectiveness.

Theorem 6 *For any $f:\mathbb{F}_2^n\to\mathbb{F}_2^n$, f is hereditarily bijective if and only if for any $x,y\in\mathbb{F}_2^n$ such that $x\neq y$, $[x,y]\not\perp[f(x),f(y)]$.*

To prove Theorem 6, we shall need the following two lemmas.

Lemma 5 *Two subcubes $x[I],y[J]$ of $\mathbb{F}_2^n$ are orthogonal if and only if $I\cap J=\varnothing$.*

Proof The vector spaces underlying $x[I]$ and $y[J]$ are spanned by the sets $\{e_i|i\in I\}$ and $\{e_j|j\in J\}$ respectively. Therefore $x[I]$ and $y[J]$ are orthogonal if and only if any two spanning vectors e_i,e_j, with $(i,j)\in I\times J$, are orthogonal. As $\langle e_i,e_j\rangle=\delta_{i,j}$, this happens exactely when $i\neq j$ for any $(i,j)\in I\times J$, *i.e.* $I\cap J=\varnothing$. $\square$

Lemma 6 *Projections onto subcubes are orthogonal projections.*

Proof By Lemma 1, it suffices to prove that the linear projection onto any linear subspace $0[I]$, with $I\subseteq\{1,\dots,n\}$, is an orthogonal projection. The null space of $\pi_{0[I]}$ is clearly the subspace $0[J]$, where $J=\{1,\dots,n\}\setminus I$. By Lemma 5, we may then conclude that the null space $0[J]$ and the range $0[I]$ of $\pi_{0[I]}$ are orthogonal, as expected. $\square$

We shall also use the following immediate consequence of Lemma 6.

Corollary 2 *Two subcubes κ,λ of $\mathbb{F}_2^n$ are orthogonal if and only if $\pi_\kappa(\lambda)$ is a singleton, if and only if $\pi_\lambda(\kappa)$ is a singleton.*

Corollary 2 and Theorem 6 are illustrated in Figure 3. We now turn to the proof of Theorem 6.

Let us first prove that if f is hereditarily bijective, then for any $x,y\in\mathbb{F}_2^n$ such that $x\neq y$, $[x,y]\not\perp[f(x),f(y)]$. Assume for a contradiction that for some $x\neq y$,

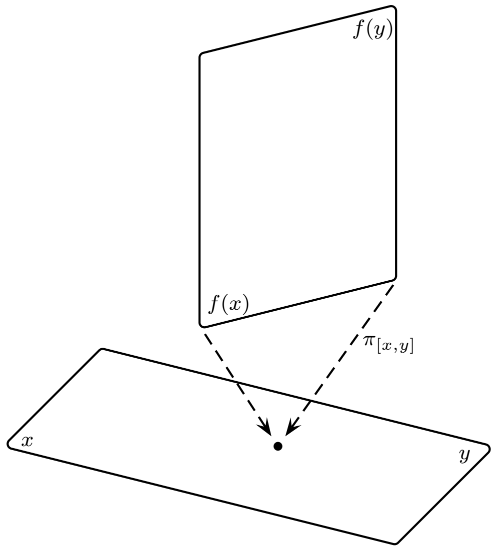

Fig. 3 According to Corollary 2, the subcubes $[x, y]$ and $[f(x), f(y)]$ of Theorem 6 are orthogonal exactly when $[f(x), f(y)]$ projects to a singleton of $[x, y]$.

$[x, y] \perp [f(x), f(y)]$. Let $\kappa = [x, y]$ and $\lambda = [f(x), f(y)]$, so that $\kappa \perp \lambda$. By Corollary 2, π_κ maps the whole subcube λ to a single point. Hence in particular, the two points $f(x)$ and $f(y)$ are mapped by π_κ to the same point $\pi_\kappa(f(x)) = \pi_\kappa(f(y))$. Therefore the two points $x, y \in \kappa$ are mapped by $f\restriction_\kappa = \pi_\kappa \circ f \circ \iota_\kappa$ to the same point, and $f\restriction_\kappa$ is not bijective: contradiction.

Conversely, if f is not hereditarily bijective, then $f\restriction_\kappa$ is not bijective for some subcube κ: there exist $x, y \in \kappa$ such that $x \neq y$ and $f\restriction_\kappa (x) = f\restriction_\kappa (y)$. In particular, we have:

$$\pi_{[x,y]}(f(x)) = f\restriction_{[x,y]} (x) = f\restriction_{[x,y]} (y) = \pi_{[x,y]}(f(y)),$$

and $\pi_{[x,y]}$ maps $f(x)$ and $f(y)$ to the same point. By Corollary 1, $\pi_{[x,y]}$ thus maps the subcube $[f(x), f(y)]$ to a single point, and by Corollary 2 we conclude that $[x, y] \perp [f(x), f(y)]$. This completes the proof of Theorem 6.

Alternatively, we may observe that Corollary 2 leads to the following characterization of mirror pairs through orthogonality.

Lemma 7 *Given $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$, a pair $(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$ is a mirror pair of f if and only if $x \neq y$ and $[x, y] \perp [f(x) + x, f(y) + y]$.*

Proof For $x \neq y$ and $\kappa = [x, y]$, (x, y) is a mirror pair when:

$$\pi_\kappa(f(x) + x) = (f + \mathrm{id})\restriction_{[x,y]} (x) = (f + \mathrm{id})\restriction_{[x,y]} (y) = \pi_\kappa(f(y) + y).$$

This holds exactly when $\pi_\kappa([f(x) + x, f(y) + y])$ is a singleton, *i.e.* when $\kappa \perp [f(x) + x, f(y) + y]$. □

The above characterization is illustrated in Figure 4 and immediately entails Theorem 6. We may summarize the above results as follows.

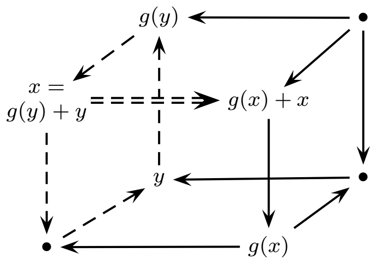

Fig. 4 The network g of Figure 2, one of its mirror pairs (x, y), and the two subcubes $[x, y]$ (dashed lines) and $[g(x) + x, g(y) + y]$ (dashed double lines). As claimed by Lemma 7, these two subcubes are orthogonal.

Corollary 3 *The following are equivalent:*

1. $f + \mathrm{id}$ *is hereditarily bijective;*
2. f *is hereditarily ufp;*
3. f *has no mirror pair;*
4. f *has no even or odd self-dual subnetwork;*
5. *for any* $x \neq y$, $[x, y] \not\perp [f(x) + x, f(y) + y]$.

Another consequence of Theorem 6 is the following preservation property of hereditary bijectiveness.

Corollary 4 *Inverses of hereditarily bijective maps are hereditarily bijective.*

Proof By Theorem 6, we have to prove that if f is hereditarily bijective, then for any $x \neq y$, $[x, y] \not\perp [f^{-1}(x), f^{-1}(y)]$. When $x \neq y$, $f^{-1}(x) \neq f^{-1}(y)$, hence, again by Theorem 6, $[f^{-1}(x), f^{-1}(y)] \not\perp [f(f^{-1}(x)), f(f^{-1}(y))] = [x, y]$. □

This property is especially interesting in view of the fact that hereditarily bijective maps do not form a category: for instance, $f : (x_1, x_2) \mapsto (x_1, x_1 + x_2)$ and $g : (x_1, x_2) \mapsto (x_1 + x_2, x_2)$ are hereditarily bijective, but their composite $g \circ f : (x_1, x_2) \mapsto (x_2, x_1 + x_2)$ is not.

5 Constructions

5.1 Constructing even or odd self-dual networks

Even self-dual networks may be constructed in full generality as follows. Given $\sigma \in \mathfrak{S}_{2^n}$, let $F : \mathbb{F}_2^{n+1} \to \mathbb{F}_2^{n+1}$ be the map defined on the subcube $0[\{1, \dots, n\}]$ by:

$$F(x, 0) = \begin{cases} (\sigma(x), 0) & \text{if } \sigma(x) \text{ is even,} \\ (\sigma(x), 1) & \text{otherwise,} \end{cases}$$

and by $F(x, 1) = F(\overline{x}, 0)$. Then $f = F + \mathrm{id}$ is an even self-dual network.

Theorem 7 *Any even self-dual network can be constructed as above.*

Proof Assume $n \geqslant 0$ and $f : \mathbb{F}_2^{n+1} \to \mathbb{F}_2^{n+1}$ is even self-dual. Let $F = f + \mathrm{id}$, $\kappa = 0[\{1, \ldots, n\}]$ and $\sigma = F{\restriction}_\kappa$. It suffices to show that σ is a permutation of κ, *i.e.* that it is surjective. By self-duality, $\mathrm{Im}(F \circ \iota_\kappa) = \mathrm{Im}(F)$, and since f is even, $\mathrm{Im}(F \circ \iota_\kappa)$ is the set E of even points of $\mathbb{F}_2^{n+1}$. Therefore $\mathrm{Im}(\sigma) = \mathrm{Im}(F{\restriction}_\kappa)$ is the image of E under π_κ, *i.e.* κ itself. □

Replacing even by odd in the above definition provides arbitrary odd self-dual networks.

As we have seen in Section 3, a Boolean network which exhibits some non-trivial dynamics (f such that $\Gamma(f)$ is not directly terminating to a unique fixed point) must contain some even or odd self-dual subnetwork. The above procedure thus provides an effective method for constructing such "non-trivial" networks:

1. construct an even or odd self-dual network $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$,
2. extend this "kernel" network f to a larger network $g : \mathbb{F}_2^{n+p} \to \mathbb{F}_2^{n+p}$ such that $g{\restriction}_{0[\{1,\ldots,n\}]} = f$.

All non-trivial networks can certainly be constructed in this way (from some extension of a kernel), although the above network g is not guaranteed to be non-trivial because the extension might trivialize the kernel f. This obviously raises the question of characterizing the extensions (of an even or odd self-dual network) which preserve some interesting dynamical property, for instance the number of fixed points, or at least the existence of several fixed points, or the existence of cyclic attrators, or more specifically attractive cycles.

5.2 Constructing hereditarily bijective maps

The case of hereditarily bijective maps is less immediate. Starting from hereditarily bijective maps $\sigma, \tau : \mathbb{F}_2^n \to \mathbb{F}_2^n$, let $f : \mathbb{F}_2^{n+1} \to \mathbb{F}_2^{n+1}$ be defined by:

$$f(x, 0) = (\sigma(x), 0) \quad \text{and} \quad f(x, 1) = (\tau(x), 1).$$

Then f is clearly a hereditarily bijective map such that $f{\restriction}_\kappa = \sigma$ and $f{\restriction}_\lambda = \tau$, where κ and λ are the two $\{1, \ldots, n\}$-subcubes.

But there exist other hereditarily bijective maps projecting to σ and τ. Let S be a subset of $\mathbb{F}_2^n$ which is invariant under the action of $\tau^{-1}\sigma$, and let g be defined by:

$$g(x,0) = \begin{cases} (\sigma(x), 1) & \text{if } x \in S \\ (\sigma(x), 0) & \text{otherwise} \end{cases} \qquad g(x,1) = \begin{cases} (\tau(x), 0) & \text{if } x \in S \\ (\tau(x), 1) & \text{otherwise.} \end{cases} \tag{2}$$

Then g is bijective because, w.r.t. f, the roles of $(x, 0)$ and $(y, 1)$ are permuted exactly when $\sigma(x) = \tau(y)$, *i.e.* $y = \tau^{-1}\sigma(x)$. (See Figure 5.) For it to be hereditarily bijective (and not merely bijective), the following theorem claims that S needs to satisfy a stronger invariance condition, which is, roughly speaking, a "hereditary" version of the invariance of S by $\tau^{-1}\sigma$: namely that S is invariant by $(\tau{\restriction}_\mu)^{-1}\sigma{\restriction}_\mu$ for all subcubes μ. Then Proposition 2 characterizes this condition through the orthogonality relation defined in Section 4.

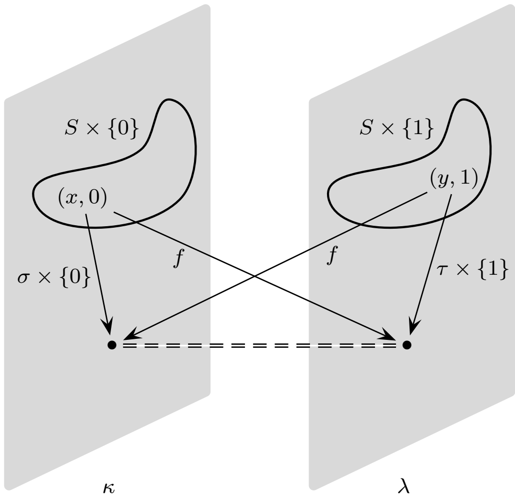

Fig. 5 Constructing a bijective map f by permuting the values of $(x, 0)$ and $(y, 1)$ when $\sigma(x) = \tau(y)$. Thus the set $S = \{x, y, \ldots\}$ of permuted tuples has to be invariant under $\tau^{-1}\sigma$.

Theorem 8 *Let $\sigma, \tau : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be two hereditarily bijective maps, and $S \subseteq \mathbb{F}_2^n$ be invariant under the action of the permutation subgroup of $\mathfrak{S}_{2^n}$ generated by $(\tau\restriction_\mu)^{-1}\sigma\restriction_\mu$ for all subcubes μ of $\mathbb{F}_2^n$. Then equations (2) define a hereditarily bijective map $g : \mathbb{F}_2^{n+1} \to \mathbb{F}_2^{n+1}$ such that $g\restriction_\kappa = \sigma$ and $g\restriction_\lambda = \tau$. Moreover, any hereditarily bijective map can be defined in this way.*

Proof Let $g : \mathbb{F}_2^{n+1} \to \mathbb{F}_2^{n+1}$ be any map such that $g\restriction_\kappa = \sigma, g\restriction_\lambda = \tau$, and S be the set of $x \in \mathbb{F}_2^n$ such that $g(x, 0) \notin \kappa$, *i.e.* $g(x, 0) = (\sigma(x), 1)$.

- Assume that g is hereditarily bijective. Then for any x, $g\restriction_{[(x,0),(x,1)]}$ has to be a bijection. By definition of S, this bijection exchanges $(x, 0)$ and $(x, 1)$ exactly when $x \in S$, therefore $g(x, 1) = (\tau(x), 0)$ if and only if $x \in S$, and g is defined as in (2).
  We now prove that S satisfies the invariance condition. (See Figure 6.) Let μ be any subcube of $\mathbb{F}_2^n$ and ν be the subcube of $\mathbb{F}_2^{n+1}$ consisting in all $(n+1)$-tuples $(x, 0)$ and $(x, 1)$ for $x \in \mu$. If $x \in S \cap \mu$, then
  $$g\restriction_\nu (x, 0) = (\sigma\restriction_\mu (x), 1).$$
  Since τ is hereditarily bijective, $\tau\restriction_\mu$ is a bijection and $y = (\tau\restriction_\mu)^{-1}\sigma\restriction_\mu (x)$ is such that $g\restriction_\nu (y, 1)$ equals either $(\tau\restriction_\mu (y), 0)$ or $(\tau\restriction_\mu (y), 1)$. But $g\restriction_\nu$ is bijective and
  $$g\restriction_\nu (x, 0) = (\sigma\restriction_\mu (x), 1) = (\tau\restriction_\mu (y), 1),$$
  thus $g\restriction_\nu (y, 1) = (\tau\restriction_\mu (y), 0)$ and $y \in S$. Therefore S satisfies the invariance condition.
- To prove the converse, assume g is defined by equations (2), with S satisfying the invariance condition. If ν is any subcube of $\mathbb{F}_2^{n+1}$, we want to show that $g\restriction_\nu$ is bijective. As σ and τ are hereditarily bijective, we may assume that ν

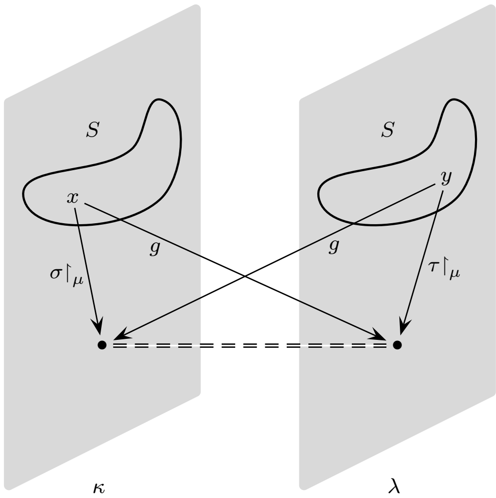

Fig. 6 Constructing a hereditarily bijective map g in Theorem 8. If $x \in S$, then $y = (\tau{\restriction_\mu})^{-1}\sigma{\restriction_\mu}(x) \in S$. The notation of Figure 5 has been simplified, e.g. by denoting $S \times \{0\}$ by S and $(x,0)$ by x on the left.

intersects both κ and λ, *i.e.* that for some subcube μ of $\mathbb{F}_2^n$, ν consists in tuples $(x,0)$ and $(x,1)$ for $x \in \mu$. Given $x, y \in \mu$, we have:

$$g{\restriction_\nu}(x,0) = g{\restriction_\nu}(y,0) \;\Rightarrow\; \sigma{\restriction_\mu}(x) = \sigma{\restriction_\mu}(y) \;\Leftrightarrow\; x = y \quad \text{and}$$
$$g{\restriction_\nu}(x,1) = g{\restriction_\nu}(y,1) \;\Rightarrow\; \tau{\restriction_\mu}(x) = \tau{\restriction_\mu}(y) \;\Leftrightarrow\; x = y.$$

To show that $g{\restriction_\nu}$ is injective, it remains to compare $g{\restriction_\nu}(x,0)$ and $g{\restriction_\nu}(y,1)$. Assume for a contradiction that they are equal. Then in particular, $\sigma{\restriction_\mu}(x) = \tau{\restriction_\mu}(y)$, and by the invariance condition:

$$\begin{array}{rl} \text{either} & x, y \in S,\; g{\restriction_\nu}(x,0) \in \lambda,\; g{\restriction_\nu}(y,1) \in \kappa \\ \text{or} & x, y \notin S,\; g{\restriction_\nu}(x,0) \in \kappa,\; g{\restriction_\nu}(y,1) \in \lambda, \end{array}$$

in contradiction with $g{\restriction_\nu}(x,0) = g{\restriction_\nu}(y,1)$.

□

Proposition 2 *For $\sigma, \tau : \mathbb{F}_2^n \to \mathbb{F}_2^n$ two hereditarily bijective maps and $S \subseteq \mathbb{F}_2^n$, the following conditions are equivalent:*

1. *S is invariant under the action of the permutation subgroup of $\mathfrak{S}_{2^n}$ generated by $(\tau{\restriction_\mu})^{-1}\sigma{\restriction_\mu}$ for all subcubes μ of $\mathbb{F}_2^n$;*
2. *S is closed under the equivalence relation generated by the following binary relation $\smile$ on $\mathbb{F}_2^n$: $x \smile y$ if and only if $[x,y] \perp [\sigma(x), \tau(y)]$.*

Proof It is sufficient to check that $x \smile y$ if and only if $y = (\tau{\restriction_\mu})^{-1}\sigma{\restriction_\mu}(x)$, *i.e.* $\sigma{\restriction_\mu}(x) = \tau{\restriction_\mu}(y)$, for some subcube $\mu \ni x, y$.

– If $x \smile y$, then letting $\mu = [x,y]$, we have $\mu \perp [\sigma(x), \tau(y)]$, therefore, by Corollary 2, $\pi_\mu([\sigma(x), \tau(y)])$ is a singleton and $\sigma{\restriction_\mu}(x) = \tau{\restriction_\mu}(y)$.

– Conversely, if $y = (\tau\restriction_\mu)^{-1}\sigma\restriction_\mu(x)$ for some $\mu \ni x, y$, then

$$\pi_\mu(\sigma(x)) = \pi_\mu(\tau(y)),$$

hence $\pi_\mu([\sigma(x)), \tau(y)])$ is a singleton by Corollary 1, and by Corollary 2, $\mu \perp [\sigma(x), \tau(y)]$. Since $x, y \in \mu$, $[x, y]$ is a subcube of μ, and this entails $[x, y] \perp [\sigma(x), \tau(y)]$.

□

Proposition 2 provides a geometric interpretation of the invariance condition in Theorem 8, but leaves quite open the question of fully characterizing the corresponding permutation subgroups of $\mathfrak{S}_{2^n}$, *i.e.* those generated by $(\tau\restriction_\mu)^{-1}\sigma\restriction_\mu$ for all subcubes μ.

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