Under consideration for publication in Math. Struct. in Comp. Science

Non-commutative logic II : sequent calculus and phase semantics

PAUL RUET[†]

Institut de Mathématiques de Luminy,
163 avenue de Luminy, Case 907, 13288 Marseille Cedex 9 (France).
E-mail : ruet@iml.univ-mrs.fr.

Received 8 November 1999

Non-commutative logic, which is a unification of commutative linear logic and cyclic linear logic, is extended to all linear connectives : additives, exponentials and constants. We give two equivalent versions of the sequent calculus — directly with the structure of order varieties, and with their presentations as partial orders —, phase semantics and a cut elimination theorem.
This involves in particular the study of the entropy relation between partial orders, and the introduction of a special class of order varieties : the series-parallel order varieties.

Dedicated to Jim Lambek for his seventy-fifth birthday.

1. INTRODUCTION

Non-commutative logic is a unification of :
— commutative linear logic (Girard 1987) and
— cyclic linear logic (Girard 1989; Yetter 1990), a classical conservative extension of the Lambek calculus (Lambek 1958).

In a previous paper with Abrusci (Abrusci and Ruet 1999) we presented the multiplicative fragment of non-commutative logic, with proof nets and a sequent calculus based on the structure of order varieties, and a sequentialization theorem. Here we consider full propositional non-commutative logic.

Non-commutative logic. Let us first review the basic ideas. Consider the purely non-commutative fragment of linear logic, obtained by removing the exchange rule entirely :

$$\frac{\vdash \Gamma, \Delta, \Sigma, \Pi}{\vdash \Gamma, \Sigma, \Delta, \Pi},$$

[†] This work has been partly carried out at LIENS-CNRS, Ecole Normale Supérieure (Paris), at McGill University (Montréal), supported by an INRIA fellowship, and at Università Roma 3, supported by a TMR post-doctoral fellowship.

and say we want to introduce commutative connectives. First, we cannot just remove exchange entirely, because we need be able to distinguish one formula, sometimes two, in a sequent, and separate it from the context ; a reasonable solution is to admit cyclic permutations† :

$$\frac{\vdash \Delta, \Sigma}{\vdash \Sigma, \Delta} ,$$

with the well-known consequence that there is a single negation.

In order to combine a commutative conjunction and a non-commutative conjunction, we are naturally led to the idea of a single conjunction on a partial order : A *times* B can have therefore three different meanings, depending on the order between A and B. We are then faced with the following problems :

1. *Entropy.* We must be able ro replace a partial order by a weaker one (or a stronger one, depending on the connective considered), for instance a totally ordered sequent by an unordered one.
2. *Contexts.* There is a difficulty in isolating a formula from its context : should we write a sequent as $\vdash \Gamma, A$ or $\vdash A, \Gamma$, not to speak about A in parallel with Γ, and other configurations ... ?
3. *Cyclicity.* Since the non-commutative fragment should be cyclic, our system has to allow some kind of cyclicity, to move a formula A from left to right in a sequent, etc. The number of negations however, remains open : there could be a commutative one and a non-commutative one ...

Order varieties. The solution is based on :
— a syntactic idea : the seesaw rule, and
— its semantic counterpart : the structure of order variety.

Order varieties — see (Ruet 1997; Abrusci and Ruet 1999) and section 3 — are structures that can be presented by *partial* orders in several ways, a good analogy being the oriented circle which becomes a total order as soon as an origin is fixed. An essential property of order varieties (proposition 2.5) is that in a sequent $\vdash \Gamma$ structured by an order variety, any formula can be isolated. For instance, if we focus on $A \in \Gamma$, the context Γ' becomes a partial order, and $\vdash \Gamma$ can be presented as a partial order in three different ways‡ : $\vdash \Gamma'; A$, $\vdash A; \Gamma'$ or $\vdash \Gamma', A$. The point is that the three presentations are equivalent (proposition 3.6) : we can forget about the choice of A (proposition 3.7) and this process of “swinging” is completely reversible.

This solves problem 2, but also problem 3 at the same time : indeed, the restriction of the sequent calculus to the class of *total* order varieties is precisely cyclic LL. This is easily visualized in the calculus on presentations (section 2), where the reversible structural rules (seesaw, and its inverse, a particular case of entropy) enable one to change a parallel composition into a serial composition and vice-versa. A significant consequence is that there is only one negation. Another consequence of the seesaw rule is that there is

† See Girard’s Appendix F in (Girard 1999) for the associativity problems in absence of cyclicity.
‡ From now on, commas and semicolons denote respectively the parallel and serial compositions of orders.

a single unit **1** for both multiplicative conjunctions (and dually a single unit $\perp$ for both multiplicative disjunctions).

By the way, we are only dealing with binary connectives here, so another class of order varieties will play an important role, namely the *series-parallel* order varieties, introduced and characterized in sections 3.2 and 3.3. They are the order varieties that can be presented by a series-parallel order.

Entropy. Finally, problem 1 : the good notion of entropy is the inclusion of *order varieties* — see section 3.5. At the level of series-parallel orders, i.e. the presentations of series-parallel order varieties, this corresponds to the ability to move arbitrarily from serial to parallel composition. Note that this is not quite what one could expect : for instance one might have expected that the right notion be something like the inclusion of orders, but this is too strong : the two conjunctions become commutative, and even equivalent. Even other intermediate notions of refinement, like "$\vdash \Gamma; (\Delta, \Sigma)$ implies $\vdash (\Gamma; \Delta), \Sigma$", lead to the same problem.

Exponentials. For the extension to exponentials, there are several possible choices :
1. ?ed formulas do not commute in a non-commutative situation : from $\vdash \Gamma[?A; B]$, we may not infer $\vdash \Gamma[B; ?A]$, even if B is itself a ?ed formula. This has been considered by Demaille in (Demaille 1999).
2. Bags of ?ed formulas commute. This has been considered in (Ruet 1997), and it is consistent with the intuition that there is basically a single par — and a single tensor — and the isomorphisms : $(?A; ?B) \cong ?(A \oplus B) \cong (?A, ?B)$.
3. ?ed formulas are central : they commute with everyone. This is the choice we make here, as it is simpler than 2, while preserving the above isomorphisms.

Sequent calculus. There are two natural ways of describing the sequent calculus : either directly on order varieties or on the presentations (partial orders) ; we shall give both descriptions (sections 2 and 4) and prove that they are equivalent, because the calculus on orders is closer to the phase semantics, and on the other hand order varieties give a better understanding of the structural rules and are necessary for the connection with proof nets.

The present sequent calculus on order varieties differs from the one given in (Abrusci and Ruet 1999) essentially by the fact that it contains an explicit rule for entropy. The absence of rule for entropy in (Abrusci and Ruet 1999) was motivated by the desired connection with proof nets, which indeed do not have entropy links. In section 4.2, we show that the two presentations are equivalent, relying on non-trivial results from sections 3.6 and 3.7, where we develop a bit further the theory of order varieties by studying :
— the appropriate notion of inf of order varieties,
— the operation of identification of two points in an order variety.

The crucial point is that it is always possible to identify two points in an order variety, and that this modifies the context in a way that can be simulated by entropy.

Phase semantics and cut elimination. Finally in section 5, we give a phase semantics. At this stage, we think it lacks a simple and natural construction ; still it enables one to prove (weak) cut elimination (section 6), using a technique of Okada (Okada 1994).

2. SEQUENT CALCULUS : ON PARTIAL ORDERS

2.1. *Language*

Definition 2.1. The *formulas (of NL)* are built from atoms $p, q, \ldots, p^\perp, q^\perp, \ldots$, constants $\mathbf{1}, \perp$ (multiplicative), $\top, \mathbf{0}$ (additive), and the following multiplicative connectives :
— non-commutative conjunction $\odot$ *(next)* and disjunction $\triangledown$ *(sequential)*,
— commutative multiplicative conjunction $\otimes$ *(times)* and disjunction $⅋$ *(par)*,
— additive conjunction $\&$ *(with)* and disjunction $\oplus$ *(plus)*,
— exponentials : $!$ *(of course)* and $?$ *(why not)*.

Definition 2.2 (Negation). *Negation* is defined by De Morgan rules :

$$\begin{array}{ll}
(p)^\perp = p^\perp & (p^\perp)^\perp = p \\
(A \odot B)^\perp = B^\perp \triangledown A^\perp & (A \triangledown B)^\perp = B^\perp \odot A^\perp \\
(A \otimes B)^\perp = B^\perp ⅋ A^\perp & (A ⅋ B)^\perp = B^\perp \otimes A^\perp \\
(A\&B)^\perp = B^\perp \oplus A^\perp & (A \oplus B)^\perp = B^\perp \& A^\perp \\
(!A)^\perp = ?A^\perp & (?A)^\perp = !A^\perp \\
\mathbf{1}^\perp = \perp & \perp^\perp = \mathbf{1} \\
\top^\perp = \mathbf{0} & \mathbf{0}^\perp = \top
\end{array}$$

Negation is then an involution : for any formula A, $A^{\perp\perp} = A$.

Definition 2.3 (Implications).

$$A \multimap B = A^\perp ⅋ B, \quad A \twoheadrightarrow B = A^\perp \triangledown B, \quad B \twoheadleftarrow A = B \triangledown A^\perp.$$

2.2. *Rules*

Definition 2.4. *Sequents* are of the form $\vdash \Gamma$, where Γ is an expression built from formulas and binary constructors $(-,-)$ and $(-;-)$.

The rules of the sequent calculus are given in table 1. In the sequel, $\Gamma[\,]$ denotes an expression with a hole (a leaf of the expression Γ), and $\Gamma[\Delta]$ is the expression obtained by filling the hole with Δ. We also use the notation $?\Gamma$ for *any* expression whose formulas are all ?ed.

Remarks.
▶ *About associativity and commutativity.* There is a bijective correspondence between the equivalence classes of sequents under the rules for the associativity of $(-,-)$ and $(-;-)$ and the commutativity of $(-,-)$, and the *series-parallel ordered sets* of occurrences of formulas — see (Mohring 1989) for a survey on series-parallel orders. Recall the definition of serial and parallel compositions of orders : let ω_1 and ω_2 be orders on disjoint sets X and Y respectively ; their *serial* and *parallel* compositions $\omega_1 < \omega_2$ and $\omega_1 \parallel \omega_2$ respectively are two orders on $X \cup Y$ defined by :
• $(\omega_1 < \omega_2)(x, y)$ iff $x <_{\omega_1} y$ or $x <_{\omega_2} y$ or ($x \in X$ and $y \in Y$),
• $(\omega_1 \parallel \omega_2)(x, y)$ iff $x <_{\omega_1} y$ or $x <_{\omega_2} y$.
In the bijective correspondence, the constructors $(-;-)$ and $(-,-)$ of the sequent

Table 1. *Sequent calculus I.*

Identity - Cut

$$\frac{}{\vdash A^{\perp}, A} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,\mathrm{cut}$$

Associativity - Commutativity

$$\frac{\vdash \Pi[\Gamma; (\Delta; \Sigma)]}{\vdash \Pi[(\Gamma; \Delta); \Sigma]}\,\mathrm{a1} \qquad \frac{\vdash \Pi[(\Gamma; \Delta); \Sigma]}{\vdash \Pi[\Gamma; (\Delta; \Sigma)]}\,\mathrm{1a}$$

$$\frac{\vdash \Pi[\Gamma, (\Delta, \Sigma)]}{\vdash \Pi[(\Gamma, \Delta), \Sigma]}\,\mathrm{a2} \qquad \frac{\vdash \Pi[(\Gamma, \Delta), \Sigma]}{\vdash \Pi[\Gamma, (\Delta, \Sigma)]}\,\mathrm{2a} \qquad \frac{\vdash \Pi[\Gamma, \Delta]}{\vdash \Pi[\Delta, \Gamma]}\,\mathrm{com}$$

Structural rules

$$\frac{\vdash \Gamma; \Delta; \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,\mathrm{entropy} \qquad \frac{\vdash \Gamma, \Delta}{\vdash \Gamma; \Delta}\,\mathrm{seesaw} \qquad \frac{\vdash ?\Gamma, \Delta, \Sigma}{\vdash ?\Gamma; \Delta; \Sigma}\,\mathrm{center}$$

Multiplicatives

$$\frac{\vdash \Gamma; A \qquad \vdash B; \Delta}{\vdash \Gamma; A \odot B; \Delta}\,\odot \qquad \frac{\vdash \Gamma; A; B}{\vdash \Gamma; A \triangledown B}\,\triangledown$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr$$

Additives

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\,\& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,\oplus 1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus 2$$

Exponentials

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,\mathrm{d} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,\mathrm{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,\mathrm{w}$$

Constants

$$\frac{}{\vdash \mathbf{1}} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp \qquad \text{(no rule for } \mathbf{0}\text{)} \qquad \frac{}{\vdash \Gamma, \top}$$

calculus are respectively interpreted by *serial* and *parallel* compositions of partial orders, $(- < -)$ and $(- \parallel -)$. We have included the rules for associativity and commutativity for sake of clarity, but we shall refer explicitly to these rules as sparingly as possible. In particular we systematically avoid useless parentheses (see the rules for $\parr$, $\triangledown$, contraction, center and entropy in table 1) and we shall freely consider — even though this is not strictly true — that the present version of the sequent calculus relies on (series-parallel) partial orders. This is why they are separated from the more interesting structural rules : seesaw, entropy and center.

▶ *About seesaw.* It is the key of the system. It is reversible and its inverse is a particular case of entropy :

$$\frac{\vdash \Gamma ; \Delta}{\vdash \Gamma , \Delta}\ \text{co-seesaw.}$$

We shall call *co-seesaw* the inverse of seesaw. Together, they imply cyclic exchange in the usual sense :

$$\cfrac{\cfrac{\cfrac{\vdash \Gamma ; \Delta}{\vdash \Gamma , \Delta}\ \text{co-seesaw}}{\vdash \Delta , \Gamma}\ \text{com}}{\vdash \Delta ; \Gamma}\ \text{seesaw.}$$

Intuitively the seesaw rule can be read as follows : proofs with at most 2 conclusions can freely pivot, thus for such subproofs of a larger proof, commutative and non-commutative compositions should be indistinguishable.

Major consequences of the combination of seesaw and co-seesaw are :
— a single negation,
— a single unit, $\mathbf{1}$ (resp. $\bot$), for both multiplicative conjunctions (resp. disjunctions) : $A \odot \mathbf{1} \dashv\vdash \mathbf{1} \odot A \dashv\vdash A \dashv\vdash A \otimes \mathbf{1}$ and $A \triangledown \bot \dashv\vdash \bot \triangledown A \dashv\vdash A \dashv\vdash A \parr \bot$,
— a single introduction rule for each connective,
— the following focusing[§] property.

Proposition 2.5 (Focusing). Let $\vdash \Gamma$ be a sequent and A be any formula of Γ. Then there is a sequence of seesaw and co-seesaw, the application of which leads from $\vdash \Gamma$ to a sequent of the form $\vdash \Gamma', A$ with the same formulas.

Proof. The expression Γ is a binary tree with nodes either $(-,-)$ or $(-;-)$. We proceed by induction on the length l of the path from the root of Γ to A. If $l = 0$ or 1, i.e., $\Gamma = A$, (Δ, A), (A, Δ), $(\Delta; A)$ or $(A; \Delta)$, the result is obtained after at most one application of the seesaw rule and/or commutativity. If $l \geq 2$, we have the following cases :
— $\Gamma = (\Delta; \Sigma); \Pi$. If A is in Σ, first apply the following rules :

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash (\Delta; \Sigma[A]); \Pi}{\vdash (\Delta; \Sigma[A]), \Pi}\ \text{co-seesaw}}{\vdash \Pi, (\Delta; \Sigma[A])}\ \text{com}}{\vdash \Pi; (\Delta; \Sigma[A])}\ \text{seesaw}}{\vdash (\Pi; \Delta); \Sigma[A]}\ \text{a1,}$$

§ The terminology should not be confused with Andreoli's homonymic property for linear logic (Andreoli 1992), which is related but not at all identical.

and then the induction hypothesis. If A is in Δ, apply one step of associativity and then the induction hypothesis.
— $\Gamma = \Pi; (\Delta; \Sigma)$ and A is in Δ or Σ : identical argument.
— $\Gamma = (\Delta, \Sigma); \Pi$: if A is in Σ, co-seesaw, commutativity and associativity of $(-,-)$ map $\vdash \Gamma$ to $\vdash (\Pi, \Delta), \Sigma$, and then the induction hypothesis applies. If A is in Δ, start with co-seesaw and associativity and commutativity of $(-,-)$.
— $\Gamma = \Pi; (\Delta, \Sigma)$: similar.
— $\Gamma = (\Delta; \Sigma), \Pi$ and A is in Σ : by commutativity, seesaw and associativity of $(-;-)$, $\vdash \Gamma$ is mapped to $\vdash (\Pi; \Delta); \Sigma$, and then the induction hypothesis applies.
— $\Gamma = \Pi, (\Delta; \Sigma)$: identical argument.
— $\Gamma = (\Delta, \Sigma), \Pi$: similar.
— $\Gamma = \Pi, (\Delta, \Sigma)$: identical argument.
Note also that the result is unique modulo the associativity and commutativity rules, i.e. all the expressions Γ' we obtain from a given sequent $\vdash \Gamma$ have the same associated partial order. □

▶ *About exponentials and the center rule.* As we shall verify below (see the examples or the phase semantics : proposition 5.13 (viii)), the sequent calculus enjoys the essential properties of exponentials :

$!A \odot !B \dashv\vdash !(A \& B) \dashv\vdash !A \otimes !B,$

which express the fact that commutativity constraints are irrelevant between ?ed formulas. Besides, for cut elimination, it is necessary to add at least the rule

$$\frac{\vdash \Gamma[?\Delta, ?\Sigma]}{\vdash \Gamma[?\Delta; ?\Sigma]},$$

which, in presence of seesaw, is equivalent to the simpler

$$\frac{\vdash \Gamma, ?\Delta, ?\Sigma}{\vdash \Gamma; ?\Delta; ?\Sigma}.$$

(For instance in the proof of $!A \odot !B \dashv\vdash !B \odot !A$ obtained by composing the proofs of $!A \odot !B \dashv\vdash !(A \& B)$ and $!(A \& B) \dashv\vdash !B \odot !A$ given below, the final cut indeed cannot be eliminated without this rule.)

Note that the above two equations do not imply that a ?ed formula commutes with any context, only with a ?ed context. However it is cheap — and much simpler when we shall move to order varieties — to make this additionnal assumption, which is expressed by the *center* rule : hence as in (Yetter 1990), ?ed formulas are central (proposition 2.6). In order to make the terminology as clear as possible and avoid the word entropy in the reversible cases, we shall adopt the same convention as for the seesaw rule and call *co-center* rule the inverse of the center rule (particular case of entropy).

Proposition 2.6 (Central exponentials). Let $\vdash \Gamma[?A]$ be a sequent where $?A$ is any formula in the expression Γ. There is a sequence of seesaw, co-seesaw, center and co-center, the application of which leads from $\vdash \Gamma[?A]$ to the sequent $\vdash \Gamma[\,], ?A$ with the same formulas.

Proof. As in the proof of proposition 2.5, we proceed by induction on the length l of the path from the root of Γ to $?A$. If $l = 0$ or 1, $\Gamma = ?A$, $(\Delta, ?A)$, $(?A, \Delta)$, $(\Delta; ?A)$ or

$(A;\Delta)$, and the result is obtained after at most one application of the center rule and/or commutativity. If $l \geq 2$, we have the same cases as in the proof of proposition 2.5. Let us consider the case $\Gamma = (\Delta; \Sigma[?A]); \Pi$, the other ones being similar :

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash (\Delta; \Sigma[?A]); \Pi}{\vdash (\Pi; \Delta); \Sigma[?A]}\,\text{cyclicity}}{\vdash ((\Pi;\Delta); \Sigma[\,]), ?A}\,\text{induction hypothesis}}{\vdash \Pi; (\Delta; \Sigma[\,]); ?A}\,\text{seesaw}}{\vdash \Pi, (\Delta; \Sigma[\,]), ?A}\,\text{co-center}}{\vdash (\Delta; \Sigma[\,]), \Pi, ?A}\,\text{com}}{\vdash (\Delta; \Sigma[\,]); \Pi; ?A}\,\text{center}}{\vdash ((\Delta; \Sigma[\,]); \Pi), ?A}\,\text{co-seesaw.}
$$

$\square$

Corollary 2.7. Let $\vdash \Pi$ be a sequent and $?A_1, \ldots, ?A_n$ be all occurrences of ?ed formulas in the set $|\Pi|$. Let $X = |\Pi| \setminus \{?A_1, \ldots, ?A_n\}$. Then given any expression Π' on the set $|\Pi|$ such that $\Pi' \upharpoonright_X = \Pi \upharpoonright_X$, there is a sequence of seesaw, co-seesaw, center and co-center, the application of which leads from $\vdash \Pi$ to $\vdash \Pi'$. In particular, taking $\Pi' = \Pi \upharpoonright_X, ?A_1, \ldots, ?A_n$ amounts to extract all ?ed formulas from Π.

Remark that a formula of MANL (the multiplicative additive fragment of NL) is provable iff it is provable without the center rule : this is an immediate consequence of the cut elimination theorem 6.1 and the subformula property 2.8.

Examples.

▶ In the first two examples, we still mention explicitly the commutativity rule for $(-,-)$. Here is a proof of $A \otimes B \vdash A \odot B$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash A^\perp, A}}{\vdash A^\perp; A}\,\text{seesaw} \qquad \dfrac{\overline{\vdash B, B^\perp}}{\vdash B; B^\perp}\,\text{seesaw}}{\vdash A^\perp; A \odot B; B^\perp}\,\odot}{\vdash A^\perp, A \odot B, B^\perp}\,\text{entropy}}{\vdash A \odot B, B^\perp, A^\perp}\,\text{com}}{\vdash A \odot B, B^\perp \parr A^\perp}\,\parr}{\vdash B^\perp \parr A^\perp, A \odot B}\,\text{com}
$$

▶ Here is a proof of $A \otimes (A \multimapdot B) \vdash B$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash B, B^\perp}}{\vdash B; B^\perp}\,\text{seesaw} \qquad \dfrac{\overline{\vdash A, A^\perp}}{\vdash A; A^\perp}\,\text{seesaw}}{\vdash B; B^\perp \odot A; A^\perp}\,\odot}{\vdash B, B^\perp \odot A, A^\perp}\,\text{entropy}}{\vdash B, (B^\perp \odot A) \parr A^\perp}\,\parr}{\vdash (B^\perp \odot A) \parr A^\perp, B}\,\text{com}
$$

▶ Here are proofs of $!(A \& B) \dashv\vdash !A \odot !B$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}}{\vdash ?A^\perp, A}\,\text{d}}{\vdash ?A^\perp ; A}\,\text{seesaw}
}{\vdash ?B^\perp, (?A^\perp ; A)}\,\text{w}
}{\vdash ?B^\perp ; ?A^\perp ; A}\,\text{seesaw}
}{\vdash (?B^\perp ; ?A^\perp), A}\,\text{co-seesaw}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}}{\vdash B, ?B^\perp}\,\text{d}}{\vdash B ; ?B^\perp}\,\text{seesaw}
}{\vdash (B ; ?B^\perp), ?A^\perp}\,\text{w}
}{\vdash B ; ?B^\perp ; ?A^\perp}\,\text{seesaw}
}{\vdash (?B^\perp ; ?A^\perp), B}\,\text{co-seesaw}
}{\vdash (?B^\perp ; ?A^\perp), A \& B}\,\&
}{\vdash (?B^\perp ; ?A^\perp), !(A \& B)}\,!
}{\vdash !(A \& B) ; ?B^\perp ; ?A^\perp}\,\text{seesaw}
}{\vdash !(A \& B) ; ?B^\perp \triangledown ?A^\perp}\,\triangledown
}{\vdash ?B^\perp \triangledown ?A^\perp, !(A \& B)}\,\text{co-seesaw}
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}}{\vdash B^\perp \oplus A^\perp, A}\,\oplus 2}{\vdash ?(B^\perp \oplus A^\perp), A}\,\text{d}
}{\vdash ?(B^\perp \oplus A^\perp), !A}\,!
}{\vdash ?(B^\perp \oplus A^\perp) ; !A}\,\text{seesaw}
\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}}{\vdash B, B^\perp \oplus A^\perp}\,\oplus 1}{\vdash B, ?(B^\perp \oplus A^\perp)}\,\text{d}
}{\vdash !B, ?(B^\perp \oplus A^\perp)}\,!
}{\vdash !B ; ?(B^\perp \oplus A^\perp)}\,\text{seesaw}
}{\vdash ?(B^\perp \oplus A^\perp) ; !A \odot !B ; ?(B^\perp \oplus A^\perp)}\,\odot
}{\vdash ?(B^\perp \oplus A^\perp), !A \odot !B, ?(B^\perp \oplus A^\perp)}\,\text{entropy}
}{\vdash ?(B^\perp \oplus A^\perp), !A \odot !B}\,\text{c}
$$

▶ Here are proofs of $A \odot \mathbf{1} \dashv\vdash A$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\vdash A, A^\perp}}{\vdash A^\perp ; A}\,\text{seesaw}
}{\vdash \perp, (A^\perp ; A)}\,\perp
}{\vdash \perp ; A^\perp ; A}\,\text{seesaw}
}{\vdash A, (\perp ; A^\perp)}\,\text{co-seesaw}
}{\vdash A ; \perp ; A^\perp}\,\text{seesaw}
}{\vdash A ; \perp \triangledown A^\perp}\,\triangledown
}{\vdash \perp \triangledown A^\perp, A}\,\text{co-seesaw}
\qquad\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\vdash A^\perp, A}}{\vdash A^\perp ; A}\,\text{seesaw}
\qquad
\dfrac{}{\vdash \mathbf{1}}
}{\vdash A^\perp ; A \odot \mathbf{1}}\,\odot
}{\vdash A^\perp, A \odot \mathbf{1}}\,\text{co-seesaw}
$$

▶ Here is a proof of $!A \odot B \vdash B \odot !A$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\vdash B^\perp, B}}{\vdash B^\perp ; B}\,\text{seesaw}
\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\vdash A, A^\perp}}{\vdash A, ?A^\perp}\,\text{d}
}{\vdash !A, ?A^\perp}\,!
}{\vdash !A ; ?A^\perp}\,\text{seesaw}
}{\vdash B^\perp ; B \odot !A ; ?A^\perp}\,\odot
}{\vdash B^\perp ; (?A^\perp, B \odot !A)}\,\text{co-center}
}{\vdash B^\perp ; ?A^\perp ; B \odot !A}\,\text{center}
}{\vdash B^\perp \triangledown ?A^\perp, B \odot !A}\,\triangledown
$$

Clearly, the sequent calculus enjoys the subformula property :

Proposition 2.8 (Subformula property). If $\mathcal{D}$ is a cut-free proof of $\vdash \Gamma$, then the formulas occurring in $\mathcal{D}$ are subformulas of the formulas in Γ.

Cut elimination will be proved using the phase semantics (theorem 6.1).

2.3. *Invariants*

The sequent calculus with all the structural rules explicit (previous section) is not entirely satisfactory, essentially because :
— a proof of a sequent containing only non-commutative connectives may use the commutative composition $(-,-)$, for instance :

$$\dfrac{\dfrac{\overline{\vdash A^\perp, A}}{\vdash A^\perp ; A}\,\text{seesaw}}{\vdash A^\perp \triangledown A}\,\triangledown,$$

— for sake of stability by cut elimination (in particular commutation with seesaw), the cut rule has to appear in 4 different forms (premisses with $(-,-)$ or $(-;-)$) which are clearly equivalent modulo seesaw and co-seesaw :

$$\dfrac{\dfrac{\vdash \Gamma ; A}{\vdash \Gamma, A}\,\text{seesaw} \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\text{cut} \quad \rightarrow \quad \dfrac{\vdash \Gamma ; A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\text{cut}.$$

This raises the question of determining the invariant of sequents under seesaw and co-seesaw. Otherwise said, we are looking for the sequent calculus without seesaw and co-seesaw, corresponding to the calculus of the present section. One might expect to solve the problem by adding $\vdash A^\perp ; A$ as an axiom, and other logical rules, with nested contexts, like :

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Delta[B]}{\vdash \Delta[\Gamma ; A \odot B]}\ \ldots$$

and by eliminating the seesaw rule, having again in mind a calculus relying on orders (reminiscences of the intuitionistic calculus, see (de Groote 1996)). But in fact to prove the associativity of multiplicative connectives, the seesaw rule is essential, which means that the mathematical structure underlying a sequent, invariant by the rules seesaw and co-seesaw, is not an order.

This is our reason for introducing order varieties. Order varieties (section 3) are structures that can be presented by *partial* orders in several ways, a good analogy being the oriented circle which becomes a total order as soon as an origin is fixed : provided a point of view (an element x in the base set), an order variety can be seen as a partial order on the complement of $\{x\}$. This reflects precisely proposition 2.5, which enables one to change the presentation of a sequent (the associated order) in a completely reversible way, e.g. by "pulling out" any formula A. Order varieties can therefore be presented in different ways by changing the viewpoint, of course they are invariant under the change of presentation.

In section 4 we shall present the version of the sequent calculus without seesaw and co-seesaw, using order varieties. We first present the definition and some properties of order varieties (some of them are just recalled from (Abrusci and Ruet 1999)), and introduce *series-parallel* order varieties.

3. ORDER VARIETIES

3.1. *Order varieties and orders*

Definition 3.1 (Order varieties). Let X be a set. An *order variety* on X is a ternary relation $\boldsymbol{\alpha}$ which is :

$$\begin{cases} \textit{cyclic}: & \forall x,y,z \in X, \boldsymbol{\alpha}(x,y,z) \Rightarrow \boldsymbol{\alpha}(y,z,x), \\ \textit{anti-reflexive}: & \forall x,y \in X, \neg\boldsymbol{\alpha}(x,x,y) \\ \textit{transitive}: & \forall x,y,z,t \in X, \boldsymbol{\alpha}(x,y,z) \text{ and } \boldsymbol{\alpha}(z,t,x) \Rightarrow \boldsymbol{\alpha}(y,z,t), \\ \textit{spreading}: & \forall x,y,z,t \in X, \boldsymbol{\alpha}(x,y,z) \Rightarrow \boldsymbol{\alpha}(t,y,z) \text{ or } \boldsymbol{\alpha}(x,t,z) \text{ or } \boldsymbol{\alpha}(x,y,t). \end{cases}$$

An order variety $\boldsymbol{\alpha}$ on X is said to be *total* when $\forall x,y,z \in X, x \neq y \neq z \neq x \Rightarrow \boldsymbol{\alpha}(x,y,z)$ or $\boldsymbol{\alpha}(z,y,x)$.

Ternary relations satisfying the first three axioms have been studied by Novák (Novák 1982) and called cyclic orders.

A few elementary properties and examples of order varieties (for proofs, see (Abrusci and Ruet 1999)) :

Remarks.
▶ If $\boldsymbol{\alpha}$ is a total order variety, $\boldsymbol{\alpha}(x,y,z)$ can be read as "y is between x and z".

Definition 3.2. An order variety $\boldsymbol{\alpha}$ on X induces a binary relation $\rightarrow_{\boldsymbol{\alpha}}$ on X by $x \rightarrow_{\boldsymbol{\alpha}} y$ iff $\forall z \in X$:

$$z \neq x \text{ and } z \neq y \Rightarrow \boldsymbol{\alpha}(x,y,z).$$

One verifies easily that, when $\boldsymbol{\alpha}$ is total, $\rightarrow_{\boldsymbol{\alpha}}$ is an oriented cycle. In section 4, the relation $\rightarrow_{\boldsymbol{\alpha}}$ will be used in the introduction rule for $\triangledown$, for arbitrary order varieties.

Conversely, any oriented cycle G induces a ternary $r(G)$ on $|G|$ by : $r(G)(x,y,z)$ iff y is between x and z in G ; then the set of finite oriented cycles is isomorphic to the set of finite total order varieties, by $\rightarrow_{r(G)}= G$ and $r(\rightarrow_{\boldsymbol{\alpha}}) = \boldsymbol{\alpha}$.

▶ The empty ternary relation on any set X is an order variety on X, called the *empty order variety* on X and denoted by $\varnothing_X$, or simply $\varnothing$ if there is no ambiguity.

▶ The cyclic closure of $\{(a,b,c)\}$ is not an order variety on $\{a,b,c,d\}$ as it does not enjoy the spreading condition. It is an order variety on $\{a,b,c\}$.

Notation. The finite total order variety corresponding to the oriented cycle $a_1 \rightarrow \cdots \rightarrow a_n \rightarrow a_1$ will be simply denoted $(a_1 \ldots a_n)$.

Definition 3.3.
(i) Let $\boldsymbol{\alpha}$ be an order variety on X and $x \in X$. Define the binary relation $\boldsymbol{\alpha}_x$ on $X \setminus \{x\}$ by : $\boldsymbol{\alpha}_x(y,z)$ iff $\boldsymbol{\alpha}(x,y,z)$.
(ii) Let $\omega = (X,<)$ be a (strict) partial order on X and $z \in X$. Define the binary relation $\overset{z}{<}$ by :

— $x \overset{z}{<} y$ iff $x < y$ and z is comparable with neither x nor y,
and the ternary relation $\overline{\omega}$ on X by :

$$\text{— } \overline{\omega}(x,y,z) \quad \text{iff} \quad \begin{array}{lllllll} x<y<z & \text{or} & y<z<x & \text{or} & z<x<y & \text{or} \\ x \overset{z}{<} y & \text{or} & y \overset{x}{<} z & \text{or} & z \overset{y}{<} x. & \end{array}$$

Proposition 3.4. (i) If $\boldsymbol{\alpha}$ is an order variety on X and $x \in X$, then $\boldsymbol{\alpha}_x$ is a partial order on $X \setminus \{x\}$. It is called the order *induced by* $\boldsymbol{\alpha}$ *and* x.
(ii) If (X, ω) is a partial order and $z \in X$, then $\overline{\omega}$ is an order variety on X.

Proposition 3.4 expresses the possibility of focusing on an arbitrary element x in an order variety ($\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}_x \parallel x$) to perform operations (the usual operations on binary orders) and then come back to an order variety ($\omega \mapsto \overline{\omega}$). Note the following properties of commutation with restriction :

Facts 3.5. (i) Let ω be an order on X and $Y \subseteq X$. Then $(\overline{\omega})\restriction_Y = \overline{\omega\restriction_Y}$.
(ii) Let $\boldsymbol{\alpha}$ be an order variety on X, $x \in X$ and $Y \subseteq X \setminus \{x\}$. Then $\boldsymbol{\alpha}_x \restriction_Y = (\boldsymbol{\alpha}\restriction_{Y \uplus \{x\}})_x$.

Proposition and Definition 3.6 (Glueing). If ω and τ are two partial orders on disjoints sets, then $\overline{\omega < \tau} = \overline{\omega \parallel \tau} = \overline{\tau < \omega}$. Define :

$$\boxed{\omega * \tau \;=\; \overline{\omega < \tau} \;=\; \overline{\omega \parallel \tau} \;=\; \overline{\tau < \omega}}\,.$$

For instance, one may easily check as an exercise that if $\omega * \tau$ is a total order variety, with $|\omega|$ and $|\tau|$ non empty, then both ω and τ are total orders. The following proposition establishes a reversible relation between order varieties and partial orders :

Proposition 3.7. Let $\boldsymbol{\alpha}$ be an order variety on a set X, $x \in X$ and ω a partial order on $X \setminus \{x\}$. Then :

$$\boxed{\boldsymbol{\alpha}_x * x \;=\; \boldsymbol{\alpha}} \quad \text{and} \quad \boxed{(\omega * x)_x \;=\; \omega}\,.$$

An order variety is indeed a glueing of order structures, as its name implies, but a very strict kind of glueing, more like the one-point compactification of the plane than general manifold glueing.

3.2. *Series-parallel order varieties*

Recall that the class of so-called *series-parallel* orders is the least class of finite orders containing empty orders on singletons and closed by serial and parallel compositions. See (Mohring 1989) for a survey on series-parallel orders.

In this section, we define the corresponding class of order varieties, called *series-parallel order varieties*. In practice the sequent calculus of section 4 will rely on series-parallel order varieties.

Definition 3.8. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on the sets X and Y respectively, with $X \cap Y = \{x\}$. Define :

$$\begin{cases} \boldsymbol{\alpha} \odot_x \boldsymbol{\beta} \;=\; \overline{\boldsymbol{\alpha}_x < x < \boldsymbol{\beta}_x} \;=\; (\boldsymbol{\beta}_x < \boldsymbol{\alpha}_x) * x \\ \boldsymbol{\alpha} \otimes_x \boldsymbol{\beta} \;=\; \overline{\boldsymbol{\alpha}_x \parallel x \parallel \boldsymbol{\beta}_x} \;=\; (\boldsymbol{\alpha}_x \parallel \boldsymbol{\beta}_x) * x. \end{cases}$$

Clearly, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are order varieties on the sets X and Y respectively, with $X \cap Y = \{x\}$, then $\boldsymbol{\alpha} \odot_x \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta}$ are order varieties on $X \cup Y$.

Example. if $X \cap Y = \{x\}$, $\varnothing_X \otimes_x \varnothing_Y = \varnothing_{X \cup Y}$, but $\varnothing_X \odot_x \varnothing_Y \neq \varnothing_{X \cup Y}$.

Definition 3.9 (Series-parallel order varieties). Given a set X, the class of *series-parallel order varieties* on X is the least class of order varieties containing the empty order varieties $\varnothing_{\{a\}}$ and $\varnothing_{\{a,b\}}$ on singletons and pairs ($a, b \in X$), and closed by $\odot_x$ and $\otimes_x$.

For instance total and empty order varieties are series-parallel order varieties. The following is a straightforward calculation :

Lemma 3.10. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on the sets X and Y respectively, with $X \cap Y = \{x\}$, and let $y \in X \setminus \{x\}$, $z \in Y \setminus \{x\}$.

$$\begin{array}{rclrcl}
(\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_x & = & \boldsymbol{\beta}_x < \boldsymbol{\alpha}_x & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_x & = & \boldsymbol{\beta}_x \parallel \boldsymbol{\alpha}_x \\
(\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_y & = & \boldsymbol{\alpha}_y[(x < \boldsymbol{\beta}_x)/x] & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_y & = & \boldsymbol{\alpha}_y[(x \parallel \boldsymbol{\beta}_x)/x] \\
(\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_z & = & \boldsymbol{\beta}_z[(\boldsymbol{\alpha}_x < x)/x] & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_z & = & \boldsymbol{\beta}_z[(\boldsymbol{\alpha}_x \parallel x)/x]
\end{array}$$

Lemma 3.11. If $\boldsymbol{\alpha}$ is a series-parallel order variety on a non-empty set X, and $x \in X$, then $\boldsymbol{\alpha}_x$ is a series-parallel order on $X \setminus \{x\}$.

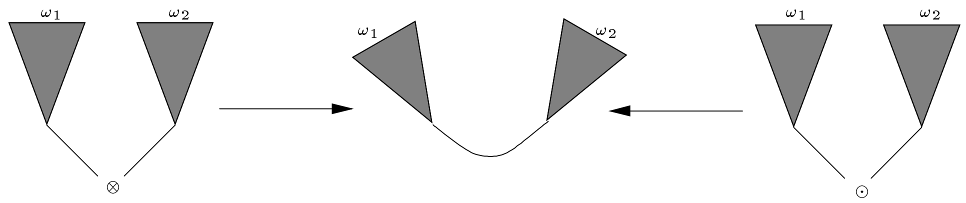
Proof. By proposition 3.4 (i), $\boldsymbol{\alpha}_x$ is an order on $X \setminus \{x\}$. To show that it is series-parallel, proceed by induction on the construction of series-parallel order varieties : for singletons and pairs, it is obvious ; let $\boldsymbol{\gamma} = \boldsymbol{\alpha} \odot_t \boldsymbol{\beta}$ or $\boldsymbol{\alpha} \otimes_t \boldsymbol{\beta}$, by induction hypothesis $\boldsymbol{\alpha}_x$, $\boldsymbol{\alpha}_t$, $\boldsymbol{\beta}_x$ and $\boldsymbol{\beta}_t$ are series-parallel orders, so by lemma 3.10, $(\boldsymbol{\alpha} \odot_t \boldsymbol{\beta})_x$ and $(\boldsymbol{\alpha} \otimes_t \boldsymbol{\beta})_x$ are series-parallel, qed. ☐

Proposition 3.12. Let $\boldsymbol{\alpha}$ be an order variety on a non-empty set X. $\boldsymbol{\alpha}$ is series-parallel iff there exists a series-parallel order ω on X such that $\overline{\omega} = \boldsymbol{\alpha}$.

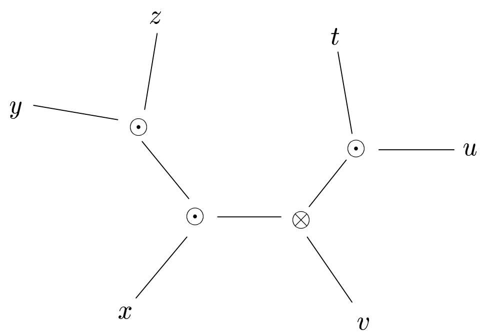
Proof. If $\boldsymbol{\alpha}$ is series-parallel, then let $x \in X$: by lemma 3.11, $(x \parallel \boldsymbol{\alpha}_x)$ is a series-parallel order, and by proposition 3.7, $\boldsymbol{\alpha} = \overline{x \parallel \boldsymbol{\alpha}_x}$.

Conversely, if X has one or two elements, it is obvious, otherwise take $x \in X$: by lemma 3.11, $\boldsymbol{\alpha}_x$ is a series-parallel order on the set $X \setminus \{x\}$ which has at least two elements, so $\boldsymbol{\alpha}_x = (\omega_1 \parallel \omega_2)$ or $(\omega_1 < \omega_2)$. By proposition 3.7, $\boldsymbol{\alpha} = x * \boldsymbol{\alpha}_x$ hence $\boldsymbol{\alpha} = x * (\omega_1 \parallel \omega_2)$ or $x * (\omega_1 < \omega_2)$. In the first case, $\boldsymbol{\alpha} = (x * \omega_1) \otimes_x (x * \omega_2)$ is series-parallel, and in the second case, $\boldsymbol{\alpha} = (x * \omega_1) \odot_x (x * \omega_2)$ is series-parallel as well, qed. ☐

Propositions 3.12, 3.6 and 3.7 suggest the possibility of visualizing series-parallel order varieties on a set X as rootless trees with leaves labeled by elements of X and ternary nodes labeled by $\otimes$ or $\odot$. Given an order variety $\boldsymbol{\alpha}$ on X ($\sharp X \geq 2$), $\boldsymbol{\alpha} = \overline{\omega}$ for some (non-unique) series-parallel order ω : write ω as a (non-unique) binary tree t with leaves labeled by elements of X, and root and nodes labeled by $\otimes$ (in case of parallel composition) or $\odot$ (serial composition) ; then remove the root of t.



For instance $\overline{(x<y<z) \parallel v \parallel (t<u)}$ can be represented by :



To read the tree, take three leaves a, b, c ; then (a, b, c) is in the order variety iff :
— the node $\bullet$ at the intersection of the three paths ab, bc and ca is labeled by $\odot$ and
— the paths $a\bullet$, $b\bullet$ and $c\bullet$ are in this cyclic order while moving clockwise around $\bullet$. Note that the law of spreading, in particular, is easy to check in the tree representation. Note also how the removal of the root corresponds to the glueing operation of proposition 3.6.

The resulting tree is obviously not unique (change $x \odot (y \odot z)$ for $(x \odot y) \odot z$ in the above example) ; however if we quotient the set of such trees by associativity of $\otimes$ and $\odot$ and commutativity of $\otimes$ then the result is unique (in particular it is independent from the choice of ω) : see proposition 3.17.

3.3. *Characterization of series-parallel order varieties*

There is a characterization of series-parallel orders as those orders on finite sets X whose restriction to any 4-elements subset $\{a, b, c, d\}$ is different from the order $N(a, b, c, d) = \{(a, b), (c, b), (c, d)\}$. There is a similar characterization of series-parallel order varieties.

Definition and Lemma 3.13. $G(a, b, c, d, e)$ denotes the ternary relation $(a, b, d) \cup (a, c, d) \cup (c, b, e) \cup (c, d, e) \cup (a, b, e)$. It is an order variety on $\{a, b, c, d, e\}$.

Note that $G(a, b, c, d, e)_e = \{(a, b), (c, b), (c, d)\} = N(a, b, c, d)$ is not a series-parallel order, so according to lemma 3.11, $G(a, b, c, d, e)$ is not a series-parallel order variety. Moreover :

Proposition 3.14. Let $\boldsymbol{\alpha}$ be an order variety on a non-empty finite set X. The following are equivalent :
(i) $\boldsymbol{\alpha}$ is series-parallel,
(ii) there exists a series-parallel order ω on X such that $\overline{\omega} = \boldsymbol{\alpha}$,

(iii) the restriction of $\boldsymbol{\alpha}$ to every 5-element subset $\{a,b,c,d,e\}$ of X is different from $G(a,b,c,d,e)$.

Proof. (i $\Leftrightarrow$ ii) : cf. proposition 3.12.

(ii $\Rightarrow$ iii) : assume that $\overline{\omega} = \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \restriction_Y = G(a,b,c,d,e)$ for some 5-element subset $Y = \{a,b,c,d,e\}$ of X. Then $\overline{\omega \restriction_Y} = (\overline{\omega}) \restriction_Y = G(a,b,c,d,e)$, therefore $\overline{\omega \restriction_Y}$ is not a series-parallel order variety and $\omega \restriction_Y$ is not a series-parallel order by proposition 3.12, henceforth ω is not a series-parallel order either : contradiction.

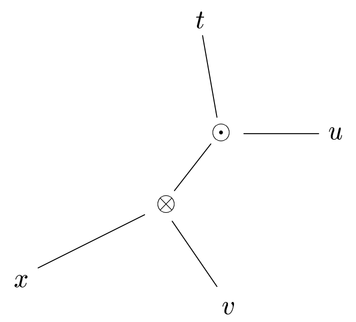
(iii $\Rightarrow$ ii) : let $e \in X$. We can concentrate on sets of at least 5 elements (for smaller sets the result is obvious). If (iii), then $\boldsymbol{\alpha}_e$ is a series-parallel order : indeed, otherwise $\boldsymbol{\alpha}_e \restriction_{\{a,b,c,d\}} = (\boldsymbol{\alpha} \restriction_{\{a,b,c,d,e\}})_e = N(a,b,c,d)$, thus $\boldsymbol{\alpha} \restriction_{\{a,b,c,d,e\}} = N(a,b,c,d) * e = G(a,b,c,d,e)$, contradiction. Therefore there exists a series-parallel order ω on X such that $\overline{\omega} = \boldsymbol{\alpha}$, namely $\omega = (e \,\|\, \boldsymbol{\alpha}_e)$, qed. $\square$

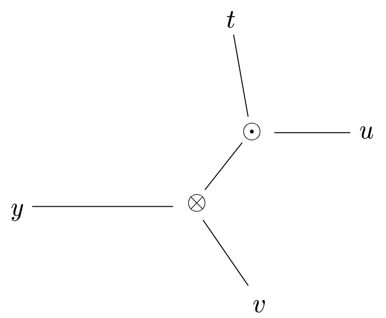
In particular, we have the following consequence :

Proposition 3.15 (Restriction).
(i) Let ω be an order on X and $Y \subseteq X$. Then $(\overline{\omega}) \restriction_Y = \overline{\omega \restriction_Y}$.
(i) If $\boldsymbol{\alpha}$ is an order variety on a set X and $Y \subseteq X$, then the restriction $\boldsymbol{\alpha} \restriction_Y$ of $\boldsymbol{\alpha}$ to Y (as a set of triples) is an order variety on Y. Moreover if $Y \neq \varnothing$ and $\boldsymbol{\alpha}$ is series-parallel, then so is $\boldsymbol{\alpha} \restriction_Y$.
(ii) If $\boldsymbol{\alpha}$ is an order variety on $X \cup \{x\}$ with $x \notin X$, then $\boldsymbol{\alpha} \restriction_X = \overline{\boldsymbol{\alpha}_x}$.

Proof. (i) Immediate consequences of (i), proposition 3.7 and proposition 3.14.
(ii) Let $\boldsymbol{\alpha}$ be an order variety on $X \cup \{x\}$ with $x \notin X$. $\boldsymbol{\alpha} \restriction_X$ and $\overline{\boldsymbol{\alpha}_x}$ are both order varieties on X, and by proposition 3.7, $\boldsymbol{\alpha} \restriction_X = (\overline{\boldsymbol{\alpha}_x \,\|\, x}) \restriction_X = \overline{\boldsymbol{\alpha}_x}$, qed. $\square$

In the tree representation for series-parallel order varieties, the restriction to a subset Y is obtained by cancelling all nodes and edges which are not on a simple path between two leaves of Y : in particular the leaves outside Y and the adjacent edges are cancelled, etc. For instance, the restrictions of $\overline{(x<y<z) \,\|\, v \,\|\, (t<u)}$ to the sets $\{x,t,u,v\}$ and $\{y,t,u,v\}$ are respectively :





3.4. *Seesaw*

The equivalence of series-parallel orders induced by the seesaw rule of section 2 is the same as the equality of the associated order varieties :

Definition 3.16 (Seesaw). $\sim$ is the equivalence relation between partial orders on the same set, defined by :

$$\boxed{\omega \sim \sigma \ \text{ iff } \ \overline{\omega} = \overline{\sigma}}.$$

Proposition 3.17. The restriction of $\sim$ to series-parallel orders is the least equivalence relation between series-parallel orders on the same set such that :

$$(\omega_1 \,\|\, \omega_2) \sim (\omega_1 < \omega_2).$$

Proof. One direction is just proposition 3.6.

Conversely, let $\approx$ be the equivalence relation defined by $(\omega_1 \,\|\, \omega_2) \approx (\omega_1 < \omega_2)$. The result is obvious if $X = \varnothing$. If $\sharp X > 0$, let $x \in X$. One proves that $\omega \approx (x \,\|\, (\overline{\omega})_x)$ by induction on $h(x,\omega) = \min\{$lenght of the path from x to the root of $t \mid t$ tree representing $\omega\}$:

— if $h(x,\omega) = 0$ or 1, it is clear ;

— if $h(x,\omega) > 1$, then ω is a ternary combination of series-parallel orders with $\|$ and $<$: let us consider the case $\omega = (\omega_1[x] \,\|\, \omega_2) < \omega_3$, the other cases being similar. Let $\omega'' = \omega_1[x] \,\|\, (\omega_2 \,\|\, \omega_3)$. On one hand $\omega \approx \omega''$, so $\omega \sim \omega''$ and $\overline{\omega} = \overline{\omega''}$, and on the other hand $\omega'' \approx (x \,\|\, (\overline{\omega''})_x)$ because $h(x,\omega'') = h(x,\omega) - 1$, hence $\omega \approx \omega'' \approx (x \,\|\, (\overline{\omega''})_x) = (x \,\|\, (\overline{\omega})_x)$, qed.

Therefore, if $\omega \sim \sigma$, i.e. $\overline{\omega} = \overline{\sigma}$, then $\omega \approx (x \,\|\, (\overline{\omega})_x) = (x \,\|\, (\overline{\sigma})_x) \approx \sigma$. □

3.5. *Entropy*

Definition 3.18 (Entropy). $\trianglelefteq$ is the relation defined on the set of all partial orders for a fixed set by :

$$\boxed{\omega \trianglelefteq \sigma \ \text{ iff } \ \omega \subseteq \sigma \ \text{ and } \ \overline{\omega} \subseteq \overline{\sigma}}.$$

Obviously :

Facts 3.19. (i) $\trianglelefteq$ is a partial order.
(ii) $\trianglelefteq$ is compatible with restriction.

Lemma 3.20. $\trianglelefteq$ is compatible with the compositions $\|$ and $<$ of partial orders.

Proof. If $\omega \trianglelefteq \omega'$ and $\sigma \trianglelefteq \sigma'$, then $(\omega \,\|\, \sigma) \subseteq (\omega' \,\|\, \sigma')$, and furthermore $\overline{\omega \,\|\, \sigma} \subseteq \overline{\omega' \,\|\, \sigma'}$. Indeed :

— $\overline{\omega \,\|\, \sigma}\restriction_{|\omega|} = \overline{\omega} = \overline{\omega' \,\|\, \sigma'}\restriction_{|\omega|}$ and $\overline{\omega \,\|\, \sigma}\restriction_{|\sigma|} = \overline{\sigma} = \overline{\omega' \,\|\, \sigma'}\restriction_{|\sigma|}$;

— if $a, b \in |\omega|$ and $c \in |\sigma|$, then $\overline{\omega \,\|\, \sigma}(a,b,c) \Leftrightarrow \omega(a,b) \Rightarrow \omega'(a,b) \Leftrightarrow \overline{\omega' \,\|\, \sigma'}(a,b,c)$; idem for $a, b \in |\sigma|$ and $c \in |\omega|$.

Idem for $<$. □

Corollary 3.21. If $\omega \trianglelefteq \omega'$ and $\sigma \trianglelefteq \sigma'$, then $\omega * \sigma \subseteq \omega' * \sigma'$.

In proposition 3.24, we will see that $\trianglelefteq$ is a generalization to arbitrary partial orders of the relation considered in the sequent calculus of section 2 in the series-parallel case : moving from serial composition to parallel composition. The fact that $\trianglelefteq$ corresponds to the *inclusion* of order varieties (proposition 3.22) confirms that it is indeed a good choice, and it will serve as a basis for defining a calculus on order varieties in section 4.

Proposition 3.22. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on X and $x \in X$. Then the following are equivalent :
(i) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$,
(ii) $\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x$,
(iii) there exist partial orders ω and σ on X such that $\boldsymbol{\alpha} = \overline{\omega}$, $\boldsymbol{\beta} = \overline{\sigma}$ and $\omega \trianglelefteq \sigma$.

Proof. (i) $\Rightarrow$ (ii). If $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$, then $\boldsymbol{\alpha}_x \subseteq \boldsymbol{\beta}_x$, and on the other hand $\overline{\boldsymbol{\alpha}_x} = \boldsymbol{\alpha}\lceil_{X\setminus\{x\}} \subseteq \boldsymbol{\beta}\lceil_{X\setminus\{x\}} = \overline{\boldsymbol{\beta}_x}$ by proposition 3.15, therefore $\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x$.

(ii) $\Rightarrow$ (iii). $\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x$ implies the existence of the required partial orders : take $\omega = (\boldsymbol{\alpha}_x \,\|\, x)$ and $\sigma = (\boldsymbol{\beta}_x \,\|\, x)$; by lemma 3.20, $\omega \trianglelefteq \sigma$.

(iii) $\Rightarrow$ (i). If there exist partial orders ω and σ on X such that $\boldsymbol{\alpha} = \overline{\omega}$, $\boldsymbol{\beta} = \overline{\sigma}$ and $\omega \trianglelefteq \sigma$, then by definition of $\trianglelefteq$, $\overline{\omega} \subseteq \overline{\sigma}$, i.e. $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$. □

Lemma 3.23. $\omega[\omega_1 \,\|\, \omega_2] \trianglelefteq \omega[\omega_1 < \omega_2]$.

Proof. If $\omega = \omega[\omega_1 \,\|\, \omega_2]$ and $\sigma = \omega[\omega_1 < \omega_2]$, then
— clearly $\omega \subseteq \sigma$,
— $\overline{\omega} \subseteq \overline{\sigma}$: indeed, consider $a, b, c \in X$ such that $\overline{\omega}(a, b, c)$; let $X = |\omega|$, $X_1 = |\omega_1|$, $X_2 = |\omega_2|$, $X_3 = X \setminus (X_1 \cup X_2)$ and I be the least set of indices $\subseteq \{1, 2, 3\}$ such that $\{a, b, c\} \subseteq \bigcup_{i \in I} X_i$; if $I = \{i\}$, then $\overline{\sigma}(a, b, c)$ holds trivially ; if $I = \{1, 2\}$, then say $a, b \in X_1$ and $c \in X_2$, whence $\overline{\omega}(a, b, c)$ iff $\omega_1(a, b)$ iff $\overline{\sigma}(a, b, c)$; if $I = \{1, 3\}$, then $\overline{\omega}(a, b, c)$ iff $\overline{\omega\lceil_{X_1 \cup X_3}}(a, b, c)$ iff $\overline{\sigma\lceil_{X_1 \cup X_3}}(a, b, c)$ iff $\overline{\sigma}(a, b, c)$; the case $I = \{2, 3\}$ is similar ; finally, $I = \{1, 2, 3\}$ simply contradicts $\overline{\omega}(a, b, c)$.
Hence $\omega \trianglelefteq \sigma$. □

Proposition 3.24. The restriction of $\trianglelefteq$ to series-parallel orders is the least reflexive transitive relation between series-parallel orders on the same set such that :

$$\omega[\omega_1 \,\|\, \omega_2] \trianglelefteq \omega[\omega_1 < \omega_2].$$

Proof. By lemma 3.23, the above rule is sound. Conversely, let ω and σ be two series-parallel orders on the same set such that $\omega \trianglelefteq \sigma$. We use the axiomatisation of the *inclusion* between series-parallel orders given by (Bechet, de Groote and Retoré 1997) ; the result stated in propositions 3.2 and 4.1 of (Bechet, de Groote and Retoré 1997) is the following : $\omega \subseteq \sigma$ iff ω is obtained from σ by means of the following reflexive and transitive congruence :

$$\begin{array}{llcl}
(1) & \tau < \theta & \to & \tau \parallel \theta \\
(2) & \tau < (\theta_1 \parallel \theta_2) & \to & (\tau < \theta_1) \parallel \theta_2 \\
(3) & (\theta_1 \parallel \theta_2) < \tau & \to & \theta_1 \parallel (\theta_2 < \tau) \\
(4) & (\tau_1 \parallel \tau_2) < (\theta_1 \parallel \theta_2) & \to & (\tau_1 < \theta_1) \parallel (\tau_2 < \theta_2).
\end{array}$$

Now $\omega \trianglelefteq \sigma$, thus $\omega \subseteq \sigma$. We have to use condition $\overline{\omega} \subseteq \overline{\sigma}$ to restrict ourselves to the rewrite rule (1). We shall write $\stackrel{1}{\to}$ for the congruence induced by (1). We proceed by induction on the cardinality of $\sigma \setminus \omega$. If $\sigma \setminus \omega = \varnothing$, then the result is trivial. If $\sigma \setminus \omega \neq \varnothing$, then there is a derivation $\omega \leftarrow \sigma$ of $\omega \subseteq \sigma$. There are four cases :

— The first rule applied is (1), i.e. $\omega \leftarrow \rho[\tau \parallel \theta] \leftarrow \rho[\tau < \theta] = \sigma$. Then clearly $\rho[\tau \parallel \theta] \stackrel{1}{\leftarrow} \sigma$ and $\omega \subseteq \rho[\tau \parallel \theta]$. On the other hand $\overline{\omega} \subseteq \overline{\rho[\tau \parallel \theta]}$: indeed

$$\begin{array}{rcl}
\overline{\omega}\restriction_{|\rho|} & \subseteq & \overline{\sigma}\restriction_{|\rho|} \\
 & = & \overline{\rho[\tau \parallel \theta]}\restriction_{|\rho|} \quad \text{and} \\
\overline{\omega}\restriction_{X\setminus|\rho|} & \subseteq & \overline{\sigma}\restriction_{X\setminus|\rho|} \\
 & = & \overline{\tau < \theta} \\
 & = & \overline{\tau \parallel \theta} \\
 & = & \overline{\rho[\tau \parallel \theta]}\restriction_{X\setminus|\rho|} \quad ;
\end{array}$$

besides, $\overline{\omega}(x,a,b)$ for some $x \in |\rho|$ and $a,b \in X \setminus |\rho|$ implies $\overline{\sigma}(x,a,b)$ and then $a,b \in |\tau|$ or $a,b \in |\theta|$ because $\omega \subseteq \rho[\tau \parallel \theta]$, whence $\overline{\rho[\tau \parallel \theta]}(x,a,b)$ as well ; and finally $\overline{\omega}(x,y,a)$ for some $x,y \in |\rho|$ and $a \in X \setminus |\rho|$ implies $\overline{\sigma}(x,y,a)$ and clearly $\overline{\rho[\tau \parallel \theta]}(x,y,a)$ holds. Hence $\omega \trianglelefteq \rho(\tau \parallel \theta)$, and by induction $\omega \stackrel{1}{\leftarrow} \rho(\tau \parallel \theta)$, therefore $\omega \stackrel{1}{\leftarrow} \sigma$ by transitivity of $\stackrel{1}{\leftarrow}$, qed.

— The first rule applied is (2) : $\omega \leftarrow \rho[(\tau < \theta_1) \parallel \theta_2] \leftarrow \rho[\tau < (\theta_1 \parallel \theta_2)] = \sigma$. Since $\overline{\omega} \subseteq \overline{\sigma}$, for all $a \in |\tau|, b_1 \in |\theta_1|, b_2 \in |\theta_2|$, we have $\neg\overline{\omega}(a,b_1,b_2)$ and $\neg\overline{\omega}(a,b_2,b_1)$. By $\omega \subseteq \rho[(\tau < \theta_1) \parallel \theta_2]$, we conclude that a and b_1 are incomparable in ω, henceforth

$$\omega \subseteq \rho[\tau \parallel \theta_1 \parallel \theta_2].$$

We prove $\overline{\omega} \subseteq \overline{\rho[\tau \parallel \theta_1 \parallel \theta_2]}$ as avove, by replacing θ by $\theta_1 \parallel \theta_2$. Therefore $\omega \trianglelefteq \rho[\tau \parallel \theta_1 \parallel \theta_2]$ and by induction $\omega \stackrel{1}{\leftarrow} \rho[\tau \parallel \theta_1 \parallel \theta_2]$. Now obviously $\rho[\tau \parallel \theta_1 \parallel \theta_2] \stackrel{1}{\leftarrow} \sigma$, therefore $\omega \stackrel{1}{\leftarrow} \sigma$ by transitivity of $\stackrel{1}{\leftarrow}$.

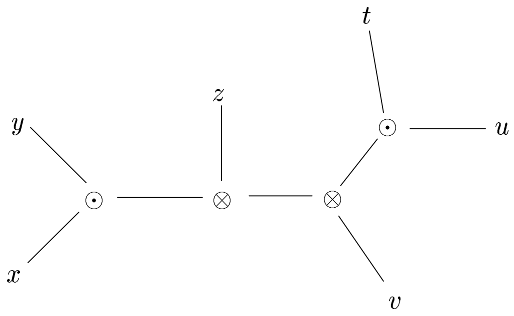
— The first rule applied is (3) : similar.

— The first rule applied is (4) :

$$\omega \leftarrow \rho[(\tau_1 < \theta_1) \parallel (\tau_2 < \theta_2)] \leftarrow \rho[(\tau_1 \parallel \tau_2) < (\theta_1 \parallel \theta_2)] = \sigma.$$

Since $\overline{\omega} \subseteq \overline{\sigma}$ and $\omega \subseteq \rho[(\tau_1 < \theta_1) \parallel (\tau_2 < \theta_2)]$, we can prove, as in the case of rule (2), that $\omega \subseteq \rho[\tau_1 \parallel \theta_1 \parallel \tau_2 \parallel \theta_2]$. Then by replacing τ by $\tau_1 \parallel \tau_2$ and θ by $\theta_1 \parallel \theta_2$ in the argument used for rule (1), we get $\overline{\omega} \subseteq \overline{\rho[\tau_1 \parallel \theta_1 \parallel \tau_2 \parallel \theta_2]}$, and we conclude by using $\rho[\tau_1 \parallel \theta_1 \parallel \tau_2 \parallel \theta_2] \stackrel{1}{\leftarrow} \sigma$.
□

In the tree representation for series-parallel order varieties, entropy is performed by changing some $\odot$-nodes into $\otimes$-nodes. For instance, the following order variety is obtained by entropy from $\overline{(x < y < z) \parallel v \parallel (t < u)}$:



Lemma 3.25. (i) The partial orders ω such that $\omega \trianglerighteq N(a,b,c,d)$ are $N(a,b,c,d)$, $a<c<b<d$ and $c<d<a<b$.
(ii) The order varieties $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \supseteq G(a,b,c,d,e)$ are $G(a,b,c,d,e)$ and the total order varieties (a,c,b,d,e), and (c,d,a,b,e).
(iii) The partial orders ω such that $\omega \trianglelefteq N(a,b,c,d)$ are $N(a,b,c,d)$ and $a \parallel b \parallel c \parallel d$.
(iv) The order varieties $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \subseteq G(a,b,c,d,e)$ are $G(a,b,c,d,e)$ and $\varnothing_{\{a,b,c,d,e\}}$.

Proof. (i) Let ω be a partial order such that $\omega \trianglerighteq N(a,b,c,d)$, and assume $\omega \neq N(a,b,c,d)$. Then in particular, $\omega \supset N(a,b,c,d)$ and is therefore one of the following 12 non-trivial extensions of $N(a,b,c,d)$:

$$\begin{array}{ll}
(a \parallel c) < (b \parallel d) & (a \parallel c) < b < d \\
a<c<b<d & (a \parallel c) < d < b \\
a<c<d<b & (a \parallel (c<d)) < b \\
c<a<b<d & a<c<(b \parallel d) \\
c<a<d<b & c<a<(b \parallel d) \\
c<d<a<b & c<((a<b) \parallel d).
\end{array}$$

It is easy to check that the additionnal condition $\overline{\varpi} \supseteq \overline{N(a,b,c,d)} = (a,b,d) \cup (a,c,d)$ implies the result.

(ii) By proposition 3.22, $\boldsymbol{\alpha} \supseteq G(a,b,c,d,e)$ iff $\boldsymbol{\alpha}_e \trianglerighteq G(a,b,c,d,e)_e = N(a,b,c,d)$, whence the result.

(iii) Among the 8 partial orders $\omega \subseteq N(a,b,c,d)$, only $N(a,b,c,d)$ and $a \parallel b \parallel c \parallel d$ enjoy $\overline{\varpi} \subseteq \overline{N(a,b,c,d)}$.

(iv) Same argument as in (ii). □

3.6. *Interior and wedges*

Before turning to the sequent calculus on order varieties, we develop a bit further the theory of order varieties by studying :
— the appropriate notion of inf of order varieties,
— the operation of identification of two points in an order variety (next section).

The reason for these two sections is to show (section 4) that the forthcoming calculus on order varieties is consistent with the multiplicative sequent calculus given in (Abrusci and Ruet 1999). They can therefore be skipped in a first reading.

Intersections of order varieties are obviously cyclic orders but not necessarily order varieties. As we are definitely dealing with order varieties, we need a way to transform cyclic orders into order varieties. This motivates the following definitions :

Definition 3.26 (Interior). Let $\boldsymbol{\alpha}$ be a cyclic order on X (i.e. an order variety without the spreading condition). Define its *interior* $\natural\boldsymbol{\alpha}$ by :

$$\boxed{\natural\boldsymbol{\alpha} \ = \ \bigcap_{x\in X}(\boldsymbol{\alpha}_x * x)}.$$

(The definition of the partial order $\boldsymbol{\alpha}_x$ on $X \setminus \{x\}$ is the same as for order varieties.) For instance if $\boldsymbol{\alpha} = (x,y,z,t) \cup (x,y,u)$, then $\natural\boldsymbol{\alpha} = (x,y,z) \cup (x,y,t) \cup (x,y,u)$. As shown in (Abrusci and Ruet 1999), the interior of a cyclic order satisfies the following properties :

Proposition 3.27. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be cyclic orders on X.
(i) $\natural\boldsymbol{\alpha}$ is an order variety on X.
(ii) $\natural\boldsymbol{\alpha} \subseteq \boldsymbol{\alpha}$.
(iii) $\natural\natural\boldsymbol{\alpha} = \natural\boldsymbol{\alpha}$.
(iv) $\natural\boldsymbol{\alpha}$ is the largest order variety included in $\boldsymbol{\alpha}$.
(v) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta} \Rightarrow \natural\boldsymbol{\alpha} \subseteq \natural\boldsymbol{\beta}$.
(vi) If $Y \subseteq X$ then $(\natural\boldsymbol{\alpha})\restriction_Y \subseteq \natural(\boldsymbol{\alpha}\restriction_Y)$.
(vii) $\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \subseteq \natural\boldsymbol{\alpha} \cap \natural\boldsymbol{\beta}$.
(viii) $\natural(\natural\boldsymbol{\alpha} \cap \natural\boldsymbol{\beta}) = \natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$.
(ix) $\rightarrow_{\natural\boldsymbol{\alpha}} = \rightarrow_{\boldsymbol{\alpha}}$.

Definition 3.28 (Wedge of order varieties). Let $\boldsymbol{\alpha}_i, i \in I$, be order varieties on X. Define :

$$\boxed{\bigwedge \boldsymbol{\alpha}_i \ = \ \natural\bigcap \boldsymbol{\alpha}_i}.$$

If I has cardinality 2, we write $\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2$.

Proposition 3.29. (i) $\wedge$ is commutative and associative.
(ii) $\bigwedge \boldsymbol{\alpha}_i$ is the largest order variety included in all the $\boldsymbol{\alpha}_i$.

Proof. (i) Commutativity is clear. For associativity :

$$\begin{aligned}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma} \ &= \ \natural(\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \cap \boldsymbol{\gamma}) \\ &= \ \natural(\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \cap \natural\boldsymbol{\gamma}) \\ &= \ \natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta} \cap \boldsymbol{\gamma}).\end{aligned}$$

The second equality holds because $\boldsymbol{\gamma}$ is an order variety and the third one is a consequence of proposition 3.27 (viii). Similarly $\boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) = \natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta} \cap \boldsymbol{\gamma})$, qed.
(ii) Obvious application of proposition 3.27. □

Corollary 3.30. Order varieties on a given set form a complete inf-semi-lattice for inclusion and wedge.

Definition 3.31 (Wedge of orders). Let $\omega_i, i \in I$, be partial orders on X. Define :

$$\boxed{\bigwedge \omega_i \ = \ (\bigwedge \omega_i * x)_x},$$

where $x \notin X$. If I has cardinality 2, we write $\omega_1 \wedge \omega_2$.

Lemma 3.32. (i) If $Y \subseteq |\boldsymbol{\alpha}_i|$, we have $(\bigwedge \boldsymbol{\alpha}_i)\restriction_Y \subseteq \bigwedge \boldsymbol{\alpha}_i\restriction_Y$.
(ii) If $Y \subseteq |\omega_i|$, we have $(\bigwedge \omega_i)\restriction_Y \trianglelefteq \bigwedge \omega_i\restriction_Y$.
(iii) Wedge commutes with focusing : $(\bigwedge \boldsymbol{\alpha}_i)_x = \bigwedge(\boldsymbol{\alpha}_i)_x$ for any $x \in |\boldsymbol{\alpha}_i|$.
(iv) Wedge commutes with glueing : $(\bigwedge \omega_i) * x = \bigwedge(\omega_i * x)$ for any $x \notin |\omega_i|$.
(v) More generally, $(\bigwedge_{i\in I} \omega_i) * (\bigwedge_{j\in J} \tau_j) = \bigwedge_{i\in I, j\in J}(\omega_i * \tau_j)$ when $|\omega_i| \cap |\tau_j| = \varnothing$.

Proof. (i) follows from proposition 3.27 (vi). (ii) follows from :

$$\begin{aligned}
(\bigwedge \omega_i)\restriction_Y &= ((\bigwedge \omega_i * x)_x)\restriction_Y \\
&\trianglelefteq (\bigwedge \omega_i\restriction_Y * x)_x \quad \text{by (i) and proposition 3.22} \\
&= \bigwedge \omega_i\restriction_Y .
\end{aligned}$$

(iii) follows from the equality $\bigwedge(\boldsymbol{\alpha}_i)_x = [\bigwedge((\boldsymbol{\alpha}_i)_x * x)]_x = (\bigwedge \boldsymbol{\alpha}_i)_x$, a consequence of proposition 3.6. (iv) is analogous.

For (v), it is enough to prove that $(\bigwedge_{i\in I} \omega_i) * \tau = \bigwedge_{i\in I}(\omega_i * \tau)$. The inclusion $(\bigwedge \omega_i) * \tau \subseteq \bigwedge(\omega_i * \tau)$ follows from $\bigwedge \omega_i \trianglelefteq \omega_i$ and corollary 3.21. For the converse, let $X = |\omega_i|$, $Y = |\tau|$, and consider $a, b, c \in X \cup Y$ such that $\bigwedge(\omega_i * \tau)\ (a, b, c)$. Either the a, b, c are all in X or all in Y, or one of them is in one set and the other two are in the other, but anyway we can apply (iv). For instance, if $a, b, c \in X$, take $y \in Y$; by (i) and (iv) we have :

$$\begin{aligned}
(\bigwedge(\omega_i * \tau))\restriction_{X\cup\{y\}} &\subseteq \bigwedge(\omega_i * \tau)\restriction_{X\cup\{y\}} \\
&= \bigwedge(\omega_i * y) \\
&= (\bigwedge \omega_i) * y,
\end{aligned}$$

therefore $\bigwedge(\omega_i * \tau)\ (a, b, c)$ implies $((\bigwedge \omega_i) * y)\ (a, b, c)$, and henceforth $((\bigwedge \omega_i) * \tau)\ (a, b, c)$. If on the other hand $a, b \in X$ and $c \in Y$, we apply the same property with $y = c$. $\square$

Proposition 3.33. (i) $\wedge$ is a commutative and associative operation on partial orders.
(ii) $\bigwedge \omega_i$ is the largest order ω (w.r.t. $\trianglelefteq$) such that $\omega \trianglelefteq \omega_i$ for all i.

Proof. (i) Commutativity is obvious. Associativity is a consequence of propositions 3.29 and 3.6. As for (ii) :

$$\begin{array}{lll r}
\tau \trianglelefteq \text{all the } \omega_i & \text{iff} & \tau * x \subseteq \bigcap(\omega_i * x) & \text{by proposition 3.22} \\
& \text{iff} & \tau * x \subseteq \bigwedge(\omega_i * x) & \text{because } \tau * x \text{ is an order variety} \\
& \text{iff} & (\tau * x)_x \trianglelefteq [\bigwedge(\omega_i * x)]_x & \text{by proposition 3.22 again} \\
& \text{iff} & (\tau * x)_x \trianglelefteq \bigwedge(\omega_i * x)_x & \text{by lemma 3.32} \\
& \text{iff} & \tau \trianglelefteq \bigwedge \omega_i & \text{by proposition 3.6.}
\end{array}$$

$\square$

Remarks.
▶ $\omega \wedge \tau \subset \omega \cap \tau$ but the inclusion is strict in general. Consider for instance $\omega = (a \parallel b) < c$

and $\tau = (a < c) \parallel b$. Since $\tau \subseteq \omega$, we have $\omega \cap \tau = \tau$. On the other hand :

$$\begin{aligned}
\omega \wedge \tau &= (\omega * x \wedge \tau * x)_x \\
&= [((a,c,x) \cup (b,c,x)) \wedge ((a,c,b) \cup (a,c,x))]_x \\
&= [\natural(a,c,x)]_x \\
&= (\varnothing_{\{a,b,c,x\}})_x \\
&= \varnothing,
\end{aligned}$$

because $\natural(a,c,x) \subseteq (a,c,x)_b * b = \varnothing_{\{a,b,c,x\}}$.

▶ $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ may be series-parallel and not $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$: take $\boldsymbol{\alpha} = (a,c,b,d,e)$ and $\boldsymbol{\beta} = (c,d,a,b,e)$, they are even total, but $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = G(a,b,c,d,e) = \boldsymbol{\alpha} \cap \boldsymbol{\beta}$.

▶ For similar reasons, ω and τ may be series-parallel but not $\omega \wedge \tau$:

$$\begin{aligned}
(c < d < a < b) \wedge (a < c < b < d) &= ((c < d < a < b) * e \wedge (a < c < b < d) * e)_e \\
&= ((c,d,a,b,e) \wedge (a,c,b,d,e))_e \\
&= G(a,b,c,d,e)_e \\
&= N(a,b,c,d).
\end{aligned}$$

By the way, note that the wedge of two order varieties equals their intersection as soon as this intersection is an order variety, whereas the intersection of partial orders is always an order and we cannot conclude anything about their wedge.

3.7. *Identification*

Definition 3.34 (Identification). Let $\boldsymbol{\alpha}$ be an order variety on a set $X \cup \{x,y\}$, with $x, y \notin X$, $x \neq y$, and let $z \notin X$. Define the *identification* $\boldsymbol{\alpha}[z/x,y]$ of x and y into z in $\boldsymbol{\alpha}$ by :

$$\boxed{\boldsymbol{\alpha}[z/x,y] \;=\; \boldsymbol{\alpha}{\restriction_{X \cup \{x\}}} [z/x] \;\wedge\; \boldsymbol{\alpha}{\restriction_{X \cup \{y\}}} [z/y]}.$$

Clearly, if $\boldsymbol{\alpha}$ is an order variety, then so is $\boldsymbol{\alpha}[z/x,y]$.

Lemma 3.35. (i) $\boldsymbol{\alpha}[z/x,y]_z = (\overline{\boldsymbol{\alpha}_x})_y \wedge (\overline{\boldsymbol{\alpha}_y})_x$.
(ii) $\boldsymbol{\alpha}[z/x,y]_z * (x \parallel y) \subseteq \boldsymbol{\alpha}$.
(iii) Let $\boldsymbol{\alpha}$ be an order variety on $X \cup \{x,y\}$, with x, y different and not in X, and ω be a partial order on X such that $\omega * (x \parallel y) \subseteq \boldsymbol{\alpha}$. Then $\omega * (x \parallel y) \subseteq \boldsymbol{\alpha}[z/x,y]_z * (x \parallel y)$, or equivalently $\omega \trianglelefteq \boldsymbol{\alpha}[z/x,y]_z$.

Proof. (i) By lemma 3.32 :

$$\begin{aligned}
\boldsymbol{\alpha}[z/x,y]_z &= \boldsymbol{\alpha}{\restriction_{X \cup \{x\}}} [z/x]_z \wedge \boldsymbol{\alpha}{\restriction_{X \cup \{y\}}} [z/y]_z \\
&= (\boldsymbol{\alpha}{\restriction_{X \cup \{x\}}})_x \wedge (\boldsymbol{\alpha}{\restriction_{X \cup \{y\}}})_y \\
&= (\overline{\boldsymbol{\alpha}_y})_x \wedge (\overline{\boldsymbol{\alpha}_x})_y.
\end{aligned}$$

(ii) Let $a, b, c \in X \cup \{x,y\}$ such that $\boldsymbol{\alpha}[z/x,y]_z * (x \parallel y)\ (a,b,c)$. First, $\{a,b,c\}$ does not contain both x and y, so assume $y \notin \{a,b,c\}$, whence $(\boldsymbol{\alpha}[z/x,y]_z * (x \parallel y)) \restriction_{X\cup\{x\}} (a,b,c)$, i.e. $(\boldsymbol{\alpha}[z/x,y]_z * x)(a,b,c)$. By (i) and proposition 3.33 :

$$\begin{array}{rcl} \boldsymbol{\alpha}[z/x,y]_z * x & = & (\overline{\boldsymbol{\alpha}_x})_y * x \wedge (\overline{\boldsymbol{\alpha}_y})_x * x \\ & \subseteq & (\overline{\boldsymbol{\alpha}_y})_x * x \\ & = & \overline{\boldsymbol{\alpha}_y} \\ & = & \boldsymbol{\alpha} \restriction_{X\cup\{x\}}, \end{array}$$

therefore $\boldsymbol{\alpha}(a,b,c)$. The case where $x \notin \{a,b,c\}$ is similar.

(iii) Let us prove that $\omega * z \subseteq \boldsymbol{\alpha}[z/x,y]$, i.e. $\omega * z \subseteq \boldsymbol{\alpha}\restriction_{X\cup\{x\}} [z/x]$ and $\omega * z \subseteq \boldsymbol{\alpha}\restriction_{X\cup\{y\}} [z/y]$. This is immediate since by hypothesis $\omega * x \subseteq \boldsymbol{\alpha}\restriction_{X\cup\{x\}}$ and $\omega * y \subseteq \boldsymbol{\alpha}\restriction_{X\cup\{y\}}$. □

Proposition 3.36. (i) Inclusion is compatible with identification in order varieties.
(ii) If $\boldsymbol{\alpha}$ is a series-parallel order variety, so is $\boldsymbol{\alpha}[z/x,y]$.

Proof. (i) We have to show that $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$ implies $\boldsymbol{\alpha}[z/x,y] \subseteq \boldsymbol{\beta}[z/x,y]$. By lemma 3.35 (ii), $\boldsymbol{\alpha}[z/x,y]_z * (x \parallel y) \subseteq \boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$. Therefore by lemma 3.35 (iii) applied to $\omega = \boldsymbol{\alpha}[z/x,y]_z$, we have $\boldsymbol{\alpha}[z/x,y]_z \trianglelefteq \boldsymbol{\beta}[z/x,y]_z$, qed.

(ii) Let $|\boldsymbol{\alpha}| = X \uplus \{x,y\}$, and assume for a contradiction that $\boldsymbol{\alpha}[z/x,y]$ is not series-parallel, i.e. that $\boldsymbol{\alpha}[z/x,y]_z$ is not a series-parallel order : $\boldsymbol{\alpha}[z/x,y]_z \restriction_Y = N(a,b,c,d)$ for some $Y = \{a,b,c,d\} \subseteq X$. By lemma 3.35 (i), $\boldsymbol{\alpha}[z/x,y]_z = (\overline{\boldsymbol{\alpha}_x})_y \wedge (\overline{\boldsymbol{\alpha}_y})_x$, hence by lemma 3.32 (ii) :

$$\begin{array}{rcl} \boldsymbol{\alpha}[z/x,y]_z \restriction_Y & \trianglelefteq & (\overline{\boldsymbol{\alpha}_x})_y \restriction_Y \wedge (\overline{\boldsymbol{\alpha}_y})_x \restriction_Y \\ & = & (\overline{\boldsymbol{\beta}_x})_y \wedge (\overline{\boldsymbol{\beta}_y})_x, \end{array}$$

with $\boldsymbol{\beta} = \boldsymbol{\alpha}\restriction_{Y\uplus\{x,y\}} = \boldsymbol{\alpha}\restriction_{\{a,b,c,d,x,y\}}$. Therefore $N(a,b,c,d) \trianglelefteq (\overline{\boldsymbol{\beta}_x})_y$ and $N(a,b,c,d) \trianglelefteq (\overline{\boldsymbol{\beta}_y})_x$. Moreover, $(\overline{\boldsymbol{\beta}_x})_y$ and $(\overline{\boldsymbol{\beta}_y})_x$ are series-parallel order varieties because so is $\boldsymbol{\beta}$. Hence by lemma 3.25, $(\overline{\boldsymbol{\beta}_x})_y$ and $(\overline{\boldsymbol{\beta}_y})_x$ are either $(a<c<b<d)$ or $(c<d<a<b)$. They cannot be equal, because otherwise their wedge, $\boldsymbol{\alpha}[z/x,y]_z \restriction_Y$, would equal their common value. So we may assume without loss of generality that :

$$\begin{array}{rcl} (\overline{\boldsymbol{\beta}_x})_y & = & (a<c<b<d) \\ (\overline{\boldsymbol{\beta}_y})_x & = & (c<d<a<b), \end{array}$$

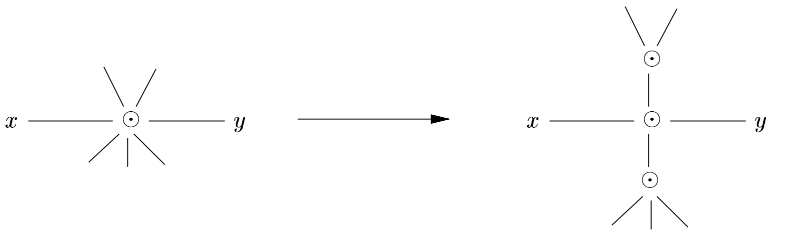
i.e.

$$\begin{array}{rcccl} \boldsymbol{\alpha}\restriction_{\{a,b,c,d,y\}} & = & \overline{\boldsymbol{\beta}_x} & = & (a,c,b,d,y) \\ \boldsymbol{\alpha}\restriction_{\{a,b,c,d,x\}} & = & \overline{\boldsymbol{\beta}_y} & = & (c,d,a,b,x). \end{array}$$

We got a contradiction because the first equality implies $\boldsymbol{\alpha}(a,c,b)$ whereas the second one implies $\boldsymbol{\alpha}(a,b,c)$. □

In the series-parallel case, the identification can be performed directly on the tree representation introduced in section 3.2. Take a representation T of $\boldsymbol{\alpha}$, an order variety on $X \cup \{x,y\}$, and let p be the path between x and y in T. Then "normalize" T by :
— collapsing neighboring nodes of the same type ($\odot$ or $\otimes$) in p, so that in the resulting tree T', nodes of the path p' between x and y have alternating types ;

— transforming every $\odot$-node in p' into 2 or 3 adjacent $\odot$-nodes in the following way :



It is obvious that the normal tree still represents $\boldsymbol{\alpha}$, and the identification $\boldsymbol{\alpha}[z/x, y]$ is obtained by changing all the $\odot$-nodes into $\otimes$-nodes and by identifying x and y.

To see that this is the largest series-parallel order variety obtained by identifying x and y (in fact the largest order variety thanks to proposition 3.36), first recall that, in general, entropy between series-parallel order varieties is done indeed in the tree representation by changing some $\odot$-nodes into $\otimes$-nodes. Of course, there is some choice in where to perform entropies. In the case of identification, however, we know from lemma 3.35 that x and y have to be in parallel at the end. The above "fusions" and "fissions" are then done to minimize the number of $\odot$-nodes between x and y to which entropies are applied : indeed these operations commute with entropy.

4. SEQUENT CALCULUS : ON ORDER VARIETIES

4.1. *Rules*

Definition 4.1. A *sequent* $\vdash \boldsymbol{\alpha} \mid \Gamma$ consists of an order variety $\boldsymbol{\alpha}$ of formula occurrences, and a set Γ of formula occurrences (with no additional structure). Γ is required to be disjoint from the support of $\boldsymbol{\alpha}$.

The rules of the sequent calculus are given in table 2. When there is no ambiguity, i.e. when $\boldsymbol{\alpha}$ or Γ is empty and the notation enables one to decide whether it is an order variety or a set, we avoid the stoup $\mid$. If ω is an order on X, then we denote X by $|\omega|$.

Example. Here is a proof of $!A \odot !B \vdash !(A \& B)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash A * A^\perp}}{\vdash A \mid A^\perp}\,\text{center}}{\vdash A \mid A^\perp, B^\perp}\,\text{w} \qquad \dfrac{\dfrac{\overline{\vdash B * B^\perp}}{\vdash B \mid B^\perp}\,\text{center}}{\vdash B \mid B^\perp, A^\perp}\,\text{w}}{\vdash A\&B \mid B^\perp, A^\perp}\,\&}{\vdash !(A\&B) \mid B^\perp, A^\perp}\,!}{\vdash \overline{!(A\&B) < ?B^\perp} \mid A^\perp}\,\text{d}}{\vdash (!(A\&B) < ?B^\perp) * ?A^\perp}\,\text{d}$$

$$\begin{array}{c} \vdash (!(A\&B) < ?B^\perp) * ?A^\perp \\ \hdashline \vdash !(A\&B) * (?B^\perp < ?A^\perp) \end{array}$$

$$\dfrac{\vdash !(A\&B) * (?B^\perp < ?A^\perp)}{\vdash !(A\&B) * (?B^\perp \triangledown ?A^\perp)}\,\triangledown$$

The dotted line corresponds to a change in the presentation of the series-parallel order variety.

Table 2. *Sequent calculus II.*

All order varieties and partial orders here are series-parallel. In particular, in dereliction and the $\bot$-rule, ω is series-parallel, and in the entropy rule, $\boldsymbol{\alpha}$ is series-parallel.

Identity - Cut

$$\frac{}{\vdash A^{\perp} * A} \qquad \left\{ \begin{array}{l} \dfrac{\vdash \omega * A \mid \Gamma \qquad \vdash \omega' * A^{\perp} \mid \Gamma'}{\vdash \omega * \omega' \mid \Gamma, \Gamma'}\ \text{cut} \\[2ex] \dfrac{\vdash \boldsymbol{\alpha} \mid A, \Gamma \qquad \vdash !A^{\perp} \mid \Gamma'}{\vdash \boldsymbol{\alpha} \mid \Gamma, \Gamma'}\ \text{cut}_! \end{array} \right.$$

Structural rules

$$\frac{\vdash \omega * A \mid \Gamma}{\vdash \overline{\omega} \mid A, \Gamma}\ \text{center} \qquad \frac{\vdash \boldsymbol{\beta} \mid \Gamma}{\vdash \boldsymbol{\alpha} \mid \Gamma}\ \text{entropy}, \boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$$

Multiplicatives

$$\frac{\vdash \omega * A \mid \Gamma \qquad \vdash \omega' * B \mid \Gamma'}{\vdash (\omega' < \omega) * A \odot B \mid \Gamma, \Gamma'}\ \odot \qquad \frac{\vdash \omega * (A < B) \mid \Gamma}{\vdash \omega * A \triangledown B \mid \Gamma}\ \triangledown$$

$$\frac{\vdash \omega * A \mid \Gamma \qquad \vdash \omega' * B \mid \Gamma'}{\vdash (\omega \parallel \omega') * A \otimes B \mid \Gamma, \Gamma'}\ \otimes \qquad \frac{\vdash \omega * (A \parallel B) \mid \Gamma}{\vdash \omega * A \parr B \mid \Gamma}\ \parr$$

Additives

$$\frac{\vdash \omega * A \mid \Gamma \qquad \vdash \omega * B \mid \Gamma}{\vdash \omega * A \& B \mid \Gamma}\ \& \qquad \left\{ \begin{array}{l} \dfrac{\vdash \omega * A \mid \Gamma}{\vdash \omega * A \oplus B \mid \Gamma}\ \oplus 1 \\[2ex] \dfrac{\vdash \omega * B \mid \Gamma}{\vdash \omega * A \oplus B \mid \Gamma}\ \oplus 2 \end{array} \right.$$

Exponentials

$$\frac{\vdash \overline{\omega} \mid A, \Gamma}{\vdash \omega * ?A \mid \Gamma}\ \text{d} \qquad \frac{\vdash A \mid \Gamma}{\vdash !A \mid \Gamma}\ ! \qquad \frac{\vdash \boldsymbol{\alpha} \mid A, A, \Gamma}{\vdash \boldsymbol{\alpha} \mid A, \Gamma}\ \text{c} \qquad \frac{\vdash \boldsymbol{\alpha} \mid \Gamma}{\vdash \boldsymbol{\alpha} \mid A, \Gamma}\ \text{w}$$

Constants

$$\frac{}{\vdash \mathbf{1}} \qquad \frac{\vdash \overline{\omega} \mid \Gamma}{\vdash \omega * \bot \mid \Gamma}\ \bot \qquad \text{(no rule for } \mathbf{0}\text{)} \qquad \frac{}{\vdash \omega * \top \mid \Gamma}$$

Remarks.

▶ By proposition 3.7, there is no loss of generality in taking order varieties of the form $\omega * A$, since any order variety $\boldsymbol{\alpha}$ can be written as $\boldsymbol{\alpha}_A * A$.

Note that the condition of series-parallelism in entropy, dereliction and the $\perp$-rule suffices to preserve series-parallelism along the construction of a proof :

Lemma 4.2. If $\vdash \boldsymbol{\alpha} \mid \Gamma$ is a provable sequent, then $\boldsymbol{\alpha}$ is series-parallel.

On the other hand these conditions are necessary since a partial order ω may not be series-parallel, even when $\overline{\omega}$ is (for instance, $\overline{N(a,b,c,d)} = (a,b,d) \cup (a,c,d)$), and a non-series-parallel order variety may be included in a series-parallel one (for instance, $G(a,b,c,d,e) \subseteq (a,c,b,d,e)$ by lemma 3.25).

▶ *About the stoup.* The center rule and the rule for weakening take formulas out of the scope of the order variety. Conversely, on its way back to earth — the scope of the order variety —, a formula can be placed in any position that does not affect the current structure $\boldsymbol{\alpha}$ on X, i.e. the extended order variety $\boldsymbol{\alpha}'$ on $X + x$ is *any* series-parallel order variety enjoying $\boldsymbol{\alpha}' \upharpoonright_X = \boldsymbol{\alpha}$. Indeed, if $\boldsymbol{\alpha}'$ is such an order variety, take $\omega = \boldsymbol{\alpha}'_x$: by proposition 3.15, we have $\boldsymbol{\alpha} = \boldsymbol{\alpha}' \upharpoonright_X = \overline{\boldsymbol{\alpha}'_x} = \overline{\omega}$.

This corresponds to the previous center rule in the calculus of section 2. Note that in absence of exponentials, the "set" part of sequents remains empty (consequence of the subformula property) and can henceforth be forgotten.

▶ *About the* $\triangledown$*-rule.* The rule for $\triangledown$ corresponding to the one given in (Abrusci and Ruet 1999) would be :

$$\frac{\vdash \boldsymbol{\alpha}[A,B] \mid \Gamma}{\vdash \boldsymbol{\alpha}[A \triangledown B/A,B] \mid \Gamma}\ \triangledown\star,\ \text{if } A \to_{\boldsymbol{\alpha}} B$$

where the notation $\boldsymbol{\alpha}[A,B,\dots]$ stands for an order variety on a set X such that $A, B, \dots \in X$. In table 2, we prefered to replace it by another one, where the relation $\to_{\boldsymbol{\alpha}}$ is expressed more simply. They are equivalent. Indeed, if $A \to_{\boldsymbol{\alpha}} B$, consider the partial order $\boldsymbol{\alpha}_B$: every C is less or equal than A, therefore $\boldsymbol{\alpha}_B = (\omega < A)$, hence $\boldsymbol{\alpha} = \omega * (A < B)$. Conversely, it is clear that $A \to_{\omega*(A<B)} B$.

In order to simplify the notation, we consider that the sequent calculus of section 2 relies on partial orders.

Theorem 4.3 (Equivalence between the two sequent calculi).
1. Given a proof of $\vdash \omega$ in the calculus on orders, let $?X$ be the set of $?$ed formulas in $|\omega|$ and $Y = |\omega| \setminus ?X$. We construct, by forgetting the seesaw rule, a proof of $\vdash \boldsymbol{\alpha} \mid X$ in the calculus on order varieties, for $\boldsymbol{\alpha} = \overline{\omega} \upharpoonright_Y$. The notation X means that the main ? has been removed in all formulas of $?X$.
2. Conversely, given a proof of $\vdash \boldsymbol{\alpha} \mid X$, where the support of $\boldsymbol{\alpha}$ is Y, we construct, by expliciting some structural steps, a proof of $\vdash \omega$ for any series-parallel order ω on $Y \cup ?X$ such that $\overline{\omega} \upharpoonright_Y = \boldsymbol{\alpha}$.
3. The mappings (1) and (2) preserve the absence of cuts.

Proof. 1. Proceed by induction on a proof of $\vdash \omega$. We consider the last rule of the proof :
— Axiom : at most one of the formulas is ?ed, so the translation is either just an axiom or an axiom followed by center.
— Cut rule :

$$\frac{\vdash \omega, A \qquad \vdash A^{\perp}, \omega'}{\vdash \omega, \omega'}\,\text{cut}$$

If neither A nor $A^{\perp}$ is ?ed, then the induction hypothesis gives proofs of $\vdash \tau * A \mid X$ and $\vdash \tau' * A^{\perp} \mid X'$, where $|\omega| = ?X \uplus Y$, $|\omega'| = ?X' \uplus Y'$, $\tau * A = (\omega * A)\restriction_Y$ and $\tau' * A^{\perp} = (\omega' * A^{\perp})\restriction_{Y'}$. An application of cut then gives a proof of $\vdash \tau * \tau' \mid X, X'$. We have $\tau = \omega\restriction_Y$ and $\tau' = \omega'\restriction_{Y'}$ so $\tau * \tau' = (\omega * \omega')\restriction_{Y \cup Y'}$, qed.

In case A or $A^{\perp}$ is ?ed, say A, then the induction hypothesis gives proofs of $\vdash \boldsymbol{\alpha} \mid A, X$ and $\vdash \tau' * !A^{\perp} \mid X'$ with $\boldsymbol{\alpha} = \overline{\omega}\restriction_Y$ and $\tau' * !A^{\perp} = (\omega' * !A^{\perp})\restriction_{Y'}$. Taking $\tau = \omega\restriction_Y$, we have $\boldsymbol{\alpha} = \overline{\tau}$, so by an application of dereliction and a cut, we get a proof of $\vdash \tau * \tau' \mid X, X'$ enjoying $\tau * \tau' = (\omega * \omega')\restriction_{Y \cup Y'}$:

$$\frac{\vdash \omega, ?A \qquad \vdash !A^{\perp}, \omega'}{\vdash \omega, \omega'}\,\text{cut} \quad \mapsto \quad \frac{\dfrac{\dfrac{\vdash \boldsymbol{\alpha} \mid A, X}{\vdash \overline{\tau} \mid A, X}}{\vdash \tau * ?A \mid X}\,\text{d} \qquad \vdash \tau' * !A^{\perp} \mid X'}{\vdash \tau * \tau' \mid X, X'}\,\text{cut}.$$

— Entropy : by lemma 3.23, $\overline{\omega \parallel \tau \parallel \sigma} \subseteq \overline{\omega < \tau < \sigma}$.
— Seesaw and center : by proposition 3.6, $\overline{\omega < \tau} = \overline{\omega \parallel \tau}$.
— The cases of the rules for $\odot$, $\otimes$, $\triangledown$, $⅋$, $\&$, $\oplus$ and ! are handled in a similar way to the cut rule, by using the dereliction rule when an active formula is ?ed.
— For dereliction, either the active formula A is not ?ed and the translation of the rule is a center rule, or it is already ?ed and we apply dereliction followed by center.
— The cases of contraction, weakening and the rules for $\mathbf{1}$, $\perp$ and $\top$ are trivial.

2. We do not need to prove that much : it is enough to exhibit a proof of $\vdash \omega$ for *some* series-parallel order ω on $Y \cup ?X$ such that $\overline{\omega}\restriction_Y = \boldsymbol{\alpha}$. Indeed by corollary 2.7 we can then move arbitrarily the formulas in $?X$ without changing the order on Y, and by proposition 3.17 we can change ω for another arbitrary series-parallel order ω' such that $\overline{\omega'\restriction_Y} = \boldsymbol{\alpha}$.

Proceed by induction on a proof of $\vdash \boldsymbol{\alpha} \mid X$. The above remark makes the result trivial, but for the following cases :
— The center rule is translated by a step of dereliction.
— In the case of entropy, the induction hypothesis gives in particular a proof of $\vdash \omega \parallel A$ for some formula $A \in |\boldsymbol{\beta}| = Y$ and $\omega\restriction_Y = \boldsymbol{\beta}_A$. Now $\boldsymbol{\alpha}_A \trianglelefteq \boldsymbol{\beta}_A$ by proposition 3.22, so by proposition 3.24, we get, by applications of entropy, a proof of $\vdash \tau \parallel A$ where $\tau\restriction_Y = \boldsymbol{\alpha}_A$, qed.
— In the cases of dereliction and the $\perp$-rule, the point is to choose a proof where the restriction of the order to $|\omega|$ is precisely ω.

3. This is just a straighforward remark. □

4.2. *Discussion about entropy and the par-rule*

The $⅋$-rule corresponding to the one given in (Abrusci and Ruet 1999) would be :

$$\frac{\vdash \boldsymbol{\alpha}[A, B] \mid \Gamma}{\vdash \boldsymbol{\alpha}[A⅋B/A, B] \mid \Gamma} ⅋\star$$

where $\boldsymbol{\alpha}[A⅋B/A, B]$ is the identification defined in section 3.7. In (Abrusci and Ruet 1999) there is no explicit rule for entropy : entropy is hidden in the $⅋\star$-rule through the absence of condition on the order variety. In the multiplicative fragment, the two versions are equivalent (hence our presentation is consistent with (Abrusci and Ruet 1999)) in the following sense :
— by lemma 3.35 (ii), $\boldsymbol{\alpha}[A⅋B/A, B]_{A⅋B} * (A \parallel B) \subseteq \boldsymbol{\alpha}$, so our new pair of rules, entropy and the $⅋$-rule, can mimic the $⅋\star$-rule ;
— by lemma 3.35 (iii), $\omega * (A \parallel B) \subseteq \boldsymbol{\alpha}$ implies $\omega * (A \parallel B) \subseteq \boldsymbol{\alpha}[A⅋B/A, B]_{A⅋B} * (A \parallel B)$, so $⅋\star$ is an optimized version of $⅋$-rule, where entropy has been minimized.

Removing the rule of entropy from the sequent calculus would carry us closer to proof nets since proof nets do not have entropy links, but it is not clear how to do it in full NL. In the multiplicative-additive fragment MANL, it is possible to remove the rule of entropy and define an optimized rule for & as well. Because entropy is implicit, it is then necessary to allow different order varieties $\omega * x$ and $\tau * x$ (of equal support, of course) in the two premisses :

$$\frac{\vdash \omega * A \mid \Gamma \qquad \vdash \tau * B \mid \Gamma}{\vdash (\omega \wedge \tau) * A\&B \mid \Gamma} \&\star.$$

Note that $\omega \wedge \tau$ may not be series-parallel, even if ω and τ are (see section 3.6), so the situation becomes subtler.

In presence of exponentials, the problem is with entropy, dereliction and the $\bot$-rule, where points move into or out of the order variety. The point is that it is possible to have a partial order ω on X such that $\overline{\omega} \subseteq \boldsymbol{\alpha}$ and no partial order τ satisfying both $\overline{\tau} = \boldsymbol{\alpha}$ and $\omega \trianglelefteq \tau$: otherwise said, it becomes more difficult to relate the calculus on order varieties to the one on partial orders because inclusion of order varieties may not be simulated by entropy for any given choice of presentation ; this is also true in the series-parallel case. For instance, take $\omega = (a \parallel b \parallel d) < (c \parallel e)$ and $\boldsymbol{\alpha} = a * [(b \parallel c) < (d \parallel e)]$: $\overline{\omega} = \varnothing$, but the two conditions $(b \parallel d) < (c \parallel e) = \omega \restriction_{b,c,d,e} \subseteq \tau \restriction_{b,c,d,e}$ and $\overline{\tau \restriction_{b,c,d,e}} = \boldsymbol{\alpha} \restriction_{b,c,d,e} = \varnothing$ are incompatible.

So we leave this study — essentially general proof nets for NL — to further work and keep explicit entropy in our calculus.

5. PHASE SEMANTICS

5.1. *Phase spaces*

Definition 5.1. A *phase space* is a sextuplet $P = (P, \cdot, \star, 1, \leq, \bot)$ such that :
1. $(P, \cdot, 1)$ is a monoid,
2. $(P, \star, 1)$ is a commutative monoid,
3. $\leq$ is a partial order on P, compatible with both monoidal structures and such that $x \star y \leq x \cdot y, \forall x, y \in P$,

4. $\perp \subseteq P$ is an order ideal such that $\forall x, y \in P, x \cdot y \in \perp \Leftrightarrow y \cdot x \in \perp \Leftrightarrow x \star y \in \perp$.
The elements of P are called *phases*, the elements of $\perp$ are called the *antiphases*.

The compatibility condition for $\leq$ means that $x \leq x'$ and $y \leq y' \Rightarrow x \cdot y \leq x' \cdot y'$ and $x \star y \leq x' \star y'$. As an order ideal $\perp$ satisfies : $x \in \perp$ and $y \leq x \Rightarrow y \in \perp$.

Examples.
▶ If $(P, \star, 1, \perp)$ is a commutative phase space in the sense of (Girard 1987), then $(P, \star, \star, 1, =, \perp)$ is a phase space.

▶ Let $(P, \cdot, 1, \leq, \wedge, \vee)$ be a lattice-ordered monoid (a good reference is (Fuchs 1963)), i.e., a monoid $(P, \cdot, 1)$ together with a lattice structure $(\leq, \wedge, \vee)$ compatible with multiplication. The product $x \star y = xy \wedge yx$ is obviously commutative with unit 1, and satisfies $x \star y \leq xy$. If for all $x, y, z \in P$,

$$xyz \wedge zyx = yzx \wedge xzy$$

i.e. the quantity $xyz \wedge zyx$ is invariant by any permutation of x, y, z, then $x \star y$ is associative.

Definition 5.2. If G is a subset of P, its *dual* is defined by :

$$G^\perp = \{p \in P \mid \forall q \in G, p \star q \in \perp\}.$$

Alternatively $G^\perp = \{p \in P \mid \forall q \in G, p \cdot q \in \perp\} = \{p \in P \mid \forall q \in G, q \cdot p \in \perp\}$. For G, H subsets of P, define :

$$\begin{aligned} G \cdot H &= \{p \cdot q \mid p \in G, q \in H\} \\ G \star H &= \{p \star q \mid p \in G, q \in H\}. \end{aligned}$$

Definition 5.3 (Fact). A *fact* is a subset A of P such that $A^{\perp\perp} = A$.

As for usual commutative and cyclic linear logic, we have the following items (i)-(v) and (vii). Item (vi) is specific to NL.

Proposition 5.4. (i) For any $G \subseteq P$, $G \subseteq G^{\perp\perp}$.
(ii) For any $G, H \subseteq P$, $G \subseteq H \Rightarrow H^\perp \subseteq G^\perp$.
(iii) $G \subseteq P$ is a fact iff it is of the form $H^\perp$ for some $H \subseteq P$.
(iv) If G is any subset of P, then $G^{\perp\perp}$ is the smallest fact containing G.
(v) $\perp$ is a fact since $\perp = \{1\}^\perp$.
(vi) Facts are order ideals.
(vii) Facts are closed under arbitrary intersections.

Proof. (i) to (v) are immediate.
For (vi), let G be a fact, and take $x \in G$ and $y \leq x$. If $z \in G^\perp$, then $x \cdot z \in \perp$ so $y \cdot z \in \perp$ (monotonicity of $\cdot$). Therefore $y \in G^{\perp\perp} = G$.
For (vii) it suffices to verify that if $(G_i)_{i \in I}$ is a family of facts, then $\bigcap G_i = (\bigcup G_i^\perp)^\perp$: if $x \in \bigcap G_i$ then for all $i \in I$, $x \in G_i$, now if $y \in \bigcup G_i^\perp$, then $y \in G_{i_0}^\perp$ for some $i_0 \in I$, so $x \cdot y \in \perp$; conversely if $x \in (\bigcup G_i^\perp)^\perp$, then for all $i \in I$ and all $y \in G_i^\perp$, $x \cdot y \in \perp$, so $x \in G_i^{\perp\perp} = G_i$, qed. □

Definition 5.5. A few notable facts : the largest one $\top = \varnothing^\perp = P$ (w.r.t. inclusion), the smallest one $\mathbf{0} = \top^\perp$, and $\mathbf{1} = \perp^\perp$.

Definition 5.6. Define the following operations on facts A, B :

$$\begin{array}{rcl} A \odot B &=& (A \cdot B)^{\perp\perp} \\ A \triangledown B &=& (B^\perp \cdot A^\perp)^\perp \\ A \otimes B &=& (A \star B)^{\perp\perp} \\ A ⅋ B &=& (B^\perp \star A^\perp)^\perp \\ A \& B &=& A \cap B \\ A \oplus B &=& (A \cup B)^{\perp\perp}. \end{array}$$

Phase spaces enjoy the following fundamental property :

Lemma 5.7. For any subsets F and G of P, $F^{\perp\perp} \cdot G^{\perp\perp} \subseteq (F \cdot G)^{\perp\perp}$ and $F^{\perp\perp} \star G^{\perp\perp} \subseteq (F \star G)^{\perp\perp}$.

Proof. Consider the case of $\cdot$. Let $p \in F^{\perp\perp}$ and $q \in G^{\perp\perp}$. If $v \in (F \cdot G)^\perp$ then for all $f \in F$ and $g \in G$, $v \star (f \cdot g) \in \perp$, so $v \cdot (f \cdot g) = (v \cdot f) \cdot g \in \perp$ — because $v \star v' \in \perp$ iff $v \cdot v' \in \perp$ —, so for all $f \in F$, $v \cdot f \in G^\perp = G^{\perp\perp\perp}$, and $q \cdot (v \cdot f) = (q \cdot v) \cdot f \in \perp$, whence $q \cdot v \in F^\perp = F^{\perp\perp\perp}$. Therefore $p \cdot q \cdot v \in \perp$, qed.

For $\star$, apply a similar argument : if $v \in (F \star G)^\perp$ then for all $f \in F$ and $g \in G$, $v \cdot (f \star g) \in \perp$, so $v \star (f \star g) = (v \star f) \star g \in \perp$, and $(v \star f) \cdot g \in \perp$. Therefore for all $f \in F$, $v \star f \in G^\perp = G^{\perp\perp\perp}$, so $q \star (v \star f) = (q \star v) \star f \in \perp$, whence $q \star v \in F^\perp = F^{\perp\perp\perp}$. Therefore $p \star q \star v \in \perp$, qed. □

Proposition 5.8. (i) De Morgan laws hold for $\odot$ and $\triangledown$, $\otimes$ and ⅋, & and $\oplus$. Moreover these 6 operations are associative, $\otimes$, ⅋, & and $\oplus$ are commutative, $\mathbf{1}$ is neutral for $\odot$ and $\otimes$, $\perp$ is neutral for $\triangledown$ and ⅋, $\top$ and $\mathbf{0}$ are respectively neutral for & and $\oplus$. Distributivity properties hold for $\odot$ and $\oplus$, $\otimes$ and $\oplus$, $\triangledown$ and &, ⅋ and &.
(ii) Let A and B be any facts :

$$A \otimes B \subseteq A \odot B \text{ (dually } A \triangledown B \subseteq A ⅋ B).$$

Proof. Only the following deserve attention :
— Associativity of the multiplicatives (by duality, we just consider the conjunctions) : using lemma 5.7, we have $(A \odot B) \odot C = ((A \cdot B)^{\perp\perp} \cdot C)^{\perp\perp} = ((A \cdot B)^{\perp\perp} \cdot C^{\perp\perp})^{\perp\perp} \subseteq (A \cdot B \cdot C)^{\perp\perp}$, and $(A \cdot B \cdot C)^{\perp\perp} \subseteq (A \odot B) \odot C$ is immediate. Hence $(A \odot B) \odot C = (A \cdot B \cdot C)^{\perp\perp}$, qed. The case of $\otimes$ is similar.
— Neutrality : these properties rely on the neutrality of $1 \in \mathbf{1}$ for both $\cdot$ and $\star$.
— Distributivities : the proofs are exactly the same as for commutative LL.
— $A \otimes B \subseteq A \odot B$: it is enough to show that $A \star B \subseteq A \odot B = (A \cdot B)^{\perp\perp}$. If $a \in A$ and $b \in B$, then $a \cdot b \in A \cdot B \subseteq (A \cdot B)^{\perp\perp}$. $(A \cdot B)^{\perp\perp}$ is a fact and $a \star b \leq a \cdot b$ so by proposition 5.4 (vi), $a \star b \in (A \cdot B)^{\perp\perp}$. □

Definition 5.9.

$$\begin{aligned} A \multimap B &= \{x \in P \mid \forall a \in A, a \star x \in B\} \\ A \multimap\!\!\bullet\, B &= \{x \in P \mid \forall a \in A, a \cdot x \in B\} \\ B \bullet\!\!\!- A &= \{x \in P \mid \forall a \in A, x \cdot a \in B\}. \end{aligned}$$

Proposition 5.10. Let A and B be any facts. Then :

$$\begin{gathered} A \multimap\!\!\bullet\, B = A^{\perp} \triangledown B \\ B \bullet\!\!\!- A = B \triangledown A^{\perp} \\ A \multimap B = A^{\perp} ⅋ B \\ A \multimap \perp = A \multimap\!\!\bullet \perp = \perp \bullet\!\!\!- A = A^{\perp}. \end{gathered}$$

Hence in particular $A \multimap\!\!\bullet\, B$, $B \bullet\!\!\!- A$ and $A \multimap B$ are facts.

Proof. Here we use again the fact that for all $x, y \in P$, $x \cdot y \in \perp$ iff $x \star y \in \perp$. Let us just consider the case of $\multimap\!\!\bullet$ (the others are similar). Assume $x \in A \multimap\!\!\bullet\, B$. Let $a \in A$ and $y \in B^{\perp}$; $a \cdot x \in B$ so $y \cdot a \cdot x \in \perp$, thus $x \in A^{\perp} \triangledown B = (B^{\perp} \cdot A)^{\perp}$. Conversely, assume $x \in A^{\perp} \triangledown B$, and take $a \in A$. For all $y \in B^{\perp}$, $y \cdot a \cdot x \in \perp$, thus $a \cdot x \in B^{\perp\perp} = B$, whence $x \in A \multimap\!\!\bullet\, B$. □

As in (Girard 1995; Lafont 1997), we extend the semantics to exponential connectives. If P is a phase space, define $J(P) = \{x \in \mathbf{1} \mid \ x \in \{x \star x\}^{\perp\perp}\}$. Note that $x \in J(P) \Rightarrow x \in \{x \cdot x\}^{\perp\perp}$, because $\{x \star x\}^{\perp\perp} \subseteq \{x \cdot x\}^{\perp\perp}$.

Definition 5.11. An *enriched phase space* consists of a phase space P and a subset K of $J(P)$ such that :
— for any $x \in K$ and $y \in P$, we have $\{x \cdot y\}^{\perp\perp} = \{y \cdot x\}^{\perp\perp} = \{x \star y\}^{\perp\perp}$,
— both $(K, \star, 1)$ and $(K, \cdot, 1)$ are monoids.
The enriched phase space will still be denoted by P.

Definition 5.12. Let A be a fact, define :

$$\begin{aligned} ?A &= (A^{\perp} \cap K)^{\perp} \\ !A &= (A \cap K)^{\perp\perp}. \end{aligned}$$

Proposition 5.13. (i) For every fact A, $?A$ and $!A$ are facts.
(ii) ! and ? satisfy the De Morgan laws.
(iii) For every facts A and B, $A \subseteq B \Rightarrow\, !A \subseteq\, !B$,
(iv) $!!A = !A \subseteq A$,
(v) $!A \otimes !B \subseteq !(A \otimes B)$ and $!A \odot !B \subseteq !(A \odot B)$,
(vi) $!A \subseteq \mathbf{1}$, $!A = !A \otimes !A$ and $!A = !A \odot !A$,
(vii) $!A \otimes B = !A \odot B = B \odot !A$,
(viii) $!A \otimes !B = !(A \& B) = !A \odot !B = !B \odot !A$.

Proof. (i), (ii) and (iii) are immediate.

(iv) For a fact A, one has clearly $!A \subseteq A^{\perp\perp} = A$; in particular $!!A \subseteq !A$, and besides

$A \cap K \subseteq (A \cap K)^{\perp\perp} = !A$ and $A \cap K \subseteq K$ thus $A \cap K \subseteq !A \cap K \subseteq (!A \cap K)^{\perp\perp}$, whence $!A = (A \cap K)^{\perp\perp} \subseteq (!A \cap K)^{\perp\perp} = !!A$.

(v) $!A \otimes !B = ((A \cap K)^{\perp\perp} \star (B \cap K)^{\perp\perp})^{\perp\perp} = ((A \cap K) \star (B \cap K))^{\perp\perp} \subseteq ((A \star B) \cap K)^{\perp\perp}$ since $(K, \star)$ is a monoid, therefore $!A \otimes !B \subseteq ((A \star B)^{\perp\perp} \cap K)^{\perp\perp} = !(A \otimes B)$. Using the fact that $(K, \cdot)$ is a monoid, one proves that $!A \odot !B \subseteq !(A \odot B)$.

(vi) The first inclusion is obvious. If $x \in A \cap K$, then $x \in K$ thus $x \in \{x \star x\}^{\perp\perp} \subseteq ((A \cap K) \star (A \cap K))^{\perp\perp}$, therefore $A \cap K \subseteq ((A \cap K) \star (A \cap K))^{\perp\perp} = (!A \star !A)^{\perp\perp}$ according to lemma 5.7, whence $!A = (A \cap K)^{\perp\perp} \subseteq !A \otimes !A$. One proceeds similarly for $\odot$.

(vii) As $B = B^{\perp\perp}$, $!A \otimes B = ((A \cap K)^{\perp\perp} \star B)^{\perp\perp} = ((A \cap K) \star B)^{\perp\perp} = ((A \cap K) \cdot B)^{\perp\perp} = (B \cdot (A \cap K))^{\perp\perp}$ by definition of K, qed.

(viii) In view of (vii) it is enough to prove that $!A \otimes !B = !(A \& B)$. According to (vi), $!(A \& B) \subseteq !(A \& B) \otimes !(A \& B)$, now $A \& B = A \cap B \subseteq A$, whence by (iii), $!(A \& B) \subseteq !A$ and similarly $!(A \& B) \subseteq !B$, thus $!(A \& B) \subseteq !A \otimes !B$. Conversely, $!A \otimes !B \subseteq !A \otimes \mathbf{1} = !A \subseteq A$, and similarly $!A \otimes !B \subseteq B$, thus $!A \otimes !B \subseteq A \& B$. Hence, according to (iv) and (v), $!A \otimes !B = !!A \otimes !!B \subseteq !(!A \otimes !B) \subseteq !(A \& B)$. $\square$

The phase semantics we have defined, when restricted to the connectives $⅋$ and $\otimes$ (resp. $\triangledown$ and $\odot$), is the phase semantics of commutative (resp. cyclic) linear logic.

5.2. *Soundness*

Definition 5.14 (Phase structure, Validity). A *phase structure* (P, S) is an enriched phase space P, together with a *valuation* that assigns a fact $S(p)$ to any positive propositional symbol p.
Given a phase structure, one defines inductively the *interpretation* $S(A)$ of a formula A in the obvious way. The interpretation of a context Γ is defined by induction : $S(()) = \perp$, $S(\Gamma; \Delta) = S(\Gamma) \triangledown S(\Delta)$ and $S(\Gamma, \Delta) = S(\Gamma) ⅋ S(\Delta)$.
Let A be a formula : A is said to be *valid in* S when $1 \in S(A)$. A is a *tautology* if it is valid in every phase structure. A sequent $\vdash \Gamma$ is valid if $1 \in S(\Gamma)$ for every phase structure S.

Theorem 5.15. If a sequent is provable in the sequent calculus, then it is valid.

Proof. We use the sequent calculus on orders (section 2). Let (P, S) be a phase structure, and $\vdash \Gamma$ be a sequent provable in the sequent calculus. First note that implicit associativity of $(-,-)$ and $(-;-)$ and commutativity of $(-,-)$ in the sequent calculus are sound, because of the associativity of $⅋$ and $\triangledown$ and of the commutativity of $⅋$ (proposition 5.8 (i)). Then proceed by induction on a proof of $\vdash \Gamma$:
— The proof is an axiom $\vdash A^\perp, A$: the interpretation is $S(A^\perp, A) = S(A^\perp ⅋ A) = S(A \multimap A)$ by proposition 5.10, thus $S(A^\perp, A) = S(A) \multimap S(A)$ and $1 \in S(A^\perp, A)$.
— The proof is an axiom $\vdash \mathbf{1}$: $1 \in \mathbf{1} = \perp^\perp$.
— The proof is an axiom $\vdash \Gamma, \top$: the interpretation is $S(\Gamma, \top) = S(\Gamma) ⅋ S(\top) = S(\top) ⅋ S(\Gamma) = S(\mathbf{0}) \multimap S(\Gamma) = \mathbf{0} \multimap S(\Gamma)$, and $\mathbf{0}$ is the least fact, thus $\mathbf{0} \subseteq S(\Gamma)$ and $1 \in \mathbf{0} \multimap S(\Gamma)$.

— The proof ends with a cut rule : by induction hypothesis $1 \in S(\Gamma)⅋S(A)$ and $1 \in S(A^{\perp})⅋S(\Delta)$, what means $S(\Gamma)^{\perp} \subseteq S(A)$ and $S(A^{\perp})^{\perp} = S(A) \subseteq S(\Delta)$, qed.
— The proof ends with an entropy : follows from proposition 5.8 (ii).
— The proof ends with the seesaw rule : by induction hypothesis, $1 \in S(\Gamma) ⅋ S(\Delta)$, i.e. $S(\Gamma)^{\perp} \subseteq S(\Delta)$; as 1 is neutral for $\cdot$, it is equivalent to $1 \in S(\Gamma)^{\perp} \multimap S(\Delta) = S(\Gamma) \triangledown S(\Delta)$.
— The proof ends with the center rule : follows from proposition 5.13 (vii).
— The proof ends with the &-rule : immediate consequence of $A\&B = A \cap B$.
— The proof ends with a $\oplus$-rule : immediate consequence of $A \oplus B = (A \cup B)^{\perp\perp}$ and of proposition 5.4 (v).
— The proof ends with a $\triangledown$ or ⅋-rule : immediate.
— The proof ends with the $\odot$-rule : by induction $S(\Gamma)^{\perp} \subseteq S(A)$ and $S(\Delta)^{\perp} \subseteq S(B)$, thus $(S(\Delta) \triangledown S(\Gamma))^{\perp} = S(\Gamma)^{\perp} \odot S(\Delta)^{\perp} \subseteq S(A) \odot S(B)$, i.e. $1 \in (S(\Delta) \triangledown S(\Gamma)) \triangledown S(A \odot B) = S(\Delta; \Gamma; A \odot B)$, therefore by seesaw $1 \in S(\Gamma; A \odot B; \Delta)$.
— The proof ends with the $\otimes$-rule : similar argument, using the fact that $G^{\perp} \subseteq H$ iff $1 \in G⅋H$.
— The proof ends with the dereliction rule : by induction $S(\Gamma)^{\perp} \subseteq S(A)$, and by proposition 5.13 (iv) (translated in dual terms of "?") $S(A) \subseteq S(?A)$, qed.
— The proof ends with the promotion rule : immediate consequence of the monotonicity of ! (proposition 5.13 (iii)-(v)).
— The proof ends with the contraction rule : immediate because $S(?A) = S(?A⅋?A)$ (proposition 5.13 (vi)).
— The proof ends with the weakening rule : immediate consequence of $\perp \subseteq S(?A)$ (proposition 5.13 (vi)), and $\perp$ neutral for ⅋ (proposition 5.8 (i)).
— The proof ends with an introduction of $\perp$: like for weakening. □

5.3. *Completeness*

Theorem 5.16. If a sequent is valid, then it is provable in the sequent calculus.

Proof. One defines the following enriched phase space :
— P is the set of contexts, $\Gamma \cdot \Delta$ is sequential composition $(\Gamma; \Delta)$, $\Gamma \star \Delta$ is parallel composition (Γ, Δ), the neutral is $()$,
— the order $\leq$ is the least order such that : $(\Gamma, \Delta) \leq (\Gamma; \Delta)$ for every $\Gamma, \Delta \in P$; and $\Gamma \leq \Gamma'$ and $\Delta \leq \Delta'$ imply $(\Gamma; \Delta) \leq (\Gamma'; \Delta')$ and $(\Gamma, \Delta) \leq (\Gamma', \Delta')$,
— $\perp$ is the set of contexts Γ such that $\vdash \Gamma$ is provable in the sequent calculus,
— K is the set of contexts of the form $?\Gamma$ (where Γ is an arbitrary context).

By entropy and seesaw, $\perp$ satisfies the axioms of definition 5.1. Besides K satisfies the axioms of definition 5.11 :
— K contains $1 = ()$ and is closed by $\cdot$ and $\star$;
— by the weakening rule, $?\Gamma \in \perp^{\perp} = \mathbf{1}$;
— by the contraction rule, $?\Gamma \in \{?\Gamma \star ?\Gamma\}^{\perp\perp}$;
— by the center and co-center rules, $\{?\Gamma \star \Delta\}^{\perp\perp} = \{?\Gamma \cdot \Delta\}^{\perp\perp} = \{\Delta \cdot ?\Gamma\}^{\perp\perp}$.

Thus what we have defined is an enriched phase space.

We have $\{A\}^\perp = \{\Gamma \in P \mid \vdash \Gamma, A \text{ is provable in the sequent calculus}\}$.
The $\{A\}^\perp$'s are facts, more precisely $\{A\}^\perp = \{A^\perp\}^{\perp\perp}$ (the proof is the same as in (Girard 1987)). Define a phase structure S by letting $S(p) = \{p\}^\perp$ for every positive propositional symbol p. One then proves easily as in the commutative case, by induction on A, that $S(A) = \{A\}^\perp$: it amounts to prove commutations of the type $\{A \otimes B\}^\perp = \{A\}^\perp \otimes \{B\}^\perp$.
Let us consider the case of the exponentials :
— $?\{A\}^\perp = (\{A\}^{\perp\perp} \cap K)^\perp = (\{A^\perp\}^\perp \cap K)^\perp$. Let $\Gamma \in \{?A\}^\perp$ and $?\Delta \in \{A^\perp\}^\perp$: one has $?\Delta \in \{!A^\perp\}^\perp$ by the promotion rule, whence by the cut rule $\Gamma, ?\Delta \in \perp$, what shows that $\{?A\}^\perp \subseteq ?\{A\}^\perp$. Conversely by the dereliction rule, $?A \in \{A^\perp\}^\perp$, and besides $?A \in K$, thus if $\Gamma \in ?\{A\}^\perp$, then $\Gamma, ?A \in \perp$, i.e. $\Gamma \in \{?A\}^\perp$, qed.
— $S(!A)^\perp = ?S(A^\perp) = ?\{A^\perp\}^\perp = \{?A^\perp\}^\perp = \{(!A)^\perp\}^\perp = \{!A\}^{\perp\perp}$, thus $S(!A) = \{!A\}^\perp$.

Finally if $\vdash \Gamma$ is a valid sequent, $1 = () \in S(\Gamma) = \{\Gamma\}^\perp$ and thus $\vdash \Gamma$ is provable. $\square$

6. CUT ELIMINATION

As in (Okada 1994), one can use the phase semantics to prove cut elimination.

Theorem 6.1. If a sequent is valid, then it is provable in the sequent calculus without cuts.

Proof. Define the following enriched phase space P' : P' is defined like the phase space P in the proof of theorem 5.16 except for $\perp$; $\perp$ is the set of contexts Γ such that $\vdash \Gamma$ is provable in the sequent calculus *without cuts*. $\perp$ satisfies again the axioms of definition 5.1, thus the above is a well defined enriched phase space.

$\{A\}^\perp = \{\Gamma \in P' \mid \vdash \Gamma, A$ is provable in the sequent calculus *without cut*$\}$.

The $\{A\}^\perp$'s are facts, thus one can define a phase structure S' by letting $S'(p) = \{p\}^\perp$ for every positive propositional symbol p. One then proves by induction on A, that $S'(A) \subseteq \{A\}^\perp$:
— for a positive propositional symbol p, it is clear,
— for the dual $p^\perp$ of a propositional symbol, one has $S'(p^\perp) = S'(p)^\perp = \{p\}^{\perp\perp} \subseteq \{p^\perp\}^\perp$ because $p \in \{p^\perp\}^\perp$ (axiom),
— $S'(A \parr B) = S'(A) \parr S'(B) = (S'(B)^\perp \star S'(A)^\perp)^\perp \subseteq (\{B\}^{\perp\perp} \star \{A\}^{\perp\perp})^\perp \subseteq (\{B\} \star \{A\})^\perp$ by lemma 5.7, so $S'(A \parr B) \subseteq \{A \parr B\}^\perp$ by the $\parr$-rule,
— similarly $S'(A \triangledown B) \subseteq \{A \triangledown B\}^\perp$ by the $\triangledown$-rule,
— $S'(A \otimes B) = (S'(A) \star S'(B))^{\perp\perp} \subseteq (\{A\}^\perp \star \{B\}^\perp)^{\perp\perp} \subseteq \{A \otimes B\}^\perp$ because $\{A\}^\perp \star \{B\}^\perp \subseteq \{A \otimes B\}^\perp$ by the $\otimes$-rule,
— similarly $S'(A \odot B) \subseteq \{A \odot B\}^\perp$ by the $\odot$-rule,
— $S'(A \& B) = S'(A) \cap S'(B) \subseteq \{A\}^\perp \cap \{B\}^\perp \subseteq \{A \& B\}^\perp$ by the &-rule,
— $S'(A \oplus B) = (S'(A) \cup S'(B))^{\perp\perp} \subseteq (\{A\}^\perp \cup \{B\}^\perp)^{\perp\perp} \subseteq \{A \oplus B\}^\perp$ because $\{A\}^\perp \cup \{B\}^\perp \subseteq \{A \oplus B\}^\perp$ by the $\oplus$-rules,
— $S'(\perp) = \perp \subseteq \{\perp\}^\perp$ by the $\perp$-rule,
— $S'(\mathbf{1}) = \perp^\perp \subseteq \{\mathbf{1}\}^\perp$ because $\mathbf{1} \in \perp$ by the axiom for $\mathbf{1}$,

— $S'(\top) = P' = \{\top\}^\perp$ by the axiom for $\top$,
— $S'(\mathbf{0}) = P'^\perp \subseteq \{\mathbf{0}\}^\perp$,
— $S'(?A) = (S'(A)^\perp \cap K)^\perp \subseteq (\{A\}^{\perp\perp} \cap K)^\perp \subseteq \{?A\}^\perp$ because $?A \in K$ and $\{A\}^\perp \subseteq \{?A\}^\perp$ by the dereliction rule,
— $S'(!A) = (S'(A) \cap K)^{\perp\perp} \subseteq (\{A\}^\perp \cap K)^{\perp\perp} \subseteq \{!A\}^\perp$ because $\{A\}^\perp \cap K \subseteq \{!A\}^\perp$ by the promotion rule.

Finally if $\vdash \Gamma$ is a valid sequent, $1 = () \in S'(\Gamma) \subseteq \{\Gamma\}^\perp$ and thus $\vdash \Gamma$ is provable in the sequent calculus without cuts. □

Corollary 6.2. If a sequent is provable in the sequent calculus, then it is provable without the cut rule.

Proof. Immediate consequence of the soundness of the phase semantics (theorem 5.15) and of its completeness w.r.t. to the sequent calculus without cuts (theorem 6.1). □

Acknowledgements.
First, I am indebt to V. Michele Abrusci and Jean-Yves Girard for their numerous stimulating remarks. In particular, V. Michele Abrusci noticed a mistake in a preliminary version of the additive rules, and Jean-Yves Girard gave the spreading condition for order varieties and proposition 3.14. I thank Christian Retoré who first mentionned to me the existence of cyclic orders, and Rick Blute, Philippe de Groote and François Lamarche for fruitful discussions on this work.

References

V. M. Abrusci and P. Ruet. Non-commutative logic I : the multiplicative fragment. *To appear in Annals Pure Appl. Logic*, 1999.

J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *J. Logic and Comp.*, 2(3), 1992.

D. Bechet, Ph. de Groote, and Ch. Retoré. A complete axiomatisation for the inclusion of series-parallel orders. In *Proc. Int. Conf. on Rewriting Techniques and Applications*. Springer LNCS, 1997.

Ph. de Groote. Partially commutative linear logic : sequent calculus and phase semantics. In V. M. Abrusci and C. Casadio, editors, *Proofs and Linguistic Categories, Proc. 1996 Roma Workshop*. Cooperativa Libraria Universitaria Editrice Bologna, 1996.

A. Demaille. *Logique linéaires hybrides et leurs modalités : théories et applications.* PhD thesis, Ec. Nat. Sup. Télécommunications, 1999.

L. Fuchs. *Partially ordered algebraic systems.* Pergamon Press, 1963.

J.-Y. Girard. Linear logic. *Theoret. Comp. Sci.*, 50(1):1–102, 1987.

J.-Y. Girard. Towards a geometry of interaction. *Contemp. Math.*, 92:69–108, 1989.

J.-Y. Girard. Linear logic : its syntax and semantics. In *Advances in Linear Logic*, pages 1–42. Cambridge University Press, LMSLNS 222, 1995.

J.-Y. Girard. On the meaning of logical rules I : syntax vs. semantics. In U. Berger and H. Schwichtenberg, editors, *Computational Logic*, NATO Series F 165. Springer, 1999.

Y. Lafont. The finite model property for various fragments of linear logic. *J. Symb. Logic*, 62(4):1202–1208, 1997.

J. Lambek. The mathematics of sentence structure. *Amer. Math. Monthly*, 65(3):154–170, 1958.

R. Mohring. *Computationally tractable classes of ordered sets.* ASI Series 222. NATO, 1989.

V. Novák. Cyclically ordered sets. *Czech. Math. J.*, 32(107):460–473, 1982.

M. Okada. Girard's phase semantics and a higher-order cut-elimination proof. *Preprint Institut de Mathématiques de Luminy*, 1994.

P. Ruet. *Logique non-commutative et programmation concurrente par contraintes.* PhD thesis, Université Denis Diderot, Paris 7, 1997.

D.N. Yetter. Quantales and (non-commutative) linear logic. *J. Symb. Logic*, 55(1), 1990.