Non-commutative logic III: focusing proofs*

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Abstract

It is now well-established that the so-called focalization property plays a central role in the design of programming languages based on proof search, and more generally in the proof theory of linear logic.

We present here a sequent calculus for non-commutative logic (NL) which enjoys the focalization property. In the multiplicative case, we give a *focalized sequentialization* theorem, and in the general case, we show that our focalized sequent calculus is equivalent to the original one by studying the permutabilities of rules for NL and showing that all permutabilities of linear logic involved in focalization can be lifted to NL permutabilities. These results are based on a study of the partitions of partially ordered sets *modulo entropy*.

1 Introduction

Non-commutative logic, NL for short, was introduced by Abrusci and the second author in [1, 14] (see also Section 3). It unifies commutative linear logic [7] and cyclic linear logic [15], a classical conservative extension of the Lambek calculus [9]. The present paper investigates the "focalization" property for non-commutative logic.

1.1 The property of focalization

The rules of the sequent calculus for linear logic, LL, are well-known to split into two categories according to their deterministic or non-deterministic behaviour in proof construction: irreversible rules, like the $\otimes$ and $\oplus$-rules:

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$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \oplus 1,$$

which introduce non-deterministic choices in proof search, and reversible rules, like:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B} \&,$$

which can be applied immediately in the (bottom-up) process of proof construction, because having a proof of the conclusion is equivalent to having proof(s) of the premise(s).

Here, the connectives whose introduction rules are reversible are dual to the connectives whose introduction rules are irreversible, and this extends to the other connectives of LL: call *negative, or asynchronous*, formulas with reversible main connective, and *positive, or synchronous*, their duals.

A priori, deterministic and non-deterministic steps can be interleaved in a very complex way in proof search, but the *property of focalization*, introduced by Andreoli in [2], enables to significantly reduce this complexity: when you have negative formulas in a sequent, you can decompose them deterministically — in parallel — up to positive subformulas; on the other hand, when you have no negative formula, you need to make the choice of a positive formula, but then, you can proceed similarly and deterministically decompose the chosen positive formula — the *focus* — systematically up to negative subformulas. This is a constructive statement: any proof can be transformed — by appropriate permutations of rules — into a *focalized proof*, i.e. a proof where negative and positive inferences are grouped as explained above. A *focalized sequent calculus* is defined in [2] for LL and proven equivalent to the standard one. The idea is essentially to consider two shapes of sequents: when decomposing a negative formula, the sequent is a usual one $\vdash \Delta$, where Δ is a set of formula occurrences, and negative rules are similar to the usual ones; when decomposing a positive formula, the sequent is of the form $\vdash \Delta \mid A$, where A is a formula (the focus) and Δ contains only atoms or positive formulas. Positive rules keep the focus: for instance, the rule for $\otimes$ is:

$$\frac{\vdash \Gamma \mid A \qquad \vdash \Delta \mid B}{\vdash \Gamma, \Delta \mid A \otimes B} \otimes,$$

and two structural rules enable to start focusing on a formula (decision), and leave the focus (reaction) when positive rules have been applied obsessionally up to negative subformulas.

$$\frac{\vdash \Delta \mid P}{\vdash \Delta, P} \textit{ decision}, P \text{ positive} \qquad \frac{\vdash \Delta, N}{\vdash \Delta \mid N} \textit{ reaction}, N \text{ negative}.$$

For instance, there is only one focalized proof of the following sequent:

$$\vdash A^{\perp}, E^{\perp}, D^{\perp}, A \otimes (B \otimes C), (C^{\perp} \parr B^{\perp}) \otimes (D \otimes E),$$

where A, B, C, D, E are positive atomic formulas.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\vdash A^\perp \mid A$}
\AxiomC{}
\UnaryInfC{$\vdash B^\perp \mid B$}
\AxiomC{}
\UnaryInfC{$\vdash C^\perp \mid C$}
\BinaryInfC{$\vdash C^\perp, B^\perp \mid (B \otimes C)$}
\BinaryInfC{$\vdash A^\perp, C^\perp, B^\perp \mid A \otimes (B \otimes C)$}
\UnaryInfC{$\vdash A^\perp, A \otimes (B \otimes C), C^\perp, B^\perp$}
\UnaryInfC{$\vdash A^\perp, A \otimes (B \otimes C), C^\perp \parr B^\perp$}
\UnaryInfC{$\vdash A^\perp, A \otimes (B \otimes C) \mid C^\perp \parr B^\perp$}
\AxiomC{}
\UnaryInfC{$\vdash E^\perp \mid E$}
\AxiomC{}
\UnaryInfC{$\vdash D^\perp \mid D$}
\BinaryInfC{$\vdash E^\perp, D^\perp \mid D \otimes E$}
\BinaryInfC{$\vdash A^\perp, E^\perp, D^\perp, A \otimes (B \otimes C) \mid (C^\perp \parr B^\perp) \otimes (D \otimes E)$}
\UnaryInfC{$\vdash A^\perp, E^\perp, D^\perp, A \otimes (B \otimes C), (C^\perp \parr B^\perp) \otimes (D \otimes E)$}
\end{prooftree}
$$

The proof obtained by taking $A \otimes (B \otimes C)$ as the lowest active formula is not focalized because then, $B \otimes C$ cannot be active, for $C^\perp$ and $B^\perp$ are not formulas of the sequent (they still appear as subformulas of $(C^\perp \parr B^\perp) \otimes (D \otimes E)$).

In practice, the advantage of the focalized calculus is that it reduces the intrinsic non-determinism of proof search, and for this reason, it has been applied to the design of programming languages based on proof search, see e.g. [4]. From a more theoretical viewpoint, focalization is also now a central property in ludics [8].

It is worth noting that the paradigm of *uniform proofs*, introduced by Miller, Nadathur, Pfenning and Scedrov in [12], was an important precursor of focusing. In fact, it has been the first attempt towards a foundation of logic programming based on the sequent calculus instead of the Robinson's resolution method. A uniform proof is essentially an intuitionistic proof. This approach was interesting from a computational point of view because it manages the important distinction between *permanent* and *temporary* information. But intuitionistic proofs did not allow any theoretical exhaustive foundation of the proof construction paradigm. In other words, uniform proofs identify a restricted class of proofs which is not complete w.r.t. the class of intuitionistic proofs, but only w.r.t. the fragment of the hereditary Harrop formulas, whereas focusing proofs are complete w.r.t full LL — and we shall extend this to non-commutative logic in the present paper.

1.2 Problems in the non-commutative case

Let us first concentrate on the multiplicative fragment MNL, which contains the main difficulties.

In [3], Andreoli and the first author showed that the focalization property in *sequent calculus* corresponds to a "hereditary splitting" lemma for *proof nets*, i.e. a refinement of Girard's original splitting lemma [7] — a central lemma towards the sequentialization theorem (see Section 4). In particular, MNL proof nets [1] do enjoy the focalization property, because of the correctness criterium and sequentialization theorem given in [1]. This abstract notion of focalization (hereditary splitting) implies that there should be a sequentialization into a focalized sequent calculus, but it does not give explicitely such a sequent calculus at all.

The two main difficulties towards the definition of a focalized sequent calculus for MNL are the following:

▶ *Par.* According to [1], if π is an MNL proof net with conclusion $\Gamma, A \parr B$, then the proof structure obtained by removing the link of conclusion $A \parr B$ is still a proof net. This implies that the commutative multiplicative disjonction, *par* $\parr$, should be negative, like in the commutative case. However, in the sequent calculus given in [1] and relying on order varieties (see below and Section 2), the introduction rule for $\parr$:

$$\frac{\vdash \boldsymbol{\alpha}[A, B]}{\vdash \boldsymbol{\alpha}[A \parr B/A, B]}$$

erases information. Here, $\boldsymbol{\alpha}$ denotes an order variety, and the problem is that from an order variety $\boldsymbol{\alpha}[z/x, y]$, where the points x and y have been identified, there is no way to recover the original variety $\boldsymbol{\alpha}$ in general.

▶ *Entropy.* In the calculus given by [14] relying on partial orders, this drawback disappears, i.e. the $\parr$-rule is indeed reversible:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B},$$

but then there is a problem with another rule, entropy:

$$\frac{\vdash \Gamma; \Delta; \Sigma}{\vdash \Gamma, \Delta, \Sigma},$$

which enables to increase the partial order underlying a sequent by replacing parallel sums (commas) by serial sums (semicolons), and is clearly highly non-deterministic. Non-determinism should be reduced as much as possible when defining a proof search oriented sequent calculus.

The solution relies on the following ideas:

- Taking the reversible structural rules for $\parr$ and $\triangledown$:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \frac{\vdash \Gamma; A; B}{\vdash \Gamma; A \triangledown B}.$$

  A consequence is that entropy, which was originally contained in the $\parr$-rule [1], should appear somewhere else.

- Pushing entropy towards $\otimes$ and $\odot$-rules. For these rules, we consider context splitting plus some entropy. For instance, such an inference is admissible:

$$\cfrac{\cfrac{\vdash \Delta', A \qquad \vdash \Delta, B}{\vdash \Delta; \Delta'; A \odot B}}{\vdash (\Delta, \Delta'); A \odot B}.$$

> It is not obvious that, given any proof, one may systematically permute rules so as to push entropy in this way, but it will turn out to be a consequence of the sequentialization theorem for multiplicative NL in Section 7 — and more generally it follows from NL permutabilities given in Section 8.
>
> Anyway, how much entropy should be used? As said above, this rule is highly non-deterministic and its use should be minimized.

This leads us to the problem of partitionning contexts, actually series-parallel orders, *up to* entropy.

1.3 Partitions of series-parallel orders

Given a fixed series-parallel order ω and a fixed partition (X, Y) of its support $|\omega|$, we give in Sections 5 and 6:

- ▶ *Admissibility:* conditions under which ω can be "partitionned modulo entropy" into two orders on X and Y, i.e. the equation

$$\omega \trianglelefteqslant (\omega_X < \omega_Y)$$

  admits a solution, with ω_X, ω_Y series-parallel orders on X, Y respectively; here, $\trianglelefteqslant$ denotes the *entropy* relation between partial orders, recalled in Section 2;

- ▶ *Optimality:* optimal solutions to this partitionning problem, i.e. minimal such orders when they exist.

There are two cases, depending on how the two orders are composed: parallel sum (which corresponds to the connective $\otimes$) and serial sum (which corresponds to the connective $\odot$), respectively studied in Sections 5 and 6. The case of parallel sum is almost trivial, while the case of serial sum is significantly more difficult, and involves in particular the combinatorics of series-parallel orders and their representations as special trees, which we call *spines*.

It is worth noting that the binary splitting of orders has been extended, in [5], to the more general splitting of orders in n pieces, $n \geqslant 2$, with motivations rooted in ludics.

1.4 Focalized sequentialization

The results on partitions of orders enable us to define in Section 7 a focalized sequent calculus for MNL, where sequents are:

- either order varieties, essentially when applying reversible rules,
- or pointed order varieties (an order plus a focus), for the positive rules.

We show it equivalent to the original sequent calculus (recalled in Appendix A) via focalized sequentialization from proof nets, as advocated at the beginning.

1.5 Focalized sequent calculus for full NL

In Section 9 (see also [11]), we give a focalized sequent calculus for full NL, generalizing the calculus of Section 7. We prove the equivalence with the original sequent calculus by studying the permutabilities of rules for NL (Section 8) and showing that all LL permutabilities involved in focalization can be "lifted" to NL permutabilities.

2 Order varieties

2.1 Series-parallel orders

Let us recall the definition of series-parallel orders. Let ω and τ be partial orders on disjoint sets X and Y respectively; their *serial* and *parallel* sums $\omega < \tau$ and $\omega \parallel \tau$ respectively are two partial orders on $X \uplus Y$ defined by:

- $(\omega_1 < \omega_2)(x, y)$ if, and only if, $x <_{\omega_1} y$ or $x <_{\omega_2} y$ or ($x \in X$ and $y \in Y$),
- $(\omega_1 \parallel \omega_2)(x, y)$ if, and only if, $x <_{\omega_1} y$ or $x <_{\omega_2} y$.

The class of *series-parallel orders* on a given set X is the least class of partial orders containing the unique orders on singletons and closed by serial and parallel sums. For a more substantial survey, see [13].

2.2 Order varieties

An *order variety* [1, 14] on a given set X is a ternary relation $\boldsymbol{\alpha}$ which is:

$$\begin{cases} \textit{cyclic}: & \forall x,y,z \in X, & \boldsymbol{\alpha}(x,y,z) \Rightarrow \boldsymbol{\alpha}(y,z,x), \\ \textit{anti-reflexive}: & \forall x,y \in X, & \neg\boldsymbol{\alpha}(x,x,y) \\ \textit{transitive}: & \forall x,y,z,t \in X, & \boldsymbol{\alpha}(x,y,z) \text{ and } \boldsymbol{\alpha}(z,t,x) \Rightarrow \boldsymbol{\alpha}(y,z,t), \\ \textit{spreading}: & \forall x,y,z,t \in X, & \boldsymbol{\alpha}(x,y,z) \Rightarrow \\ & & \boldsymbol{\alpha}(t,y,z) \text{ or } \boldsymbol{\alpha}(x,t,z) \text{ or } \boldsymbol{\alpha}(x,y,t). \end{cases}$$

An order variety $\boldsymbol{\alpha}$ on X is said *total* when $\forall x,y,z \in X$, $x \neq y \neq z \neq x \Rightarrow \boldsymbol{\alpha}(x,y,z)$ or $\boldsymbol{\alpha}(z,y,x)$. For instance, any oriented cycle $G = (a_1 \to \cdots \to a_n \to a_1)$ induces a total order variety $r(G)$ on the set of vertices by: $r(G)(x,y,z)$ if, and only if, y is between x and z in G; it will be denoted by $(a_1 a_2 \ldots a_n) = (a_2 \ldots a_n a_1)$, etc. Note that (abc) is an order variety on $\{a,b,c\}$ but not on $\{a,b,c,d\}$ as it does not enjoy the spreading condition. Spreading enables to systematically give "presentations" of order varieties as partial orders in a reversible way, as follows.

Given an order variety $\boldsymbol{\alpha}$ on X and $x \in X$, we may define a partial order $\boldsymbol{\alpha}_x$ on $X \setminus \{x\}$ by: $\boldsymbol{\alpha}_x(y,z)$ if, and only if, $\boldsymbol{\alpha}(x,y,z)$. Conversely, given a partial order $\omega = (X, <)$ and $z \in X$, let $\overset{z}{<}$ denote the binary relation: $x \overset{z}{<} y$ if, and only if, $x < y$ and z is comparable with neither x nor y; then we may define an order variety $\overline{\omega}$ on X, the *closure* of ω by $\overline{\omega}(x,y,z)$ if, and only if,: $x < y < z$ or $y < z < x$ or $z < x < y$ or $x \overset{z}{<} y$ or $y \overset{x}{<} z$ or $z \overset{y}{<} x$. When $\overline{\omega} = \boldsymbol{\alpha}$, we say that ω *presents* $\boldsymbol{\alpha}$. The closure of partial orders identifies serial and parallel sums:

$$\overline{\omega < \tau} = \overline{\omega \parallel \tau} = \overline{\tau < \omega}.$$

The above order variety is denoted

$$\omega * \tau$$

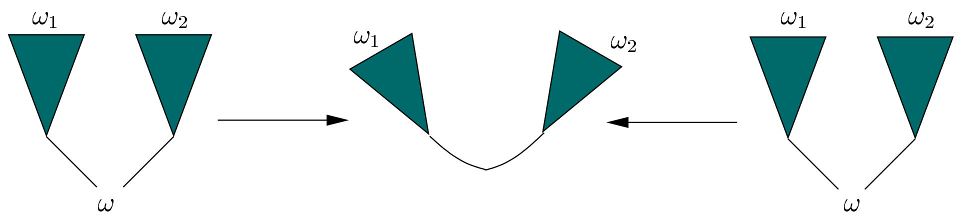
and called the *gluing* of ω and τ. Hence the operation $*$ takes two partial orders and produces an order variety.

The two processes of fixing a point in an order variety and gluing orders are related by the following equations:

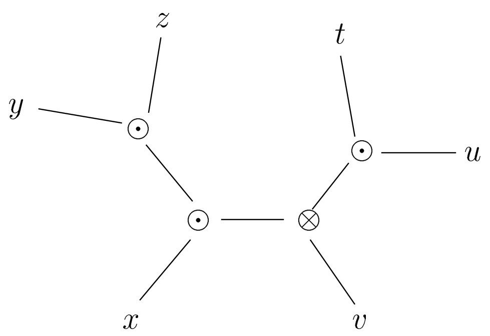
$$\boldsymbol{\alpha}_x * x \;=\; \boldsymbol{\alpha} \quad \text{and} \quad (\omega * x)_x \;=\; \omega,$$

for $\boldsymbol{\alpha}$ an order variety on a set X, $x \in X$ and ω a partial order on $X \setminus \{x\}$. So an order variety is — as its name implies — a kind of gluing of order structures.

Series-parallel order varieties are precisely those order varieties which can be presented by a series-parallel order. A series-parallel order variety $\boldsymbol{\alpha}$ on a set X can be representated by a *rootless planar tree* with leaves labeled by elements of X and ternary nodes labeled by $\otimes$ or $\odot$: take an arbitrary presentation of $\boldsymbol{\alpha}$ as a series-parallel order ω, write ω as a — non-unique (associativity, commutativity) — planar binary tree t with leaves labeled by elements of X, and root and nodes labeled by $\otimes$ for parallel sum, or $\odot$ for serial sum; then remove the root of t.



For instance $\overline{(x<y<z) \parallel v \parallel (t<u)}$ can be represented by:



To read the tree, take three leaves a, b, c; then (a, b, c) is in the order variety if, and only if,:

- the node $\circ$ at the intersection of the three paths ab, bc and ca is labeled by $\odot$ and
- the paths $a\odot, b\odot$ and $c\odot$ are in this cyclic order while moving clockwise around $\odot$.

The *supports* of a partial order ω or an order variety $\boldsymbol{\alpha}$ are denoted $|\omega|$, $|\boldsymbol{\alpha}|$. *Restrictions* to a subset X of the support are denoted $\omega\restriction_X$, $\boldsymbol{\alpha}\restriction_X$; restriction preserves the structures of order and order variety, and preserves series-parallelism.

2.3 Seesaw and entropy

Seesaw is the equivalence relation between series-parallel orders ω and τ on a same given set, defined by $\overline{\omega} = \overline{\tau}$. In case ω and τ are series-parallel, it turns out to be precisely the equivalence $\sim$ given by:

$$(\omega_1 \,\|\, \omega_2) \sim (\omega_1 < \omega_2).$$

Entropy $\trianglelefteq$ is the relation between series-parallel orders on the same given set defined by $\omega \trianglelefteq \tau$ if, and only if, $\omega \subseteq \tau$ and $\overline{\omega} \subseteq \overline{\tau}$. In the series-parallel case, it turns out to be precisely the least reflexive transitive relation such that:

$$\omega[\omega_1 \,\|\, \omega_2] \trianglelefteq \omega[\omega_1 < \omega_2].$$

Entropy is clearly a partial order, compatible with restriction:

$$\omega \trianglelefteq \tau \quad \Rightarrow \quad \omega\restriction_Y \trianglelefteq \tau\restriction_Y$$

and with the serial and parallel sums of orders:

$$\omega_1 \trianglelefteq \tau_1 \quad \text{and} \quad \omega_2 \trianglelefteq \tau_2 \quad \Rightarrow \quad \omega_1 \,\|\, \omega_2 \trianglelefteq \tau_1 \,\|\, \tau_2 \quad \text{and} \quad \omega_1 < \omega_2 \trianglelefteq \tau_1 < \tau_2.$$

More importantly, entropy between orders corresponds to inclusion of order varieties: given two order varieties $\boldsymbol{\alpha}, \boldsymbol{\beta}$ on X and $x \in X$, we have

$$\boldsymbol{\alpha} \subseteq \boldsymbol{\beta} \quad \text{if, and only if,} \quad \boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x.$$

This is independent from the choice of x: hence $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$ if, and only if, there exists $x \in X$ such that $\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x$, and given $x, y \in X$, we have $\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x$ if, and only if, $\boldsymbol{\alpha}_y \trianglelefteq \boldsymbol{\beta}_y$.

In the tree representation for series-parallel order varieties, entropy is performed by changing some $\odot$-nodes into $\otimes$-nodes.

2.4 Wedge

Let I be a non-empty set. Given partial orders $\omega_i, i \in I$, on a given set X, there is a largest partial order ω (w.r.t. $\trianglelefteq$) such that $\omega \trianglelefteq \omega_i$ for all i. It is denoted $\bigwedge \omega_i$, the *wedge* of the family $(\omega_i)_{i \in I}$; when I has cardinality 2, we write $\omega_1 \wedge \omega_2$. Partial orders on a given set form a complete inf-semi-lattice for entropy and wedge. The wedge commutes "partially" with restriction, i.e. if $Y \subseteq |\omega_i|$, then:

$$\left(\bigwedge \omega_i\right)\restriction_Y \;\trianglelefteq\; \bigwedge \omega_i\restriction_Y.$$

The wedge is not inclusion in general! For instance, $(a < (b \,\|\, c)) \cap ((a < b) \,\|\, c) = ((a < b) \,\|\, c)$ whereas $(a < (b \,\|\, c)) \wedge ((a < b) \,\|\, c) = (a \,\|\, b \,\|\, c)$, because $((a < b) \,\|\, c) \ntrianglelefteq (a < (b \,\|\, c))$. However, $\bigwedge \omega_i \subseteq \bigcap \omega_i$. Also, in general, $\bigwedge \omega_i$ may not be series-parallel even if all the ω_i are series-parallel. For instance,

$$(c < d < a < b) \wedge (a < c < b < d) = N(a, b, c, d) = \{(a, b), (c, b), (c, d)\}.$$

If on the other hand $\boldsymbol{\alpha}_i, i \in I$, are order varieties on X, their *wedge* $\bigwedge \boldsymbol{\alpha}_i$ is

$$\left(\bigwedge (\boldsymbol{\alpha}_i)_x\right) * x$$

for an arbitrary $x \in X$. (This is independent from the choice of x.) The wedge of order varieties commutes partially with restriction as above, and the two notions of wedge are related by:

$$\left(\bigwedge \boldsymbol{\alpha}_i\right)_x = \bigwedge (\boldsymbol{\alpha}_i)_x \quad \text{and} \quad \left(\bigwedge \omega_i\right) * x = \bigwedge (\omega_i * x).$$

There is a particular case of wedge which we shall use in the sequentialization theorem: let $\boldsymbol{\alpha}$ be an order variety on a set $X \uplus \{x\} \uplus \{y\}$, and let $z \notin X \uplus \{x\} \uplus \{y\}$; define the *identification* $\boldsymbol{\alpha}[z/x, y]$ of x and y into z in $\boldsymbol{\alpha}$ by:

$$\boldsymbol{\alpha}[z/x, y] \;=\; \boldsymbol{\alpha}\!\restriction_{X \cup \{x\}} [z/x] \;\wedge\; \boldsymbol{\alpha}\!\restriction_{X \cup \{y\}} [z/y].$$

Identification is clearly monotonic ($\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$ implies $\boldsymbol{\alpha}[z/x, y] \subseteq \boldsymbol{\beta}[z/x, y]$) and, as proved in [14], we have $\boldsymbol{\alpha}[z/x, y]_z * (x \parallel y) \subseteq \boldsymbol{\alpha}$. A more detailed introduction to order varieties can be found in [14].

3 Non-commutative logic

3.1 Language of NL

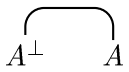
Formulas of NL are built from atoms $p, q, \ldots$ *(positive)*, $p^\perp, q^\perp, \ldots$ *(negative)* and the following connectives and constants:

	Positive		Negative	
Multiplicatives	$\otimes$	*times*	$⅋$	*par*
	$\odot$	*next*	$\triangledown$	*sequential*
	$\mathbf{1}$	*one*	$\perp$	*bot*
Additives	$\oplus$	*plus*	$\&$	*with*
	$\mathbf{0}$	*zero*	$\top$	*top*
Exponentials	$!$	*of course*	$?$	*why not*

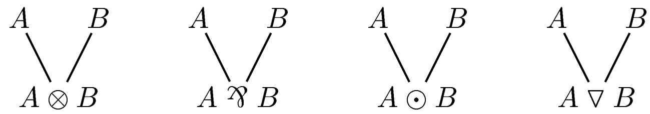
Negation is defined by usual De Morgan rules exchanging positive and negative connectives of a same row. For instance, $(A \odot B)^\perp = B^\perp \triangledown A^\perp$. A compound formula is said positive (resp. negative) when its main connective is positive (resp. negative). Therefore, a formula is either an atom (positive or negative), or a positive compound formula, or a negative compound formula.

3.2 Proof nets of MNL

Cut-free proof structures. As we are interested in proof search, we only deal with *cut-free* proof structures, but there are proof structures with cut links as well. Proof nets for multiplicative NL have been introduced in [1]. The *links* are the following graphs where the vertices are labeled by formulas of MNL:
— *identity* links:



with two *conclusions* $A^\perp$ and A and no *premise*;
— $\otimes$, $\parr$, $\odot$ and $\triangledown$-links:

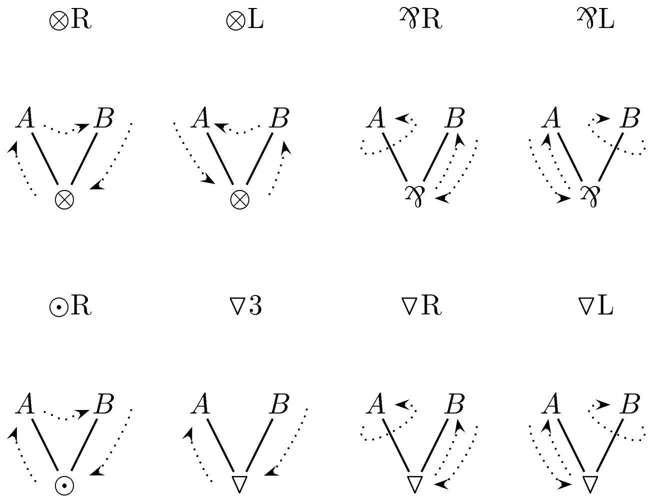


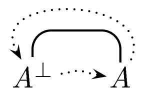
where the formula A is the *first premise*, the formula B is the *second premise* and the third formula is the *conclusion* of the link.

A *proof structure (of MNL)* is a graph built from links of MNL such that every occurrence of formula is the conclusion of exactly one link of MNL and the premise of at most one link. If π is a proof structure of MNL, the *conclusions* of π are the occurrences of formulas in π which are not premises of a link.

Switchings. We consider formulas with *decorations*: $\uparrow$ (question) or $\downarrow$ (answer). A *decorated formula* is of the form $A^\uparrow$ or $A^\downarrow$, where A is a formula of MNL. Define $\bar{\uparrow} = \downarrow$, $\bar{\downarrow} = \uparrow$. For each link l of MNL, we can consider two sets of decorated formulas:
— l^{in} is the set of all decorated formulas A^x, where A is a premise of l and x is $\downarrow$, or A is a conclusion of l and x is $\uparrow$;
— l^{out} is the set of all A^x, where A is a premise of l and x is $\uparrow$, or A is a conclusion of l and x is $\downarrow$.

For each link l of MNL we define a set $S(l)$ of (partial) functions from l^{in} to l^{out}, called the *switching positions* of l, as follows:





Given a proof structure π, a *switching* for π is a function s such that for every link l of π, $s(l) \in S(l)$. A switching s for π is *$\nabla 3$-free* if for every ∇-link l, $s(l) \neq \nabla 3$.

Let π be a proof structure and s a switching for π. The *switched proof structure* $s(\pi)$ is the oriented graph with vertices the decorated formulas labeling π, and with an oriented edge from A^x to B^y if, and only if, either $B^y = s(l)(A^x)$ for some link l in π, or $A^x = C^\downarrow$ and $B^y = C^\uparrow$ for some conclusion C of π.

A *trip* in $s(\pi)$ is a cycle or a maximal path in $s(\pi)$.

Correctness criterium. Let π be a proof structure of MNL and s a switching for π. A cycle v in $s(\pi)$ is *bilateral* if v is not of the form:

$$A^x, \ldots, B^y, \ldots, A^{\bar{x}}, \ldots, B^{\bar{y}}, \ldots, A^x$$

where A, B are occurrences of formulas in π and x, y are decorations. A proof structure π of MNL is a *proof net* if, and only if, for every switching s for π:

1. there is exactly one cycle σ in $s(\pi)$,
2. σ contains all the conclusions,
3. σ is bilateral.

The order variety associated to a proof net. Let π be a proof net of MNL with conclusion Γ. If s is a switching for π, the conclusions in the unique cycle of $s(\pi)$ appear by pairs $C^\downarrow, C^\uparrow$, so the two occurrences of a conclusion can be identified and $s(\pi)$ induces a cycle (a total order variety) on the set of unlabeled conclusions. Denote by $\alpha_{\pi,s}$ this total order variety on the set Γ of unlabeled conclusions. Then the order variety associated to π is $\alpha_\pi = \bigwedge_s \alpha_{\pi,s}$, an order variety on Γ. It follows from [1] that α_π is a series-parallel order variety.

In particular, if a proof net π is obtained from a proof net π' by adding a $\parr$ or ∇-link $\frac{A\ B}{C}$, then

$$\alpha_\pi = \alpha_{\pi'}[C/A, B].$$

See Section 2 for the definition of the identification of two points in an order variety. In the case of a ∇-link, the correctness criterium implies $\alpha_{\pi'} = \omega * (A < B)$ for some order ω, and thus $\alpha_\pi = \omega * C$.

4 Focusing MNL proof nets

If π is a proof net of MNL and A is one of its positive conclusions, then A is said a *splitting conclusion of* π, written $A \in \mathrm{Split}(\pi)$, when removing from π the link $\frac{A_1\ \ A_2}{A}$ introducing A leads to two disjoint proof structures π_1 and π_2. Moreover in that case π_1 and π_2 are proof nets. The essential content of the sequentialization theorem for MNL [1] is the following: if π is a proof net of MNL with no negative conclusion, then $\mathrm{Split}(\pi) \neq \varnothing$.

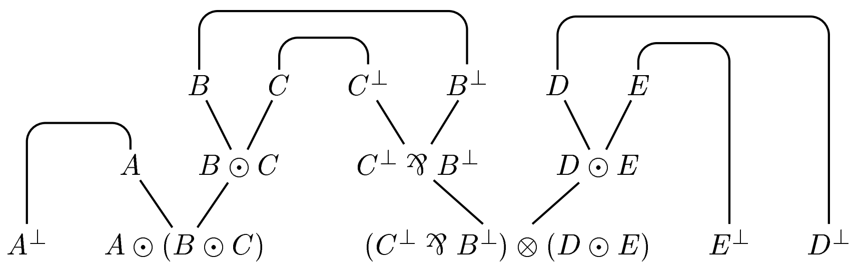
Now let π be a proof net of MNL and A be one of its conclusions. Then A is a said to be a *focusing conclusion of* π, written $A \in \mathrm{Foc}(\pi)$, when one of the following holds:

- A is a positive atom and π is an axiom link,

- $A \in \mathrm{Split}(\pi)$, π is split at $A = A_1 \odot A_2$ or $A_1 \otimes A_2$ into the two proof nets π_1 and π_2, and

$$\begin{cases} A_1 \text{ is negative or } A_1 \in \mathrm{Foc}(\pi_1), \text{ and} \\ A_2 \text{ is negative or } A_2 \in \mathrm{Foc}(\pi_2). \end{cases}$$

Therefore, when π is not reduced to an axiom link, $\mathrm{Foc}(\pi) \subseteq \mathrm{Split}(\pi)$. For instance, in the following non-commutative proof net, $(C^\perp \nabla B^\perp) \otimes (D \odot E)$ is focusing, whereas $A \odot (B \odot C)$ is only a splitting conclusion.



The splitting property can be refined as follows [3]:

Theorem 4.1 (Focusing) *If π is a proof net of MNL with no non-atomic negative conclusion, then* $\mathrm{Foc}(\pi) \neq \varnothing$.

Focusing is a form of hereditary splitting. We shall now make more precise the meaning of focusing in terms of a splitting strategy in the sequentialization procedure.

5 Partitions of series-parallel orders: parallel sum

In the present section and the next one, we consider a fixed series-parallel order ω and a fixed partition (X, Y) of $|\omega|$, and we give:

- conditions under which ω can be "partitionned modulo entropy" into two orders on X and Y,

- optimal solutions to the partitionning problem, i.e. minimal such orders when they exist.

There are two cases, depending on how the two orders are composed : parallel sum (which corresponds to the connective $\otimes$) and serial sum (which corresponds to the connective $\odot$). In this section, we consider the case of the parallel sum.

Lemma 5.1 *The following are equivalent:*

1. $\omega = \omega\restriction_X \parallel \omega\restriction_Y$.
2. *There exist series-parallel orders ω_X, ω_Y on X, Y respectively, such that $\omega \trianglelefteq (\omega_X \parallel \omega_Y)$.*

PROOF — (i) $\Rightarrow$ (ii) is trivial. Conversely, if $\omega \trianglelefteq (\omega_X \parallel \omega_Y)$, then $\omega \subseteq (\omega_X \parallel \omega_Y)$, therefore in ω any two elements, one in X one in Y, are incomparable. This implies $\omega = \omega\restriction_X \parallel \omega\restriction_Y$, since $\omega = (\omega_1 < \omega_2)$, with $|\omega_1| \neq \varnothing$ and $|\omega_2| \neq \varnothing$, implies $|X| = \varnothing$ or $|Y| = \varnothing$. □

When the above condition is satisfied, $\omega\restriction_X$ and $\omega\restriction_Y$ are clearly optimal:

Lemma 5.2 *For any pair of series-parallel orders ω_X, ω_Y on X, Y respectively, such that $\omega \trianglelefteq (\omega_X \parallel \omega_Y)$, we have $\omega\restriction_X \trianglelefteq \omega_X$ and $\omega\restriction_Y \trianglelefteq \omega_Y$.*

PROOF — Obvious consequence of the commutation of $\trianglelefteq$ with restriction (see Section 2). □

6 Partitions of series-parallel orders: serial sum

6.1 Admissibility

Definition 6.1 (Admissible partition) *Let ω be a series-parallel order on $X \uplus Y$. (X, Y) is said* admissible *when it enjoys the following two properties:*

(i)	*for all* $x \in X, y \in Y$,	$y \not< x$,
(ii)	*for all* $x_1, x_2 \in X, y_1, y_2 \in Y$,	$\omega\restriction_{x_1,x_2,y_1,y_2} \neq (x_1 < y_1) \parallel (x_2 < y_2)$.

The rest of the section is devoted to:

- show that this notion of admissibility characterizes those partitions $X \uplus Y$ of the support of ω for which the equation

$$\omega \trianglelefteq (\omega_X < \omega_Y)$$

  with ω_X, ω_Y on X, Y respectively, admits a solution;
- give optimal solutions to the above equation.

We first prove the following lemma:

Lemma 6.2 *Let ω and σ be series-parallel orders on $X \uplus Y$, such that (X, Y) is admissible for σ and $\omega \trianglelefteq \sigma$. Then (X, Y) is admissible for ω.*

PROOF — $\omega \trianglelefteq \sigma$ implies $\omega \subseteq \sigma$, therefore:

1. If $\omega(y,x)$ for some $x \in X, y \in Y$, then $\sigma(y,x)$.
2. If $\omega\restriction_{x_1,x_2,y_1,y_2} = (x_1 < y_1) \,\|\, (x_2 < y_2)$, then from $\omega\restriction_{x_1,x_2,y_1,y_2} \trianglelefteq \sigma\restriction_{x_1,x_2,y_1,y_2}$, we get $\sigma\restriction_{x_1,x_2,y_1,y_2} = (x_1 < y_1) \,\|\, (x_2 < y_2)$ or $(x_1 < y_1 < x_2 < y_2)$ or $(x_2 < y_2 < x_1 < y_1)$.

Both conclusions contradict the assumption that (X,Y) is admissible for σ. □

Definition 6.3 (Well-teasing) *ω is said* well-teased *by (X,Y) when either $|\omega|$ is empty, or $\omega = \omega_1 \,\|\, \omega_2$ or $\omega_1 < \omega_2$, and one of the following holds:*

$$\begin{cases} |\omega_1| \subseteq X & \text{and} \quad \omega_2 \text{ is well-teased by } (X,Y), \text{ or} \\ |\omega_2| \subseteq Y & \text{and} \quad \omega_1 \text{ is well-teased by } (X,Y). \end{cases}$$

For instance, the partial order $\omega = (x_1 < y_1) \,\|\, (x_2 \,\|\, y_2)$ with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, is well-teased: indeed $\omega = x_2 \,\|\, (x_1 < y_1) \,\|\, y_2$. On the other hand $(x_1 < y_1) \,\|\, (x_2 < y_2)$ is not well-teased. Essentially, ω is well-teased by (X,Y) if, and only if, ω can be represented by a binary tree (with nodes labelled by $<, \|$) where all leaves in X (resp. in Y) are on the left side (resp. right side).

Lemma 6.4 *The following are equivalent :*
(i) (X,Y) is admissible for ω.
(ii) ω is well-teased by (X,Y).
(iii) There exist series-parallel orders ω_X, ω_Y on X, Y respectively, such that $\omega \trianglelefteq (\omega_X < \omega_Y)$.

PROOF — (i) $\Rightarrow$ (ii). By induction on ω. If ω is empty, any partition is both admissible and well-teasing. Otherwise, $\omega = (\omega_1 < \omega_2)$ or $(\omega_1 \,\|\, \omega_2)$, with $|\omega_1|$ and $|\omega_2|$ both non empty.

▶ $\omega = (\omega_1 < \omega_2)$. Since (X,Y) is admissible for ω, we have:

- either $X \subseteq |\omega_1|$ and $Y = Y_1 \uplus Y_2$, with $Y_1 \subseteq |\omega_1|$ and $Y_2 = |\omega_2|$,
- or $Y \subseteq |\omega_2|$ and $X = X_1 \uplus X_2$, with $X_1 = |\omega_1|$ and $X_2 \subseteq |\omega_2|$.

In the first case, (X, Y_1) is admissible for ω_1, so by induction ω_1 is well-teased by (X, Y_1), i.e. ω_1 can be represented by a binary tree T (with nodes labelled by $<, \|$) with leaves in X on the left and leaves in Y on the right. Thus $\omega = (\omega_1 < \omega_2)$ is represented by the grafting of T and any tree-representation of ω_2, which is well-teased because $|\omega_2| \subseteq Y$. The second case is symmetric.

▶ $\omega = (\omega_1 \,\|\, \omega_2)$. If some $x_1 \in X \cap |\omega_1|$ and $y_1 \in Y \cap |\omega_1|$ have $x_1 < y_1$, then, by definition of admissibility (second condition), $\omega_2 = (\omega_2\restriction_X \,\|\, \omega_2\restriction_Y)$. By induction, ω_1 is well-teased, so $\omega = (\omega_2\restriction_X \,\|\, \omega_1 \,\|\, \omega_2\restriction_Y)$ is also well-teased. If on the other hand, some $x_2 \in X \cap |\omega_2|$ and $y_2 \in Y \cap |\omega_2|$ have $x_2 < y_2$, the argument is similar.

(ii) $\Rightarrow$ (iii). In a tree-representation of ω with leaves in X on the left and leaves in Y on the right, replace all $\|$-labels by $<$-labels: this gives a new tree which represents a total order $\omega_X < \omega_Y$.
(iii) $\Rightarrow$ (i). In $(\omega_X < \omega_Y)$, every element in X is less than every element in Y, therefore $(\omega_X < \omega_Y)$ enjoys properties 1 and 2 of Definition 6.1, hence (X, Y) is admissible for $(\omega_X < \omega_Y)$. By Lemma 6.2 this implies that (X, Y) is admissible for ω, since $\omega \trianglelefteqslant (\omega_X < \omega_Y)$. $\square$

6.2 Optimality

Definition 6.5 (Δ) *Given (X, Y) an admissible partition for ω, let $O(\omega, X, Y)$ be the set of series-parallel orders $(\omega_X < \omega_Y)$, with ω_X on X and ω_Y on Y, such that $\omega \trianglelefteqslant (\omega_X < \omega_Y)$. Define $\Delta_{X,Y}(\omega)$ to be the wedge $\bigwedge O(\omega, X, Y)$.*

By Lemma 6.4, $O(\omega, X, Y)$ is non-empty when (X, Y) admissible for ω, so $\Delta_{X,Y}(\omega)$ is well-defined.

Facts 6.6

1. *$\omega \trianglelefteqslant \Delta_{X,Y}(\omega)$.*
2. *If $X = \varnothing$ or $Y = \varnothing$ then $\Delta_{X,Y}(\omega) = \omega$.*
3. *$\Delta_{X,Y}(\omega_X < \omega_Y) = \omega_X < \omega_Y$ when $|\omega_X| = X$ and $|\omega_Y| = Y$.*

Lemma 6.7 (Δ_X, Δ_Y) *$\Delta_{X,Y}(\omega)$ is of the form $\Delta_X(\omega) < \Delta_Y(\omega)$ with $\Delta_X(\omega)$ and $\Delta_Y(\omega)$ partial orders on X, Y respectively.*

Proof — For any $(\omega_X < \omega_Y) \in O(\omega, X, Y)$ we have $(\varnothing_X < \varnothing_Y) \trianglelefteqslant (\omega_X < \omega_Y)$, so $(\varnothing_X < \varnothing_Y) \trianglelefteqslant \bigwedge O(\omega, X, Y)$, in particular $(\varnothing_X < \varnothing_Y) \subseteq \Delta_{X,Y}(\omega)$, and this implies $\Delta_{X,Y}(\omega)(x, y)$ for any $x \in X$ and $y \in Y$, qed. $\square$

Lemma 6.8 *Let ω be a series-parallel order on the disjoint union $X \uplus Y$. Then for any series-parallel order τ on $X \uplus Y$, such that (X, Y) is admissible for τ and $\omega \trianglelefteqslant \tau$, we have $\Delta_{X,Y}(\omega) \trianglelefteqslant\ \Delta_{X,Y}(\tau)$.*

Proof — $\omega \trianglelefteqslant \tau$ implies $O(\tau, X, Y) \subseteq O(\omega, X, Y)$, hence $O(\omega, X, Y) = O(\tau, X, Y) \uplus\ Z$ for some set Z of orders on $X \uplus Y$. Then:

$$
\begin{aligned}
\Delta_{X,Y}(\omega) &= \bigwedge\Big(O(\tau, X, Y) \uplus Z\Big) \\
&= \Big(\bigwedge O(\tau, X, Y)\Big) \wedge \Big(\bigwedge Z\Big) \\
&\trianglelefteqslant \bigwedge O(\tau, X, Y) \\
&= \Delta_{X,Y}(\tau).
\end{aligned}
$$

$\square$

Theorem 6.9 (Optimality) *If (X, Y) is an admissible partition for ω, then for any pair of series-parallel orders ω_X, ω_Y on X, Y respectively, such that $\omega \trianglelefteqslant (\omega_X < \omega_Y)$, we have $\Delta_X(\omega) \trianglelefteqslant \omega_X$ and $\Delta_Y(\omega) \trianglelefteqslant \omega_Y$.*

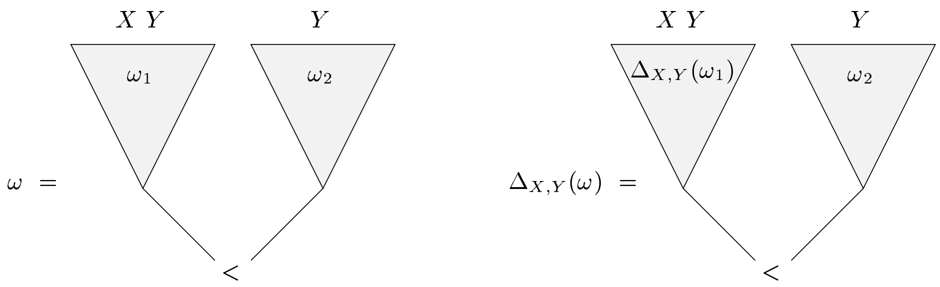
Proof — By Lemma 6.8, $\omega \trianglelefteqslant (\omega_X < \omega_Y)$ implies $\Delta_{X,Y}(\omega) \trianglelefteqslant \Delta_{X,Y}(\omega_X < \omega_Y)$, and by Facts 6.6, $\Delta_{X,Y}(\omega_X < \omega_Y) = (\omega_X < \omega_Y)$, whence $\Delta_X(\omega) \trianglelefteqslant \omega_X$ and $\Delta_Y(\omega) \trianglelefteqslant \omega_Y$ by Lemma 6.7. $\square$

6.3 Preservation of series-parallelism

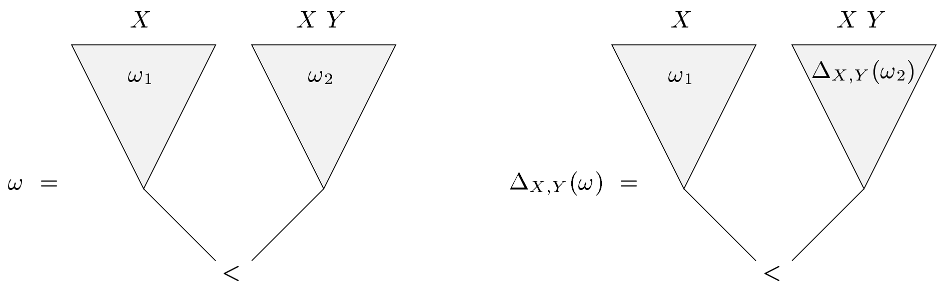
We show that if ω is series-parallel, so is $\Delta_{X,Y}(\omega)$. It is not obvious since the wedge of series-parallel orders may not be series-parallel, as noticed in Section 2. However, by induction, it is a consequence of the following explicit formulas for $\Delta_{X,Y}$:

Lemma 6.10 *Let (X, Y) be an admissible partition of a series-parallel order $\omega = (\omega_1 < \omega_2)$. Then:*

- either $X \subseteq |\omega_1|$, $|\omega_2| \subseteq Y$ *and* $\Delta_{X,Y}(\omega) = (\Delta_{X,Y}(\omega_1) < \omega_2)$,



- or $Y \subseteq |\omega_2|$, $|\omega_1| \subseteq X$ *and* $\Delta_{X,Y}(\omega) = (\omega_1 < \Delta_{X,Y}(\omega_2))$.



Proof — Note that the statements about the decomposition of $|\omega|$ are the same as the cases in the proof of Lemma 6.4 (ii $\Rightarrow$ iii) and directly follow from the definition of admissibility. Let us show the equations concerning $\Delta_{X,Y}(\omega)$. By symmetry we simply consider the first case. We have $\omega = (\omega_1 < \omega_2) \trianglelefteq (\Delta_{X,Y}(\omega_1) < \omega_2)$, so by Theorem 6.9, we just have to show that $(\Delta_{X,Y}(\omega_1) < \omega_2)$ is indeed optimal. To this end, take series-parallel orders ω_X, ω_Y on X, Y respectively, such that $\omega \trianglelefteq (\omega_X < \omega_Y)$. Since $\omega = (\omega_1 < \omega_2) \subseteq (\omega_X < \omega_Y)$ and on the other hand $X \subseteq |\omega_1|$ and $|\omega_2| \subseteq Y$, we may conclude that

$$(\omega_X < \omega_Y) = (\omega_{1X} < \omega_{1Y} < \omega_2'),$$

with $|\omega_{1X} < \omega_{1Y}| = |\omega_1|$ and $|\omega_2'| = |\omega_2|$, as the notation suggests. Restriction to $|\omega_2|$ gives

$$\omega_2 \trianglelefteq \omega_Y\restriction_{|\omega_2|} = \omega_2'$$

and restriction to $|\omega_1|$ gives

$$\omega_1 \trianglelefteq (\omega_{1X} < \omega_{1Y}).$$

Hence $\Delta_{X,Y}(\omega_1) \trianglelefteq \Delta_{X,Y}(\omega_{1X} < \omega_{1Y}) = (\omega_{1X} < \omega_{1Y})$ by Lemma 6.8 and Facts 6.6, and $(\Delta_{X,Y}(\omega_1) < \omega_2) \trianglelefteq (\omega_{1X} < \omega_{1Y} < \omega_2') = (\omega_X < \omega_Y)$, qed. $\square$

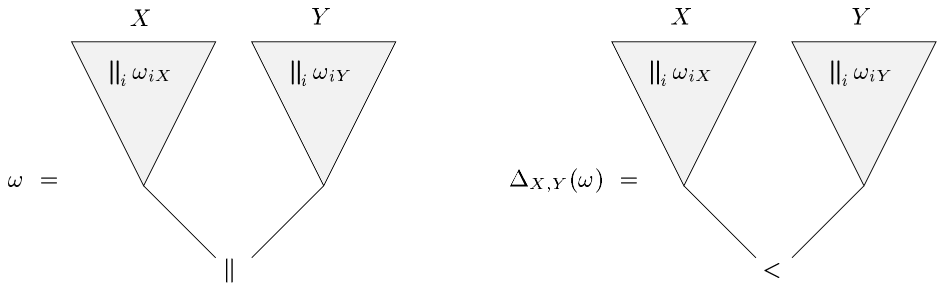
Lemma 6.11 *Let (X, Y) be an admissible partition of a series-parallel order $\omega = (\omega_1 \| \cdots \| \omega_n)$. If there exist $x_i \in X \cap |\omega_i|$ and $y_i \in Y \cap |\omega_i|$ such that $x_i < y_i$, then for any $j \neq i$, $\omega_j = (\omega_{jX} \| \omega_{jY})$, with $|\omega_{jX}| \subseteq X$ and $|\omega_{jY}| \subseteq Y$.*

Proof — Immediate consequence of the definition of admissibility. $\square$

Lemma 6.12 *Let (X, Y) be an admissible partition of a series-parallel order $\omega = (\omega_1 \| \cdots \| \omega_n) = \|_{i=1}^{n} \omega_i$, each ω_i is either a singleton, or of the form $\sigma_i < \tau_i$ with $|\sigma_i|$ and $|\tau_i|$ non-empty. Then:*

- either *for any i, $\omega_i = (\omega_{iX} \| \omega_{iY})$ with $|\omega_{iX}| \subseteq X$ and $|\omega_{iY}| \subseteq Y$, and:*

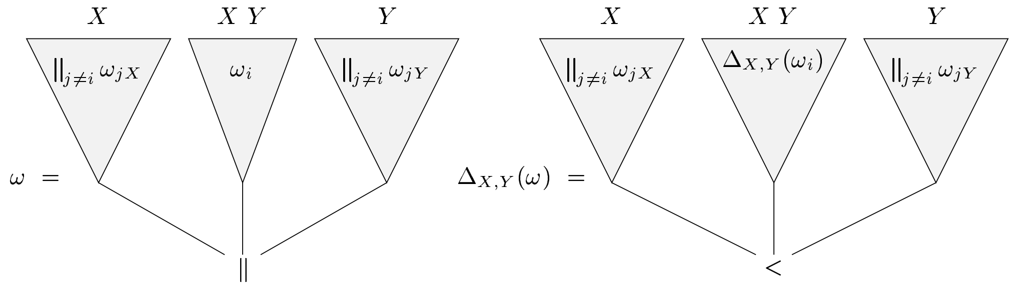
$$\begin{aligned} \omega &= \Big(\big\|_{i=1}^{n} \omega_{iX}\Big) \, \| \, \Big(\big\|_{i=1}^{n} \omega_{iY}\Big) \\ \Delta_{X,Y}(\omega) &= \Big(\big\|_{i=1}^{n} \omega_{iX}\Big) < \Big(\big\|_{i=1}^{n} \omega_{iY}\Big), \end{aligned}$$



- or *for some i, there exist $x_i \in X \cap |\omega_i|$ and $y_i \in Y \cap |\omega_i|$ such that $x_i < y_i$, and then:*

$$\begin{aligned} \omega &= \Big(\big\|_{j \neq i} \omega_{jX}\Big) \, \| \, \omega_i \, \| \Big(\big\|_{j \neq i} \omega_{jY}\Big) \\ \Delta_{X,Y}(\omega) &= \Big(\big\|_{j \neq i} \omega_{jX}\Big) < \Delta_{X,Y}(\omega_i) < \Big(\big\|_{j \neq i} \omega_{jY}\Big), \end{aligned}$$

with for any $j \neq i$, $|\omega_{jX}| \subseteq X$ and $|\omega_{jY}| \subseteq Y$.



Proof — The first case is obvious, so let us consider the second case. The equation for ω is an immediate consequence of Lemma 6.11. Call $\delta_{X,Y}(\omega)$ the right hand side of the equation for $\Delta_{X,Y}(\omega)$: clearly $\omega \trianglelefteq \delta_{X,Y}(\omega)$ so we just have to show that $\delta_{X,Y}(\omega)$ is optimal in order to conclude that $\Delta_{X,Y}(\omega) = \delta_{X,Y}(\omega)$. Take series-parallel orders ω_X, ω_Y on X, Y respectively, such that $\omega \trianglelefteq (\omega_X < \omega_Y)$.

We first show that $\delta_{X,Y}(\omega) \subseteq (\omega_X < \omega_Y)$. Let $a, b, \in X \uplus Y$ be such that $\delta_{X,Y}(\omega)(a,b)$. If $\omega(a,b)$, we are done because $\omega \subseteq (\omega_X < \omega_Y)$. Otherwise, disinguish between the possible positions of a and b in $|\omega|$:

- *$a, b \in |\omega_i|$:* we have $(a,b) \in \delta_{X,Y}(\omega)\restriction_{|\omega_i|} = \Delta_{X,Y}(\omega_i)$, and by Theorem 6.9, from $\omega_i \trianglelefteq (\omega_X < \omega_Y)\restriction_{|\omega_i|}$ follows $\Delta_{X,Y}(\omega_i) \trianglelefteq (\omega_X < \omega_Y)\restriction_{|\omega_i|}$, thus $(a,b) \in (\omega_X < \omega_Y)$;

- *$a \in \biguplus_{j\neq i} |\omega_{jX}|$ and $b \in |\omega_i|$:* by hypothesis, ω_i is either a singleton, or of the form $\sigma_i < \tau_i$ with $|\sigma_i|$ and $|\tau_i|$ non-empty; it cannot be a singleton for it contains at least x_i and y_i, so it is of the form $\sigma_i < \tau_i$; then by Lemma 6.10 either $|\tau_i| \subseteq Y$ or $|\sigma_i| \subseteq X$; if $|\tau_i| \subseteq Y$, then $a \in X$ so $(a,b) \in (\omega_X < \omega_Y)$ whenever $b \in |\tau_i|$, and when $b \in |\sigma_i|$, any $y \in |\tau_i|$ gives

$$\begin{aligned} (a,b,y) &\in \overline{\omega} \\ (b,y) &\in (\omega_X < \omega_Y) \\ (a,y) &\in (\omega_X < \omega_Y), \end{aligned}$$

  so $(a,b) \in (\omega_X < \omega_Y)$ because $\overline{\omega} \subseteq \overline{\omega_X < \omega_Y}$; if $|\sigma_i| \subseteq X$, we have by hypothesis a $y_i \in Y \cap |\omega_i| \subseteq |\tau_i|$, so

$$(a, y_i) \in (\omega_X < \omega_Y),$$

  and again, taking any $x \in |\sigma_i| \subseteq X$ leads to

$$\begin{aligned} (a,x,y_i) &\in \overline{\omega} \\ (x,y_i) &\in (\omega_X < \omega_Y), \end{aligned}$$

  hence $(a,x) \in (\omega_X < \omega_Y)$, and then $(\omega_X < \omega_Y)(a,b)$ for $b \in |\tau_i|$ by transitivity;

- *$a \in |\omega_i|$ and $b \in \biguplus_{j\neq i} |\omega_{jY}|$:* symmetric;

- *$a \in \biguplus_{j\neq i} |\omega_{jX}|$ and $b \in \biguplus_{j\neq i} |\omega_{jY}|$:* $(\omega_X < \omega_Y)(a,b)$ follows from the previous two items by transitivity, since $|\omega_i| \neq \varnothing$.

Moreover, $\overline{\delta_{X,Y}(\omega)} \subseteq \omega_X * \omega_Y$. Indeed, by definition of the order variety associated to a partial order, $(a,b,c) \in \overline{\delta_{X,Y}(\omega)}$ implies $\delta_{X,Y}(\omega)\restriction_{a,b,c} = (a<b<c)$ or $((a<b) \parallel c)$ or a cyclic permutation of the above orders. Since $\delta_{X,Y}(\omega) \subseteq (\omega_X < \omega_Y)$, $\delta_{X,Y}(\omega)\restriction_{a,b,c} = (a<b<c)$ implies $(\omega_X < \omega_Y)\restriction_{a,b,c} = (a<b<c)$, hence $(a,b,c) \in \omega_X * \omega_Y$. Now, assume that $\delta_{X,Y}(\omega)\restriction_{a,b,c} = ((a<b) \parallel c)$. As $\omega \subseteq \delta_{X,Y}(\omega)$, the only possibility for $(a,b,c) \notin \omega_X * \omega_Y$ is when $\omega\restriction_{a,b,c} = (a \parallel b \parallel c)$. It is clear that the only possible positions of a,b,c in $|\omega|$ are a,b,c all in $|\omega_i|$, and by Theorem 6.9, $\delta_{X,Y}(\omega)\restriction_{|\omega_i|} = \Delta_{X,Y}(\omega_i) \trianglelefteq (\omega_X < \omega_Y)\restriction_{|\omega_i|}$, so $(a,b,c) \in \overline{(\omega_X < \omega_Y)\restriction_{|\omega_i|}} = (\omega_X * \omega_Y)\restriction_{|\omega_i|}$, contradiction.

To sum up, $\delta_{X,Y}(\omega) \subseteq (\omega_X < \omega_Y)$ and $\overline{\delta_{X,Y}(\omega)} \subseteq \omega_X * \omega_Y$, so $\delta_{X,Y}(\omega) \trianglelefteq (\omega_X < \omega_Y)$. This holds for any ω_X, ω_Y such that $\omega \trianglelefteq (\omega_X < \omega_Y)$, hence $\delta_{X,Y}(\omega) = \Delta_{X,Y}(\omega)$, qed. $\square$

Corollary 6.13 *If ω is series-parallel, then so is $\Delta_{X,Y}(\omega)$.*

Example 6.14 *Let $\omega = (a \parallel b) < (e \parallel (c < d))$, $X = \{a,b,c\}$, $Y = \{d,e\}$. Then $\Delta_{X,Y}(\omega) = ((a \parallel b) < c) < (d < e)$.*

6.4 Spines

Consider a series-parallel order ω with an admissible partition (X,Y) (or equivalently, a well-teasing partition, by Lemma 6.4), and a tree-representation of ω with leaves in X on the left and leaves in Y on the right. Using associativity, the middle path from the root to the point of separation between X and Y can be compressed into an "alternating path" where two consecutive nodes have different labels $(<, \parallel)$.

$$
\begin{array}{rcl}
X & & Y \\
& \vdots & \\
\sigma_k - & < & \\
& | & \\
& < & -\tau_k \\
& \vdots & \\
\sigma_{i-1} - & \parallel & \\
& | & \\
\sigma_i - & \parallel & \\
& \vdots &
\end{array}
\quad \longrightarrow \quad
\begin{array}{rcl}
X & & Y \\
& \vdots & \\
\sigma_k - & < & -\tau_k \\
& \vdots & \\
(\sigma_{i-1} \parallel \sigma_i) - & \parallel & \\
& \vdots &
\end{array}
$$

This compression may create ternary nodes on the middle path, as in the figure below, where $\bullet$ and $\circ$ represent alternating labels (resp. $<$ and $\parallel$, or $\parallel$ and $<$). The result of this compression is called the *spine* of (ω, X, Y).

$$
\begin{array}{ccc}
\begin{array}{c}
\sigma_n - \circ - \tau_n \\
| \\
\sigma_{n-1} - \bullet - \tau_{n-1} \\
\vdots \\
\sigma_2 - \circ - \tau_2 \\
| \\
\sigma_1 - \bullet - \tau_1 \\
\text{if } n \text{ is even}
\end{array}
& \qquad &
\begin{array}{c}
\sigma_n - \bullet - \tau_n \\
| \\
\sigma_{n-1} - \circ - \tau_{n-1} \\
\vdots \\
\sigma_2 - \circ - \tau_2 \\
| \\
\sigma_1 - \bullet - \tau_1 \\
\text{if } n \text{ is odd}
\end{array}
\end{array}
$$

Definition 6.15 (Spine) *Assume ω is well-teased by (X, Y). With the notations of figure 6.4 we call $[((\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n)), \xi]$ the* spine *of (ω, X, Y).*

- *The σ_i (resp. τ_i), $1 \leqslant i \leqslant n$, are called the left (resp. right)* stings *and are series-parallel orders on disjoint subsets X_i of X (resp., $Y_i \subseteq Y$),*
- *$\xi \in \{<, \|\}$ is the label of the root,*
- *n is called the* height *of the spine.*

It should satisfy the following requirements:

- *both $|\sigma_n|$ and $|\tau_n|$, respectively, top-left and top-right sting, are not empty;*
- *$|\sigma_i| \uplus |\tau_i| \neq \varnothing$, for any i such that $1 \leqslant i \leqslant n$.*

Lemma 6.16 *Assume ω is well-teased by (X, Y), and let $[((\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n)), \xi]$ be the spine of (ω, X, Y). Then $\Delta_X(\omega) = (\sigma_1 < \cdots < \sigma_n)$ and $\Delta_Y(\omega) = (\tau_n < \cdots < \tau_1)$.*

$$
\begin{array}{ccc}
\begin{array}{c}
X \qquad Y \\
\vdots \\
\sigma_{i+1} - \| - \tau_{i+1} \\
| \\
\sigma_i - < - \tau_i \\
| \\
\sigma_{i-1} - \| - \tau_{i-1} \\
\vdots \\
\omega
\end{array}
& \qquad\qquad &
\begin{array}{c}
X \qquad Y \\
\vdots \\
\sigma_{i+1} - < - \tau_{i+1} \\
| \\
\sigma_i - < - \tau_i \\
| \\
\sigma_{i-1} - < - \tau_{i-1} \\
\vdots \\
\Delta_{X,Y}(\omega)
\end{array}
\end{array}
$$

PROOF — This is just a reformulation of the equations in Lemmas 6.10 and 6.12. □

7 Focalized calculus and sequentialization for MNL

Definition 7.1 *A* sequent *is of the form either $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is an order variety of formula occurrences, or $\gamma \mid A$, where A is a formula occurrence and γ is a series-parallel order of formula occurrences. In the latter case, A is called the* focus.

Identity

$$\frac{}{p^{\perp} \mid p}\; p \text{ positive atom}$$

Reaction: N is negative

$$\frac{\gamma * N}{\gamma \mid N}\;\text{reaction}$$

Decision: P is positive and γ contains only atoms or positive formulas

$$\frac{\gamma \mid P}{\gamma * P}\;\text{decision}$$

Multiplicative connectives

$$\frac{\Delta_Y(\gamma) \mid A \qquad \Delta_X(\gamma) \mid B}{\gamma \mid A \odot B}\;\odot \qquad (X, Y) \text{ admissible partition for } \gamma$$

$$\frac{\gamma' \mid A \qquad \gamma'' \mid B}{(\gamma' \parallel \gamma'') \mid A \otimes B}\;\otimes \qquad \frac{\gamma * (A < B)}{\gamma * A \triangledown B}\;\triangledown \qquad \frac{\gamma * (A \parallel B)}{\gamma * A \parr B}\;\parr$$

Table 1: Focalized sequent calculus for MNL.

In other terms, a sequent is an order variety or a pointed order variety. Note that we omit the symbol $\vdash$ at the beginning of sequents, since it is useless in one-sided sequents. The rules of the sequent calculus are given in Table 1. As we are interested in proof search, we only deal with *cut-free* sequent calculus.

Observe that a crucial rule of NL, entropy, does not appear explicitely in Table 1. As we have already said in the introduction, entropy is a source of non-determinism in proof search. In Table 1, it is included in the rule for $\odot$, the only place where it is actually necessary: this is not trivial, but a consequence of the results in the previous section, and the rest of the present section is devoted to proving that this "optimized" sequent calculus is actually equivalent to the original one in [1, 14] or in Appendix A. We do this by proving adequacy and sequentialization w.r.t. proof nets.

Example 7.2 *Similarly to the example given in the introduction, there is only one focusing proof for the sequent with order variety* $\boldsymbol{\alpha} = \varnothing$ *on*

$$\{A^{\perp}, E^{\perp}, D^{\perp}, A \odot (B \odot C), (C^{\perp} \triangledown B^{\perp}) \otimes (D \odot E)\}$$

with A, B, C, D, E *positive:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A^\perp \mid A} \qquad \dfrac{\overline{B^\perp \mid B} \qquad \overline{C^\perp \mid C}}{(C^\perp < B^\perp) \mid B \odot C}\,\odot}{(A^\perp \parallel (C^\perp < B^\perp)) \mid A \odot (B \odot C)}\,\odot}{(A^\perp \parallel A \odot (B \odot C)) * (C^\perp < B^\perp)}\,\mathit{decision}}{(A^\perp \parallel A \odot (B \odot C)) * C^\perp \triangledown B^\perp}\,\triangledown}{(A^\perp \parallel A \odot (B \odot C)) \mid C^\perp \triangledown B^\perp}\,\mathit{reaction} \qquad \dfrac{\overline{D^\perp \mid D} \qquad \overline{E^\perp \mid E}}{(E^\perp \parallel D^\perp) \mid D \odot E}\,\odot}{(A^\perp \parallel E^\perp \parallel D^\perp \parallel A \odot (B \odot C)) \mid ((C^\perp \triangledown B^\perp) \otimes (D \odot E))}\,\otimes}{(A^\perp \parallel E^\perp \parallel D^\perp \parallel A \odot (B \odot C)) * ((C^\perp \triangledown B^\perp) \otimes (D \odot E))}\,\mathit{decision}$$

Note that, for instance, entropy is used for the $\odot$-rule on the right: $(E^\perp \parallel D^\perp) \trianglelefteq (E^\perp < D^\perp)$ and $\Delta_{D^\perp, E^\perp}(E^\perp \parallel D^\perp) = (E^\perp < D^\perp)$.

Example 7.3 *Non-commutativity can be used to reduce part of the non-determinism of context splitting, because some failures in commutative LL can be avoided just by using the* structural properties *of the context. An obvious example is given by (η-expanded) proofs of $B^\perp, A^\perp, (A \text{ tensor } B)$. In the non-commutative case, there is a single possible partition, and it leads to an actual proof:*

$$\dfrac{\begin{matrix}\vdots \\ A^\perp \mid A\end{matrix} \qquad \begin{matrix}\vdots \\ B^\perp \mid B\end{matrix}}{(B^\perp < A^\perp) \mid A \odot B},$$

whereas in the commutative case, there are two possible partitions, one leading to a proof as above, the other one leading to an arbitrarily long failure in general:

$$\dfrac{\begin{matrix}\vdots \\ A^\perp \mid A\end{matrix} \qquad \begin{matrix}\vdots \\ B^\perp \mid B\end{matrix}}{(B^\perp \parallel A^\perp) \mid A \otimes B} \qquad \dfrac{\begin{matrix}\mathit{failure} \\ \vdots \\ B^\perp \mid A\end{matrix} \qquad \begin{matrix}\mathit{failure} \\ \vdots \\ A^\perp \mid B\end{matrix}}{(B^\perp \parallel A^\perp) \mid A \otimes B}.$$

Facts 7.4 *Let $\mathcal{D}$ be a sequent calculus proof of a sequent C with a single conclusion formula. In a sequent of the form $\gamma \mid A$, only atoms or positive formulas occur in γ.*

Theorem 7.5 (Adequacy) *To any proof $\mathcal{D}$ with conclusion $\boldsymbol{\alpha}$ (resp. $\omega \mid A$) in the focalized sequent calculus we associate a cut-free proof net $\mathcal{D}^-$ of MNL with same conclusion and associated order variety $\alpha_{\mathcal{D}^-}$ enjoying $\alpha_{\mathcal{D}^-} \supseteq \boldsymbol{\alpha}$ (resp. $\alpha_{\mathcal{D}^-} \supseteq \omega * A$).*

PROOF — By induction on $\mathcal{D}$. The cases of identity, reaction and decision are trivial. For $\triangledown$, we have by induction a proof net $\mathcal{D}'^-$ associated to the proof $\mathcal{D}'$ of $\gamma * (A < B)$, with $\alpha_{\mathcal{D}'^-} \supseteq \gamma * (A < B)$: this implies the absence of conclusion between $B^\wedge$ and $A^\vee$ for any switching in $\mathcal{D}'$, so the proof structure $\mathcal{D}^-$ obtained by adding a $\triangledown$-link between A and B is a proof net, and

$$\alpha_{\mathcal{D}^-} = \alpha_{\mathcal{D}'^-}[A \triangledown B / A, B] \supseteq (\gamma * (A < B))[A \triangledown B / A, B] = \gamma * A \triangledown B$$

by monotonicity of identification in order varieties. For $\parr$, the argument just forgets the point with switchings which isn't to be checked. The cases of $\otimes$ and $\odot$ are immediate. $\square$

Theorem 7.6 (Sequentialization) *Let π be a cut-free proof net of MNL with conclusion Γ and $\boldsymbol{\alpha}$ be any order variety on Γ such that $\boldsymbol{\alpha} \subseteq \alpha_\pi$.*

- *If Γ contains non-atomic negative formulas, then there is a proof $\mathcal{D}$ with conclusion $\boldsymbol{\alpha}$ in the focusing sequent calculus such that $\mathcal{D}^- = \pi$.*
- *Otherwise for any focusing conclusion $A \in \mathrm{Foc}(\pi) \subseteq \Gamma$, there is a proof $\mathcal{D}$ with conclusion $\boldsymbol{\alpha}_A \mid A$ in the focusing sequent calculus such that $\mathcal{D}^- = \pi$.*

PROOF — By induction on the size of π.

▶ *π contains only atoms.* Then π is an axiom link with conclusions $p, p^\perp$, with p the positive one, and we can easily sequentialize π with the identity:

$$\frac{}{p^\perp \mid p}$$

▶ *Γ contains at least one non-atomic negative conclusion.* Then we need to consider two cases.

1. π is obtained from π' by adding a conclusion link $A \triangledown B$, and $\Gamma = \Gamma' \uplus \{A \triangledown B\}$.
   Let $\boldsymbol{\alpha}$ be an order variety such that $\boldsymbol{\alpha} \subseteq \alpha_\pi$. By definition, $\alpha_\pi = \alpha_{\pi'}[A \triangledown B/A, B]$ and moreover

   $$\alpha_{\pi'} = (\alpha_\pi)_{A \triangledown B} * (A < B)$$

   since the correctness criterium ensures that, for any swiching s, $s(\pi')$ contains no conclusion between $B^\uparrow$ and $A^\downarrow$. Now, from $\boldsymbol{\alpha}_{A \triangledown B} \trianglelefteq (\alpha_\pi)_{A \triangledown B}$, we get $(\boldsymbol{\alpha}_{A \triangledown B} < A < B) \trianglelefteq ((\alpha_\pi)_{A \triangledown B} < A < B)$, whence $\boldsymbol{\alpha}_{A \triangledown B} * (A < B) \subseteq (\alpha_\pi)_{A \triangledown B} * (A < B) = \alpha_{\pi'}$ by definition of entropy. Now consider the proof net π', and distinguish between the following two cases:

   (a) Among the conclusions Γ', A, B of π', there is still a non-atomic negative one. We know that $\boldsymbol{\alpha}' = \boldsymbol{\alpha}_{A \triangledown B} * (A < B) \subseteq \alpha_{\pi'}$ so we can apply the induction hypothesis and get a focalized sequential proof $\mathcal{D}'$ of $\boldsymbol{\alpha}'$ such that $\mathcal{D}'^- = \pi'$, whence a proof $\mathcal{D}$ by application of a $\triangledown$-rule:

   $$\dfrac{\begin{array}{c}\mathcal{D}' \\ \boldsymbol{\alpha}_{A \triangledown B} * (A < B)\end{array}}{\boldsymbol{\alpha}_{A \triangledown B} * A \triangledown B}\ \triangledown$$

   with conclusion $\boldsymbol{\alpha}_{A \triangledown B} * A \triangledown B = \boldsymbol{\alpha}$, qed.

   (b) Otherwise, π' contains no non-atomic negative conclusion and at least a positive one. By Theorem 4.1, there exists a focusing conclusion F. In this case we can apply the induction hypothesis and get a proof $\mathcal{D}'$ of $\boldsymbol{\alpha}'_F \mid F$ such that $\mathcal{D}'^- = \pi'$. W.r.t. the previous case, it is sufficient to add an instance of the Decision rule in order to get the focusing sequent proof $\mathcal{D}$:

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ \boldsymbol{\alpha}'_F \mid F\end{array}}{\boldsymbol{\alpha}'}$$

2. π is obtained from π' by adding a conclusion link $A \parr B$, and $\Gamma = \Gamma' \uplus \{A \parr B\}$.

   Let $\boldsymbol{\alpha}$ be an order variety such that $\boldsymbol{\alpha} \subseteq \alpha_\pi$. We have $\alpha_\pi = \alpha_{\pi'}[A \parr B/A, B]$, so

   $$(\alpha_\pi)_{A \parr B} * (A \,\|\, B) \;\subseteq\; \alpha_{\pi'}$$

   follows from Section 2. Again, from $\boldsymbol{\alpha}_{A\parr B} \trianglelefteq (\alpha_\pi)_{A\parr B}$, we get $\boldsymbol{\alpha}_{A\parr B} * (A \,\|\, B) \subseteq (\alpha_\pi)_{A\parr B} * (A \,\|\, B) \subseteq \alpha_{\pi'}$. Consider the proof net π', and distinguish between the following two cases:

   (a) Among Γ', A, B there is a non-atomic negative conclusion. Since $\boldsymbol{\alpha}' = \boldsymbol{\alpha}_{A\parr B} * (A \,\|\, B) \subseteq \alpha_{\pi'}$, we can apply the induction hypothesis and get a focusing sequential proof $\mathcal{D}'$ of $\boldsymbol{\alpha}'$ such that $\mathcal{D}'^{-} = \pi'$, whence a proof $\mathcal{D}$ by application of a $\parr$-rule:

   $$\dfrac{\begin{array}{c}\mathcal{D}' \\ \boldsymbol{\alpha}_{A\parr B} * (A \,\|\, B)\end{array}}{\boldsymbol{\alpha}_{A\parr B} * A \parr B}$$

   with conclusion $\boldsymbol{\alpha}_{A\parr B} * A \parr B = \boldsymbol{\alpha}$, qed.

   (b) Otherwise, if π' contains no non-atomic negative conclusion and at least a positive one, we proceed as in the case of $A \triangledown B$, by adding Decision rule.

▶ Γ *contains only atoms and non-atomic positive conclusions.* By Theorem 4.1, $\mathrm{Foc}(\pi) \neq \varnothing$, and there are two possibilities.

1. $\mathrm{Foc}(\pi)$ contains a formula $A \odot B$ and π is obtained from π_1 (with conclusion $\Gamma_1 \uplus \{A\}$) π_2 (with conclusion $\Gamma_2 \uplus \{B\}$) by adding a conclusion link $A \odot B$.

   (a) Both premises of the selected focusing link are focusing as well, i.e. $A \in \mathrm{Foc}(\pi_1)$ and $B \in \mathrm{Foc}(\pi_2)$. Let $\boldsymbol{\alpha}$ be an order variety such that $\boldsymbol{\alpha} \subseteq \alpha_\pi$. We have $\alpha_\pi = \overline{(\alpha_{\pi_1})_A < A \odot B < (\alpha_{\pi_2})_B}$. Let $\gamma_1 = (\alpha_{\pi_1})_A$, $\gamma_2 = (\alpha_{\pi_2})_B$, two series-parallel orders on Γ_1, Γ_2 respectively, and let $\sigma = \boldsymbol{\alpha}_{A\odot B}$. We have

   $$\sigma \trianglelefteq (\gamma_2 < \gamma_1).$$

   Since the partition (Γ_1, Γ_2) is clearly admissible for $(\gamma_2 < \gamma_1)$, it follows from Lemma 6.2 that (Γ_1, Γ_2) is admissible for σ as well. Hence, by Theorem 6.9 (optimality), $\Delta_{\Gamma_1}(\sigma) \trianglelefteq \gamma_1$ and $\Delta_{\Gamma_2}(\sigma) \trianglelefteq \gamma_2$, so

$$\begin{array}{rcccll} \Delta_{\Gamma_1}(\sigma) * A & \subseteq & \gamma_1 * A & = & \alpha_{\pi_1} & \text{and} \\ \Delta_{\Gamma_2}(\sigma) * B & \subseteq & \gamma_2 * B & = & \alpha_{\pi_2}. & \end{array}$$

We can therefore apply then induction hypothesis and get two focalized proofs, $\mathcal{D}_1$ of $\Delta_{\Gamma_1}(\sigma) \mid A$ and $\mathcal{D}_2$ of $\Delta_{\Gamma_2}(\sigma) \mid B$, whence a focalized proof $\mathcal{D}$ by application of a $\odot$-rule:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_{\Gamma_1}(\sigma) \mid A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta_{\Gamma_2}(\sigma) \mid B\end{array}}{\sigma \mid A \odot B}\ \odot$$

(b) One premise of the selected focusing link is not focusing, say e.g., $A \notin \mathrm{Foc}(\pi_1)$ and $B \in \mathrm{Foc}(\pi_2)$. W.r.t. the previous case, the differences are the use of the other induction hypothesis and the addition of an instance of the Reaction rule (A is indeed negative in this case):

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_{\Gamma_1}(\sigma) * A\end{array}}{\Delta_{\Gamma_1}(\sigma) \mid A}\ \text{reaction} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta_{\Gamma_2}(\sigma) \mid B\end{array}}{\sigma \mid A \odot B}\ \odot$$

2. $\mathrm{Foc}(\pi)$ contains a formula $A \otimes B$ and π is obtained from π_1 (with conclusion Γ_1, A) π_2 (with conclusion Γ_2, B) by adding a conclusion link $A \otimes B$.

(a) Both premises of the selected focusing link are focusing. Let $\underline{\boldsymbol{\alpha} \text{ be an order variety}}$ such that $\boldsymbol{\alpha} \subseteq \alpha_\pi$. We have $\alpha_\pi = (\alpha_{\pi_1})_A \parallel A \odot B \parallel (\alpha_{\pi_2})_B$, and

$$\sigma = \boldsymbol{\alpha}_{A \odot B} \trianglelefteq (\alpha_{\pi_1})_A \parallel (\alpha_{\pi_2})_B,$$

so $\sigma = (\sigma\restriction_{\Gamma_1}) \parallel (\sigma\restriction_{\Gamma_2})$ by Lemma 5.1, and by restrictions, $\sigma\restriction_{\Gamma_1} \trianglelefteq (\alpha_{\pi_1})_A$ and $\sigma\restriction_{\Gamma_2} \trianglelefteq (\alpha_{\pi_2})_B$. By definition of entropy, we conclude:

$$\begin{array}{rcccll} \sigma\restriction_{\Gamma_1} * A & \subseteq & (\alpha_{\pi_1})_A * A & = & \alpha_{\pi_1} & \text{and} \\ \sigma\restriction_{\Gamma_2} * B & \subseteq & (\alpha_{\pi_2})_B * B & = & \alpha_{\pi_2}. & \end{array}$$

We can now apply the induction hypothesis and get two focalized proofs $\mathcal{D}_1$ of $\sigma\restriction_{\Gamma_1} |A$, and $\mathcal{D}_2$ of $\sigma\restriction_{\Gamma_2} |B$, whence the following focalized proof $\mathcal{D}$ by an application of the $\otimes$-rule:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \sigma\restriction_{\Gamma_1} |A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \sigma\restriction_{\Gamma_2} |B\end{array}}{\sigma \mid A \otimes B}\ \otimes$$

(b) If one premise of the link is not focusing, the argument is the same as with $A \odot B$.

□

$R_2 \backslash R_1$	$\odot$	$\otimes$	$\oplus$	$\triangledown$	$\parr$	&	$\perp$	!	?	abs	entropy
$\odot$	-	-	-	-	-	-	-	$\times$	-	-	-
$\otimes$	-	-	-	-	-	-	-	$\times$	-	-	-
$\oplus$	-	-	-	-	-	-	-	$\times$	-	-	-
$\triangledown$	1	-	-	-	-	-	-	$\times$	-	-	-
$\parr$	1	1	-	-	-	-	-	$\times$	-	-	-
&	2	2	3	-	-	-	-	$\times$	-	4	5
$\perp$	-	-	-	-	-	-	-	6	-	-	7
!	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	8	$\times$
?	-	-	-	-	-	-	-	9	-	-	10
absorption	-	-	-	-	-	-	-	$\times$	-	-	-
entropy	11	-	-	-	-	-	-	$\times$	-	12	-

Table 2: Permutabilities in NL.

1	$\varnothing : A \parr B * B^\perp \otimes A^\perp$
2	$\varnothing : A \,\&\, B * (\perp \otimes \top \parallel B^\perp \parallel A^\perp)$
3	$\varnothing : A \,\&\, B * B^\perp \oplus A^\perp$
4	$A^\perp : (!A) \,\&\, A$
5	$\varnothing : (B^\perp \parallel A^\perp) * (A \otimes B) \,\&\, (A \odot B)$
6	$A^\perp : !A * \perp$
7	$\varnothing : ((A \parallel B^\perp \parallel D^\perp) < (C^\perp \parallel E^\perp)) * \perp$
8	$A^\perp : !(A \oplus B)$
9	$\varnothing : !A * ?A^\perp$
10	$\varnothing : ((A \parallel B^\perp \parallel D^\perp) < (C^\perp \parallel E^\perp)) * ?\mathbf{0}$
11	$\varnothing : (B^\perp \parallel A^\perp) * A \odot B$
12	$A : (B^\perp \parallel E^\perp) * (C^\perp < D^\perp)$

Table 3: Exceptions to permutabilities in NL.

8 Permutabilities in full NL

The original sequent calculus for NL is recalled in Appendix A. For the exponentials, there is a slight difference between our presentation and the sequent calculus on order varieties given in [14]: here, applications of weakening are grouped at the level of identities and the rule for $\mathbf{1}$, and contractions are systematically applied to the implicitely ?ed part of the sequent (Θ) in the $\otimes$ and $\odot$-rules; this enables us to get rid of explicit rules for weakening and contraction; on the other hand the *centre* rule of [14] becomes the *absorption* rule, where again, a contraction is applied.

Two rules R_1 and R_2 are said to be in a *situation of permutability* if there is a proof in which R_2 is applied just after R_1 and the conclusion of R_1 is not a premise of R_2. In that case, R_1 is said *impermutable* below R_2 if there is a sequent that can only be proved with R_2 below R_1; otherwise R_1 is said *permutable* below R_2.

Permutabilities in NL are summarized in Tables 2 and 3. The little bar $-$ means: "the rule of column R_1 can always permute below the rule of row R_2". The cross $\times$ means that R_1 and R_2 are not in a situation of permutability. A numeral in the table means impermutability of R_1 below R_2, and we exhibit some counter-examples in Table 3; most of them are taken or adapted from [6, 10]. For impermutability 7 (10 is similar), $A = (B \otimes C) \odot (D \otimes E)$ and the correct proof is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varnothing : B^\perp * B} \qquad \dfrac{}{\varnothing : C^\perp * C}}{\varnothing : B \otimes C * (B^\perp \parallel C^\perp)}\otimes \qquad \dfrac{\dfrac{}{\varnothing : D^\perp * D} \qquad \dfrac{}{\varnothing : E^\perp * E}}{\varnothing : D \otimes E * (D^\perp \parallel E^\perp)}\otimes}{\varnothing : A * ((B^\perp \parallel C^\perp) < (D^\perp \parallel E^\perp))}\odot}{\varnothing : (A \parallel B^\perp \parallel D^\perp) * (C^\perp \parallel E^\perp)}\text{entropy}}{\varnothing : ((A \parallel B^\perp \parallel D^\perp) < (C^\perp \parallel E^\perp)) * \perp}\perp.$$

The reason for the impermutability is that there is no series-parallel order τ on $A, B^\perp, C^\perp, D^\perp, E^\perp$ such that $\overline{\tau} = A * ((B^\perp \parallel C^\perp) < (D^\perp \parallel E^\perp))$ and $((A \parallel B^\perp \parallel D^\perp) < (C^\perp \parallel E^\perp)) \trianglelefteq \tau$, as noticed in [14], Section 4.2. For impermutability 12, $A = (E \odot D) \otimes (C \odot B)$ and the correct proof is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A : E^\perp * E} \qquad \dfrac{}{A : D^\perp * D}}{A : E \odot D * (D^\perp < E^\perp)}\odot \qquad \dfrac{\dfrac{}{A : C^\perp * C} \qquad \dfrac{}{A : B^\perp * B}}{A : C \odot B * (B^\perp < C^\perp)}\odot}{A : ((B^\perp < C^\perp) \parallel (D^\perp < E^\perp)) * A}\otimes}{A : \overline{(B^\perp < C^\perp) \parallel (D^\perp < E^\perp)}}\text{absorption}}{A : (B^\perp \parallel E^\perp) * (C^\perp < D^\perp)}\text{entropy.}$$

The reason for the impermutability here is similar: there is no series-parallel order τ on $B^\perp, C^\perp, D^\perp, E^\perp$ such that $\overline{\tau} = (B^\perp \parallel E^\perp) * (C^\perp < D^\perp)$ and $\tau \trianglelefteq (B^\perp < C^\perp) \parallel (D^\perp < E^\perp)$.

Let us now comment on the non-trivial permutations in Table 2. The $\otimes$-rule is permutable below $\triangledown$ because if the premise of a $\triangledown$-rule is the conclusion of a $\otimes$-rule, the two subformulas A and B of the formula $A \triangledown B$ introduced are necessarily in the same premise of $\otimes$, i.e. A and B are either both in $|\omega|$ or both in $|\tau|$:

$$\dfrac{\dfrac{\Theta : \omega * C \qquad \Theta : \tau * D}{\Theta : (\omega \parallel \tau)[A < B] * C \otimes D}\otimes}{\Theta : (\omega \parallel \tau)[A \triangledown B] * C \otimes D}\triangledown.$$

Permutation of entropy below $\triangledown$ holds because $\omega * (x < y) \subseteq \boldsymbol{\alpha}$ implies $\boldsymbol{\alpha}$ is of the form $\tau * (x < y)$ with $\omega \trianglelefteq \tau$. Permutation of entropy below $\parr$ holds by monotonicity of identification: $\omega * (x \parallel y) \subseteq \boldsymbol{\alpha}$ implies $\omega * z \subseteq \boldsymbol{\alpha}[z/x, y]$. Permutation of $\otimes$ below entropy holds because $\tau \trianglelefteq (\omega_1 \parallel \omega_2)$ implies τ is of the form $(\tau_1 \parallel \tau_2)$ with $\tau_i \trianglelefteq \omega_i, i = 1, 2$.

A relevant property of Table 2 is expressed by the following lemma.

Lemma 8.1 *Positive ($\odot, \otimes, \oplus, !$) and negative ($\triangledown, \parr, \&, \perp, ?$) rules are permutable below positive rules. All rules are permutable below absorption. All rules but $\odot$ and absorption are permutable below entropy.*

Now, there is an evident forgetful functor from NL to (commutative) LL mapping:

- a formula A of NL to a formula A° of LL, taking the connectives $\odot$ and ∇ respectively to $\otimes$ and $\parr$;
- a proof $\mathcal{D}$ of a sequent $(\Theta : \boldsymbol{\alpha})$ in the original calculus for NL to a proof $\mathcal{D}^\circ$ of the sequent $(\Theta^\circ : |\boldsymbol{\alpha}|^\circ)$ in LL, by forgetting the order variety and the entropy rule.

Table 2 implies that all relevant LL permutabilities are available in NL:

Lemma 8.2 *Let R_1, R_2 be two NL rules, with $(R_1, R_2) \notin \{\odot, absorption\} \times \{\bot, ?\}$, and R_1°, R_2° the corresponding LL rules. If R_1° is permutable below R_2° in LL, then R_1 is permutable below R_2 in NL and:*

$$\begin{cases} \textit{either entropy is permutable below } R_2 \textit{ in NL} \\ \textit{or } R_1 \textit{ is permutable below entropy in NL.} \end{cases}$$

As a consequence, if $(R_1, R_2) \notin \{\odot, absorption\} \times \{\bot, ?\}$ and $\mathcal{D}$ is an NL proof of an NL sequent $(\Theta : \boldsymbol{\alpha})$ such that $\mathcal{D}^\circ$ ends with R_1° above R_2° and R_1° is permutable below R_2° in LL, then $\mathcal{D}$ ends with:

$$\cfrac{\cfrac{\cfrac{\cdots}{\cdots}\, R_1}{\cdots}\, \textit{entropy}}{\Theta : \boldsymbol{\alpha}}\, R_2$$

and R_1 can be permuted below R_2 in $\mathcal{D}$.

Proof — LL permutations are recalled in Appendix B. The only assertion which is not straightforward is the one about entropy, and a simple inspection shows that the only 4 possibly problematic cases are precisely $\odot/\bot, \odot/?$, absorption/$\bot$ and absorption/?. $\square$

9 Focalized sequent calculus for full NL

A *sequent* is of one of the following forms:

- ▶ $(\Theta : \boldsymbol{\alpha})$, where Θ is a set of occurrences of formulas and $\boldsymbol{\alpha}$ an order variety of occurrences of formulas,
- ▶ $(\Theta : \omega \mid A)$, where Θ is a set of occurrences of formulas, ω a partial order of occurrences of formulas, and A is an occurrence of formula.

We omit the symbol $\vdash$ at the beginning of sequents, as it is useless in one-sided sequents. The sequent calculus is given in Table 4.

Theorem 9.1 (Adequacy) *To any proof of $(\Theta : \boldsymbol{\alpha})$ or $(\Theta : \boldsymbol{\alpha}_A \mid A)$ in the focalized sequent calculus, we associate a proof of $(\Theta : \boldsymbol{\alpha})$ in the original sequent calculus.*

Identity: a is a negative atom

$$\frac{}{\Theta : a \mid a^{\perp}} \qquad \frac{}{\Theta, a : \mid a^{\perp}}$$

Positive rules

$$\frac{\Theta : \Delta_Y(\omega) \mid A \qquad \Theta : \Delta_X(\omega) \mid B}{\Theta : \omega \mid A \odot B} \odot\text{, if } (X, Y) \text{ is admissible for } \omega$$

$$\frac{\Theta : \omega_1 \mid A \qquad \Theta : \omega_2 \mid B}{\Theta : (\omega_1 \parallel \omega_2) \mid A \otimes B} \otimes \qquad \frac{\Theta : \mid A}{\Theta : \mid !A} \,!$$

(no rule for $\mathbf{0}$) $$\frac{}{\Theta : \mid \mathbf{1}} \,\mathbf{1}$$

$$\frac{\Theta : \omega \mid A}{\Theta : \omega \mid A \oplus B} \oplus 1 \qquad \frac{\Theta : \omega \mid B}{\Theta : \omega \mid A \oplus B} \oplus 2$$

Negative rules

$$\frac{\Theta : \omega * (A < B)}{\Theta : \omega * A \triangledown B} \triangledown \qquad \frac{\Theta : \omega * (A \parallel B)}{\Theta : \omega * A \parr B} \parr \qquad \frac{\Theta, A : \overline{\omega}}{\Theta : \omega * ?A} \,?$$

$$\frac{\Theta : \overline{\omega}}{\Theta : \omega * \bot} \bot \qquad \frac{}{\Theta : \omega * \top} \top \qquad \frac{\Theta : \omega * A \qquad \Theta : \omega * B}{\Theta : \omega * A \,\&\, B} \,\&$$

Reaction: N is negative

$$\frac{\Theta : \omega * N}{\Theta : \omega \mid N} \text{reaction}$$

Decision: P is positive and ω is a series-parallel order containing only atoms or positive formulas

$$\frac{\Theta : \omega \mid P}{\Theta : \omega * P} \text{decision 1} \qquad \frac{\Theta, A : \omega \mid A}{\Theta, A : \overline{\omega}} \text{decision 2}$$

Table 4: Focalized sequent calculus for full NL.

PROOF — Forget focalization (hence reaction and decision 1), and add entropy to the $\odot$-rule ($\omega \trianglelefteq \Delta_Y(\omega) < \Delta_X(\omega)$):

$$\dfrac{\dfrac{\Theta : \Delta_Y(\omega) * A \qquad \Theta : \Delta_X(\omega) * B}{\Theta : \Delta_Y(\omega) < \Delta_X(\omega) * A \odot B}\,\odot}{\Theta : \omega * A \odot B}\,\text{entropy.}$$

Decision 2 becomes absorption. □

Theorem 9.2 (Completeness) *To any proof of $(\Theta : \boldsymbol{\alpha})$ in the original sequent calculus, we associate a proof in the focalized sequent calculus by distinguishing between two cases:*

- *If $|\boldsymbol{\alpha}|$ contains non-atomic negative formulas, then the focalized proof has conclusion $(\Theta : \boldsymbol{\alpha})$.*
- *Otherwise the focalized proof has conclusion $(\Theta : \boldsymbol{\alpha}_P \mid P)$ for some positive formula P in $|\boldsymbol{\alpha}|$.*

PROOF — The functor from NL to commutative LL of Section 8 can be extended to the focalized case so as to map a focalized NL proof of $(\Theta : \boldsymbol{\alpha})$ or $(\Theta : \boldsymbol{\alpha}_P \mid P)$ to a focalized LL proof of $(\Theta^\circ : |\boldsymbol{\alpha}|^\circ)$ or $(\Theta^\circ : |\boldsymbol{\alpha}_P|^\circ \mid P^\circ)$, by simply forgetting the order variety. By [2], commutative LL proofs can be focalized, by permuting logical steps and adding a decision or a reaction between groups of rules of the same polarity, so the situation can be summarized in the following diagram:

$$\begin{tikzcd} \text{NL} \arrow[r, dotted] \arrow[d, "(-)^\circ"'] & \text{focalized NL} \arrow[d, "(-)^\circ"] \\ \text{LL} \arrow[r] & \text{focalized LL} \end{tikzcd}$$

and we want to construct the top dotted arrow. Let $\mathcal{D}$ be an NL proof. We proceed by induction on the number of permutations applied in order to focalize $\mathcal{D}^\circ$ (bottom arrow). By Lemma 8.2, each LL permutation is simulated by one or two NL permutations, except the LL permutations $\otimes/\bot, \otimes/?,$ absorption$/\bot$ and absorption$/?$, but these two cases never occur during focalization because, more generally, focalization never uses permutations of rules *below negative rules.* So we get an NL proof $\mathcal{D}'$ whose tranlation $(\mathcal{D}')^\circ$ is the focalized LL proof $(\mathcal{D}^\circ)'$ obtained from $\mathcal{D}^\circ$, and now we just have to deal with entropy: by Lemma 8.1, entropy is permutable above all NL rules but $\odot$, so we get an NL proof where entropy is concentrated below $\odot$-rules. But we can do better and move up all the entropy which bears on each premise of each $\odot$-rule separately. Indeed given

$$\dfrac{\dfrac{\Theta : \omega * A \qquad \Theta : \omega' * B}{\Theta : (\omega' < \omega) * A \odot B}\,\odot}{\Theta : \tau * A \odot B}\,\text{entropy,}$$

let $Y = |\omega|$ and $X = |\omega'|$: we have $\tau \trianglelefteq (\omega' < \omega)$, so by Theorem 6.9, we may infer $\Delta_X(\tau) \trianglelefteq \omega'$ and $\Delta_Y(\tau) \trianglelefteq \omega$; therefore we may rewrite the above piece of proof as:

$$\dfrac{\dfrac{\Theta : \omega * A}{\Theta : \Delta_Y(\tau) * A}\,\text{entropy} \qquad \dfrac{\Theta : \omega' * B}{\Theta : \Delta_X(\tau) * B}\,\text{entropy}}{\Theta : \tau * A \odot B}\,\odot + \text{entropy},$$

and by permuting entropy up in the proof as above, we reach a proof $\mathcal{D}''$ where the only entropy applied is the one contained in the focalized $\odot$-rule of Table 4. Its translation is still $(\mathcal{D}')^\circ = (\mathcal{D}^\circ)'$, hence it is the required focalized proof. □

References

[1] V. M. Abrusci and P. Ruet. Non-commutative logic I: the multiplicative fragment. *Annals Pure Appl. Logic*, 101(1):29–64, 2000.

[2] J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *J. Logic and Comp.*, 2(3), 1992.

[3] J.-M. Andreoli and R. Maieli. Focusing and proof nets in linear and non-commutative logic. In *Proc. Int. Conf. on Logic for Programming and Automated Reasoning*. Springer LNAI 1581, 1999.

[4] J.-M. Andreoli and R. Pareschi. Linear objects: logical processes with built-in inheritance. *New Generation Computing*, 9, 1991.

[5] C. Faggian. Proof construction and non-commutativity: a cluster calculus. In *Proc. Int. Conf. on Principles and Practice of Declarative Programming*. ACM Press, 2000.

[6] D. Galmiche and G. Perrier. On proof normalization in linear logic. *Theoret. Comp. Sci.*, 135(1):67–110, 1994.

[7] J.-Y. Girard. Linear logic. *Theoret. Comp. Sci.*, 50(1):1–102, 1987.

[8] J.-Y. Girard. Locus solum. *Math. Struct. in Comp. Sci.*, 11(3), 2001.

[9] J. Lambek. The mathematics of sentence structure. *Amer. Math. Monthly*, 65(3):154–170, 1958.

[10] P. Lincoln and N. Shankar. Proof search in first-order linear logic and other cut-free sequent calculi. In *Proc. IEEE Int. Conf. on Logic in Computer Science*, pages 282–291. IEEE, 1994.

[11] R. Maieli. *Focalizzazione delle prove in logica non-commutativa.* PhD thesis, Università Roma 3, 2000.

[12] D. Miller, G. Nadathur, F. Pfenning, and A. Scedrov. Uniform proofs as a foundation for logic programming. *Annals Pure Appl. Logic*, 51:125–157, 1991.

[13] R. Mohring. *Computationally tractable classes of ordered sets.* ASI Series 222. NATO, 1989.

[14] P. Ruet. Non-commutative logic II: sequent calculus and phase semantics. *Math. Struct. in Comp. Sci.*, 10(2):277–312, 2000.

[15] D.N. Yetter. Quantales and (non-commutative) linear logic. *J. Symb. Logic*, 55(1), 1990.

A Appendix: Original sequent calculus for non-commutative logic

We recall the sequent calculus for NL introduced in [14]. Sequents $(\Theta:\boldsymbol{\alpha})$ consist of an order variety $\boldsymbol{\alpha}$ of formula occurrences, and a set Θ of formula occurrences (with no additional structure). Θ is disjoint from $|\boldsymbol{\alpha}|$. In the following sequent calculus, all order varieties and orders are assumed series-parallel. In particular, in dereliction and the $\perp$-rule, ω is series-parallel, and in the entropy rule, $\boldsymbol{\alpha}$ is series-parallel.

Identity

$$\frac{}{\Theta : A^{\perp} * A}$$

Structural rules

$$\frac{\Theta, A : \omega * A}{\Theta, A : \overline{\omega}}\text{ absorption} \qquad \frac{\Theta : \boldsymbol{\beta}}{\Theta : \boldsymbol{\alpha}}\text{ entropy, } \boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$$

Multiplicatives

$$\frac{\Theta : \omega * A \qquad \Theta : \omega' * B}{\Theta : (\omega' < \omega) * A \odot B}\odot \qquad \frac{\Theta : \omega * (A < B)}{\Theta : \omega * A \triangledown B}\triangledown$$

$$\frac{\Theta : \omega * A \qquad \Theta : \omega' * B}{\Theta : (\omega \,\|\, \omega') * A \otimes B}\otimes \qquad \frac{\Theta : \omega * (A \,\|\, B)}{\Theta : \omega * A \parr B}\parr$$

Additives

$$\frac{\Theta : \omega * A \qquad \Theta : \omega * B}{\Theta : \omega * A \,\&\, B}\,\& \qquad \frac{\Theta : \omega * A}{\Theta : \omega * A \oplus B}\oplus 1 \qquad \frac{\Theta : \omega * B}{\Theta : \omega * A \oplus B}\oplus 2$$

Exponentials

$$\frac{\Theta, A : \overline{\omega}}{\Theta : \omega * ?A}\,? \qquad \frac{\Theta : A}{\Theta : !A}\,!$$

Constants

$$\frac{}{\Theta : \mathbf{1}} \qquad \frac{\Theta : \overline{\omega}}{\Theta : \omega * \perp}\perp \qquad \text{(no rule for } \mathbf{0}\text{)} \qquad \frac{}{\Theta : \omega * \top}$$

B Appendix: permutabilities in linear logic

The conventions are the same as in Section 8. The numeral 0 in the table means impermutability. References are [6, 10].

$R_2 \backslash R_1$	$\otimes$	$\oplus$	$⅋$	$\&$	$\perp$	!	?	abs
$\otimes$	-	-	-	-	-	$\times$	-	-
$\oplus$	-	-	-	-	-	$\times$	-	-
$⅋$	0	-	-	-	-	$\times$	-	-
$\&$	0	0	-	-	-	$\times$	-	0
$\perp$	-	-	-	-	-	0	-	-
!	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0
?	-	-	-	-	-	0	-	-
absorption	-	-	-	-	-	$\times$	-	-