Self-adjoint negation

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Abstract

If X is a cartesian closed category with an initial object, the following three conditions are equivalent: (1) involutiveness of the negation functor $\neg$ up to isomorphism, (2) self-adjointness of $\neg$, (3) existence of a natural transformation from $\neg\neg$ to id_X. This strengthens a result of Joyal saying that condition (1) makes X a preorder.

If X is a cartesian closed category with an initial object $\perp$, $\neg$ denotes the functor, called negation, from X to X^{op} (or from X^{op} to X) with object function given by $\neg x = \perp^x$.

It is a well-known result of Joyal that in such a category X, there is at most one arrow from any given object x to $\perp$: see, e.g., Proposition 8.3 in [1]. As an immediate consequence, if the canonical natural transformation from id_X to $\neg\neg$ is a natural isomorphism, then X is a preorder. This is why the straightforward extension of intuitionistic denotational semantics to classical logic identifies all proofs of a given formula. However, the requirement to be a cartesian closed category with an initial object is just a matter of adjoints, so instead of asking negation to be involutive (up to isomorphism), it is more natural to ask it to be *self-adjoint*. The functor $\neg : X \to X^{\mathrm{op}}$ is always left-adjoint to $\neg : X^{\mathrm{op}} \to X$: indeed, $X^{\mathrm{op}}(\neg x, y) = X(y, \neg x) \cong X(y \times x, \perp) \cong X(x \times y, \perp) \cong X(x, \neg y)$ and all bijections are natural in x and y. The opposite adjunction, namely that $\neg : X^{\mathrm{op}} \to X$ is left-adjoint to $\neg : X \to X^{\mathrm{op}}$

$$\begin{tikzcd} X^{\mathrm{op}} \arrow[r, bend left, "\neg"] & X \arrow[l, bend left, "\neg"] \end{tikzcd}$$

need not hold in general, and is *a priori* a weaker condition; the following lemma shows that this adjunction is actually equivalent to involutiveness, and is even equivalent to the bare existence of a natural transformation from $\neg\neg$ to id_X.

Lemma 1 *Let X be a cartesian closed category with an initial object $\perp$. The following are equivalent:*

1. *The canonical natural transformation from id_X to $\neg\neg$ is a natural isomorphism.*
2. *The functor $\neg : X^{\mathrm{op}} \to X$ is left-adjoint to $\neg : X \to X^{\mathrm{op}}$.*

3. There is a natural transformation from $\neg\neg$ *to* id_X*.*

As a consequence, either of these conditions implies that X *is a preorder.*

In the posetal case, this implies that a Heyting algebra H is Boolean if, and only if, $\neg : H^{\mathrm{op}} \to H$ is left-adjoint to $\neg : H \to H^{\mathrm{op}}$. Observe that this statement differs from the remark after Proposition I.8.4 in [2].

Proof — Obviously, condition 1 implies condition 2. To show that condition 2 implies condition 1, observe that the terminal object $\top$ is isomorphic to $\neg\bot$: indeed, the image of the second projection $\top \times \bot \to \bot$ under the bijection $X(x \times y, z) \cong X(x, z^y)$ is an arrow from $\top$ to $\neg\bot$; the composite $\top \to \neg\bot \to \top$ has to be the unique arrow $\mathrm{id}_\top$ from $\top$ to $\top$, and the composite $\neg\bot \to \top \to \neg\bot$ is $\mathrm{id}_{\neg\bot}$ as an arrow to an objet $\neg x$. As a consequence,

$$X(x,y) \cong X(\top \times x, y) \cong X(\top, y^x) \cong X(\neg\bot, y^x).$$

Now, the adjunction from $\neg : X^{\mathrm{op}} \to X$ to $\neg : X \to X^{\mathrm{op}}$ gives $X(\neg\bot, y^x) \cong X(\neg(y^x), \bot)$, so $X(x,y) \cong X(\neg(y^x), \bot)$ and $X(x,y)$ has cardinality at most one. In particular, $X(x,x) = \{\mathrm{id}_x\}$, so the composite

$$x \xrightarrow{!} \neg\neg x \xrightarrow{\epsilon_x} x$$

with $\epsilon : \neg\neg \dot{\to} \mathrm{id}_X$ the counit of the given adjunction, is the identity. Since the other composite $\neg\neg x \to x \to \neg\neg x$ is the unique arrow $\mathrm{id}_{\neg\neg x} : \neg\neg x \to \neg\neg x$, we conclude that the two natural transformations are inverses of each other.

To show that condition 2 is equivalent to condition 3, recall that an adjunction from $\neg : X^{\mathrm{op}} \to X$ to $\neg : X \to X^{\mathrm{op}}$ amounts to natural transformations ϵ from $\neg\neg$ to id_X and η from $\mathrm{id}_{X^{\mathrm{op}}}$ to $\neg\neg$ (i.e., η from $\neg\neg$ to id_X) such that $\neg\epsilon \circ \eta_\neg$ is the identity natural transformation on $\neg : X \to X^{\mathrm{op}}$ and $\epsilon_\neg \circ \neg\eta$ is the identity on $\neg : X^{\mathrm{op}} \to X$. The last two equations are always satisfied because hom-sets to an objet $\neg x$ have cardinality at most one. Hence, such an adjunction simply amounts to a pair of natural transformations ϵ and η from $\neg\neg$ to id_X. Moreover, ϵ and η have to be equal because $X(\neg\neg x, x) \cong X(\neg x, \neg x) \cong X(\neg x \times x, \bot)$ has cardinality at most one. As a consequence, an adjunction from $\neg : X^{\mathrm{op}} \to X$ to $\neg : X \to X^{\mathrm{op}}$ just amounts to one natural transformation from $\neg\neg$ to id_X. $\square$

References

[1] J. Lambek and P. J. Scott. *Introduction to higher-order categorical logic.* Cambridge Stu. in Adv. Math. Cambridge University Press, 1986.

[2] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory.* Springer, 1992.