ADMISSIBLE AND DERIVABLE RULES IN INTUITIONISTIC LOGIC

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1 Introduction

A well-known problem in intuitionistic logic is the existence of valid but not derivable rules. This problem seems to be related with some constructive features of intuitionism (disjunction and existence property) but appear also in modal logics. We study here a particular case of this phenomenon, admissible rules in propositional calculus. G.E.Mints in [Mi 72] give sufficients conditions for admissible rules to be derivable. H.Friedman in [Fr 75] states the problem of the decidability of admissibility and V.V.Rybakov solves it in [Ry 84, Ry 86] using semantical and algebraic methods and the Gödel translation of intuitionistic logic into modal logic. We proposed here another approach to the problem closed to [Mi 72] but pointing out a particular class of substitutions useful when dealing with admissibility (section 2). We make use of these substitutions in this paper to extend the results of G.E.Mints.

This approach leads to results we do not expose here. They are essentially a decidable characterization of admissibility using the intuitionistic sequent of calculus (precisely a kind of "retro-derivation" in sequent calculus), see [Ro 91].

We began this work by studying W.Dekkers conference notes. He rediscovered independently result from Mints in a particular case (admissibility and derivability are the same in the "→" fragment). These results are part of my Ph.D.Thesis supervised by M.Parigot.

1.1   Notations

Greek letters stand for propositional variables. The connectives are $\vee, \wedge, \rightarrow$ and $\perp$ is a propositional constant for false. $\neg A$ stands for $A \rightarrow \perp$. A primitive negation and false defined as $\perp = A \wedge \neg A$ lead to the same results.

$A \rightarrow B \rightarrow C$ or $A, B, \rightarrow C$ stand for $A \rightarrow (B \rightarrow C)$.

If $\Gamma = A_1, \ldots, A_n$ is a finite set of formulae, $\Gamma \rightarrow C$ stands for $A_1, \ldots, A_n, \rightarrow C$, ($C$ if $\Gamma = \emptyset$), $\wedge\Gamma$ stands for $A_1 \wedge \ldots \wedge A_n$ ($\top$ if $\Gamma = \emptyset$), $A \wedge \Gamma$ stands for $A \wedge A_1 \wedge \ldots \wedge A_n$ (A if $\Gamma = \emptyset$). Though it is ambiguous, it is harmless here since changing the order of formulae A_i gives intuitionistically equivalents formulae.

We write $\vdash$ for the intuitionistic deduction relation, $A \dashv\vdash B$ stands for $A \vdash B$ and $B \vdash A$.

1.2   Definitions

Definition 1.2.1 *The rule:*

$$\frac{\vdash_{\mathcal{L}} A_1 \ldots \vdash_{\mathcal{L}} A_n}{\vdash_{\mathcal{L}} C},$$

is said an admissible rule in $\mathcal{L}$ and written down:

$$A_1, \ldots, A_n \gg C,$$

iff the set of theorems of $\mathcal{L}$ is closed under this rule, or equivalently
iff for every substitution s of propositional formulae for propositional constants:

$$\text{if } \vdash_{\mathcal{L}} s(A_1), \ldots, \vdash_{\mathcal{L}} s(A_n), \text{ then } \vdash_{\mathcal{L}} s(C).$$

This rule is said to be a derivable rule in $\mathcal{L}$ iff:

$$\vdash_{\mathcal{L}} A_1, \ldots, A_n, \rightarrow C.$$

The following proposition is clearly true.

Proposition 1.2.1 *Every derivable rule is admissible.*

In classical propositional calculus the converse, i.e. every admissible rule is derivable, is provable (completeness). On the other hand there are well-known admissible rules which are not derivable rules in intuitionistic calculus, for instance:

$$\neg\alpha \rightarrow \beta \vee \gamma \gg (\neg\alpha \rightarrow \beta) \vee (\neg\alpha \rightarrow \gamma)$$

$$\neg\alpha \rightarrow \beta \vee \gamma \nvdash (\neg\alpha \rightarrow \beta) \vee (\neg\alpha \rightarrow \gamma) .$$

In this paper we deal with formulae or finite sets of formulae Γ which have *the same admissible and derivable consequences,* i.e. $\Gamma \gg C$ *iff* $\Gamma \vdash C$.

G.E.Mints showed in [Mi 72] that the intuitionistic "$\wedge, \vee, \neg$" and "$\wedge, \rightarrow, \neg$" fragments make no difference between admissibility and derivability. The problem of decidability of admissible rules in intuitionistic propositional calculus was pointed out by H.Friedman in [Fr 75] and solved by V.V.Rybakov by semantical and algebraic methods (see [Ry 84, Ry 86]).

1.3 Preliminaries

We make use of a version of the **Glivenko theorem**, a negated formula is intuitionistically provable if and only if it is classically provable. We also make use of the **disjunction property in intuitionistic logic**, $\vdash C \vee D \;\; iff \;\; \vdash C \; or \; \vdash D$. (see for instance [Tr 88] for the proofs).

From these two theorems we get the two items of the following lemma.

Lemma 1.3.1

- $\Gamma \gg \bot \;\; iff \;\; \Gamma \vdash_c \bot \;\; iff \;\; \Gamma \vdash \bot$;

- $\Gamma, B \vee B' \gg C \;\; iff \;\; \Gamma, B \gg C \; and \; \Gamma, B' \gg C$. *Hence the set of formulae which have the same admissible and derivable consequences is closed under disjunctions.*

2 Substitutions

We describe here particular substitutions which are useful when dealing with admissibility.

2.1 Definitions

Definition 2.1.1 *If Γ is a finite set of formulae, the substitution s will be said a Γ-identity iff for every propositional variable α:*

$$\Gamma \vdash \alpha \leftrightarrow s(\alpha) \; ,$$

or equivalently:

$$\Gamma \rightarrow \alpha \dashv\vdash \Gamma \rightarrow s(\alpha).$$

We say that a substitution s is Γ-admissible iff for every formula C in Γ the formula $s(C)$ is provable.

The main interest of these definitions is that the following lemma holds.

Lemma 2.1.1 *Let Γ be a finite set of formulae such that there exists a Γ-identity which is Γ-admissible (it will be said an admissible Γ-identity), then:*

3

(i) *if* $\Gamma \gg C \vee D$ *then* $\Gamma \vdash C$ *or* $\Gamma \vdash D$, *and in this case we will say that* Γ *has the disjunction property for admissibility.*

(ii) *if there exists an admissible* Γ-*identity, then* Γ *has the same admissible and derivable consequences.*

Proof of (i): let s be the admissible Γ-identity which is given by the hypothesis . We know that:

$$\Gamma \gg C \vee D \ \ and \ \ \vdash \wedge s(\Gamma) .$$

The definition of admissibility and the disjunction property of intuitionistic logic lead to:

$$\vdash s(C) \ or \ \vdash s(D) .$$

Then by weakening we obtain:

$$\vdash \Gamma \rightarrow s(C) \ or \ \vdash \Gamma \rightarrow s(D) .$$

Hence the definition of an Γ-identity yields:

$$\vdash \Gamma \rightarrow C \ or \ \vdash \Gamma \rightarrow D .$$

Then (ii) follows from (i) with $C = D$. ∎

Remark 1: we prove the converse of (i) in [Ro 91] (the proof is too lengthy to be given here) and the following property of Γ-identity which also relates to admissibility. Let s be an Γ-identity, then $\Gamma \gg C \ \ iff \ \ s(\Gamma) \gg s(C)$ (the proof is straightforward). This paper makes no use of these two results.

Remark 2: The disjunction property for admissibility obviously implies the disjunction property of the same formula (or finite set of formulae).

The previous lemma will be applied in the following section.

3   Principal results

3.1   Harrop and anti-Harrop formulae

Harrop formulae (Rasiowa-Harrop) were introduced also in Heyting arithmetic. They have the disjunction property and can easily be syntactically characterized. We recall their definition for the restricted case of propositional calculus.

Definition 3.1.1 *The class* **H** *of Harrop formulae is defined as the class of formulae without* $\vee$ *in their strictly positive part. We inductively define this class:*

- *every atomic formula (propositional variable or* $\perp$ *) is in* **H** *;*

4

- *if A is any formula and $B \in \mathbf{H}$ then $A \to B \in \mathbf{H}$;*

- *if $A \in \mathbf{H}$ and $B \in \mathbf{H}$ then $A \wedge B \in \mathbf{H}$.*

Lemma 3.1.1 *Every Harrop formula is equivalent to a conjunction of formulae:*

$$A_1 \to \ldots \to A_n \to \alpha$$

where α is atomic. We call these formulae primitive Harrop formulae.

Proof: straightforward induction on the definition of $\mathbf{H}$. ∎
Remark: formulae of the "$\wedge, \to, \bot$" fragment are Harrop formulae.
By analogy we call *anti-Harrop formulae* formulae which have at least a propositional variable in strictly negative part.

Definition 3.1.2 *We inductively define the class $\mathbf{aH}$ of anti-Harrop formulae:*

- *If α is any propositional variable and A any formula then $\alpha \to A \in \mathbf{aH}$;*

- *if A is any formula and $B \in \mathbf{aH}$ then $A \to B \in \mathbf{aH}$;*

- *if $A \in \mathbf{aH}$ and $B \in \mathbf{aH}$ then $A \wedge B \in \mathbf{aH}$.*

Lemma 3.1.2 *Every anti-Harrop formula is equivalent to a conjunction of formulae:*

$$\alpha \to B \ ,$$

where α is a propositional variable.
We call these formulae primitive anti-Harrop formulae.

Proof: straightforward induction on the definition of $\mathbf{aH}$. ∎

3.2   Some Γ-identities

Lemma 3.2.1 *Let Γ be a finite set of formulae. Let s be the substitution defined by $s(\alpha) = \Gamma \wedge \alpha$.*
Hence s is a Γ-identity, and for every anti-Harrop formula C :

$$s(C) \dashv\vdash \Gamma \to C \ .$$

In particular if Γ is a set of anti-Harrop formulae then s is an admissible Γ-identity.

Proof: it is obvious that s is a Γ-identity . In the proof of the second result the lemma 3.1.2 and the statement $(\top \wedge \top)$, proof of the lemma below, allow us to limit ourselves to formulae $C = \alpha \to B$ where α is a propositional variable. Therefore:

$$s(C) = s(\alpha) \to s(B) \dashv\vdash (\Gamma \wedge \alpha) \to s(B) \dashv\vdash \alpha \to \Gamma \to s(B) ,$$

and as s is a Γ-identity we obtain:

$$s(C) \dashv\vdash \alpha \to \Gamma \to B \dashv\vdash \Gamma \to \alpha \to B = \Gamma \to C . \blacksquare$$

Lemma 3.2.2 *Let Γ be a finite set of formulae. let v be a classical valuation and s_v defined by:*

- *if $v(\alpha) = \top$ then $s_v(\alpha) = \Gamma \to \alpha$;*

- *if $v(\alpha) = \bot$ then $s_v(\alpha) = \neg\neg\Gamma \wedge (\Gamma \to \alpha)$.*

Hence s_v is a Γ-identity and for every Harrop formula C :

- *if $v(C) = \top$ then $s_v(C) \dashv\vdash \Gamma \to C$;*

- *if $v(C) = \bot$ then $s_v(C) \dashv\vdash \neg\neg\Gamma \wedge (\Gamma \to C)$.*

In particular if Γ is a set of Harrop formulae and v a classical valuation such that $v(\wedge\Gamma) = \top$ then s_v is an admissible Γ-identity.

Proof: it is obvious that s_v is a Γ-identity. We prove the second part of the conclusion in two steps. We first prove it for formulae in the "$\wedge, \to, \bot$" fragment by induction on the structure of such formulae. The result is obvious for propositional variables. For $\bot$ it is easy to prove that $\neg\neg\Gamma \wedge (\Gamma \to \bot) \dashv\vdash \bot$. In the induction steps we use the following equivalences.

$$
\begin{array}{lll}
(\top \to \top) & : & (\Gamma \to A) \to (\Gamma \to B) \dashv\vdash \Gamma \to (A \to B) ; \\
(\bot \to \top) & : & [\neg\neg\Gamma \wedge (\Gamma \to A)] \to (\Gamma \to B) \dashv\vdash \Gamma \to (A \to B) ; \\
(\bot \to \bot) & : & [\neg\neg\Gamma \wedge (\Gamma \to A)] \to [\neg\neg\Gamma \wedge (\Gamma \to B)] \dashv\vdash \Gamma \to (A \to B) ; \\
(\top \to \bot) & : & (\Gamma \to A) \to [\neg\neg\Gamma \wedge (\Gamma \to B)] \dashv\vdash \neg\neg\Gamma \wedge [\Gamma \to (A \to B)] ; \\
(\top \wedge \top) & : & (\Gamma \to A) \wedge (\Gamma \to B) \dashv\vdash \Gamma \to (A \wedge B) , \\
(\bot \wedge \top) & : & [\neg\neg\Gamma \wedge (\Gamma \to A)] \wedge (\Gamma \to B) \dashv\vdash \neg\neg\Gamma \wedge [\Gamma \to (A \wedge B)] ; \\
(\bot \wedge \bot) & : & [\neg\neg\Gamma \wedge (\Gamma \to A)] \wedge [\neg\neg\Gamma \wedge (\Gamma \to B)] \dashv\vdash \neg\neg\Gamma \wedge [\Gamma \to (A \wedge B)] .
\end{array}
$$

Remark: the third equivalence $(\top \to \bot)$ is the only one that necessarily use intuitionistic and not minimal calculus. So we have to prove:

$$(\Gamma \to A) \to \neg\neg\Gamma \dashv\vdash \neg\neg\Gamma .$$

The conclusion also holds for a negated formula "$A \to \bot$" which is equivalent to a formula in the "$\wedge, \to, \bot$" fragment. This follow for instance by use of Glivenko theorem.

Henceforward we deal with all Harrop formulae. The lemma 3.1.1 and the previous equivalences $(\top \wedge \top)$, $(\top \wedge \bot)$ and $(\bot \wedge \bot)$ allow us to limit ourselves to primitive Harrop formulae $C = B_1 \to \ldots \to B_n \to \alpha = \Delta \to \alpha$ where α is atomic and $\Delta = \{B_1, \ldots, B_n\}$.

- We first assume that $v(\alpha) = \top$ and then $v(C) = \top$. Therefore $s_v(\alpha) = \Gamma \to \alpha$ and:

$$
\begin{aligned}
s_v(C) \;&=\; s_v(\Delta \to \alpha) \\
&\dashv\vdash\; s_v(\Delta) \to \Gamma \to \alpha \\
&\dashv\vdash\; \Gamma \to s_v(\Delta) \to \alpha \,,
\end{aligned}
$$

Observe that s_v is an Γ-identity and infer:

$$
s_v(C) \dashv\vdash \Gamma \to \Delta \to \alpha = \Gamma \to C \,.
$$

- We henceforth assume that $v(\alpha) = \bot$. Therefore $s_v(\alpha) \dashv\vdash \neg\neg\Gamma \wedge (\Gamma \to \alpha)$ (equality for α a propositional variable, easy equivalence for $\alpha = \bot$). We then deduce the following:

$$
\begin{aligned}
s_v(C) \;&=\; s_v(\Delta \to \alpha) \\
&\dashv\vdash\; s_v(\Delta) \to [\neg\neg\Gamma \wedge (\Gamma \to \alpha)] \\
&\dashv\vdash\; [s_v(\Delta) \to \neg\neg\Gamma] \wedge [s_v(\Delta) \to \Gamma \to \alpha] \,. \quad (\star)
\end{aligned}
$$

Consider the second part of the previous conjunction $(\star)$. As in the previous case we deduce that:

$$
[s_v(\Delta) \to \Gamma \to \alpha] \dashv\vdash \Gamma \to C \,,
$$

since s_v is a Γ-identity.

Consider the first part of the conjunction $(\star)$. From the equivalence:

$$
A \to \neg B \dashv\vdash \neg\neg A \to \neg B \,,
$$

we get:

$$
\begin{aligned}
s_v(\Delta) \to \neg\neg\Gamma \;&\dashv\vdash\; \neg\neg s_v(\Delta) \to \neg\neg\Gamma \\
&\dashv\vdash\; s_v(\neg\neg\Delta) \to \neg\neg\Gamma \,.
\end{aligned}
$$

Then observe that $\neg\neg\Delta$ is a negated formula. We have already proved the conclusion in this case i.e. $(\neg\neg\Delta \dashv\vdash \neg\neg \wedge \Delta \dashv\vdash_c \Delta)$:

- if $v(\wedge\Delta) = \top$ then $s_v(\neg\neg\Delta) \dashv\vdash \Gamma \to \neg\neg\Delta$,

- if $v(\wedge\Delta) = \bot$ then $s_v(\neg\neg\Delta) \dashv\vdash \neg\neg\Gamma \wedge (\Gamma \to \neg\neg\Delta)$.

Assume that $v(\wedge\Delta) = \top$. Reminding that $v(\alpha) = \bot$ and $C = \Delta \to \alpha$ we get $v(C) = \bot$. So:

$$s_v(\Delta) \to \neg\neg\Gamma \;\;\dashv\vdash\;\; (\Gamma \to \neg\neg\Delta) \to \neg\neg\Gamma$$
$$\dashv\vdash\;\; (\Gamma \to \neg\neg\Delta), (\Gamma \to \bot), \to \bot$$
$$\dashv\vdash\;\; \neg\neg\Gamma .$$

Applying this to $(\star)$ we obtain the result we expected for $v(C) = \bot$, i.e.

$$s_v(C) \dashv\vdash \neg\neg\Gamma \wedge \Gamma \to C ,$$

Assume $v(\wedge\Delta) = \bot$. From $v(\alpha) = \bot$ and $C = \Delta \to \alpha$ we get $v(C) = \top$, hence:

$$s_v(\Delta) \to \neg\neg\Gamma \;\;\dashv\vdash\;\; \neg\neg\Gamma \wedge (\Gamma \to \wedge\Delta) \to \neg\neg\Gamma$$
$$\dashv\vdash\;\; \top .$$

Applying this to $(\star)$ we obtained the result we expected for $v(C) = \top$ i.e.

$$s_v(C) \dashv\vdash \Gamma \to C . \;\blacksquare$$

Proposition 3.2.1

- *Harrop and anti-Harrop formulae have the disjunction property for admissibility.*

- *Harrop and anti-Harrop formulae and disjunction of such formulae have the same admissible and derivable consequences.*

Proof of the first item: lemmas 2.1.1, 3.2.1, 3.2.2 for anti-Harrop and Harrop formulae which are not classical antilogies. For a classical antilogy G using lemma 1.3.1 leads to $G \vdash \bot$ and so for every formula C $G \vdash C$ *and* $G \gg C$. $\blacksquare$
Proof of the second item: second item of the lemma 1.3.1.
Remark: a finite set or a conjunction of both Harrop and anti-Harrop formulae do not generally have the same admissible and derivable consequences. For instance:

$$\neg\alpha \to \delta, \delta \to \beta \vee \gamma \gg (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma) \;;$$

$$\neg\alpha \to \delta, \delta \to \beta \vee \gamma \nvdash (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma) .$$

3.3 Fragments "$\wedge, \rightarrow, \bot$" or "$\wedge, \vee, \neg$"

The "$\wedge, \rightarrow, \bot$" fragment is closed under the substitutions s_v defined in lemma 3.2.2. The result of Mints [Mi 72], stating that in the "$\wedge, \rightarrow, \bot$" fragment every admissible rule is derivable, is then a consequence of the previous results. The result is still true in the "$\rightarrow$" and "$\wedge, \rightarrow$" fragments with the constant valuation $v = \top$. A direct proof is much simpler.

Formulae without "$\rightarrow$" are clearly equivalent to disjunctions of conjunctions of negated formulae and propositional variables. These conjunctions are particular Harrop formulae and the proposition 3.2.1 can be applied. Hence formulae of the "$\wedge, \vee, \neg$" fragment have the same admissible and derivable consequences. But this method is not very satisfying as the result make use of an admissible Γ-identity with the "$\rightarrow$" connective. Notice that "inside" a negated formula the classical equivalences are provable. Hence it's possible to use instead of the substitution s_v of the lemma 3.2.2 the substitution s'_v defined for a classical valuation v as:

- *if $v(\alpha) = \top$ then $s'_v(\alpha) = \neg\neg(\neg\gamma \vee \alpha)$,*

- *if $v(\alpha) = \bot$ then $s'_v(\alpha) = \gamma \wedge \alpha$.*

This gives the expected result about this fragment, in particular the result of Mints ([Mi 72]).

4 Conclusion

The method which we applied here to particular cases is general. In [Ro 91] we prove that a formula G has the disjunction property for admissibility iff there exists an admissible G-identity. On the other hand we prove that G has the same admissible and derivable consequences iff G is equivalent to a disjunction of formulae which have the disjunction property for admissibility. These criteria are decidable.

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