Admissible rules and backward derivation in intuitionistic logic

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This paper is essentially a translation, with some adaptations to make it self-contained, of the second part of my thesis [Ro 92a]. The translation itself dates mainly from 1993, so there is no reference to the more recent works on the subject, and the vocabulary is not updated as it could be.

Abstract

In intuitionistic logic rules can be "valid" (we say admissible) without being derivable. We show that using intuitionistic sequent calculus in both direct way and reverse way (what we call "backward derivation") is complete versus admissibility. As a direct consequence we obtain a decidable characterisation of admissibility.

Contents

1 Introduction.

A well-known problem in intuitionistic logic is the existence of valid but not derivable rules. This problem seems to be related with some constructive features of intuitionism (disjunction and existence property) but appear also in modal logics. We study here a particular case of this phenomenon, admissible rules in intuitionistic propositional calculus. G.E.Mints in [Mi 72] give sufficient conditions for admissible rules to be derivable. H.Friedman in [Fr 75] states the problem of the decidability of admissibility and V.V. Rybakov solves it using semantical and algebraic methods, via the Gödel translation of intuitionistic logic into modal logic in [Ry 84, Ry 86].

We proposed here another approach of the problem using Intuitionistic Sequent Calculus and a particular class of substitutions defined in [Ro 92a, Ro 92b].

We made use of these substitutions in [Ro 92b] to give sufficient conditions for admissible rules to be derivable (extending results of [Mi 72]). For instance if the premises of an admissible rule are disjunctions of Harrop formulae then the rule is derivable.

Here we show that a kind of "backward derivation" in (intuitionistic) sequent calculus is complete versus to admissibility. The substitutions act as valuations in usual completeness results. Decidability of admissibility comes as a consequence of this completeness result, showing the result of Rybakov but in a very different way.

The present article is almost self contained : only well known results are used, that can be find in all manuals introducing to intuitionistic logic. In particular useful definitions from [Ro 92b] are repeated here. We slightly modify some terminology in hope more clarity. We use results from this article but in a very particular case for which the proof is very easy (and given here).

1.1 Notations.

Greek letters stand for propositional variables. The connectives are $\vee, \wedge, \rightarrow$ and $\perp$ is a propositional constant for false. $\neg A$ stands for $A \rightarrow \perp$, $\top$ stands for $\perp \rightarrow \perp$. A primitive negation and false defined as $\perp = A \wedge \neg A$ lead to the same results.
$A \rightarrow B \rightarrow C$ or $A, B, \rightarrow C$ stand for $A \rightarrow (B \rightarrow C)$.
If $\Gamma = A_1, \ldots, A_n$ is a finite set of formulae, $\Gamma \rightarrow C$ stands for $A_1, \ldots, A_n, \rightarrow C$, ($C$ if $\Gamma = \emptyset$), $\wedge\Gamma$ stands for $A_1 \wedge \ldots \wedge A_n$ ($\top$ if $\Gamma = \emptyset$), $A \wedge \Gamma$ stands for $A \wedge A_1 \wedge \ldots \wedge A_n$ (A if $\Gamma = \emptyset$). Though it is ambiguous, it is harmless here since changing the order of formulae A_i gives intuitionistically equivalents formulae.

We write $\vdash$ for the intuitionistic deduction relation, $A \equiv B$ stands for $A \vdash B$ *and* $B \vdash A$.

1.2 First definitions, exposition of the main result.

1.2.1 Admissible rules, derivable rules.

We define here admissible rules for a propositional logic $\mathcal{L}$.

Definition 1.1. *The rule:*

$$\frac{\vdash_{\mathcal{L}} A_1 \ldots \vdash_{\mathcal{L}} A_n}{\vdash_{\mathcal{L}} C},$$

is said an admissible rule *in* $\mathcal{L}$ *and written down:*

$$A_1, \ldots, A_n \gg C,$$

iff the set of theorems of $\mathcal{L}$ is closed under this rule, or equivalently iff for every substitution s of propositional formulae for propositional variables:

$$if \ \vdash_{\mathcal{L}} s(A_1), \ldots, \vdash_{\mathcal{L}} s(A_n), \ then \ \vdash_{\mathcal{L}} s(C).$$

This rule is said to be a derivable rule *in $\mathcal{L}$ iff:*

$$A_1, \ldots, A_n, \vdash_{\mathcal{L}} C.$$

Note that the notions of admissible and derivable rules we are talking about donnot depend of a particular set of rules and axioms for $\mathcal{L}$. We could talk also of admissible rules for a particular set of rules and axioms, but the methods of this paper would not be relevant in this more general case.

The following proposition is clearly true.

Proposition 1.2. *If the logic $\mathcal{L}$ is stable under substitution (for instance $\mathcal{L}$ is intuitionistic or classical logic), then every derivable rule is admissible.*

In classical propositional calculus the converse, i.e. every admissible rule is derivable, is provable (by completeness and using substitutions of propositional constants $\perp$ and $\top$ as valuations). On the other hand there are well-known admissible rules which are not derivable rules in intuitionistic calculus, for instance:

$$(\alpha \to \beta) \to \gamma \vee \delta \gg ((\alpha \to \beta) \to \alpha) \vee ((\alpha \to \beta) \to \gamma) \vee ((\alpha \to \beta) \to \delta)\ . \qquad (ad_1)$$

(see for a proof section 3.1) and as consequence (take $\beta = \perp$)

$$\neg\alpha \to \gamma \vee \delta \gg (\neg\alpha \to \gamma) \vee (\neg\alpha \to \delta)$$

but it is well known that

$$\neg\alpha \to \gamma \vee \delta \nvdash (\neg\alpha \to \gamma) \vee (\neg\alpha \to \delta)\ .$$

and then

$$(\alpha \to \beta) \to \gamma \vee \delta \nvdash ((\alpha \to \beta) \to \alpha) \vee ((\alpha \to \beta) \to \gamma) \vee ((\alpha \to \beta) \to \delta)\ .$$

We will suppose in the following that the logic $\mathcal{L}$ is stable under substitution.

1.2.2 backward derivation.

An usual way to show that rules are admissible is to use a system of rules complete for the given logic in a backward way i.e. from conclusion to premises (as in tableaux method), see examples in section 3.1. What we call backward derivation is a formalisation of this method.

We will use the following notation.

For a sequent $S = \Sigma \vdash \Theta$ let $\overrightarrow{S} = \wedge\Sigma \to \vee\Theta$, for a set of sequents $\mathcal{G} = \{S_1, \ldots, S_p\}$, let $\overrightarrow{\mathcal{G}} = \overrightarrow{S_1} \wedge \ldots \wedge \overrightarrow{S_p}$ ($\overrightarrow{\mathcal{G}} = \top$ if $\mathcal{G} = \emptyset$), for a set of sets of sequents $\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$, let $\overrightarrow{\mathcal{E}} = \overrightarrow{\mathcal{E}_1} \vee \ldots \vee \overrightarrow{\mathcal{E}_n}$ ($\overrightarrow{\mathcal{E}} = \perp$ if $\mathcal{E} = \emptyset$).

In all this paragraph let (R) be a complete system of rules and axioms using sequents for the given logic $\mathcal{L}$. We suppose that (R) is a Gentzen like sequent calculus without cuts, that is essentially:

- **i.** All rules of (R) are local (depending on sequents and not on proofs).
- **ii.** All formulae occurring in premises of a rule of (R), either occurs in the conclusion of (R) or are immediate subformulae of the same formula of the conclusion of (R). In particular the system (R) has subformula property.

In case of intuitionistic logic, (R) can be any formulation of a Gentzen sequent calculus without cuts for intuitionistic propositional logic.

A *derivation* in (R) is a tree all nodes of which correspond to correct rules. A derivation all leaves of which are axioms is a *proof*. As usually for proofs we say a derivation of a sequent for a derivation whose root is this sequent. We had supposed that all rules are local in particular each complete subtree of a proof is a proof and then each sequent in a proof is provable. In all this section proofs and derivations are always in the system (R). A derivation is said *redundant* if a branch of this derivation contain two distinct occurrences of the same sequent. Just by substituting in a proof to a subproof of a given sequent the subproof of the topmost occurrence of this sequent, it is clear (and well known[1]) that each provable sequent has a non redundant proof.

We call now *strict derivation* a derivation such that a sequent in which occurs a propositional variable is never the concluding sequent of an occurrence of a rule in this derivation. For instance a trivial (containing no instance of rules) derivation is always strict. We call *maximal strict derivation* a strict derivation all leaves of which are axioms or are sequents containing a propositional variable.

Lemma 1.3. *For any logical substitution s any proof $\mathfrak{P}$ of the sequent $s(S)$ (if there is one) "ends" with the substitute of a maximal strict derivation $\mathfrak{D}$ of the sequent S (i.e. $s(\mathfrak{D})$ is a truncature of $\mathfrak{P}$).*

Proof. All formulae occuring in $\mathfrak{P}$ are subformulae of formulae of $s(S)$ (let us say subformulae of $s(S)$ for short). Some of these subformulae (at less the formulae themselves) are of the form $s(C)$ where C is a subformula of S.

Take the derivation $\mathfrak{D}'$ obtained by truncating $\mathfrak{P}$ in the following way: cut all branches before the first sequent (starting from the root) where the rule is applied to a subformula A of formulae of $s(S)$ of the form $A = s(\alpha)$, α being any variable. Because what we suppose on (R) (clause (ii) above), all formulae occuring in sequents of $\mathfrak{D}'$ are of the form $s(C)$, where C is a subformula of S and then $\mathfrak{D}' = s(\mathfrak{D})$ where $\mathfrak{D}$ is a derivation of S, a maximal strict derivation by definition of $\mathfrak{D}'$. ■

Definition 1.4. *Let $\Gamma \vdash \Delta$ be a sequent. Let $\mathcal{E}$ be a finite set of finite sets of sequents,*

$$\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$$

such that each maximal strict derivation $\mathfrak{D}$ of $\Gamma \vdash \Delta$ in (R) either verifies for some $i \in \{1, \ldots, n\}$ that all sequents in $\mathcal{E}_i$ are leaf sequents of $\mathfrak{D}$ or is redundant.

We say that $\mathcal{E}$ is obtained by backward derivation *from $\Gamma \vdash \Delta$ in (R) and we note $(\Gamma \vdash \Delta)\ \boldsymbol{rd}_{\mathcal{L},R}\ \mathcal{E}$. We say also that a formula E is obtained by* backward derivation *from a formula F and we note $F\ \boldsymbol{rd}_{\mathcal{L},R}\ E$, if there exists a sequent S and a finite set of finite sets of sequents $\mathcal{E}$ such that $F = \overrightarrow{S}$, $E = \overrightarrow{\mathcal{E}}$, and $S\ \boldsymbol{rd}_{\mathcal{L},R}\ \mathcal{E}$.*

We note $>_{\mathcal{L},R}$ the transitive closure of the usual deduction relation and the backward derivation relation between formulae. We call it the relation of outward and backward *derivation.*

It is obvious that if $F\ \boldsymbol{rd}_{\mathcal{L},R}\ G$ then $G \vdash F$.

Lemma 1.5 (soundness). *If $F > G$ then $F \gg G$.*

[1] Restriction to no redundant derivations gives, together with subformula property, the decidability of provability in intuitionistic propositional calculus.

Proof. It is sufficient to prove (prop. 1.2) that, if $F\ \boldsymbol{rd}_{\mathcal{L},R}\ G$ then $F \gg G$. Let S be a sequent, let $\mathcal{E}$ be a finite set of finite sets of formulae such that $F = \overrightarrow{S}$ and $G = \overrightarrow{\mathcal{E}}$ and $S\ \boldsymbol{rd}_{\mathcal{L},R}\ \mathcal{E}$. Let s be a substitution such that $s(F)$ is provable, and then the sequent $s(S)$ is provable. A proof $\mathfrak{P}$ of $s(S)$ ends with the substitute of a maximal strict derivation $\mathfrak{D}$ of S. By definition of the relation $\boldsymbol{rd}_{\mathcal{L},R}$ there exists one set $\mathcal{E}_i$, $\mathcal{E}_i \in \mathcal{E}$, such that every sequent of $\mathcal{E}_i$ occurs in $\mathfrak{D}$. Hence every sequent of $s(\mathcal{E}_i)$ occurs in the proof $\mathfrak{P}$ and then is provable. Then $s(\overrightarrow{\mathcal{E}})$ is provable. ■

We can know give a more precise version of the completeness result announced in the introduction.

Theorem 1.6 (completeness). *The* outward and backward *derivation in Gentzen Intuitionistic Sequent calculus is complete with respect to admissibility in Intuitionistic Propositional calculus, i.e. (we omit in this case the subscripts of the relations "$\boldsymbol{rd}$" and "$>$")*

$$F \gg G \text{ iff } F > G\ .$$

We can even more prove that, for each formula F there exist a formula F^{ad} such that :

$$\forall C\ (F^{ad} \gg C \text{ iff } F^{ad} \vdash C) \quad ; \quad F > F^{ad} \text{ and } F^{ad} \vdash F\ .$$

As a consequence we obtain the following result that can be stand without talking of backward derivation, but seems to be new.

Proposition 1.7. *For each formula F there exist a formula F^{ad} such that*

$$\forall C\ (F^{ad} \gg C \text{ iff } F \vdash C) \quad ; \quad F \gg F^{ad} \text{ and } F^{ad} \vdash F\ .$$

We will derive decidability of the admissibility relation from the effective construction of the formula F^{ad}. Note that the decidability of the outward and backward derivation does not follow trivially from its definition.

What the proof of completeness look like? The skeleton is the usual one : first for a given saturation property construct all the saturated sets containing a given set of formulae, then associate to each saturated set a particular substitution (which play here the semantic part).

But there are two sources of troubles. The first one is that we are able to construct the useful substitution only for *finite* set of formulae. That leads us to use the clumsy definition 4.4 of saturation – the saturation is restricted to a convenient finite set of formulae that cannot be only subformulae of the original formulae – instead of the more natural definition 2.3. The second one is that we want to obtain a *finite* set of formulae whose conjunction will give the formula F^{ad} above.

2 Admissibility and derivability.

2.1 Admissible and derivable consequences.

We say that formulae or finite sets of formulae Γ have *the same admissible and derivable consequences*, iff $\Gamma \gg C$ iff $\Gamma \vdash C$.

For instance set of formulae of the intuitionistic "$\wedge, \vee, \neg$" fragment have the same admissible and derivable consequences. The same holds for the "$\wedge, \to, \neg$" fragment, for disjunction of Harrop formulae etc... (see [Ro 92a, Ro 92b]).

We recall the *Glivenko theorem*, a negated formula is intuitionistically provable if and only if it is classically provable, and the *disjunction property in intuitionistic logic*, $\vdash C \vee D$ iff $\vdash C$ *or* $\vdash D$. (see for instance [Tr 88] for the proofs). From these two theorems we get the two items of the following lemma.

Lemma 2.1.

i. $\Gamma \gg \bot$ *iff* $\Gamma \vdash_c \bot$ *iff* $\Gamma \vdash \bot$;

ii. $\Gamma, B \vee B' \gg C$ *iff* $\Gamma, B \gg C$ *and* $\Gamma, B' \gg C$.

Hence the set of formulae which have the same admissible and derivable consequences is closed under disjunctions.

2.2 Some useful substitutions.

We describe here particular substitutions which are useful when dealing with admissibility.

Definition 2.2. *If* Γ *is a finite set of formulae, the substitution* s *will be said a* Γ-identity *iff for every propositional variable* α*:*

$$\Gamma \vdash \alpha \leftrightarrow s(\alpha) \ ,$$

or equivalently:

$$\Gamma \to \alpha \equiv \Gamma \to s(\alpha).$$

We say that a substitution s *is* Γ-validating *iff for every formula* C *in* Γ *the formula* $s(C)$ *is provable. A* Γ*-validating* Γ*-identity will be said a* validating Γ-identity.

These substitutions will play the semantic part in the completeness proof. We first define a kind of saturation property for finite sets of formulae, and then a lemma that relates this to validating Γ-identity.

Definition 2.3. *We say that a set of formula* Γ *has* the disjunction property for admissibility *iff for any formulae* C *and* D *,*

$$if \ \ \Gamma \gg C \vee D \ \ then \ \ \Gamma \vdash C \ or \ \Gamma \vdash D \ .$$

Lemma 2.4. *Let* Γ *be a finite set of formulae.*

i. *If there exists an validating* Γ*-identity, then* Γ *has the disjunction property for admissibility.*

ii. *If* Γ *has the disjunction property for admissibility, then* Γ *has the same admissible and derivable consequences.*

Proof of (i). Let s be the validating Γ-identity which is given by the hypothesis. We know that:

$$\Gamma \gg C \vee D \ \ and \ \ \vdash \wedge s(\Gamma) \ .$$

The definition of admissibility and the disjunction property of intuitionistic logic lead to:

$$\vdash s(C) \ or \ \vdash s(D) \ .$$

Then by weakening we obtain:

$$\vdash \Gamma \to s(C) \ or \ \vdash \Gamma \to s(D) \ .$$

Hence the definition of a Γ-identity yields:

$$\vdash \Gamma \to C \ or \ \vdash \Gamma \to D \ . \qquad \blacksquare$$

Proof of (ii). Take $C = D$. ■

Remark 1. The converse of (i) is true (see proposition 4.8) and is the crucial point of the proof of completeness for admissibility.
Remark 2. The following property of Γ-identity also relates to admissibility. Let s be a Γ-identity, then $\Gamma \gg C$ iff $s(\Gamma) \gg s(C)$ (the proof is straightforward, see [Ro 92a]).
Remark 3. The disjunction property for admissibility for a given formula obviously implies the disjunction property for the same formula. But the converse is false, see for instance $\neg\alpha \to (\beta \vee \gamma)$.

We use the previous lemma in [Ro 92a, Ro 92b] to characterise formulae having the disjunction property for admissibility or the same admissible and derivable consequences. We only need two easy particular cases of these results, that are given now.

Let us call *simple Harrop formulae* the formulae of the form $\Delta \to \alpha$, where Δ is any finite set of formulae, and α is a propositional variable. Let us call *simple anti-Harrop formulae* the formulae of the form $\alpha \to \wedge\Delta$, where Δ is any finite set of formulae, and α is a propositional variable.

Proposition 2.5.

i. *Let Γ be a finite set of simple Harrop formulae $\Gamma = \{\Delta_i \to \alpha_i / 1 \leq i \leq n\}$, where the Δ_i are any sets of formulae and the α_i propositional variables. Let s be the substitution such that :*

if α is a propositional variable occuring in Γ, then $s(\alpha) = \Gamma \to \alpha$, else $s(\alpha) = \alpha$.

Hence s is a validating Γ-identity, and then Γ has the the disjunction property for admissibility.

ii. *Let Γ be a finite set of simple anti-Harrop formulae $\Gamma = \{\alpha_i \to \wedge\Delta_i / 1 \leq i \leq n\}$ Let s be the substitution such that :*

if α is a propositional variable occuring in Γ, then $s(\alpha) = \alpha\wedge\Gamma$, else $s(\alpha) = \alpha$.

Hence s is a validating Γ-identity, and then Γ has the the disjunction property for admissibility.

Proof. In both cases the substitution is clearly a Γ-identity. In case (i) we obtain by intuitionistic equivalences for a formula $\delta \to \alpha \in \Gamma$:

$$s(\Delta \to \alpha) = s(\Delta) \to (\Gamma \to \alpha) \equiv \Gamma \to (s(\Delta) \to (\Gamma \to \alpha)) = \Gamma \to s(\Delta \to \alpha)$$

and as s is a Γ-identity

$$s(\Delta \to \alpha) \equiv \Gamma \to (\Delta \to \alpha) \equiv \top \ .$$

In case (ii) we obtain in the same way for a formula $\alpha \to \wedge\Delta \in \Gamma$:

$$s(\alpha \to \wedge\Delta) = (\Gamma \wedge \alpha) \to s(\wedge\Delta) \equiv \Gamma \to ((\Gamma \to \alpha) \to s(\wedge\Delta) = \Gamma \to s(\alpha \to \wedge\Delta) \equiv \top$$

■

Formulae in case (i) are particular cases of Harrop Formulae but note that the above proof is very simple because the α's cannot be $\perp$.

3 backward derivation.

3.1 Preliminaries.

See the definition 1.4 of backward derivation : as for proof searching it is clear that it is easier to handle with a sequent calculus without cuts and with internalised structural rules (exchange, contraction and weakening) as in figure 1 which is also very near from a tableaux method. It is also clear that the definition of backward derivation not deeply depends of this choice: it make no difference if we quotient formulae by equivalence.

Note also that axioms are restricted to atomics formulae. Definition of maximal strict derivation is then simpler. We only need to suppose that maximal strict derivations are strict derivations all leaves of which containing a propositional variable, because in this case every derivations of an axiom contains only axioms.

Axioms $\dfrac{}{\Gamma, \alpha \vdash \alpha}$ (α is a variable or $\perp$) $\dfrac{}{\Gamma, \perp \vdash A}$

rules	left	right
$\to$	$\dfrac{\Gamma, A \to B \vdash A \qquad \Gamma, B, \boldsymbol{A \to B} \vdash C}{\Gamma, A \to B \vdash C}$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$
$\wedge$	$\dfrac{\Gamma, A, B, \boldsymbol{A \wedge B} \vdash C}{\Gamma, A \wedge B \vdash C}$	$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$
$\vee$	$\dfrac{\Gamma, A, \boldsymbol{A \vee B} \vdash C \qquad \Gamma, B, \boldsymbol{A \vee B} \vdash C}{\Gamma, A \vee B \vdash C}$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$

Note. the sequent calculus with the same rules but omitting the above bold scripted formulas is also complete. the only necessary contraction is on the left premise of the rule ($\to_l$). using the version above will simplify some proofs later.

Figure 1: sequent calculus without cuts

let us show the example of section 1.2.1. take $s(\alpha) = A,\ s(\beta) = B,\ s(\Gamma) = C,\ s(\delta) = D$. in calculus of fig 1, if the sequent $(A \to B) \vdash (C \vee D)$ is provable then the last step of its proof use of one of the 3 sets of premises below.

$$\dfrac{\dfrac{A \to B \vdash A \quad A \to B, B \vdash C \vee D}{} \qquad \dfrac{A \to B \vdash C}{} \qquad \dfrac{A \to B \vdash D}{}}{A \to B \vdash C \vee D}$$

we deduce that

$$(\alpha \to \beta) \to (\gamma \vee \delta) \gg [((\alpha \to \beta) \to \alpha) \wedge (\beta \to (\gamma \vee \delta))] \vee ((\alpha \to \beta) \to \gamma) \vee ((\alpha \to \beta) \to \delta)$$

and then

$$(\alpha \to \beta) \to \gamma \vee \delta \gg ((\alpha \to \beta) \to \alpha) \vee ((\alpha \to \beta) \to \gamma) \vee ((\alpha \to \beta) \to \delta)\ . \qquad (ad_1)$$

note that $(\alpha \to \beta) \to (\gamma \vee \delta) \vdash \beta \to (\gamma \vee \delta)$.

What is different from proof searching in this method? As we work on substitute of formulae we can say nothing about possible proof of a sequent which contain a propositional variable because the proof can use a rule on the principal connective of the substitute of this formula. for instance in the example above, we can say nothing more: all the sequent in the premisses contains a propositional variable.

Let us see another example: take $A = C \vee D$ in the example above. we cannot prove the sequent $(C \vee D) \to B \vdash C \vee D$ with a rule $(\to_l)$ without redundancies. then we have only two way for proving this sequent in the system of fig 1 :

$$\dfrac{\dfrac{(C \vee D) \to B \vdash C}{} \qquad \dfrac{(C \vee D) \to B \vdash D}{}}{(C \vee D) \to B \vdash C \vee D}$$

and then

$$((\gamma \vee \delta) \to \beta) \to (\gamma \vee \delta) \gg [((\gamma \vee \delta) \to \beta) \to \gamma] \vee [((\gamma \vee \delta) \to \beta) \to \delta] \ . \qquad (ad'_1)$$

To deal with backward derivation we need a structure on the class of these particular sets of derivations used in the definition 1.4.

3.2 Describing all possible derivations.

We introduce another kind of sequents: pointed sequents which are sequents where one and only one formula is distinguished by one point, two points only if this formula is a right disjunction. The pointed formula will be the main formula of the rule whose conclusion is the given sequent. See the figure 2: when seeing the rules bottom up, only one rule can be applied to a given sequent, the disjunctive rule to an ordinary sequent, a conjunctive rule to a pointed sequent.

Let S be a sequent. Let us call *tree of possible derivations* of S the only maximal tree constructed bottom up by the rules of the figure 2.

Note that each branch of this tree that is not infinite either ends with a sequent with a pointed atom (variable or $\perp$), or ends with the $\emptyset$, considered as the empty set of premisses of the rule $\perp_l$ (we recall that the only constant is $\perp$).

It is clear that a subtree of the tree of all possible derivations of a given sequent whose all conjunctive nodes are complete and all disjunctive nodes have only one son can be identified with a derivation in the sequent calculus of figure 1.

In classical predicate calculus, because of contraction rule, it is possible to "contract" such a tree to one particular derivation which enumerate "enough" possible ways of proving a sequent. In intuitionistic case simplifications are possible, but real disjunctive choice can not be eliminated because always stay some rules without reversible[2] formulation.

In intuitionistic propositional sequent calculus with less than one formula to the left, only two rules have no reversible formulation: rules $(\to_{left})$ and $(\vee_{right})$ (see figure 1 or 2). This is clearly strongly related with the lack of contraction rule in the right part of the sequent. In the two examples of the preceding section 3.1 we obtain admissible but not derivable rules in a way illustrating this fact.

3.3 backward derivation trees.

Definition 3.1. *A* backward derivation tree *is a* finite *subtree of the tree of all possible derivations (defined in the section above) such that*

i. *all nodes are complete (truncature);*

[2] We will call *reversible* the rules whose conjunction of premisses is equivalent to the conclusion.

conjunctive rules.

Rules	left	right
$\perp$	$\dfrac{\emptyset}{\Gamma, \perp^{\cdot} \vdash A}$	$\dfrac{\Gamma \vdash \perp}{\Gamma \vdash \perp^{\cdot}}$
$\to$	$\dfrac{\Gamma, (A \to B) \vdash A \qquad \Gamma, B, A \to B \vdash C}{\Gamma, (A \to B)^{\cdot} \vdash C}$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash (A \to B)^{\cdot}}$
$\wedge$	$\dfrac{\Gamma, A, B, A \wedge B \vdash C}{\Gamma, (A \wedge B)^{\cdot} \vdash C}$	$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash (A \wedge B)^{\cdot}}$
$\vee$	$\dfrac{\Gamma, A, A \vee B \vdash C \qquad \Gamma, B, A \vee B \vdash C}{\Gamma, (A \vee B)^{\cdot} \vdash C}$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash (A \vee B)^{\cdot}} \qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash (A \vee B)^{\cdot\cdot}}$

Disjunctive rules (C is not a disjunction).

$$\frac{A_1^{\cdot}, \ldots, A_n \vdash C \ \cdots\ A_1, \ldots, A_i^{\cdot}, \ldots, A_n \vdash C \ \cdots\ A_1, \ldots, A_n^{\cdot} \vdash C \ \cdots\ A_1, \ldots, A_n \vdash C^{\cdot}}{A_1, \ldots, A_n \vdash C}$$

$$\frac{A_1^{\cdot}, \ldots, A_n \vdash E \vee F \ \cdots\ A_1, \ldots, A_n^{\cdot} \vdash E \vee F \ \cdots\ A_1, \ldots, A_n \vdash (E \vee F)^{\cdot} \ \cdots\ A_1, \ldots, A_n \vdash (E \vee F)^{\cdot\cdot}}{A_1, \ldots, A_n \vdash E \vee F}$$

Figure 2: Rules for building backward derivation trees

ii. *every sequent in the backward derivation tree which contains a propositional variable is a leaf (and then cannot be a pointed sequent);*

iii. *every pointed sequent in the backward derivation tree is not a leaf, except if the conjunctive rule whose conclusion is this sequent has one of its premisses which appear in the tree below (redundancy).*

Note that we can compose backward derivation trees (replacing a leaf by a backward derivation tree of this leaf) to obtain a new backward derivation tree, but in this composition new redundancies can appear.

Note that the two instances of the "degenerate rule" $\perp_r$ above give trivial redundancies.

In the definition 1.4 the backward derivation associate to a sequent a set of set of sequent, which is interpreted as a disjunction of conjunction of sequent. The backward derivation tree associate to its root sequent a similar but more complex *positive boolean formula of sequents* in a very direct way. First eliminate all leaf which are pointed sequent (corresponding to redundancies). Then interpret disjunctive nodes as disjunctions and conjunctive nodes as conjunctions and forget the rules labelling these nodes. We use for these positive boolean formulae of sequent the set notation: braces of odd rank are interpreted as disjunctions, of even rank as conjunctions. Note that sets corresponding to conjunctions can be empty in case of the rule $\perp_l$, and that sets corresponding to disjunctions can also be empty because of redundancies. A backward derivation tree give a complete set of possible derivations just by taking what corresponds to a disjunctive normal form of the boolean formula associated with it. We call *set of sets of sequents associated with a backward derivation tree* the disjunctive normal form obtained in the more obvious way by distributivity from this positive boolean formula (whose atoms are sequents), that is by replacing successively in the positive boolean formula any occurrence of $\{S_1, \ldots, S_n\} \cup \mathcal{E}$ where $S_1, \ldots, S_n$ are sequents, by $\{S_1\} \cup \mathcal{E}, \ldots, \{S_n\} \cup \mathcal{E}$. The following lemma is then obvious (see the definition 1.4).

Lemma 3.2. *Let $\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ be the set of sets of sequents associated to a backward derivation tree of root $\Gamma \vdash C$. Then all sequents occuring in $\mathcal{E}$ are leaves of the backward derivation tree and are ordinary not pointed sequents. Furthermore $(\Gamma \vdash C)$ **rd** $\mathcal{E}$.*

Note that when constructing the disjunctive normal form, we can obtain an empty disjunction $\emptyset$, corresponding intuitively to absurd. We can also obtain a disjunction that contains at least one empty conjunction $\{\emptyset, \mathcal{E}_1, \ldots, \mathcal{E}_n\}$, corresponding intuitively to a tautology.

Let us call now a *maximal backward derivation tree* a backward derivation tree such that all leaves are either pointed sequents, or contain a propositional variable.

Lemma 3.3. *Every sequent has a maximal backward derivation tree without redundancies.*

Proof. We know that all formulae appearing in a backward derivation tree are subformulae of this sequent and that a sequent has a finite number of subformulae. Then a sequent has a finite number of backward derivation trees without redundancies. One of these is maximal. ∎

4 Completeness.

This section is devoted to the proof of the completeness result. We first need to define a saturation property on a finite set of formulae. We define this set in the next paragraph.

4.1 Extensions of the notion of subformula.

In sequent calculus without cut and hence in a backward derivation tree, every formula is a subformula of a formula of the root sequent but more over with respect to the sign: we have to handle with sequent whose left formulae are negative subformulae of the root sequent, whose right formulae are positive subformulae of the root sequent.

Definition 4.1. *The set of the* subsequents *of a formula A, noted $\mathcal{S}(A)$, is the set of sequents whose left part contains only negative subformulae of A and whose right part contains only positive subformulae of A.*

The set of extended subformulae *of A, noted $\mathcal{F}^{\rightarrow}(A)$,is the set of formulae associated with subsequents of A.*

The set $\mathcal{F}^{\rightarrow,\wedge,\vee}(A)$, is obtained by taking disjunctions of distinct conjunctions of distinct extended subformulae of A (in other words a finite representation of the closure by disjunction and conjunction of $\mathcal{F}^{\rightarrow}(A)$ quotiented by the equivalence relation "$\equiv$ ").

These definitions are extended to finite sets of formulae by union.

The empty disjunction gives $\perp \in \mathcal{F}^{\rightarrow,\wedge,\vee}(A)$, and the empty conjunction gives $\top \in \mathcal{F}^{\rightarrow,\wedge,\vee}(A)$.

It is well known that the closure by "$\rightarrow$" connective of a finite set of formulae quotiented by intuitionistic equivalence is infinite. That is the reason why we need such a definition.

Lemma 4.2.

i. *The sets* $\mathcal{S}(A)$ *(subsequents of* A*),* $\mathcal{F}^{\rightarrow}(A)$ *(extended subformulae of* A*), and* $\mathcal{F}^{\rightarrow,\wedge,\vee}(A)$ *are finite.*

ii. *If* $B \in \mathcal{F}^{\rightarrow}(A)$*, then every formula of* $\mathcal{F}^{\rightarrow}(B)$ *is equivalent to a formula of* $\mathcal{F}^{\rightarrow}(B) \cap \mathcal{F}^{\rightarrow}(A)$*. Hence :*

$$\mathcal{F}^{\rightarrow}(\mathcal{F}^{\rightarrow}(\Gamma))/_{\equiv} \;=\; \mathcal{F}^{\rightarrow}(\Gamma)/_{\equiv} \;\; ; \;\; \mathcal{F}^{\rightarrow,\wedge,\vee}(\mathcal{F}^{\rightarrow}(\Gamma))/_{\equiv} \;=\; \mathcal{F}^{\rightarrow,\wedge,\vee}(\Gamma)/_{\equiv} \; .$$

Proof of (i). The formula A has finite sets of positive and negative subformulae. Hence the set of all subset of negative formulae of A is finite, and then the set $\mathcal{S}(A)$ of all subsequents of A is finite (recall that we choose for left-hand side of a sequent a finite set of formulae). The set $\mathcal{F}^{\rightarrow}(A)$ of extended subformulae of A is then finite and by definition $\mathcal{F}^{\rightarrow,\wedge,\vee}(A)$ is also finite. ■

Proof of (ii). Let be $B = A_1, \ldots, A_n \rightarrow C$, where $A_1, \ldots, A_n$ are negative subformulae of A, and C a positive subformula of A. The negative subformulae of B are exactly the positive subformulae of $A_1, \ldots, A_n$ and the negative subformulae of C, and then in all cases are negative subformulae of A. The positive subformulae of B are the negative subformulae of $A_1, \ldots, A_n$, the positive ones of C and the formulae :

$$A_i, \ldots, A_n \rightarrow C \quad \text{with} \quad i \in \{1, \ldots, n\} \; .$$

let S be a subsequent of B. Its left part contains only negative subformulae of A. If the right formula of S id a subformula of A, then S is a subsequent of A. Else the right formula of S is one of the formulae $A_i, \ldots, A_n, \rightarrow C$, let $S = \Gamma \vdash A_i, \ldots, A_n, \rightarrow C$. In this case the formula associated to S is the same as the one associated to the subsequent of A, $\Gamma, A_i, \ldots, A_n, \vdash C$. ■

The following lemma is a variant of subformula property for sequent calculus.

Lemma 4.3. *Every sequent occurring in a backward derivation of* $\vdash A$ *is a subsequent of* A*. Hence, if* A ***rd*** B *, then* B *is in the closure by conjunction and disjunction of* $\mathcal{F}^{\rightarrow}(A)$*, and then there exists* B' *such that* $B' \equiv B$ *and* $B' \in \mathcal{F}^{\rightarrow,\wedge,\vee}(A)$ *.*

Proof. Straightforward induction on the height of the backward derivation. ■

4.2 Saturation.

Definition 4.4. *Let* Θ *be a set of formulae. We say that a set of formulae* Γ *is* Θ-saturated *if and only if for all formulae* C *and* D *in* $\mathcal{F}^{\rightarrow,\wedge,\vee}(\Theta)$ *:*

$$\textit{if } \; \Gamma > C \vee D, \quad \textit{then } \; \Gamma \vdash C \; \textit{ or } \; \Gamma \vdash D \; .$$

We say that Γ *is* saturated *if and only if* Γ *is* Γ*-saturated.*

Lemma 4.5. *Let* Γ *be a subset of* $\mathcal{F}^{\rightarrow}(\Theta)$*. If* Γ *is* Θ*-saturated, then* Γ *is saturated.*

Proof. From the lemma 4.2 (ii) every formula of $\mathcal{F}^{\rightarrow,\wedge,\vee}(\Gamma)$, is equivalent to a formula of $\mathcal{F}^{\rightarrow,\wedge,\vee}(\Theta)$. ■

Lemma 4.6. *For every formula* A*, there exists a formula* $\overline{A}$ *such that* A ***rd*** $\overline{A}$*, and* $\overline{A}$ *is a disjunction of conjunctions of simple Harrop and anti-Harrop extended subformulae of* A*, or* $\bot$ *or* $\top$*.*

Proof. Take a maximal backward derivation of $\vdash A$ (see lemma 3.3). Let $\mathcal{E}$ be the set of set of sequents associated to this backward derivation tree, such that $(\vdash A) > \mathcal{E}$ (see lemma 3.2). If $S \in \mathcal{E}$, S is a leaf of the backward derivation tree and S is not a pointed sequent. Because the tree is maximal S contains a propositional variable and in this case $\overrightarrow{S}$ is a simple Harrop or anti-Harrop formula, and an extended subformula of A (see lemma 4.3). See the definition 1.4 of the backward derivation relation: we obtained A ***rd*** $\overrightarrow{\mathcal{E}}$, where $\overrightarrow{\mathcal{E}}$ is a disjunction of conjunction of simple Harrop and anti-Harrop extended subformulae of A (including empty disjunctions and conjunctions). We can obtain a disjunction containing at least one empty conjunction. In that case A is provable and we take $\overline{A} = \top$. We can also obtain an empty disjunction. In this last case, $A \gg \bot$ and then $A \equiv \bot$ (see lemma 2.1. i). We can naturally take $\overline{A} = \bot$. In other cases $\overline{A} = \overrightarrow{\mathcal{E}}$. ■

we deduce the following lemma.

Lemma 4.7. *Every saturated set of formulae Γ is equivalent to a saturated set Γ' of simple Harrop and anti-Harrop extended subformulae of Γ, or is contradictory, or is equivalent to a tautology. If Γ is finite, then Γ' is finite.*

Proof. We can eliminate from Γ all provable formulae. If one of the formula of Γ is contradictory, then Γ itself is contradictory. In other cases, we know by lemma 4.6 that for each formula F in Γ there exists $\overline{F}$ a disjunction of conjunction of simple Harrop and anti-Harrop extended subformulae of Γ, such that F ***rd*** $\overline{F}$. By definition of saturation, one of these conjunctions, call it $\wedge\Delta_F$, is a consequence of F. Then $F \equiv \wedge\Delta_F$. The set $\Gamma' = \bigcup_{F\in\Gamma} \Delta_F$ is equivalent to Γ and then Γ-saturated. But it is a subset of $\mathcal{F}^{\rightarrow}(\Gamma)$ and then, by lemma 4.5 a saturated set. By construction, if Γ is finite, then Γ' is finite. ■

We can now restrict ourself to saturated sets of simple Harrop and anti-Harrop formulae. It is clear that, for a finite set of formulae Γ, if $\wedge\Gamma$ has the disjunction property for admissibility, then Γ is saturated. Finite sets of simple Harrop formulae and finite sets of simple anti-Harrop formulae are then saturated (see lemma 2.5). But this result is in general false for finite set of both simple Harrop and simple anti-Harrop formulae. See for instance:

$$\neg\alpha \to \delta, \delta \to \beta \vee \gamma > (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma) \ ;$$

$$\neg\alpha \to \delta, \delta \to \beta \vee \gamma \nvdash (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma) \ .$$

Proposition 4.8. *Let Γ be any finite set of formulae. The three following propositions are equivalent.*

i. *Γ is a saturated set.*

ii. *There exists a validating Γ-identity, or $\Gamma \vdash \bot$.*

iii. *Γ has the disjunction property for admissibility.*

(iii)$\Rightarrow$(i) is trivial. (ii)$\Rightarrow$(iii) is shown in lemme 2.4. It is then sufficient to prove that (i)$\Rightarrow$(ii) The example above show that it cannot be a trivial consequence of lemma 4.7. It is in fact the crucial point of the completeness proof, and will be proved in the next section.

4.3 Construction of a validating Γ-identity for a saturated set.

We introduce first some notations. We call *id* the identical substitution.

For a formula A and a propositional variable α, we call ω_α the substitution of all occurrences of α by $\top$. $\omega_\alpha(A) = A[\top/\alpha]$. The following equivalences are trivial.

$$\alpha \to A \equiv \alpha \to \omega_\alpha(A) \quad ; \quad \alpha \wedge A \equiv \alpha \wedge \omega_\alpha(A)$$

We will need the following lemma (whose proof is immediate).

Lemma 4.9. *For any subformula B' of $\omega_\alpha(A)$, there exists a subformula B of A such that $\omega_\alpha(B) = B'$.*

In all this section Γ is the same given finite not contradictory set of formulae. We call $G = \wedge\Gamma$. Let $Var_\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ the set of all variables occuring in Γ (or in G). Note that an arbitrary order on these variables is given by the indexation.

The first (and main) step of our proof is roughly speaking to "eliminate" all the simple anti-Harrop formulae potentially in Γ (a precise formulation will be given in lemma 4.19).

We will first introduce some particular substitutions.

4.3.1 The substitutions s_i and σ_i.

Definition 4.10. *The substitutions s_i, $i \in \{1, \ldots, n\}$ are defined by induction on i and the σ_i as $\sigma_i = s_i \circ \ldots \circ s_1 \circ s_0$.*

- $s_0 = Id$,
- $s_{i+1} = [\alpha_{i+1} \wedge \omega_{\alpha_{i+1}} \circ \sigma_i(G)/\alpha_{i+1}]$.

Some straightforward consequences of the definition:

$$\begin{array}{lcl}
\text{for } i > k\ ,\ \ \sigma_k(\alpha_i) & = & \alpha_i\ ; \\
\text{for } 0 < i \leq k\ ,\ \ \sigma_k(\alpha_i) & = & s_k \circ \ldots \circ s_i(\alpha_i) \\
 & = & s_k \circ \ldots \circ s_{i+1}(\alpha_i \wedge \omega_{\alpha_i} \circ \sigma_{i-1}(G)) \\
 & \equiv & s_k \circ \ldots \circ s_i(\alpha_i \wedge \omega_{\alpha_i} \circ \sigma_{i-1}(G))\ .
\end{array}$$

Lemma 4.11. *The substitutions s_i and σ_i are Γ-identities.*

Proof. We proceed by induction on i. The base case, $s_0 = \sigma_0 = Id$, is trivial. Suppose that s_i and σ_i are Γ-identities. From σ_i is a G-identity we obtain

$$G \vdash G \leftrightarrow \sigma_i(G)\,.$$

Hence

$$G \vdash \sigma_i(G)\,.$$

But

$$\alpha_{i+1} \wedge \omega_{\alpha_{i+1}} \circ \sigma_i(G) \equiv \alpha_{i+1} \wedge \sigma_i(G)\,,$$

and then

$$G \vdash (\alpha_{i+1} \wedge \omega_{\alpha_{i+1}} \circ \sigma_i(G)) \leftrightarrow \alpha_{i+1},$$

i.e. s_{i+1} is a G-identity.

Elsewhere the stability of the set of Γ-identities by composition is a straightforward consequence of its definition, and then σ_{i+1} is a G-identity. ■

Our purpose is now to select a set of simple Harrop formulae that implies $\sigma_n(\Gamma)$ by studying maximal backward derivations of $\sigma(A)$, $A \in \Gamma$. Intending to use the saturation property we have to study subformulae of such formulae.

4.3.2 Subformulae of the $\sigma_i(A)$, $A \in \Gamma$.

There is no hope to obtain subformulae of substitute as substitute of the subformulae:

Lemma 4.12. *Let s be any substitution, A any formula. A subformula of $s(A)$ either has the form $s(B)$ where B is a subformula of A, or is a subformula of $s(\alpha)$ for a propositional variable α occurring in A.*

Proof. Straightforward induction on the complexity of A. ■

Now we introduce some notations related with the substitutions σ_i.

Definition 4.13. *Let $1 \leq i_1 < \ldots < i_l$ be a finite increasing sequence of integers. The substitution $\sigma_{i_1,\ldots,i_l;q}$ is inductively defined on q:*

- $\sigma_{i_1,\ldots,i_l;0} = \sigma_0(C) = id$;
- *if* $q+1 \notin \{i_1,\ldots,i_l\}$, *then* $\sigma_{i_1,\ldots,i_l;q+1} = s_{q+1} \circ \sigma_{i_1,\ldots,i_l;q}$;
- *if* $q+1 \in \{i_1,\ldots,i_l\}$, *then* $\sigma_{i_1,\ldots,i_l;q+1} = \omega_{\alpha_{q+1}} \circ \sigma_{i_1,\ldots,i_l;q}$.

Remarks. If $i_l > q$ and i_j is the greatest integer less than q or equal between $\{i_1,\ldots,i_l\}$, then:

$$\sigma_{i_1,\ldots,i_l;q} = \sigma_{i_1,\ldots,i_j;q} \ .$$

It is clear that for any q, $\sigma_{\emptyset;q} = \sigma_q$.

Lemma 4.14. *Let $1 \leq i_1 < \ldots < i_l$ be any increasing finite sequence of integers, then :*

$$\alpha_{i_1},\ldots,\alpha_{i_l},\sigma_{i_1,\ldots,i_l;p}(G) \vdash \sigma_p(G) \ .$$

Proof. By induction on p.
If $p = 0$, it is obvious by definition of σ_0.
Suppose that the property stand for p.

If $p+1 \in \{i_1,\ldots,i_l\}$, then $\sigma_{i_1,\ldots,i_l;p+1}(G) = \omega_{\alpha_{p+1}} \circ \sigma_{i_1,\ldots,i_l;p}(G)$. The induction hypothesis is $\alpha_{i_1},\ldots,\alpha_{i_l},\sigma_{i_1,\ldots,i_l;p}(G) \vdash \sigma_p(G)$, hence

$$\alpha_{i_1},\ldots,\alpha_{i_l},\sigma_{i_1,\ldots,i_l;p+1}(G) \vdash \sigma_p(G) \ ,$$

and then

$$\alpha_{i_1},\ldots,\alpha_{i_l},\sigma_{i_1,\ldots,i_l;p+1}(G) \vdash \omega_{\alpha_{p+1}} \circ \sigma_p(G) \ .$$

It follows that for any formula C

$$\alpha_{i_1},\ldots,\alpha_{i_l},\sigma_{i_1,\ldots,i_l;p+1}(G) \vdash s_{p+1}(C) \leftrightarrow C \ ,$$

and then the intended result:

$$\alpha_{i_1},\ldots,\alpha_{i_l},\sigma_{i_1,\ldots,i_l;p+1}(G) \vdash \sigma_{p+1}(G) \ .$$

If $p+1 \notin \{i_1,\ldots,i_l\}$, then $\sigma_{i_1,\ldots,i_l;p+1}(G) = s_{p+1}(\sigma_{i_1,\ldots,i_l;p}(G))$. The result follows by applying s_{p+1} to the induction hypothesis.

We will essentially show in lemma 4.16 that every subformula B of $\sigma_q(G)$ either is a propositional variable α_i, $i \leq q$, or is of the form $B = \sigma_{i_1,\ldots,i_l;q}(B_0)$ for a subformula B^0 in G (clear from lemma 4.12 and lemma 4.9). There is clearly one canonical choice for each occurrence of subformula in $\sigma_q(G)$, and we need to define this one for consistency reasons, what we do now.

Definition 4.15. *Let B be any occurrence of subformula of $\sigma_q(G)$. We call* type of an occurrence B of subformula in $\sigma_q(G)$, *an increasing sequence of integers $i_1 < \ldots < i_l$. We call* sequence associated with an occurrence B of subformula in $\sigma_q(G)$, *a sequence of length $q+1$ noted $B^0, \ldots, B^q = B$, such that for all $0 \le i \le q$, B^i is an occurrence of subformula of $\sigma_i(G)$. Both are inductively defined by*

- *If $q = 0$, the type of any occurrence of subformula is the empty sequence. The sequence associated with any occurrence of subformula contains only this subformula itself.*
- *Suppose that for any occurrence of subformula of $\sigma_q(G)$, the associated sequence and the type are defined. Let B be an occurrence of subformula in $\sigma_{q+1}(G)$. By lemma 4.12 three cases are possible (see also lemma 4.9).*
  - **i.** *If $B = s_{q+1}(C)$ for C a subformula of $\sigma_q(G)$, then the type of B is the type of C, the sequence associated with B is the sequence associated with C extended with B ($B^q = C$).*
  - **ii.** *If $B = \omega_{\alpha_{q+1}}(C)$ for C a subformula of $\sigma_q(G)$, Then the type of B is the type of C extended with $q+1$, the sequence associated with B is the sequence associated with C extended with B ($B^q = C$).*
  - **iii.** *$B = \alpha_{q+1}$, the type is $q+1$, the sequence associated with α_{q+1} is the constant sequence, $\underbrace{\alpha_i \ldots \alpha_i}_{q+1}$.*

We extend the notation $B^0, \ldots, B^q$ to sets of subformulae of $\sigma_q(G)$ (with eventually different types), and to sequents.

A particular case: $\sigma_q(G)$ is, as occurrence of subformula of itself, of type empty and of associated sequence $\sigma_q(G)^0 = G, \ldots, \sigma_q(G)^i = \sigma_i(G), \ldots, \sigma_q(G)^q = \sigma_q(G)$.

Note that two distinct occurrences of a same subformula could have different types and associated sequences.

Lemma 4.16. *Every subformula of $\sigma_q(G)$ either is a propositional variable α_i, $i \le q$, or is obtained from one subformula of G by one of the substitutions $\sigma_{i_1,\ldots,i_l;q}$. More precisely, let B be an occurrence of subformula in $\sigma_q(G)$ of type $i_1, \ldots, i_l$ (an increasing sequence of integers), of associated sequence $B^0, \ldots, B^q = B$.*

i. *If B is not a variable α_i for $i \le q$), then for every $j \le q$,*

$$B^j = \sigma_{i_1,\ldots,i_l;j}(B^0) ,$$

(in particular $B = \sigma_{i_1,\ldots,i_l;q}(B^0)$).

ii. *If $B = \alpha_i$, $i \le q$, then for every $0 \le j \le q$, $B^j = \alpha_i$.*

Proof of (i). By induction on q. For $q = 0$, $B = B^0$, the associated type is the empty sequence, then $B = \sigma_{\emptyset;0}(B^0)$.

Suppose now the result for q and show it for $q+1$. Let B be an occurrence of subformula in $\sigma_{q+1}(G)$, and suppose that B is not a variable α_i for $i \le q+1$. From the definition 4.15 above, we deduce that for every $j \le q$, $B^j = (B^q)^j$, and then by induction hypothesis, if $i_1, \ldots, i_l$ is the type of B^q, $B^j = \sigma_{i_1,\ldots,i_l;j}(B^0)$. The type of B is either $i_1, \ldots, i_l$, or $i_1, \ldots, i_l, q+1$.
Suppose first that $j \le q$. $B^j = \sigma_{i_1,\ldots,i_l;j}(B^0) = \sigma_{i_1,\ldots,i_l,q+1;j}(B^0)$ (see definition 4.13), and the expected result follows in both cases.
Suppose now that $j = q+1$. The occurrence of subformula B in $\sigma_{q+1}(G)$ is obtained from the subformula B^q in $\sigma_q(G)$ by applying $s_q + 1$ (definition 4.15 (i)), or $\omega_{\alpha_{q+1}}$ (definition 4.15 (ii)). In both cases, following definition 4.13,

$$B = B^{q+1} = \sigma_{i_1,\ldots,i_l,q+1;q+1}(B^0) .$$

Proof of (ii). By induction on q. Trivial for $q = 0$ because there is no α_i such that $i \leq 0$. Straightforward by applying s_{q+1} or ω_{q+1} to induction hypothesis for $q+1$ and $j \leq q$. By definition 4.15 (iii) for $j = q+1$. ■

Some easy but useful properties:

Lemma 4.17.

i. *Let B be any occurrence of subformula in $\sigma_q(G)$ such that $B^0 = E' \, c \, F'$ where c is any connective. Then $B = E \, c \, F$, and for all $j < q$, $B^j = E^j \, c \, F^j$ (in particular $E' = E^0$ et $F' = F^0$).*

ii. *Let B be any occurrence of subformula in $\sigma_q(G)$ whose type is $i_1, \ldots, i_l$ (then $B \neq \alpha_i$, $i_1 < \ldots < i_l \leq q$), and such that, for some j, $B^0 = \alpha_j$ and if $j \leq q$, $B \neq \alpha_i$.*

ii1. *if $j < i_1$,*

$$B = \sigma_{i_1,\ldots,i_l;q}(\alpha_j) = \alpha_j \wedge \sigma_{j,i_1,\ldots,i_l;q}(G)\,;$$

ii2. *if there exists k, $1 \leq k < l$ such that $i_k < j < i_{k+1}$, then*

$$B = \sigma_{i_1,\ldots,i_l;q}(\alpha_j) = \alpha_j \wedge \sigma_{j,i_{k+1},\ldots,i_l;q}(G)\,;$$

ii3. *if $i_l < j \leq q$, then*

$$B = \sigma_{i_1,\ldots,i_l;q}(\alpha_j) = \alpha_j \wedge \sigma_{j;q}(G)\,.$$

ii4. *If $j \in \{i_1, \ldots, i_l\}$, then*

$$B = \sigma_{i_1,\ldots,i_l;q}(\alpha_j) \equiv \top\,.$$

ii5. *If $j > q$, then*

$$B = \sigma_{i_1,\ldots,i_l;q}(\alpha_j) = \alpha_j\,.$$

iii. *Each occurrence of the variable α_i in $\sigma_q(G)$ such that $i \leq q$ occurs as the left son of a subformula*

$$\alpha_i \wedge \sigma_{i_1,\ldots,i_l;q}(G)\,,$$

where $i = i_1$ and $\sigma_{i_1,\ldots,i_l;q}(G)$ is effectively of type $i_1, \ldots, i_l$.

Proof of (i). By induction on q, direct consequence of definition 4.15.

Proof of (ii). By induction on q. We know that $B = \sigma_{i_1,\ldots,i_l;q}(\alpha_j)$. For $q = 0$, the type of B is the empty sequence, we are in case ii5, $B = B^0 = \alpha_j$.

Suppose now the result for q. Take $B = \sigma_{i_1,\ldots,i_l;q+1}(\alpha_j)$, $i_1, \ldots, i_l$ being the type of B, and then $i_1 < \ldots < i_l \leq q+1$ and $B^q = \sigma_{i_1,\ldots,i_l;q}(\alpha_j)$ (lemma 4.16). By definition 4.15 B^q is a subformula of $\sigma_q(G)$ and we can apply induction hypothesis. If $j > q+1$, we are in case ii5 for B and B^q. The substitution s_{q+1} do not act on α_j, the result follows then directly from induction hypothesis on B^q.
If $j \leq q$, we are in one of the cases ii1, ii2, ii3 or ii4 for B, and B^q (the same case for both).

In each case we obtain the result by applying s_{q+1} or $\omega_{\alpha_{q+1}}$ to induction hypothesis and then by using directly the definition 4.13. Let us see for instance case $q+1 = i_l$, and $j < i_1$. By induction hypothesis,

$$\sigma_{i_1,\ldots,i_{l-1};q}(\alpha_j) = \alpha_j \wedge \sigma_{j,i_1,\ldots,i_{l-1};q}(G)\,.$$

Hence, following definition 4.13,

$$\sigma_{i_1,\ldots,i_l;q+1}(\alpha_j) = \omega_{\alpha_{q+1}}(\alpha_j \wedge \sigma_{j,i_1,\ldots,i_l;q}(G)) = \alpha_j \wedge \sigma_{j,i_1,\ldots,i_l;q+1}(G)\,.$$

If $j = q+1$, then either $j = i_l$, and we deduce the result for B (case ii4) from the induction hypothesis for B^q (case ii5), or $j > i_l$, and we deduce the result for B (case ii3) from the induction hypothesis for B^q (case ii5).

Proof of (iii). By induction on q. The base case $q = 0$ is obvious, because there is no variable α_i with $i \leq 0$.
Suppose now the result for q. If the variable α_{q+1} occurs in $\sigma_{q+1}(G) = s_{q+1}(\sigma_q(G))$, it occurs in a subformula $s_{q+1}(\alpha_{q+1})$, (definition of s_{q+1}). Now:

$$s_{q+1}(\alpha_{q+1}) = \alpha_{q+1} \wedge \omega_{\alpha_q} \circ \sigma_q(G) = \alpha_{q+1} \wedge \sigma_{q+1;q+1}(G)\,.$$

The type of $\sigma_q(G)$ as subformula of itself is empty and $\sigma_q(G)^0 = G$, then $\sigma_{q+1;q+1}(G)$ is an occurrence of subformula of $\sigma_{q+1}(G)$ of type $q+1$ and $\sigma_{q+1}(G)^0 = G$ (4.15 (ii)).

Let us see now variable α_i, with $i \leq q$. The substitution s_{q+1} does not act on these variables. Using induction hypothesis, a variable α_i, $i \leq q$ occuring in $\sigma_{q+1}(G) = s_{q+1}(\sigma_q(G))$ either occurs in a subformula

$$s_{q+1}(\alpha_i \wedge \sigma_{i_1,\ldots,i_l;q}(G)) = \alpha_i \wedge \sigma_{i_1,\ldots,i_l;q+1}(G)\,,$$

or in a subformula

$$(\alpha_i \wedge \omega_{\alpha_{q+1}} \circ \sigma_{i_1,\ldots,i_l;q}(G)) = \alpha_i \wedge \sigma_{i_1,\ldots,i_l,q+1;q+1}(G)\,.$$

In both cases $i_1,\ldots,i_l$ is the type of $\sigma_{i_1,\ldots,i_l;q}(G)$, in $\sigma_q(G)$, and $\sigma_{i_1,\ldots,i_l;q}(G)^0 = G$.

In first case, by definition 4.15 (i), the type of $\sigma_{i_1,\ldots,i_l;q+1}(G)$ is $i_1,\ldots,i_l$, and

$$\sigma_{i_1,\ldots,i_l;q+1}(G)^0 = \sigma_{i_1,\ldots,i_l;q}(G)^0 = G\,.$$

In second case, by definition 4.15 (ii), the type of $\sigma_{i_1,\ldots,i_l,q+1;q+1}(G)$ is $i_1,\ldots,i_l,q+1$ and

$$\sigma_{i_1,\ldots,i_l,q+1;q+1}(G)^0 = \sigma_{i_1,\ldots,i_l;q}(G)^0 = G\,. \qquad \blacksquare$$

4.3.3 backward derivation trees of "$\sigma_p(G)$".

We set the integer p in all this paragraph.

In the following lemma we describe the particular structure of a backward derivation tree of the sequent $\vdash \sigma_p(G)$.

Lemma 4.18. *Let $\mathfrak{T}$ be a backward derivation tree of $\vdash \sigma_p(G)$ and let $\mathcal{C}$ be the associated set of sets of sequents (then $\vdash \sigma_p(G)$ $\boldsymbol{rd}\,\mathcal{C}$). Let S be a leaf occurrence of a sequent in $\mathfrak{T}$ (S occurs then in $\mathcal{C}$). Let I_S be the set of all variables that occur in the type of the formulae of S, the variables occurring in the left part of S (it there is one) excepted. Let $b(S)$ be the branch of $\mathfrak{T}$ from the root sequent to the leaf S. For each variable $\alpha_j \in I_S$, there exists two not pointed sequent S'_j and S_j,*

$$S'_j = \Sigma_j \vdash \sigma_{i_1,\ldots,i_l;p}(G) \;\; \textit{and} \;\; S_j = \Sigma_j \vdash \alpha_j \;\; \textit{with } 1 \leq i_1 < \ldots < i_l \leq p \textit{ et } j \in \{i_1,\ldots,i_l\}\,,$$

such that:

- **i.** *S'_j occurs in $b(S)$, as the premise of a rule ($\wedge$ right), the other premise of this rule being S_j. Moreover each sequent of $b(S)$ occurring between the root and S'_j, does not contain any formula of type $k_1,\ldots,k_m;p$ such that $j \in \{k_1,\ldots,k_m\}$ (in particular, the set of formulae Σ_j has the same property).*

ii. *The left part Σ_j of the sequent S_j is included in the left part of S. Moreover for each sequent S we can set an order $<_S$ on I_S: $I_S = \{\alpha_{s_1}, \ldots, \alpha_{s_t}\}$, such that if $e <_S f$, then $\Sigma_{s_e} \subset \Sigma_{s_f}$.*

iii. *For every set $\mathcal{E}$,$\mathcal{E} \in \mathcal{C}$ such that $S \in \mathcal{E}$, $\{S_j,\ \ j \in I_S\} \subset \mathcal{E}$.*

Proof of (i). Let S'_j be the first occurrence of a sequent in $b(S)$ (starting from the root), such that there exists a formula A of type $(k_1, \ldots, k_m)$ occurring in S'_j, verifying $j \in \{k_1, \ldots, k_m\}$. This formula is the the secondary formula of the rule S'_j is a premise of.

Let $B^{\cdot}$ be the main formula of the rule S'_j is a premise of. Then A is an immediate subformula of B. If B^0 is a compound formula, $B^0 = (E\ c\ F)$, by lemma 4.17 (i)), $B = \sigma_{k_1,\ldots,k_m;p}(E\ c\ F)$, a contradiction because $j \in \{k_1, \ldots, k_m\}$ and B occurs in a sequent under S'_j. We can then assume that B^0 is an atomic formula. If $B^0 = \bot$, $B = \bot$, and then $A = \bot$ (with same type), which is also a contradiction. We can then assume that B^0 is a variable. See lemma 4.17 (ii): as j does not occur in the type of B main formula of the rule, but j occurs in the type of A, and A is an immediate subformula of B, then necessarily $B^0 = \alpha_j$, and

$$B = \alpha_j \wedge \sigma_{k_1,\ldots,k_m;p}(G) \ \ ; \ \ A = \sigma_{k_1,\ldots,k_m;p}(G)\,.$$

If $B^{\cdot}$ would occur in the left part of the sequent S'_j is a premise of, then S'_j would be the leaf sequent S and al_j would occurs in S, which contradicts $\alpha_j \in I_S$. We can then assume that B occurs in the right part of the sequent S'_j is a premise of, which gives straightforwardly the expected result.

Proof of (ii). We define now the following order on I_S, $\alpha_i < \alpha_j$ if and only if S'_i occurs before S'_j in $b(S)$ (starting from the root). The result follows now from the particular formulation of the sequent calculus we gave: see figure 2, for all rules the left part of the conclusion sequent is a subset of the left part of a premise sequent.

Proof of (iii). For every $\alpha_j \in I_S$ the sequent S_j is a leaf of $\mathfrak{T}$ (because α_j is the right formula of S_j). By construction of S_j, every derivation described by $\mathfrak{T}$ and containing S (and then a branch corresponding to $b(S)$), contains S_j (following (i) of the same lemma). ■

We will suppose later that G is a saturated set. We recall that saturation property is restricted to $\mathcal{F}^{\rightarrow,\wedge,\vee}(G)$. Formulae associated with set of sets of sequent contains subformulae in $\sigma_p(G)$ are not in $\mathcal{F}^{\rightarrow,\wedge,\vee}(G)$. So we will need the following lemma.

Lemma 4.19. *Let $\mathfrak{T}$ be a backward derivation tree of $\vdash \sigma_p(G)$ and let $\mathcal{C} = \mathcal{C}_1 \vee \ldots \vee \mathcal{C}_d$ be the associated set of sets of sequents (then $\vdash \sigma_p(G)$ **rd** $\mathcal{C}$). We define $\mathcal{C}^0, \mathcal{C}^0_1, \ldots, \mathcal{C}^0_d$ as in definition 4.15 above ($\mathcal{C}^0, \mathcal{C}^0_1, \ldots, \mathcal{C}^0_d$ are sets of sequent made up with subformulae of G). Then, for all $r \in \{1, \ldots, d\}$:*

$$G \vdash \overrightarrow{\mathcal{C}_r} \leftrightarrow \overrightarrow{\mathcal{C}^0_r} \quad \textit{and} \quad G \vdash \overrightarrow{\mathcal{C}} \leftrightarrow \overrightarrow{\mathcal{C}^0}\,.$$

Proof. The second equivalence is a consequence of the first. We will prove by induction on q, $q \le p$ that :

$$G \vdash \overrightarrow{\mathcal{C}^q_r} \leftrightarrow \overrightarrow{\mathcal{C}^0_r}\,.$$

It gives the first equivalence for $q = p$, because for every subformula A of $\sigma_p(G)$, $A^p = A$ (see definition 4.15).
The result holds trivially for $q = 0$.
Suppose now the result for $q < p$ and let us show it for $q + 1$. It is sufficient to prove that $G \vdash \overrightarrow{\mathcal{C}^q_r} \leftrightarrow \overrightarrow{\mathcal{C}^{q+1}_r}$. For a given subformula A of type $(i_1, \ldots, i_l)$ occuring in $\mathcal{C}_r$, we have three possible cases (following 4.15):

(a) $q+1 \notin \{i_1, \ldots, i_l\}$, then :

$$A^{q+1} = \sigma_{i_1,\ldots,i_l;q+1}(A^0) = s_{q+1}(\sigma_{i_1,\ldots,i_l;q}(A^0)) = s_{q+1}(A^q) \ ,$$

and then, as s_{q+1} is a G-identity, $G \vdash A^q \leftrightarrow A^{q+1}$.

(b) $q+1 \in \{i_1, \ldots, i_l\}$ and $A = \alpha_{q+1}$. Then for all $i \leq p$, $A^i = \alpha_{q+1}$;

(c) $q+1 \in \{i_1, \ldots, i_l\}$, then $A^{q+1} = w_{q+1}(A^q)$, and then $\alpha_{q+1} \vdash A^{q+1} \leftrightarrow A^q$.

Let us call $\mathcal{C}_r^{q,q+1}$ the set of sequents we obtain by substituting A^{q+1} to A^q in $\mathcal{C}_r^q$ for all formulae A of type $i_1, \ldots, i_l$ such that $q+1 \notin \{i_1, \ldots, i_l\}$. Let us call $S^{q,q+1}$ the sequent in $\mathcal{C}_r^{q,q+1}$ obtained from S^q of $\mathcal{C}_r^q$. Following (a) above,

$$G \vdash \overrightarrow{\mathcal{C}_r^{q,q+1}} \leftrightarrow \overrightarrow{\mathcal{C}_r^q} \ . \tag{$*$}$$

In order to obtain $\mathcal{C}_r^q$ from $\mathcal{C}_r^{q,q+1}$ we have to replace formulae A^q such that A is of type $i_1, \ldots, i_l$ and $q+1 \in \{i_1, \ldots, i_l\}$. Every sequent $S \in \mathcal{C}_r$ falls in one of the four following cases:

(1) $S = \Sigma \vdash \alpha_{q+1}$ such that Σ does not contain formulae of type $k_1, \ldots, k_m$ with $q+1 \in \{k_1, \ldots, k_m\}$[3]. In that case $S^{q,q+1} = S^{q+1}$.

(2) S do not contain formulae of type $k_1, \ldots, k_m$ with $q+1 \in \{k_1, \ldots, k_m\}$ In that case $S^{q,q+1} = S^{q+1}$.

(3) $S = \alpha_{q+1}, A, \Delta \vdash C$. Following (b) and (c) above, $\overrightarrow{S^{q,q+1}} \equiv \overrightarrow{S^{q+1}}$.

(4) S contains no variable in right or left part, and at least one formula of type $k_1, \ldots, k_m$ with $q+1 \in \{k_1, \ldots, k_m\}$, that is $q+1 \in I_S$. We can write $S = \Sigma, \Delta \vdash E$, with Σ containing only formulae of type $k_1, \ldots, k_m$ with $q+1 \notin \{k_1, \ldots, k_m\}$, and Δ containing only formulae of type $k_1, \ldots, k_m$ with $q+1 \in \{k_1, \ldots, k_m\}$. Then

$$S^{q,q+1} = \Sigma^{q+1}, \Delta^q \vdash E^{q,q+1} \quad \text{with } E^{q,q+1} = E^q \text{ or } E^{q,q+1} = E^{q+1} \ .$$

Because $q+1 \in I_S$, by lemma 4.18 (iii) and (ii), there exists a sequent $S_{q+1} = \Sigma_{q+1} \vdash \alpha_{q+1}$ such that $S_{q+1} \in \mathcal{C}_r$ and $\Sigma_{q+1} \subset \Sigma$. Following (1) above,

$$S_{q+1}^{q,q+1} = S_{q+1}^{q+1} = \Sigma_{q+1}^{q+1} \vdash \alpha q + 1 \ .$$

Because $\Sigma_{q+1} \subset \Sigma$ and (c) above, for each subformula A of $\sigma_p(G)$:

$$\overrightarrow{S_{q+1}^{q+1}}, \Sigma^{q+1} \vdash A^q \leftrightarrow A^{q+1} \ .$$

Applying this to formulae in Δ, and to C if necessary, we obtain:

$$\overrightarrow{S_{q+1}^{q+1}} \vdash \overrightarrow{S^{q,q+1}} \leftrightarrow \overrightarrow{S^{q+1}} \ \text{ and } \ S_{q+1} \in \mathcal{C}_r \ .$$

Resuming 4 cases above,(1) necessary before (4), we obtain:

$$\overrightarrow{\mathcal{C}^{q+1}} \equiv \overrightarrow{\mathcal{C}^{q,q+1}}$$

Then using (*),

$$G \vdash \overrightarrow{\mathcal{C}^{q+1}} \leftrightarrow \overrightarrow{\mathcal{C}^{q}}$$

and by induction hypothesis

$$G \vdash \overrightarrow{\mathcal{C}^{q+1}} \leftrightarrow \overrightarrow{\mathcal{C}^{0}} \ . \qquad \blacksquare$$

[3] It is "almost" always the case for a sequent with α_{q+1} as right formula, but we will note use this fact

4.3.4 "Elimination" of anti-Harrop formulae.

We give now two consequences of lemma 4.19. The first one is a kind of extension of the saturation property to the formulae we deal with.

Lemma 4.20. *We suppose that G is saturated. Let $\mathfrak{T}$ be a backward derivation tree of $(\vdash \sigma_p(G))$ and let $\mathcal{C} = \mathcal{C}_1 \vee \ldots \vee \mathcal{C}_d$ be the associated set of sets of sequents, such that $(\vdash \sigma_p(G))$ $\boldsymbol{rd}\,\mathcal{C}$. (same notations as in lemma 4.19).*

Then, there exists $r \in \{1, \ldots, d\}$ such that:

$$G \vdash \overrightarrow{\mathcal{C}_r} \vdash \sigma_p(G) \ .$$

Proof. We know, because σ_p is a G-identity, that $G \vdash \sigma_p(G)$. We know, because definition of forward and backward relation ">", that:

$$\sigma_p(G) > \overrightarrow{\mathcal{C}_1} \vee \ldots \vee \overrightarrow{\mathcal{C}_d} \ ,$$

hence :

$$G > \overrightarrow{\mathcal{C}_1} \vee \ldots \vee \overrightarrow{\mathcal{C}_d} \ ,$$

Using lemma 4.19, we obtain:

$$G > \overrightarrow{\mathcal{C}^0_1} \vee \ldots \vee \overrightarrow{\mathcal{C}^0_d} \ .$$

But clearly, $\overrightarrow{\mathcal{C}^0} \in \mathcal{F}^{\rightarrow,\wedge,\vee}(G)$. We can then use the definition 4.4 of saturation:

$$\text{there exists } r \text{ such that } G \vdash \overrightarrow{\mathcal{C}^0_r} \ .$$

We apply another time lemma 4.19:

$$\text{there exists } r \text{ such that } G \vdash \overrightarrow{\mathcal{C}_r} \ .$$

We have $\overrightarrow{\mathcal{C}_r} \vdash \sigma_p(G)$ as a consequence of the definition of the relation "$\boldsymbol{rd}$" ($\mathcal{C}_r$ contains all leaf sequents of a derivation of $\sigma_p(G)$). ■

The following lemma allows us, in a sense, to "eliminate" anti-harrop formulae, in case G is saturated. We will use it also to show that we can recursively saturate G.

Lemma 4.21. *Let $\mathfrak{T}$ be a backward derivation tree of $(\vdash \sigma_p(G))$ and let $\mathcal{C} = \mathcal{C}_1 \vee \ldots \vee \mathcal{C}_d$ be the associated set of sets of sequents, such that $(\vdash \sigma_p(G))$ $\boldsymbol{rd}\,\mathcal{C}$. (same notations as in lemma 4.19). We suppose that there exists $r \in \{1, \ldots, d\}$ such that:*

$$G \vdash \overrightarrow{\mathcal{C}_r} \vdash \sigma_p(G) \ .$$

Then every sequent S from $\mathcal{C}_r$ with propositional variable in left part (corresponding to an anti-Harrop formula) is a consequence of the subset $\mathcal{C}_{r,d}$ of $\mathcal{C}_r$ with propositional variable in right part (corresponding to Harrop formulae):

$$\overrightarrow{\mathcal{C}_{r,d}} \vdash \overrightarrow{S} \ .$$

Proof. Let S be a sequent with propositional variable in left part, in $\mathfrak{T}$. Because restriction on backward derivation tree, S is a leaf of $\mathfrak{T}$, and S is a premise of a $(\wedge_{\text{left}})$ rule with a principal formula like $\alpha_i\ c\ A$ or $A\ c\ \alpha_i$. Following lemma 4.17(iii)we know that this formula has the form $\alpha_i \wedge \sigma_{i_1,\ldots,i_l;p}(G)$ with $i \in \{i_1, \ldots, i_l\}$. Then we can write:

$$S = (\alpha_i, \sigma_{i_1,\ldots,i_l;p}(G), \Delta \vdash C) \ .$$

Always following lemma 4.17(iii), this occurrence of $\sigma_{i_1,\ldots,i_l;p}(G)$ is effectively of type $i_1,\ldots,i_l$, and then $\sigma_{i_1,\ldots,i_l;p}(G)^0 = G$. We can then apply lemma 4.16 (ii):

$$S^q = (\alpha_i, \sigma_{i_1,\ldots,i_l;q}(G), \Delta^q \vdash C^q) \ .$$

We take the same notations as lemma 4.18: let I_S be the set of all indices occurring in types of formulae in S, but i (in particular $i_1,\ldots,i_l \in I_S$). For each $\alpha_j \in I_S$ let $S_j = \Sigma_j \vdash \alpha_j$ defined as in lemme 4.18, such that (4.18(iii)), $S_j \in \mathcal{C}_r$, and (4.18 (ii)) left parts Σ_j are totally ordered by inclusion, and included in the left part of$S$, hence in Δ (4.18 (i)). In particular, a formula Σ_j owns a type $k_1,\ldots,k_m$ such that $\{k_1,\ldots,k_m\} \subset I_S$.

We proceed (similarly to lemma 4.19) by induction on q where $q \leq p$ and p is fixed, showing that :

the sequent $\quad P_q = \{\Sigma_j^q \to \alpha_j \ , \ \ j \in I_s\}, \alpha_i, \sigma_{i_1,\ldots,i_l;q}(G), \Delta^q \vdash C^q \quad$ is provable .

For $q = 0$, in lemme 4.19, we prove that $G \vdash \mathcal{C}_r^{0\to}$, and then $S^0 = \alpha_i, G, \Delta^0 \vdash C^0$ is provable.
Suppose now the result for $q < p$ and let us show it for $q+1$. Recall that all indices in the types of formulae in (P_q) but i own to I_S. We will distinguish three cases.

First case : $q+1 \notin I_S$, and $q+1 \neq i$.
Applying substitution s_{q+1} to P_q we obtain P_{q+1}, which is then provable.

Second case : $q+1 = i$ (therefore $q+1 \notin I_S$ and $q+1 \in \{i_1,\ldots,i_l\}$).
Note that therefore α_{q+1} occurs left in P_q. We then obtain a provable sequent by changing in P_q all formulae A^q where type of A is $k_1,\ldots,k_m;p$ and $q+1 \in \{k_1,\ldots,k_m\}$, by $A^{q+1} = \omega_{\alpha_{q+1}}(A)$. We apply now s_{q+1} to this sequent. The formulae $A^{q+1} = \omega_{\alpha_{q+1}}(A)$ stay unchanged. The following sequent is then provable:

$$\{\Sigma_j^{q+1} \to \alpha_j \ , \ \ j \in I_s\}, \alpha_{q+1}, \omega_{\alpha_{q+1}} \circ \sigma_q(G), \sigma_{i_1,\ldots,i_l;q+1}(G), \Delta^{q+1} \vdash C^{q+1} \,. \qquad (*)$$

Recall that $q+1 \in \{i_1,\ldots,i_l\}$, and then $\sigma_{i_1,\ldots,i_l;q+1}(G) = \omega_{\alpha_{q+1}} \circ \sigma_{i_1,\ldots,i_l;q}(G)$.

From lemma 4.18, all Σ_j are subsets of Δ, and then for every $j \in I_s$ we can prove:

$$\{\Sigma_j^{q+1} \to \alpha_j \ , \ \ j \in I_s\}, \alpha_{q+1}, \sigma_{i_1,\ldots,i_l;q+1}(G), \Delta^{q+1} \vdash \alpha_j \,. \qquad (**)$$

Elsewhere from lemma 4.14,

$$\alpha_{i_1}, \ldots, \alpha_{i_l}, \sigma_{i_1,\ldots,i_l;q}(G) \vdash \sigma_q(G) \,,$$

therefore

$$\alpha_{i_1}, \ldots, \alpha_{i_l}, \omega_{\alpha_{q+1}} \circ \sigma_{i_1,\ldots,i_l;q}(G) \vdash \omega_{\alpha_{q+1}} \circ \sigma_q(G) \,,$$

i.e.

$$\alpha_{i_1}, \ldots, \alpha_{i_l}, \sigma_{i_1,\ldots,i_l;q+1}(G) \vdash \omega_{\alpha_{q+1}} \circ \sigma_q(G) \,.$$

From definition of I_S, $\{i_1,\ldots,i_l\} - \{q+1\} \subset I_S$, hence from assertion $(**)$ above,

$$\{\Sigma_j^{q+1} \to \alpha_j \ , \ \ j \in I_s\}, \alpha_{q+1}, \sigma_{i_1,\ldots,i_l;q+1}(G), \Delta^{q+1} \vdash \omega_{\alpha_{q+1}} \circ \sigma_q(G) \,,$$

and then, from assertion $(*)$ the required result :

$$\{\Sigma_j^{q+1} \to \alpha_j \ , \ \ j \in I_s\}, \alpha_{q+1}, \sigma_{i_1,\ldots,i_l;q+1}(G), \Delta^{q+1} \vdash C^{q+1} \,.$$

Third case : $q+1 \in I_S$ (therefore $q+1 \neq i$).
From $q+1 \in I_S$ we know that $\Sigma_{q+1} \to \alpha_{q+1}$ occurs left in P_q. We know that Σ_{q+1} does not contain formulae of type $k_1, \ldots, k_m; p$ with $q+1 \in \{k_1, \ldots, k_m\}$. We know also that $\Sigma_{q+1} \subset \Delta$, and $\Sigma_{q+1} \subset \Sigma_j$ for $j \in I_S$ (lemme 4.18). From $q+1 \in I_S$, we know that $\Sigma_{q+1} \to \alpha_{q+1}$ occurs right in P_q.

Provability if P_q is then preserved, by replacing in Σ_j^q and in S^q , formulae A^q , with A of type $k_1, \ldots, k_m; p$ and $q+1 \in \{k_1, \ldots, k_m\}$, by $A^{q+1} = \omega_{\alpha_{q+1}}(A^q)$ (this does not modify sequent $\Sigma_{q+1} \vdash \alpha_{q+1}$).

We then apply s_{q+1} to the sequent we have obtained. After decomposing $\Sigma_{q+1}^{q+1} \to \alpha_{q+1} \wedge \sigma_q(G)^{\alpha_{q+1}}$ in $\Sigma_{q+1}^{q+1} \to \alpha_{q+1}$ and $\Sigma_{q+1}^{q+1} \to \sigma_q(G)^{\alpha_{q+1}}$:

$$\{\Sigma_j^{q+1} \to \alpha_j \ , \ \ j \in I_s\}, \Sigma_{q+1}^{q+1} \to \sigma_q(G)^{\alpha_{q+1}}, \alpha_i, \sigma_{i_1,\ldots,i_l;q+1}(G), \Delta^{q+1} \vdash C^{q+1} \, .$$

Lemma 4.18 says that Σ_j's are all included in Δ. We deduce that for all $j \in I_s$:

$$\{\Sigma_j^{q+1} \to \alpha_j \ , \ \ j \in I_s\}, \alpha_i, \sigma_{i_1,\ldots,i_l;q+1}(G), \Delta^{q+1} \vdash \alpha_j \qquad (**)$$

We know, by definition of I_S, that $\{i_1, \ldots, i_l\} - \{i\} \subset I_S$. In the same manner as in preceding case, we deduce from $(**)$ and lemma 4.14 that :

$$\{\Sigma_j^{q+1} \to \alpha_j \ , \ \ j \in I_s\}, \alpha_{q+1}, \sigma_{i_1,\ldots,i_l;q+1}(G), \Delta^{q+1} \vdash \sigma_q(G)^{\alpha_{q+1}} \, ,$$

and the required result:

$$\{\Sigma_j^{q+1} \to \alpha_j \ , \ \ j \in I_s\}, \alpha_i, \sigma_{i_1,\ldots,i_l;q+1}(G), \Delta^{q+1} \vdash C^{q+1} \, . \qquad \blacksquare$$

4.3.5 Proof of completeness

In order to achieve the proof we use a syntactical equivalent of proposition 1.7.

Lemma 4.22. *For each finite set of formula Γ, there exists a finite number of finite sets of formulas $\Gamma_1, \ldots, \Gamma_n$ such that*

i. $\wedge\Gamma > (\wedge\Gamma_1) \vee \ldots \vee (\wedge\Gamma_n)$ *and* $(\wedge\Gamma_1) \vee \ldots \vee (\wedge\Gamma_n) \vdash F$

and for all Γ_i :

ii. $\Gamma_i \subset \mathcal{F}^{\to}(\Gamma)$ *;*

iii. Γ_i *is saturated ;*

iv. Γ_i *contains only simple Harrop and anti-Harrop formulas.*

Proof. By lemma 4.5 and ii, it is sufficient to show that Γ_i's are Γ-saturated to obtain iii. Lemma 4.7 will then give iv.

In order to get i, ii for Γ-saturated Γ_i's, we use iteratively the following step inside the finite set $\mathcal{F}^{\to}(\Gamma)$ (resulting formula is in $\mathcal{F}^{\to,\wedge,\vee}(A)$, see definition 4.1).

- Suppose that we obtain a finite set of subsets of $\mathcal{F}^{\to}(\Gamma)$ satisfying i, such that no one is the subset of another one, and such that at least one of them, Δ, is not Γ-saturated.
- Then for some $\Delta_i \subset \mathcal{F}^{\to}(\Gamma)$, $\Delta > \bigvee_j(\wedge\Delta_i)$. We have $\bigvee_j(\wedge\Delta \bigcup \Delta_j) \vdash \wedge\Gamma$. Replace then Δ by the sets $\{\Delta \cup \Delta_j\}$, and keep only maximal sets for subset relation.

The process will stop as $\mathcal{F}^{\to}(\Gamma)$ is finite, the step always produce at least one bigger $\Delta \bigcup \Delta_j$ than the original δ, because Δ is not saturated, and then one of the Δ_j's is not consequence (and then not subset) of Δ, and cannot be erased as subset of another set, otherwise Δ already be. $\blacksquare$

We can the prove proposition 1.7 and as a corollary theorem 1.6. Take $\Gamma = \{F\}$ and $F^{ad} = \bigvee_{i=1}^{n}(\wedge\Gamma_i)$ from lemma 4.22. As each Γ_i is saturated, it has disjunction property for admissibility by proposition 4.8, and the same admissible and derivable consequences (lemma 2.4).

Decidability is derived from the construction of F^{ad} given in proof of lemma 4.22, and decidability of the intuitionistic propositional calculus

5 Consequences

Some other consequences are given in [Ro 92a] (3rd part). The main one is an axiomatisation of admissibility by composing usual deduction and the infinite set of admissible rules :

$$\{\alpha_i \to \beta_i\}_{1\leq i\leq n} \to (\gamma \vee \delta) \gg \begin{cases} \bigvee_{j=1}^{n} (\{\alpha_i \to \beta_i\}_{1\leq i\leq n} \to \alpha_j) \\ \vee \\ (\{\alpha_i \to \beta_i\}_{1\leq i\leq n} \to \gamma) \\ \vee \\ (\{\alpha_i \to \beta_i\}_{1\leq i\leq n} \to \delta) \end{cases} \qquad (ad_n)$$

As it ad_{n+1} can not be obtained with $(ad_i)_{i<n}$, it allows to show that there is no finite axiomatisation of admissibility upon derivability. This last result is already known from Rybakov [Ry 85].

The result is shown by transformations on formal backward derivations in sequent calculus, using ideas already known for proof search in intuitionistic sequent calculus, and apparently first occuring in [Vo 58].

References

[Fr 75] H.Friedman, *102 problems in mathematical logic*, J. Symb. Logic, 40, no 2, pp. 113-130 (1975).

[Mi 72] G.E.Mints, *Derivability of admissible rules*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 32 85-89 (1972); English transl. in J. Soviet Math. 6, no 4 (1976).

[Ro 92a] P. Rozière, *Règles admissibles en calcul propositionnel intuitionniste*, Thèse de Doctorat Université Paris VII, (Mai 92).

[Ro 92b] P. Rozière, *Admissible and derivable rules* Mathematical Structures in Computer Science (accepted in september 91).

[Ry 84] V.V.Rybakov, *A criterion for admissibility of rules in the modal system S4 and the intuitionistic logic*, Algebra i Logika 23 (1984); English transl. in Algebra and logic 23 no 5, pp.369-384 (1984).

[Ry 85] V.V.Rybakov, *Bases of admissible rules of the logics S4 and Int*, Algebra i Logika 24 (1985); English transl. in Algebra and logic 24 no 1, pp.55-68 (1985).

[Ry 86] V.V.Rybakov, *Decidability of admissibility in the modal system Grz and in intuitionistic logic*, Izv. Akad. Nauf. SSSR, ser. Mat. 50 (1986); English transl. in Math. USSR Izvestia, Vol. 28, no 3 (1987).

[Tr 88] A.S.Troelsta, D.Van-Dalen, *Constructivism in Mathematics, an introduction*, Studies in Logic and the Foundations of Mathematics vol. 121, North-holland (1988).

[Vo 58] N.N. Vorob'ev, *A new derivability algorithm in the constructive propositional calculus*, In Trudy Math. Inst. Steklov, volume 52. Moscow, 1958.