Ticket Entailment is decidable

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The Logic $T_\to$ of "Ticket Entailment"

Modus ponens +

(I)	$(\alpha \to \alpha)$
(B)	$(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$
(B')	$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)))$
(W)	$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta))$

- References in Relevance Logic

  Ackermann 1956 — Anderson & Belnap 1975 [1]

  Anderson 1960 — Riche & Meyer 1999 [2]

- *Problem* (circa 1960 [1][2]): *is $T_\to$ decidable?*

- Equivalently, in Combinatory Logic + simple types:

I	:	$(\alpha \to \alpha)$	I x	$\triangleright$	x
B	:	$(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$	B $f\, g\, x$	$\triangleright$	$f\,(g\, x)$
B'	:	$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)))$	B' $g\, f\, x$	$\triangleright$	$f\,(g\, x)$
W	:	$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta))$	W $f\, x$	$\triangleright$	$f\, x\, x$

Problem (eq.): *is type inhabitation within* BB'IW *decidable?*

- Digression: this basis (and others) leads to a natural question – what kind of reasonings does it correspond to?

The Logic $T_{\to}$ – Historical background

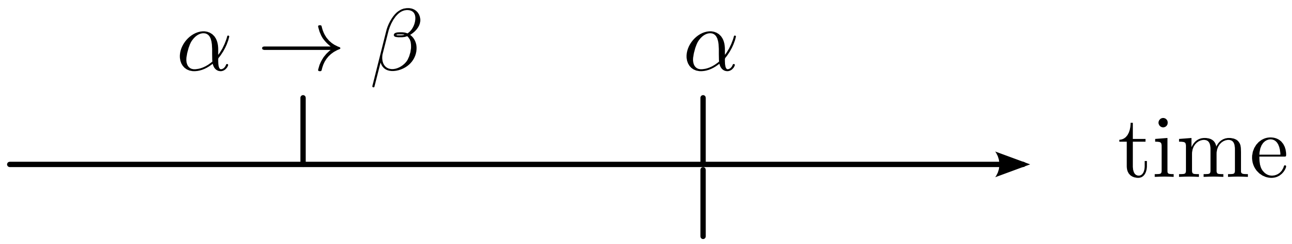
- Ackermann (1956), Anderson and Belnap ($\sim$ 1960 - 1975+).

"A law is used as, so to speak, an inference-ticket (a season ticket) which licences its possessors to move from asserting factual statements to asserting other factual statements."

(Ryle 1949) in *"Entailment: The Logic of Relevance and Necessity, Vol. 1*, (Anderson and Belnap 1975)

(does it make sense? hardly without a natural deduction)

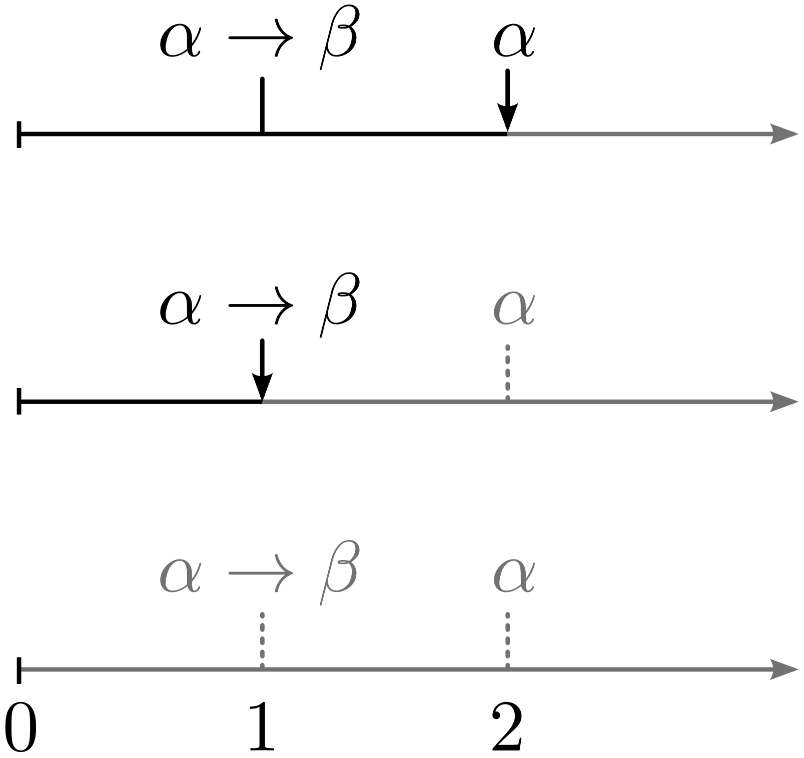
- a reasoning in $T_{\rightarrow}$ can be seen as occuring through *time*...
- β can be deduced from $\alpha \rightarrow \beta$ and α provided $\alpha \rightarrow \beta$ was introduced or proven *before* α.



hence α,
and simultaneously, β

- *all* hypothesis must be used (K is not in the basis).

- abstraction acts as a time-warp: the clock returns to the time of the last introduced hypothesis (or to 0).



$\alpha \to \beta$ (1), α (2) $\vdash \beta$ (2)

$\alpha \to \beta$ (1) $\vdash \alpha \to \beta$(1)

$\vdash (\alpha \to \beta) \to (\alpha \to \beta)$ (0)

- the last introduced hypothesis must be the first abstracted. $\alpha \to ((\alpha \to \beta) \to \beta)$ is *not* a theorem of $T_{\to}$.

- the theorems are all formulas provable at time 0.

A never-ending quest?

- *"Problem: is $T_{\to}$ decidable?"*

  Anderson and Belnap 1975, chapter 7, page 69.

- *"Warning: In the 30 years since 1975 the $T_{\to}$ problem and its combinatory equivalent have been tried by several very able workers without success."*

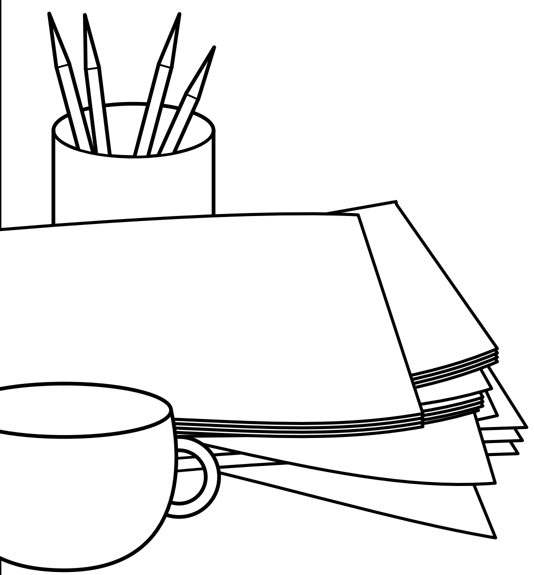
  TLCA open problems page, problem $\sharp$ 2, 2006.

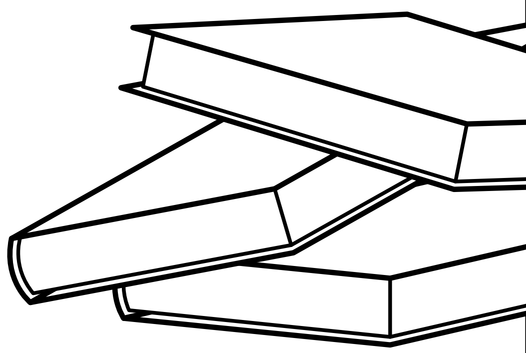
scary...
but the problem
looks *interesting*,
so why bother?

Reinventing the wheel, again and again and...

(2006-2009)

T-translations	HRM-terms! (Bunder 1996)
Kripke-like semantics	Routley & Meyer semantics! (1974)
"strange" orderings	Well quasi-orderings!
multiset theorem	Higman theorem! (1952)
stuck... any generalisation to trees?	Kruskal theorem! (1960) Melliès theorem! (1998)





- was it a waste of time? *no*. After
  - eleven versions of the proof (5178568 keys pressed),
  - 16425 hours of work,
  - 821 litres of coffee,
  - 6570 hours of chronic insomnia,

  it *worked.*

- last gaps fixed in late 2009,
  paper submitted in June 2010,
  accepted in December 2011, published in 2012.

- it's time to give more details about the proof itself...

Summary of the proof

Step 1: translation into a type inhabitation problem
in $\Lambda_\to$ + structural constraints (Bunder 1996)

Step 2: study of the properties of minimal inhabitants
(difficulty level: hum... not easy)

Step 3: an algorithm for the computation of "compact" terms
(difficulty level: hurt me plenty)

Step 4: proof of termination
(difficulty level: nightmare!).

Step 1: from BB'IW to $\Lambda_\rightarrow$

I	B	B'	W
$\lambda x.x$	$\lambda f g x.(f\,(g\,x))$	$\lambda f g x.(g\,(f\,x))$	$\lambda f x.(f\,x\,x)$

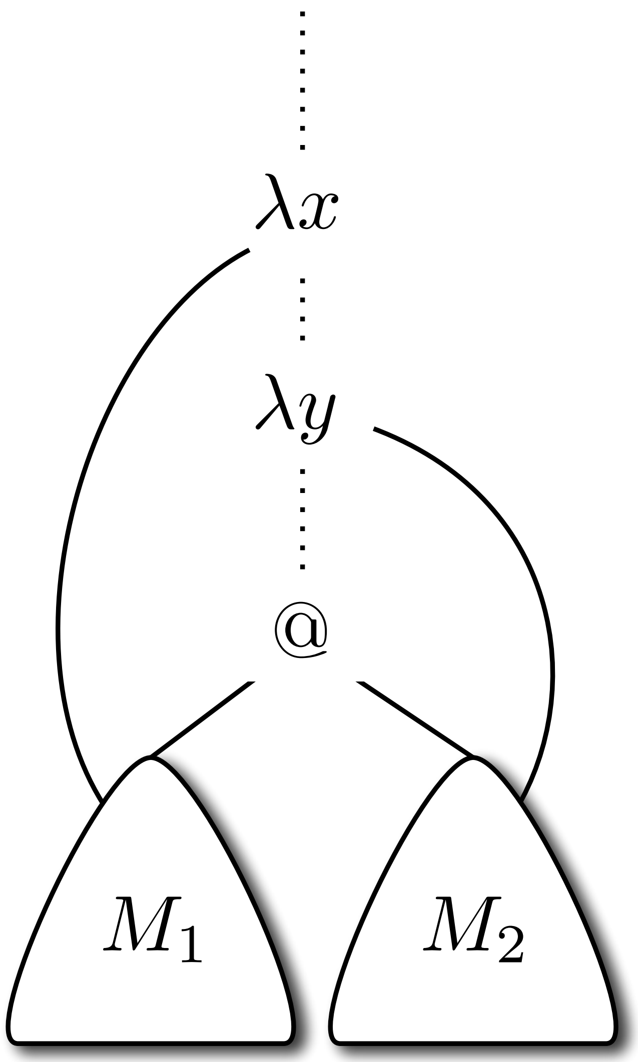
ϕ is provable in $T_\rightarrow$

$\Leftrightarrow$ ϕ is inhabited by some u within BB'IW

$\Leftrightarrow$ ϕ is inhabited by the translation of u in $\Lambda_\rightarrow$.

... fine, but if we are looking for $\Lambda_\rightarrow$-inhabitants *in normal form*, we need a characterisation of all *reducts* of translations.

Hereditarily right-maximal terms (Bunder 1996)

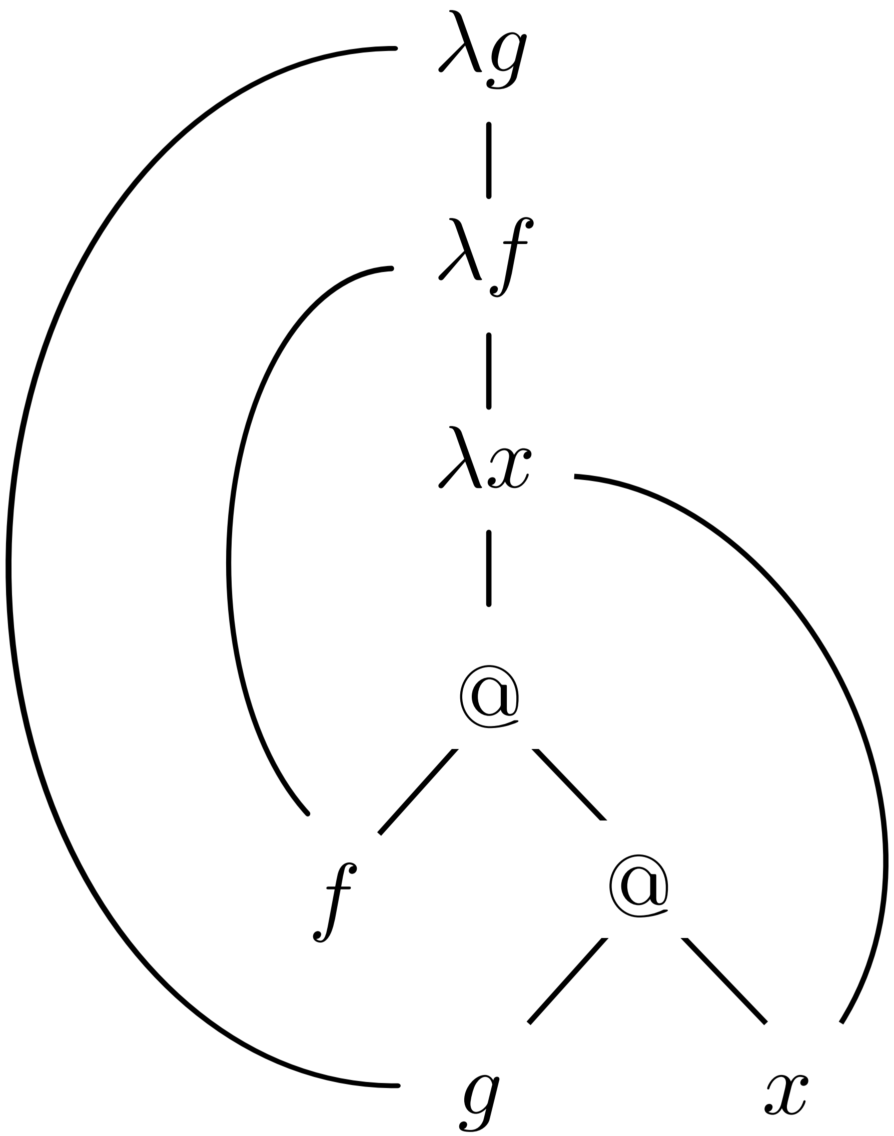


(1) no dummy λ

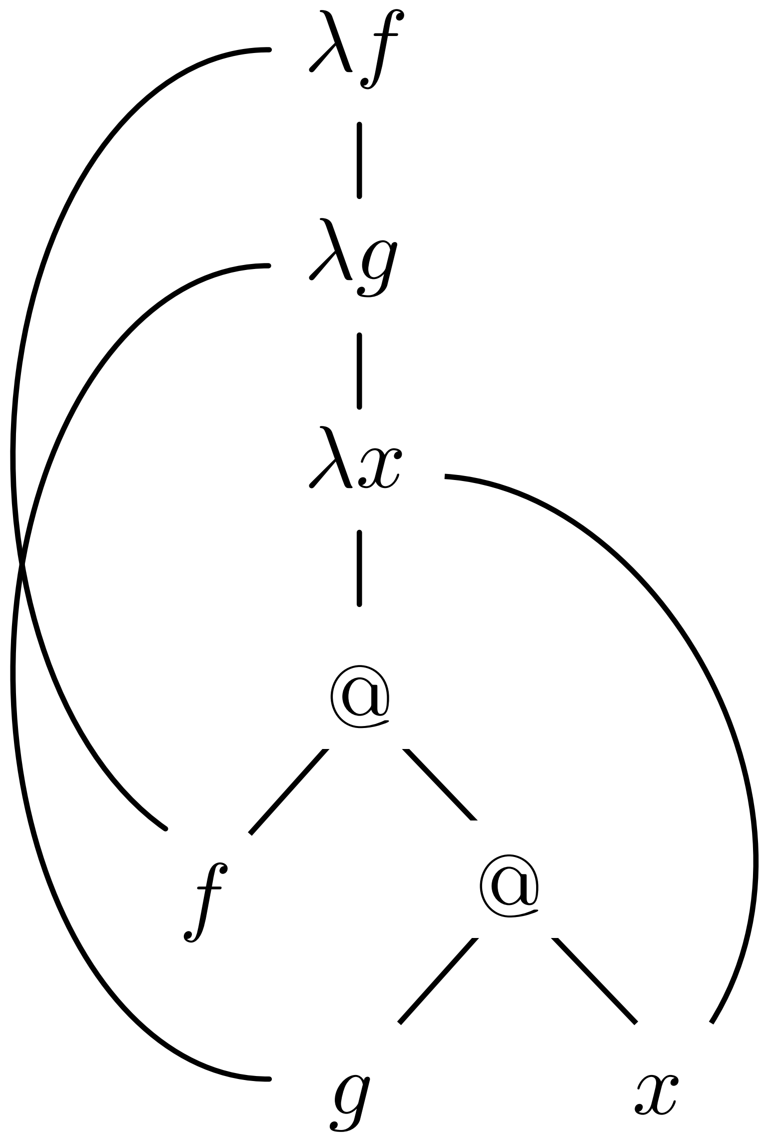
(2) M_2 closed $\Rightarrow$ M_1 closed

(3) going from the subterm to the root,

the first λ binding a variable of M_2
is below or equal to
the first λ binding a variable of M_1



$$B' = \lambda g f x.(f(g\,x))$$



$$B = \lambda f g x.(f(g\,x))$$

- Fix some order on the set of all variables:

$$x_0 < x_1 < x_2 \dots$$

  every variable is HRM.

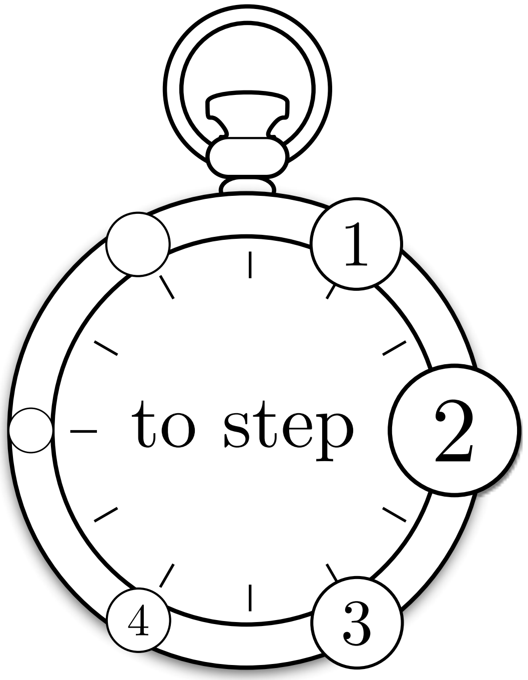
- $\lambda x.M$ – x must be the greatest free variable of M.

- (MN) – the greatest free variable of M (if any) must be less than or equal to the greatest free variable of N.

The set of HRM terms is closed under reduction:

$$\begin{array}{ll} & \phi \text{ is provable in } T_{\rightarrow} \\ \Leftrightarrow & \phi \text{ is inhabited by an HRM term in normal form.} \end{array}$$

... so, can we decide inhabitation for HRM-terms?



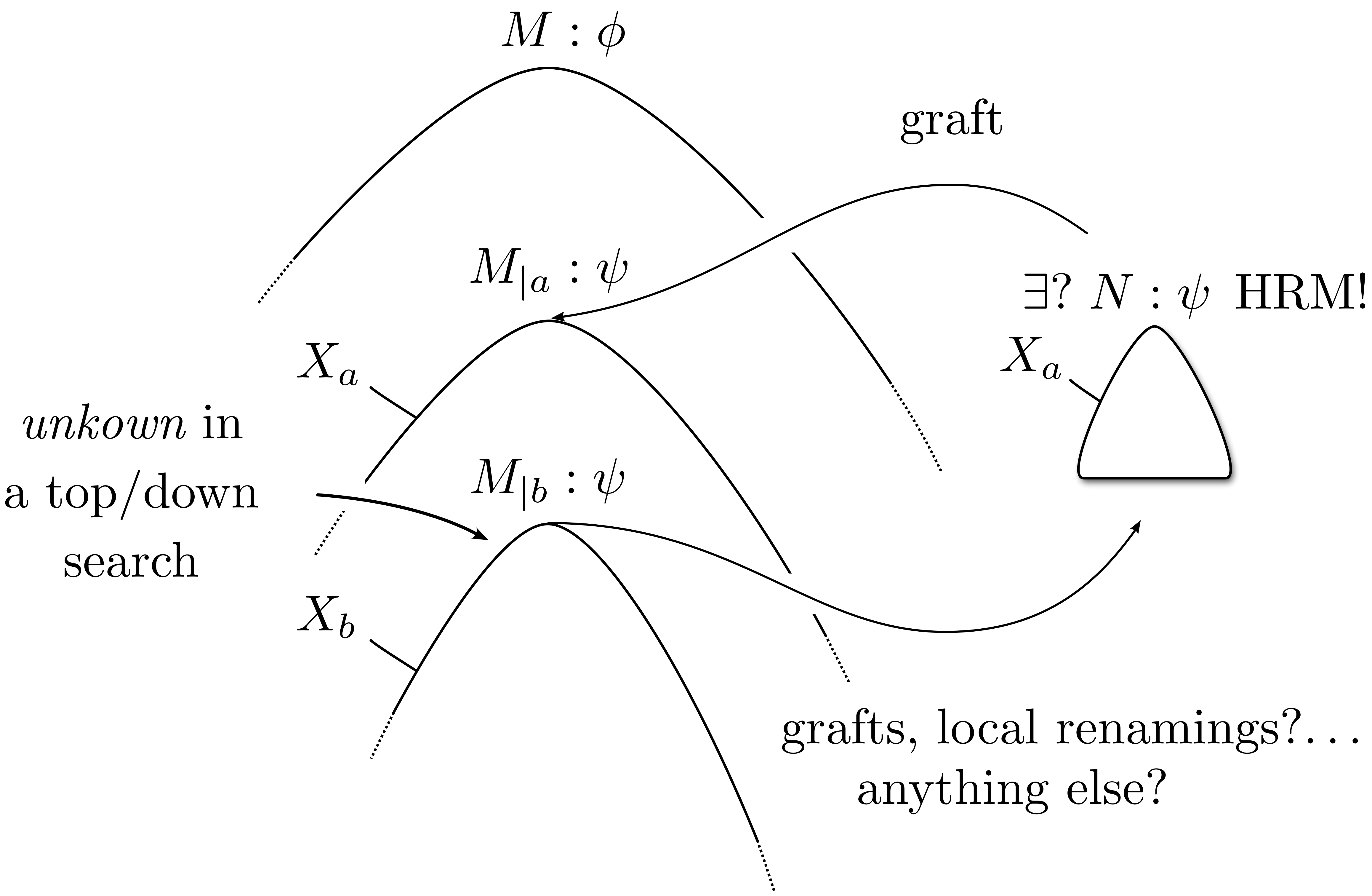
Our next goal: to compute a *minimal* inhabitant of some fixed type ϕ.

...but *why* is an inhabitant non-minimal? is there any way to decrease its size?

- Throughout steps 2 and 3, we shall study a fixed situation:
  - $M_{|a}$ is above $M_{|b}$ in $M : \phi$.
  - the subterms are of same *kind* (type, app | abs)

we ask if there is any way to decrease the size of M by transforming $M_{|b}$ into a term that can be grafted at a.

Step 2: the $M_{|a}/M_{|b}$ problem



The $M_{|a}/M_{|b}$ problem: the most obvious case

- $\mathsf{Free}(M_{|a}) = (x_1 \ldots x_n) = X_a$,
- $\mathsf{Free}(M_{|b}) = (x'_1 \ldots x'_n) = X_b$,

increasing sequences of free variables

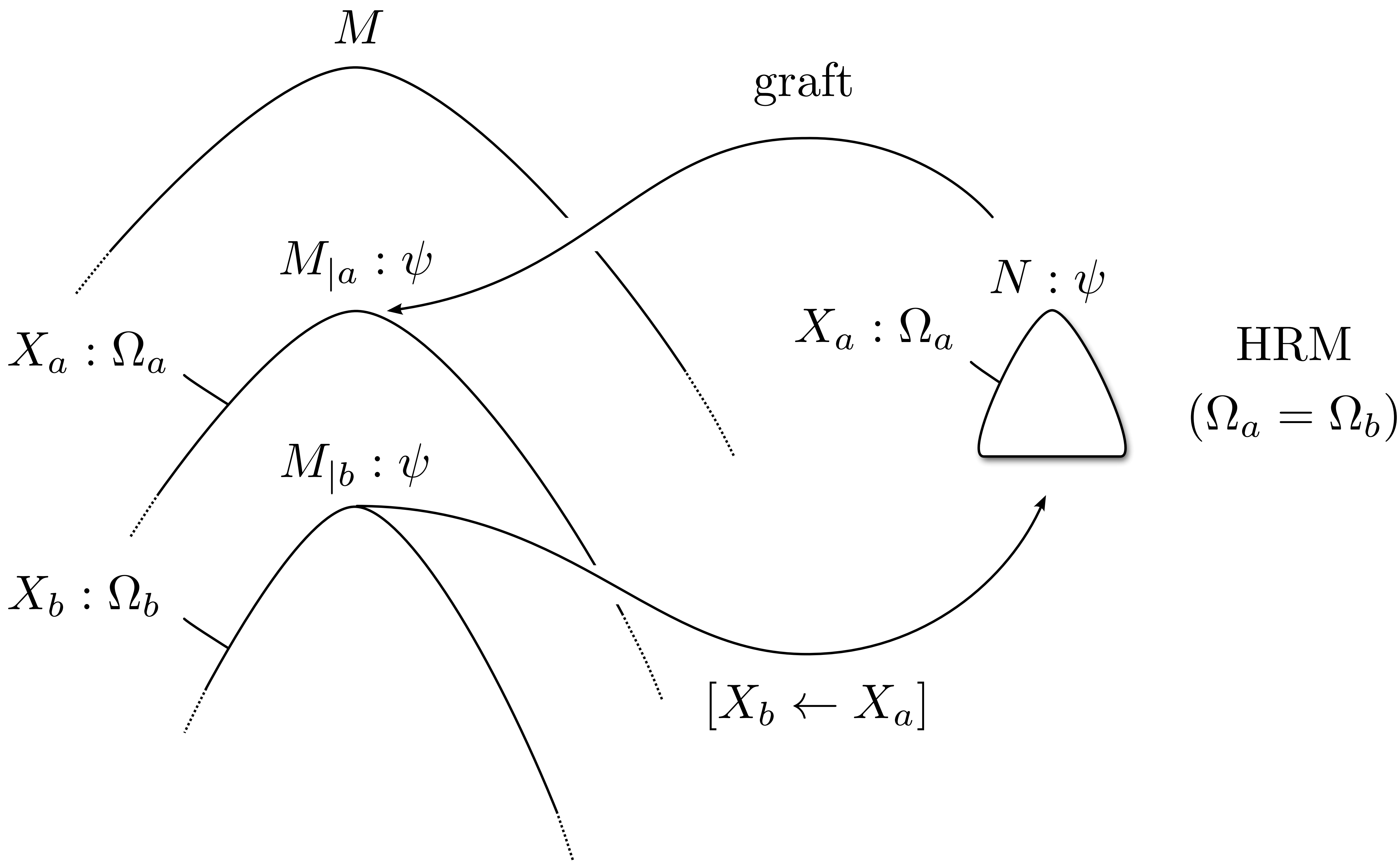
- $\mathsf{Types}(X_a) = \mathsf{Types}(X_b) = \Omega$.

sequences of types

No further information is required...

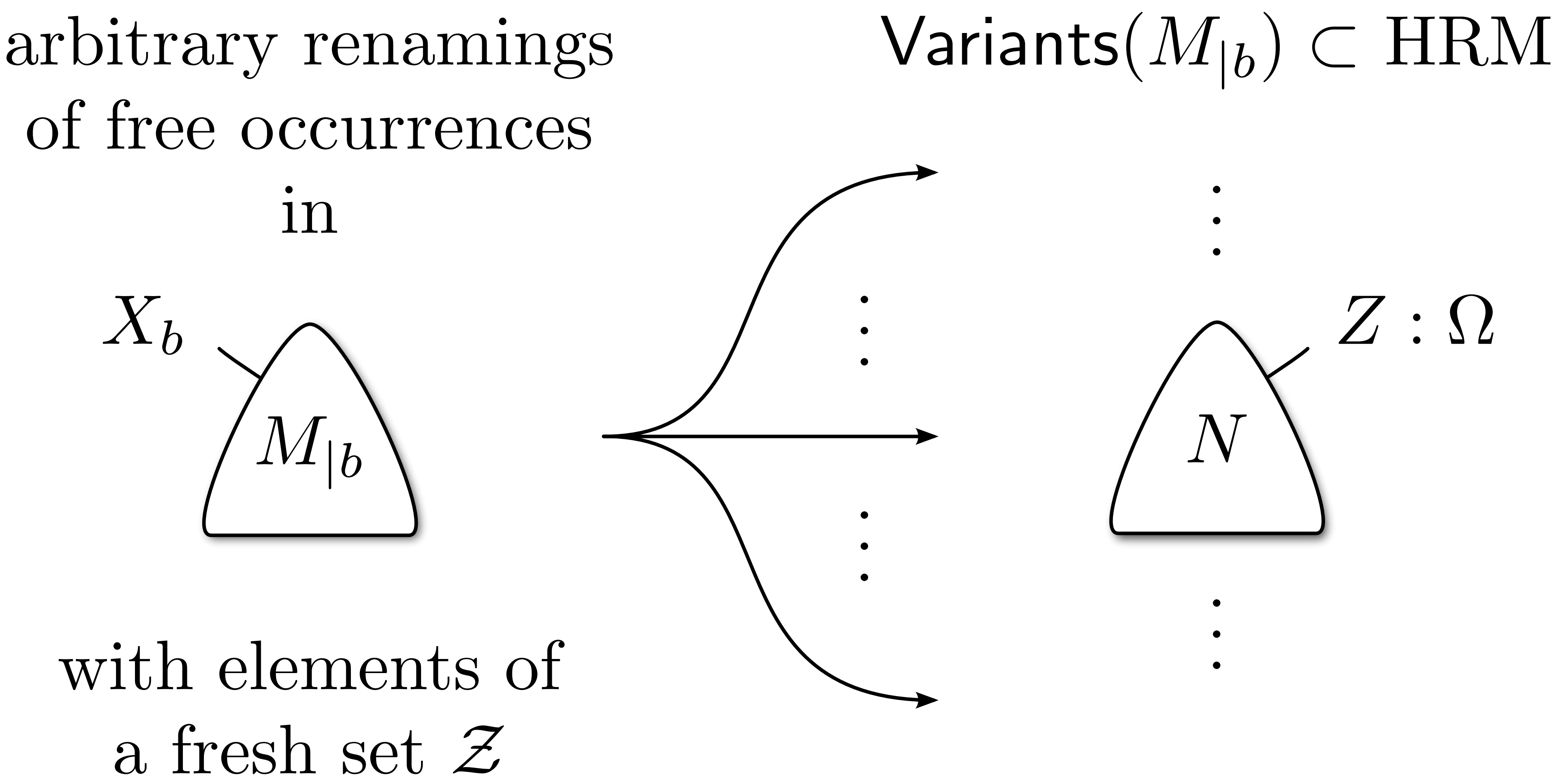
$M_{|b}[X_b \leftarrow X_a]$ is still HRM...

... so M *cannot be* of minimal size.

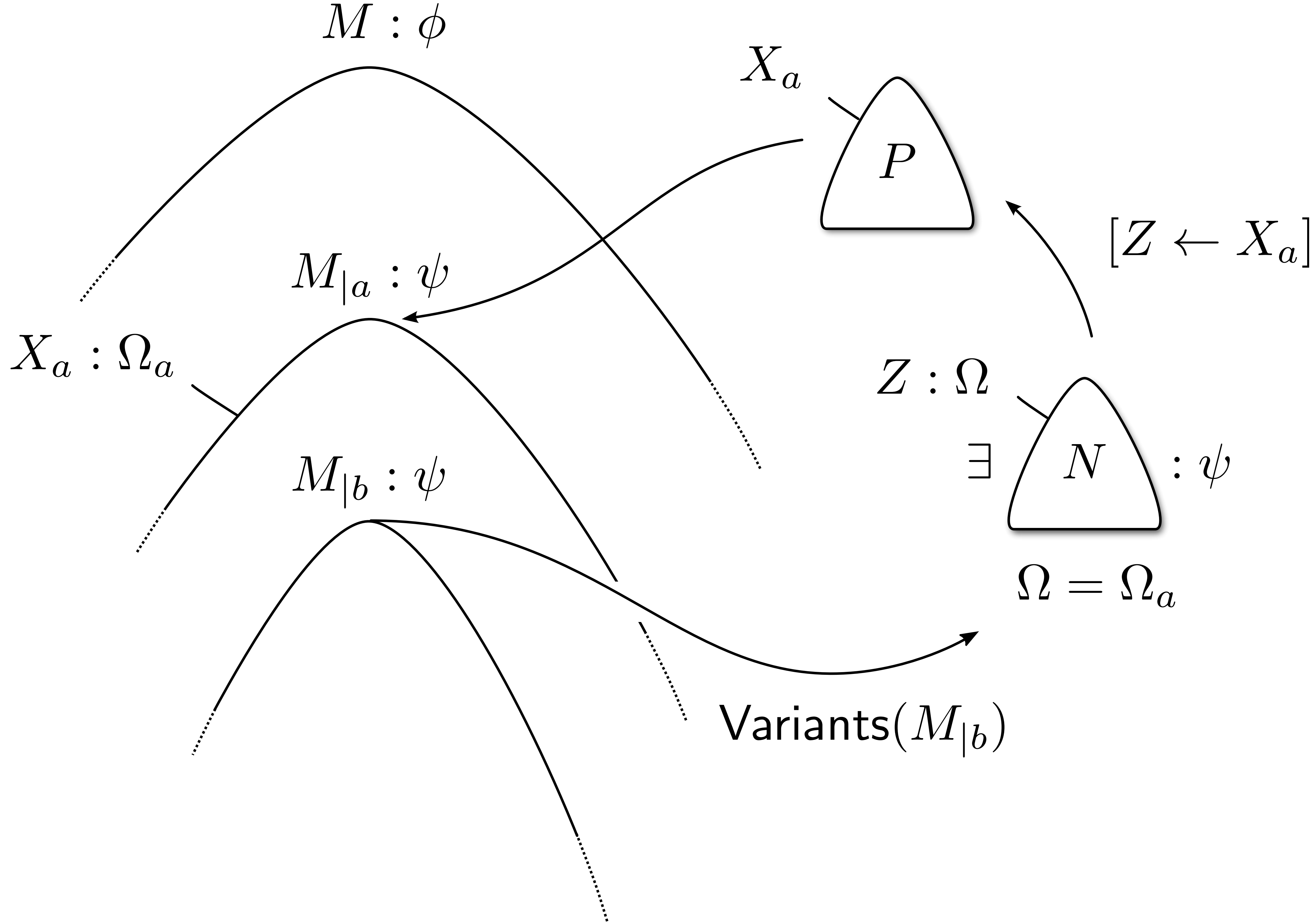


...is it sufficient to eventually gain minimality? no, of course!

A more complex transformation of $M_{|b}$



- if $\Omega = \Omega_a$ then M *cannot be* minimal.



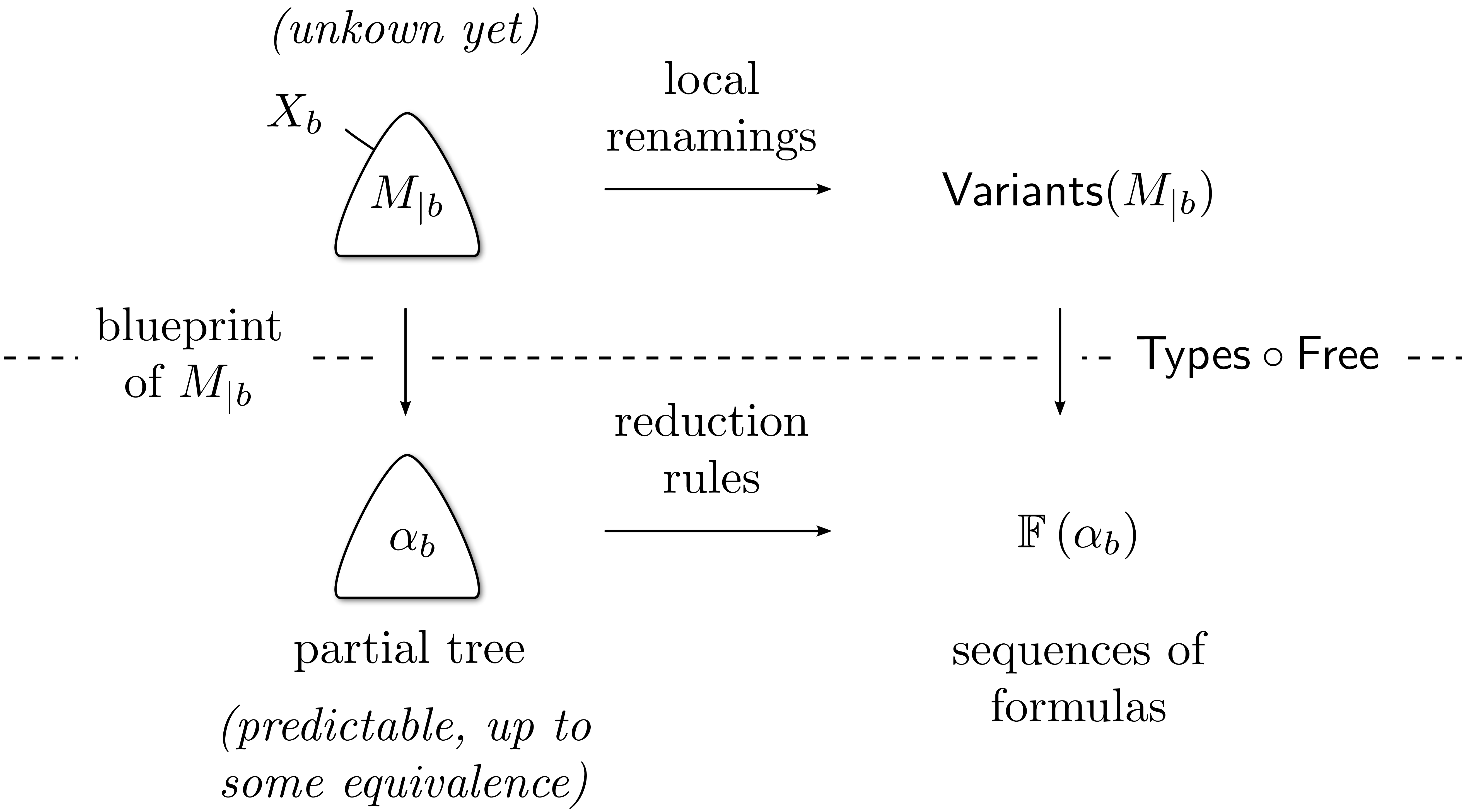
$M : \phi$
X_a
P
$[Z \leftarrow X_a]$
$M_{|a} : \psi$
$X_a : \Omega_a$
$Z : \Omega$
$\exists$
N
$: \psi$
$M_{|b} : \psi$
$\Omega = \Omega_a$
Variants$(M_{|b})$

- If we only want to detect *the existence* of such an N in $\mathsf{Variants}(M_{|b})$ what is the amount of information on $M_{|b}$ we need to know?

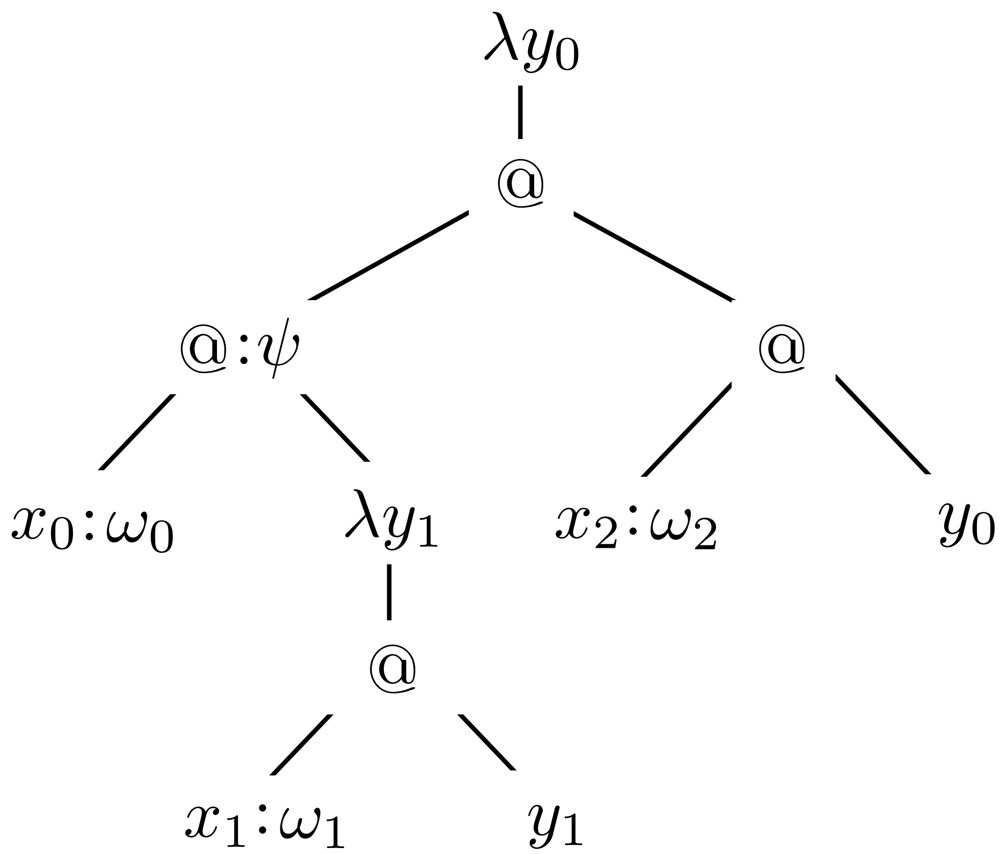
- next step: to define from $M_{|b}$ a partial tree labelled with formulas, from which one can extract *all* type sequences of the free variables of its variants.

- we call this tree *the blueprint* of $M_{|b}$.

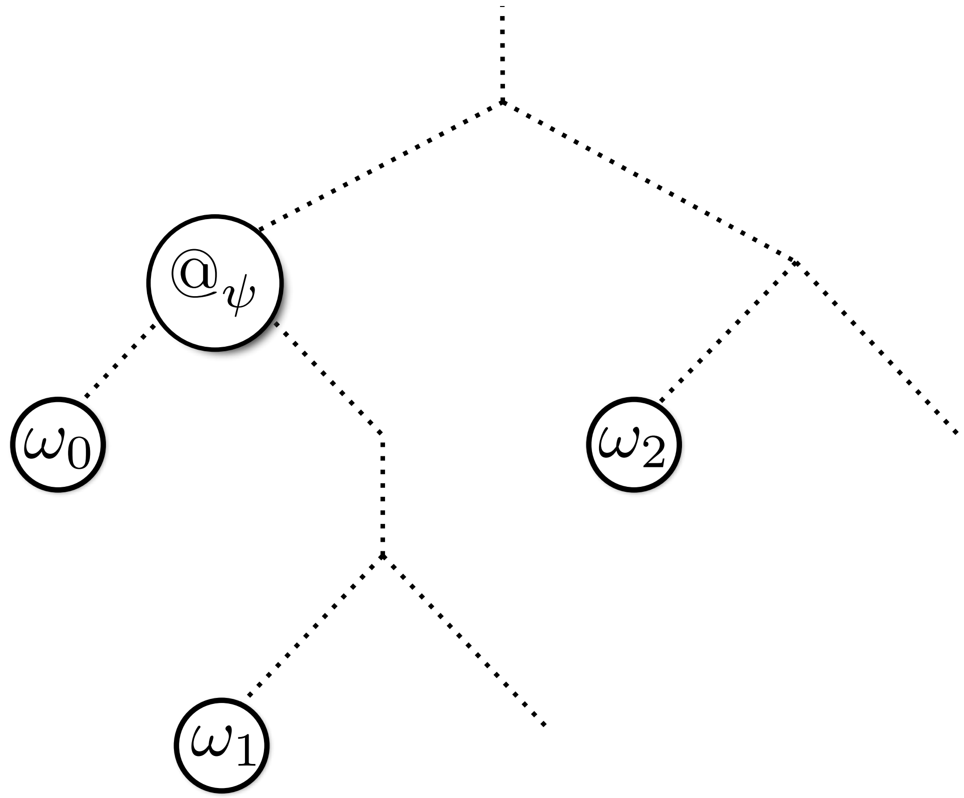
Blueprints – how to predict variants without terms



an HRM term N the blueprint of N

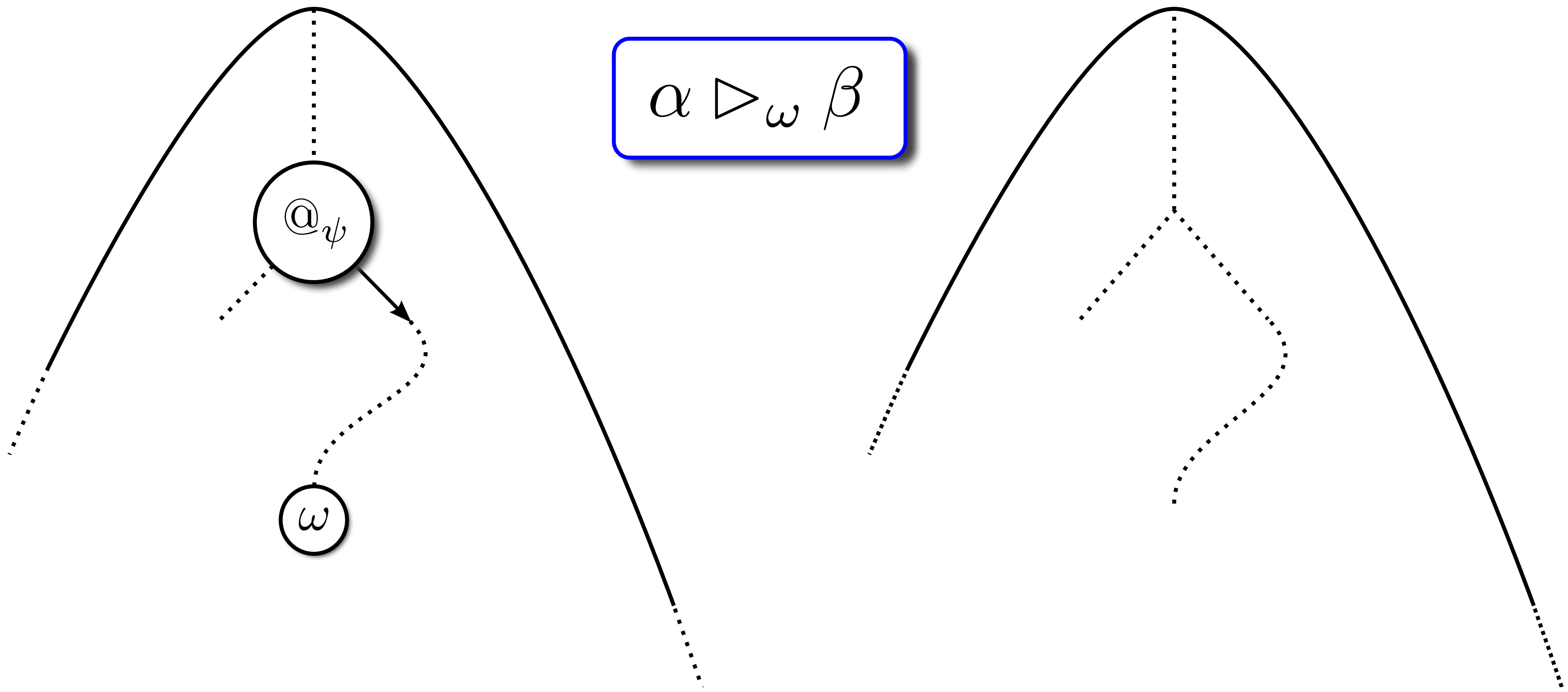


$N : \phi$



$N \Vdash \alpha$

- $\mathrm{dom}(\alpha)$ = all d such that $\mathsf{Free}(N_{|d}) \subseteq \mathsf{Free}(N)$ and $N_{|d}$ is a variable or an application.



$\alpha \rhd_\omega \beta$

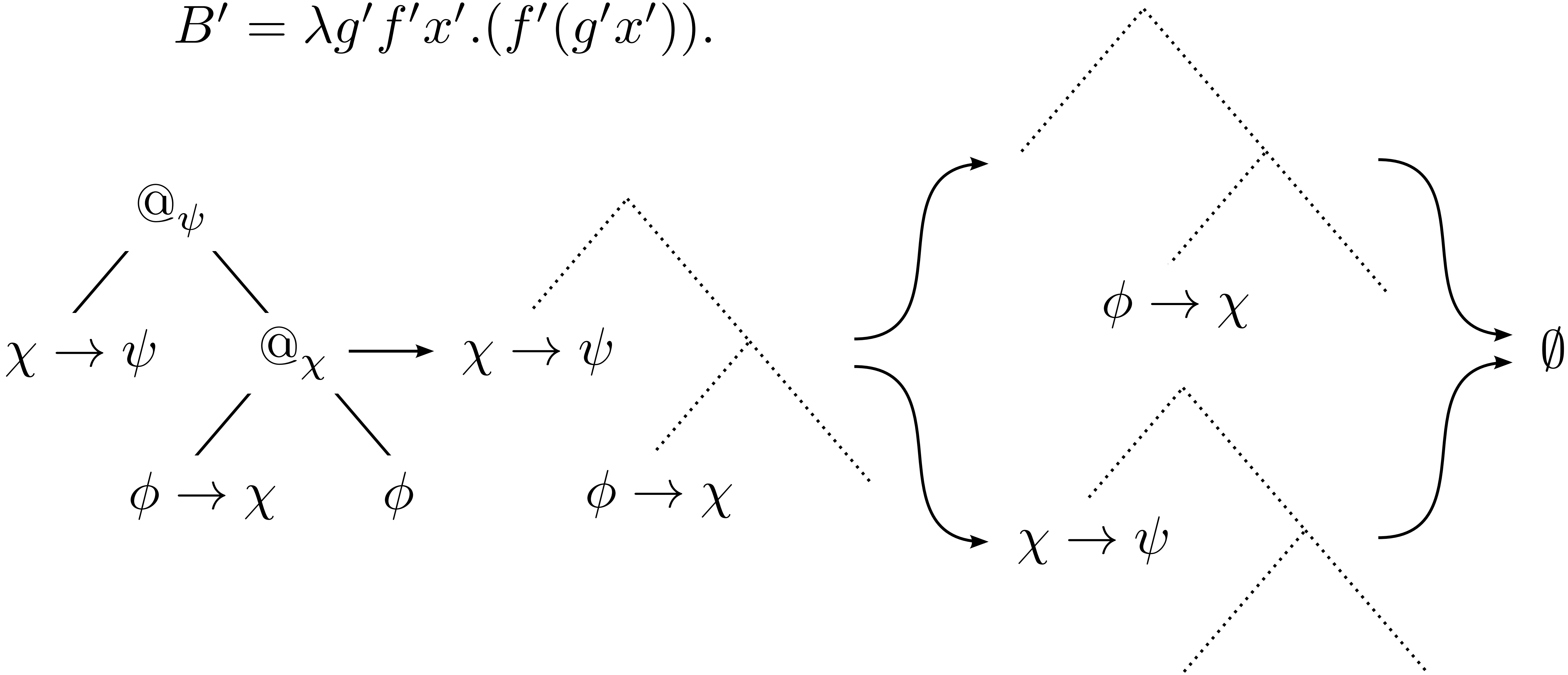
- for each @ in the path to ω, the path goes to the right.
- the reduction erases ω and all @ in this path.
- $\mathbb{F}(\alpha) =$ all $(\omega_1, \ldots, \omega_n)$ such that $\alpha \rhd^+_{\omega_n} \ldots \rhd^+_{\omega_1} \emptyset$

$$N \Vdash \alpha \Rightarrow \mathbb{F}(\alpha) = \mathsf{Types} \circ \mathsf{Free} \circ \mathsf{Variants}(N)$$

- example: inner part of

$$B \ = \lambda f\ g\ x\ .(f\ (g\ x\ ))$$

$$B' = \lambda g' f' x'.(f'(g' x')).$$

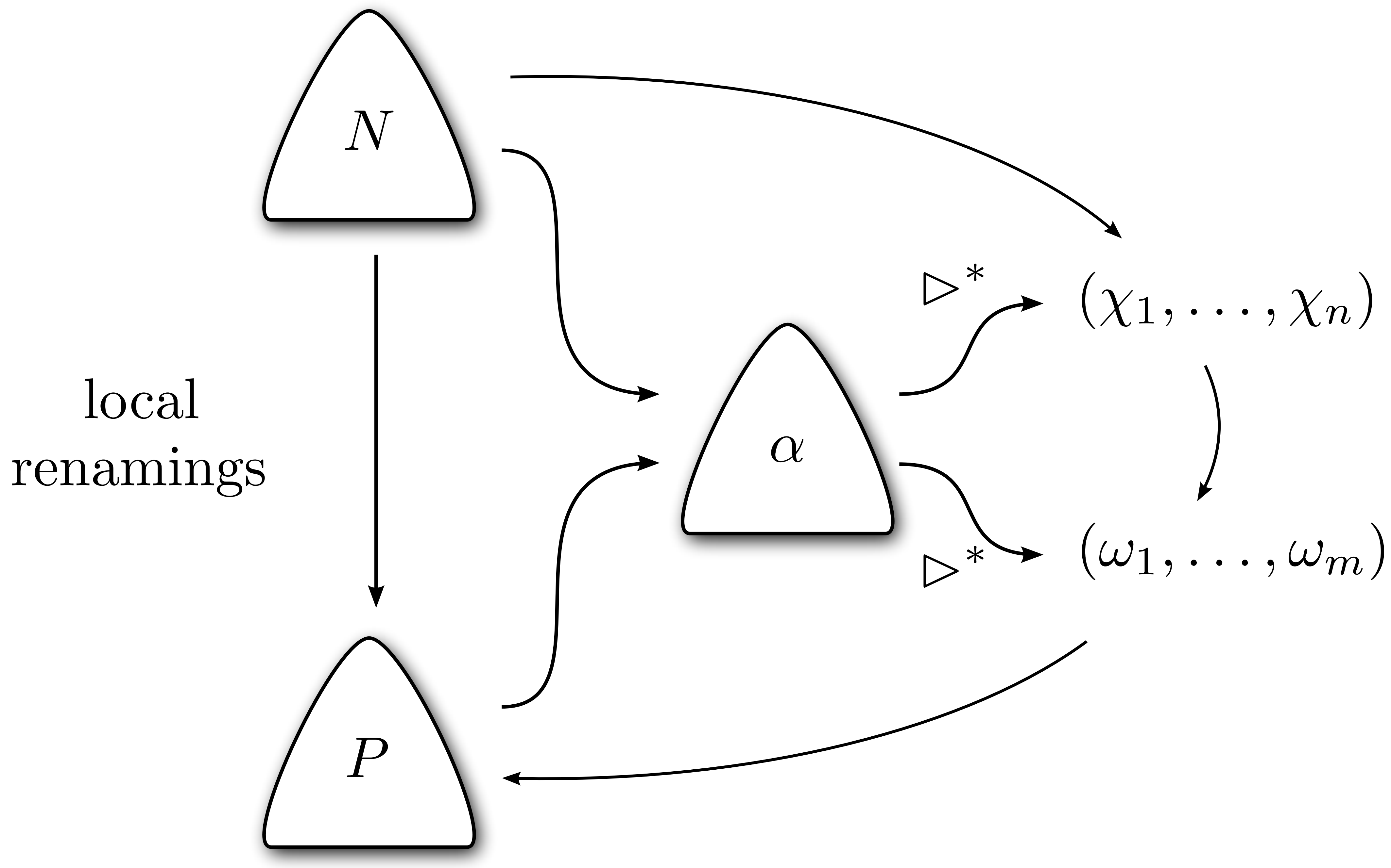


- one blueprint, two sequences...

  $(\phi \to \chi, \chi \to \psi, \phi) \quad (\chi \to \psi, \phi \to \chi, \phi)$

  two possible orderings of free occurrences.

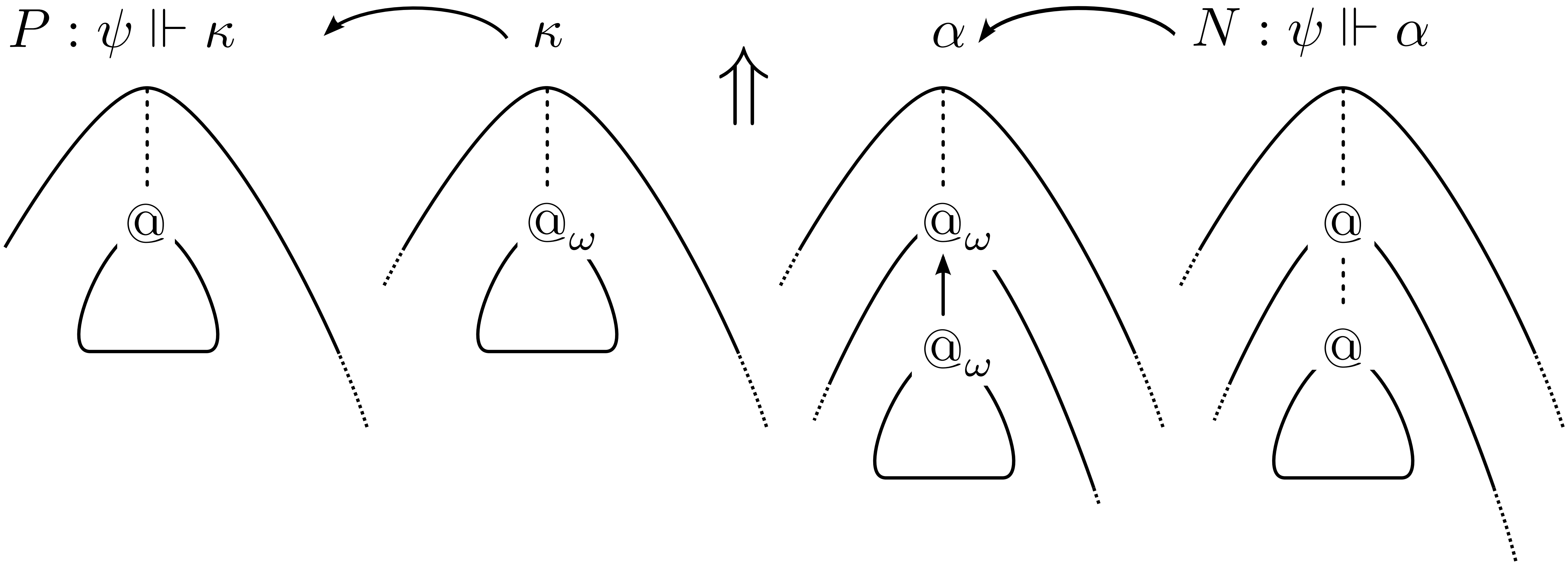
$$\mathsf{Free}(N:\psi) = (y_1 : \chi_1, \ldots, y_n : \chi_n)$$



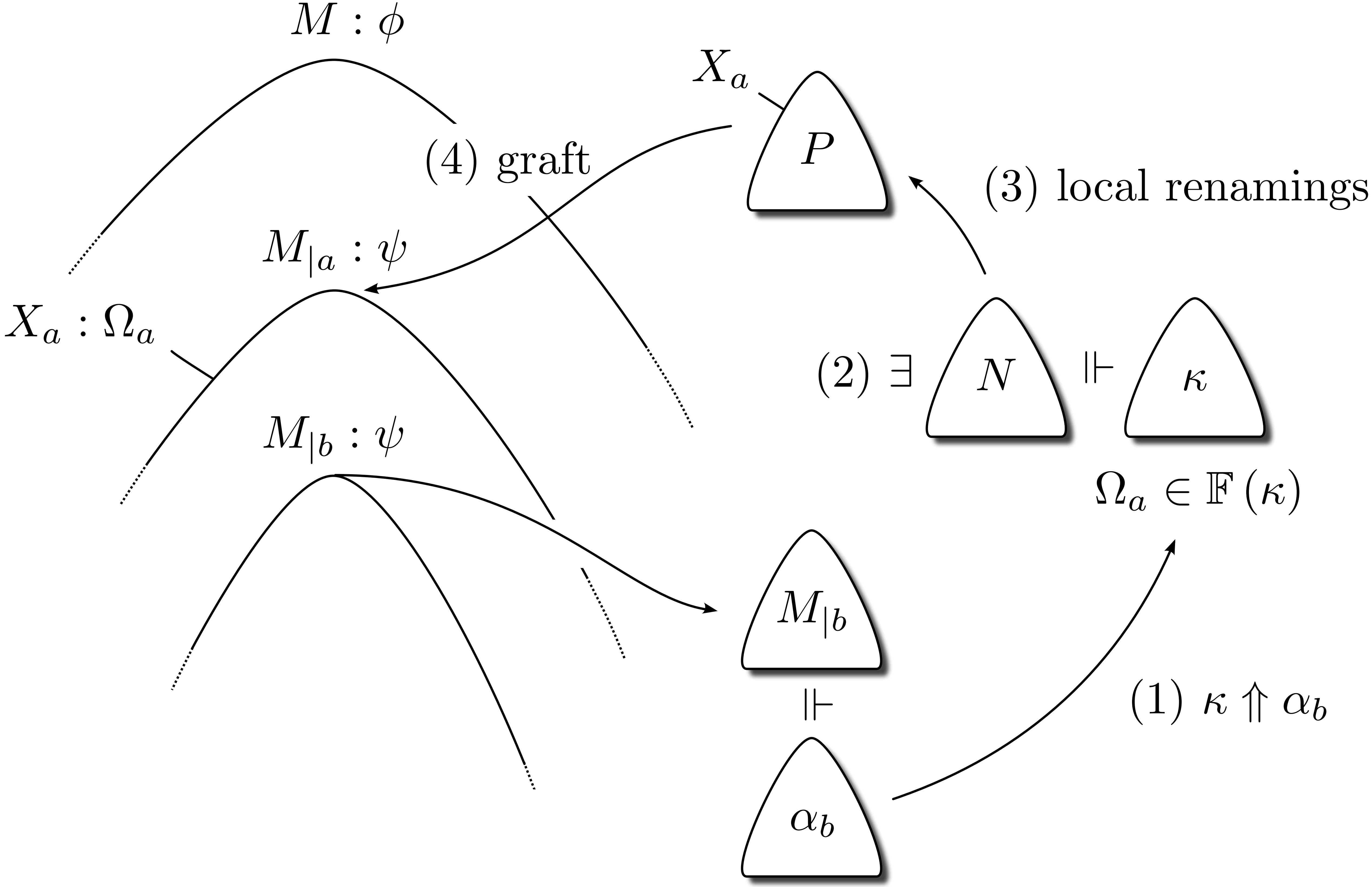
$$\mathsf{Free}(P:\psi) = (z_1 : \omega_1, \ldots, z_m : \omega_m)$$

- Thus, in the $M_{|a}/M_{|b}$ problem, the following questions are equivalent:
  - is there a variant of $M_{|b}$ whose free variables are of type sequence equal to Ω_a?
  - can Ω_a be extracted from *the blueprint* α_b of $M_{|b}$?
- can we try to extract more information from α_b?
  yes, we can try to *compress* $M_{|b}$ via its blueprint...

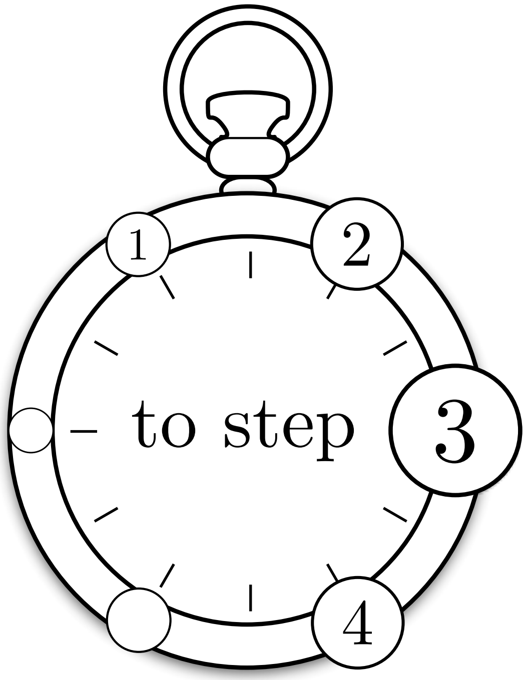
How to get even more of a blueprint: compact terms



- κ is *always* the blueprint of some HRM-term...
- this means that we can also try to extract Ω_a from *compressions* (refl + trans) of the blueprint α_b of $M_{|b}$.



(Ω_a can be extracted from a compression of α_b)



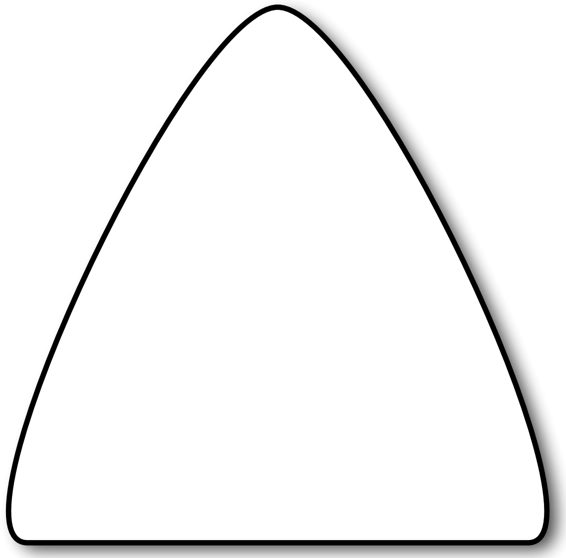
...an inhabitant in which this situation does not occur will be called *compact*.

Our next goal: to prove that the set of compact inhabitants of ϕ is a *finite* set, *computable* as a function of ϕ.

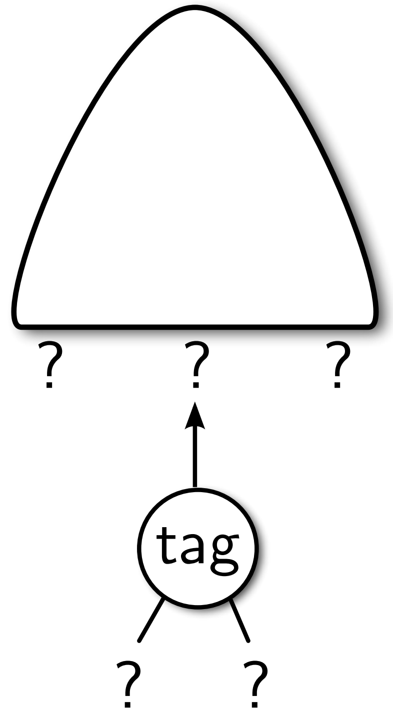
- first, we need to design some algorithm... how can we guess what the blueprints will be without the terms?

Step 3: the search for compact inhabitants

$M : \phi$
(hum... well, *maybe*)

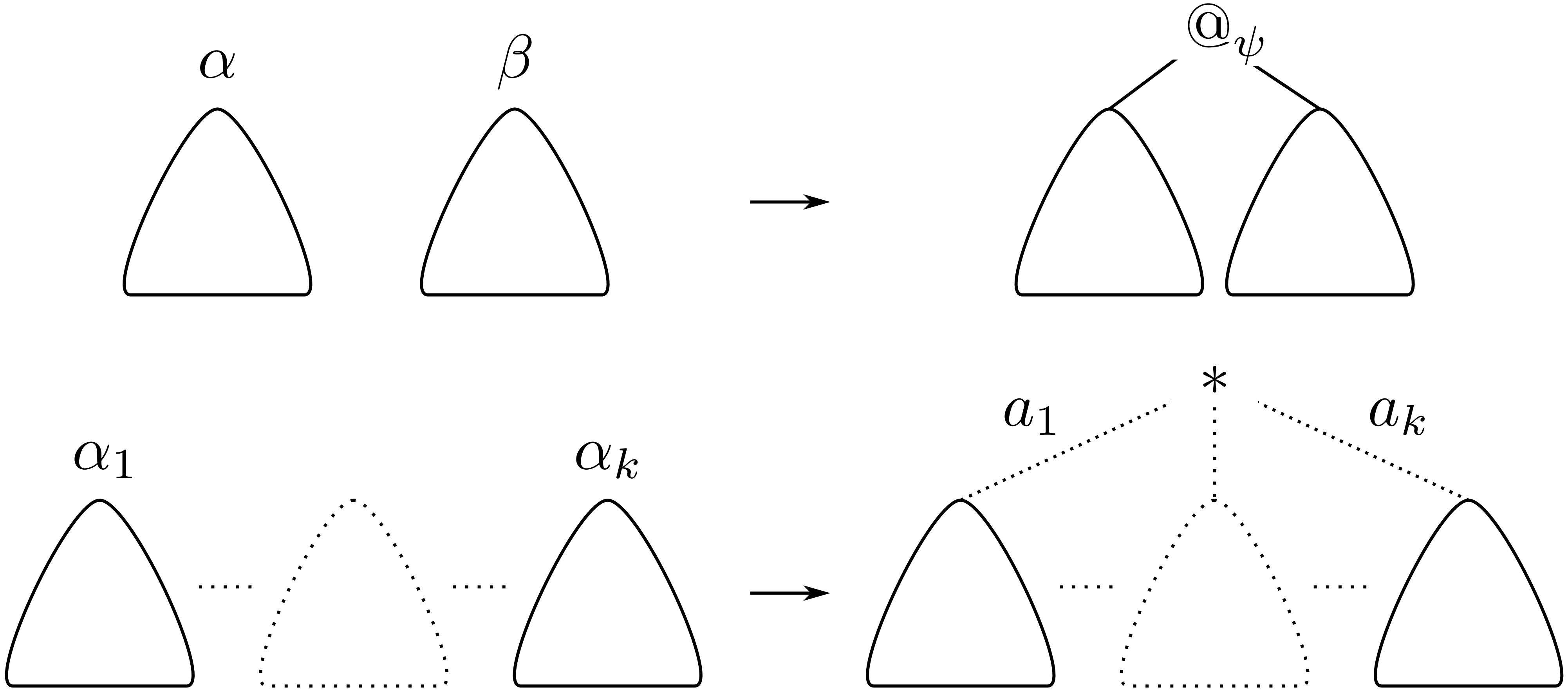


shadow of M
under construction



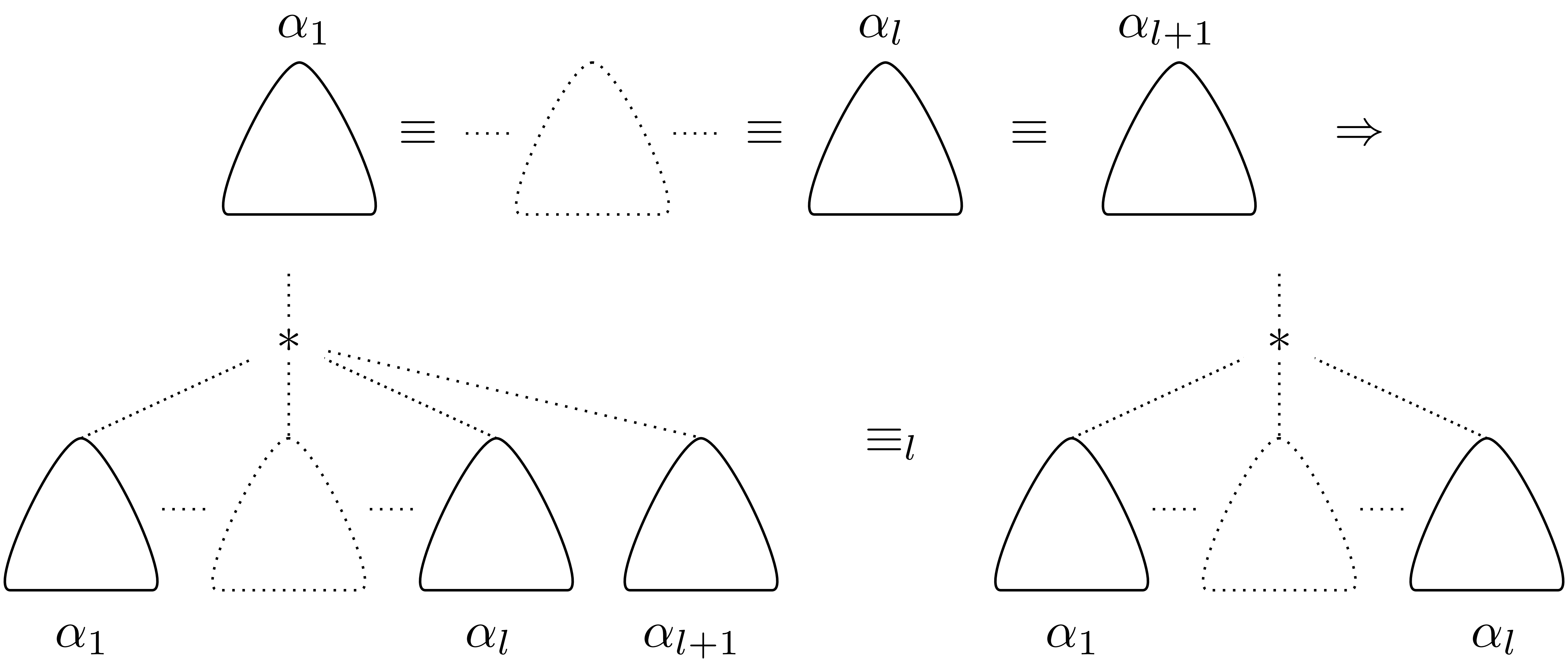
- tag $\sim$ type, types of free variables, description of a blueprint.
- descriptions must allow the detection of non-compacity.

(1) blueprints can be considered up to an equivalence that preserves their sets of extractible sequences.



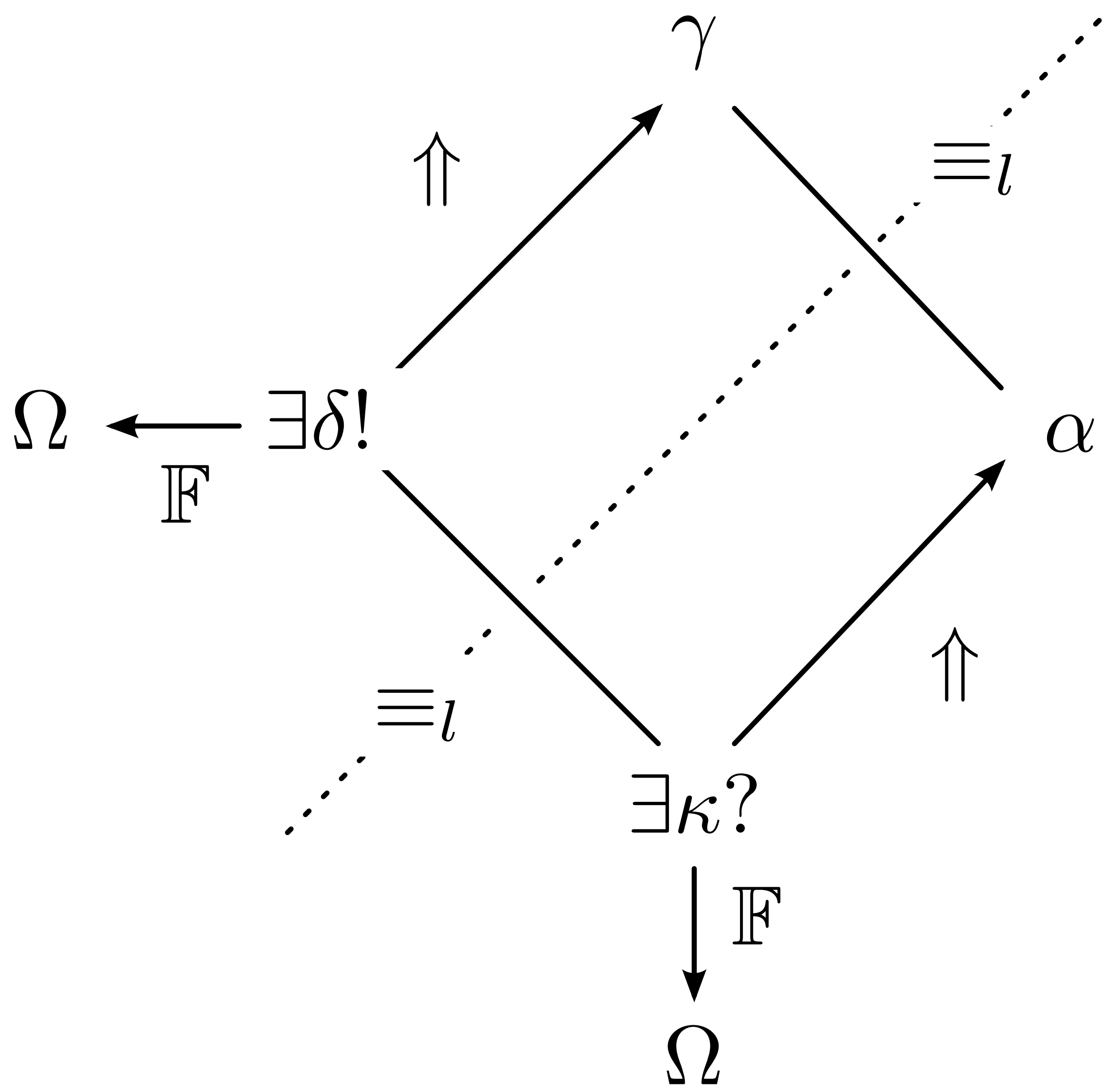
$\gamma \equiv \gamma'$ if their constructions are similar, regardless of the exact values/order of adresses in the second case.

(2) for each l, blueprints can be considered up to an equivalence $\equiv_l$ which preserves all sequences *of length at most* l.



again, regardless of the exact values/order of adresses.

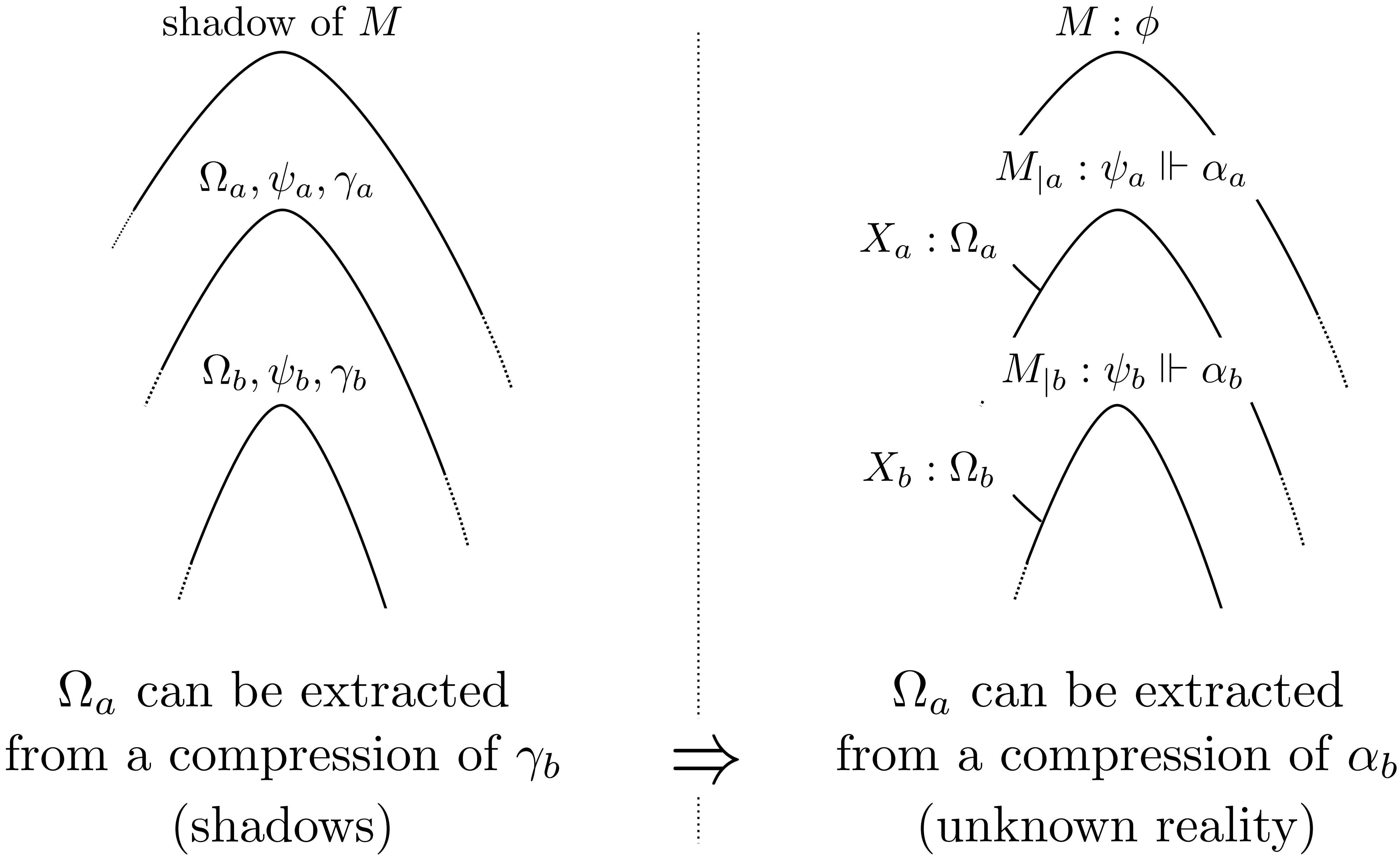
$\equiv_l$ is enough to check whether a sequence Ω of length at most l can be extracted from the compressions of a blueprint α.



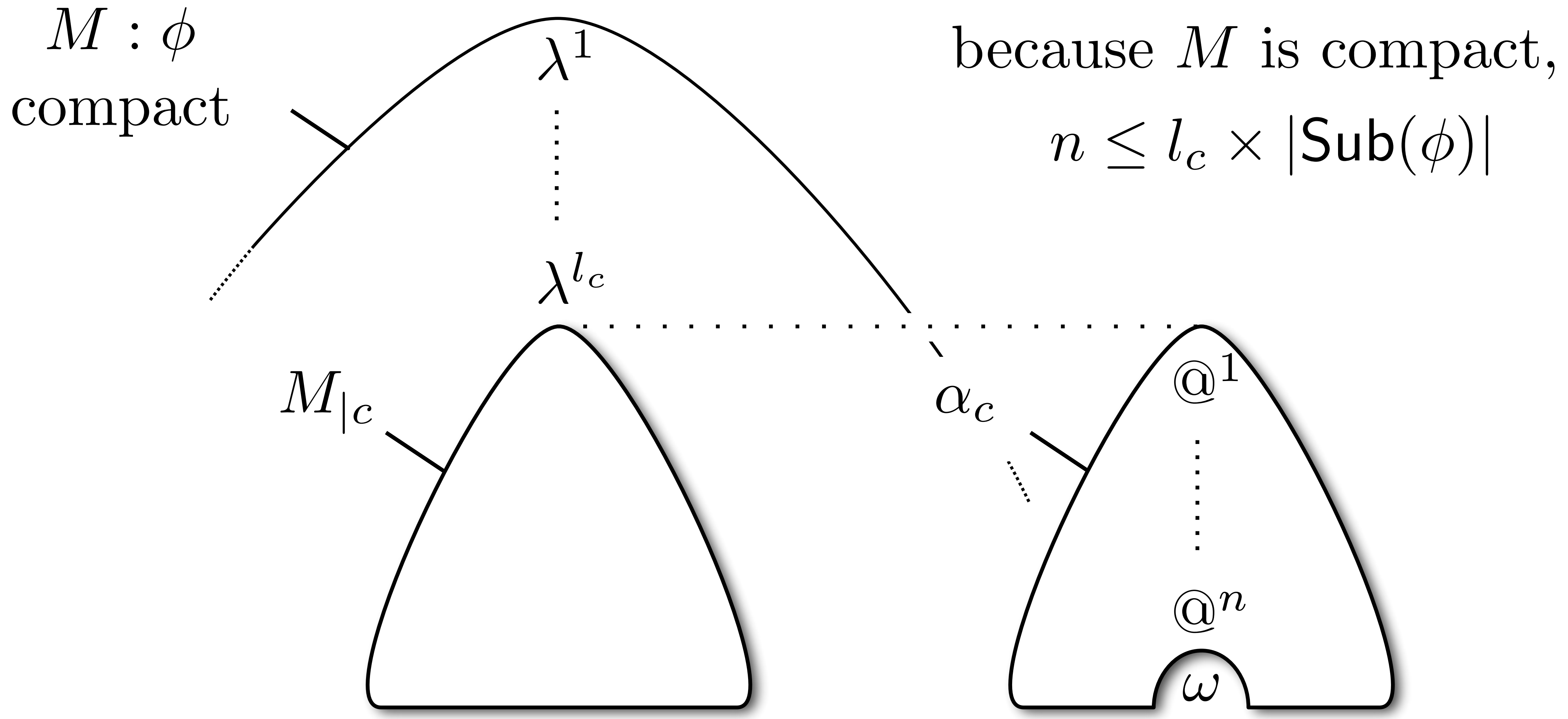
provided $|\Omega| \leq l$,
the existence of κ follows
from the existence of δ.

(4) Ω_a is of length at most l_a, where l_a is the $\sharp$ of λ's above a...

since $l_a \leq l_b$, this means that we will be able to detect non-compactness if b is tagged with *any* blueprint $\gamma_b \equiv_{l_b} \alpha_b$.



(5) moreover, for each adress c in M, the blueprint α_c of $M_{|c}$ is of "depth" at most $l_c \times |\mathsf{Sub}(\phi)|$.



- Let $\mathbb{B}(\phi, n)$ be the set of all blueprints labelled with subformulas of ϕ, of depth at most n.

Lemma. For all ϕ, n, l,

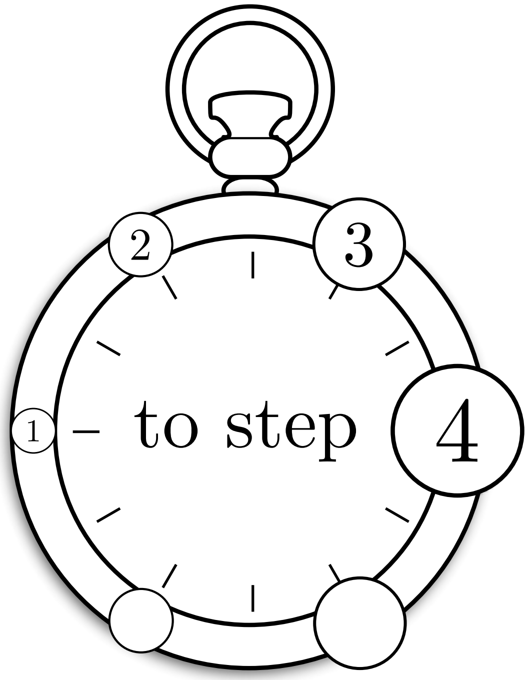
- The set $\mathbb{B}(\phi, n)/\equiv_l$ is a finite set.
- A selector $\mathbb{R}(\phi, n, l)$ for $\mathbb{B}(\phi, n)/\equiv_l$ is effectively computable from (ϕ, n, l).

- The values of $\mathbb{R}$ are the tags we're looking for!

The (naive) algorithm

Start from the empty shadow, extend it undeterministically in the following manner:

- tag a with l_a unary nodes above a with $(\Omega_a, \psi_a, \gamma_a)$, where:
  - Ω_a is a sequence over $\mathsf{Sub}(\phi)$ of length at most l_a
  - $\psi_a \in \mathsf{Sub}(\phi)$
  - $\gamma_a \in \mathbb{R}\left(\phi, l_a \times |\mathsf{Sub}(\phi)|, l_a\right)$
  - $\Omega_a \in \mathbb{F}\left(\gamma_a\right)$.
- *reject* a shadow if it's not *compact*: $a < b$, the nodes at a, b are of the same type/arity, and Ω_a can be extracted from a compression of γ_b.



This algorithm computes: a lot of garbage; *all* shadows of compact inhabitants of ϕ...

...will it terminate?

- if the answer is "yes", the problem is solved:
  - launch the algorithm.
  - for each computed shadow, check whether there is an inhabitant with the same domain.
- if the answer is "no"... hum, let's not think about it.

Step 4: Proof of termination

- Consider the following relation:

$$\begin{aligned} & \alpha \Subset \beta \\ \iff \quad & \text{for each } \Omega \in \mathbb{F}(\alpha), \\ & \text{there exists a compression } \kappa \text{ of } \beta \\ & \text{such that } \Omega \in \mathbb{F}(\kappa). \end{aligned}$$

(β is able to emulate α via its compressions.)

- Our goal: to prove that $\Subset$ is a *well quasi-ordering* over the set $\mathbb{B}(\phi)$ (all blueprints labelled with subformulas of ϕ)...

- ... *i.e.* it is impossible to find an infinite sequence

$$(\beta_0, \beta_1, \ldots, \beta_i, \ldots)$$

without two i, j such that $i < j$ and $\beta_i \Subset \beta_j$.

- if our algorithm does not terminate, then (König's lemma, etc.) it *is* possible to build such a sequence...
...hence if $\Subset$ is a WQO on $\mathbb{B}(\phi)$, the algorithm terminates.

- the proof uses an axiomatic variant of Kruskal theorem.
it is non-constructive: the resulting complexity is unknown.

The last key-lemma

- Melliès' Axiomatic Kruskal Theorem considers *an abstract decomposition system*:

$(\mathcal{T}, \preceq)$	*terms* $t, u \dots$		two relations, *e.g.*
$(\mathcal{L}, \preceq_{\mathcal{L}})$	*labels* $f, g, \dots$	$+$	$t \xrightarrow{f} T$
$(\mathcal{V}, \preceq_{\mathcal{V}})$	*vectors* $T, U \dots$		$T \vdash u$.

- intuitively (and intuitively only):
  - $t \xrightarrow{f} T$ if the root of t is labelled with f and T is the collection (sequence, multiset. . .) of its children.
  - $T \vdash u$ if u belongs to the collection T.

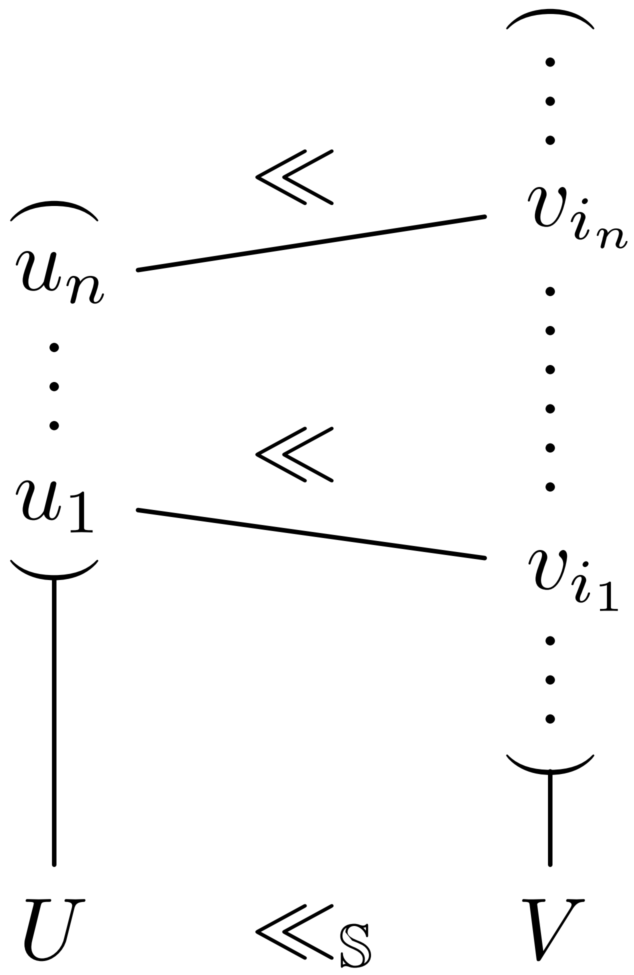
- Depending on the interpretation of "terms", vectors", "labels", the theorem can be specialized to Kruskal theorem, Higman theorem, etc.…and to the proof that $\Subset$ is a WQO on $\mathbb{B}(\phi)$.

> **Theorem** (Melliès 1998) If
>
> – $\preceq_{\mathcal{L}}$ is WQO on $\mathcal{L}$
>
> – five properties or "axioms" are satisfied.
>
> then $\preceq$ is a WQO on $\mathcal{T}$.

- just to give you the idea, our (purely ad-hoc) interpretation is:
  - $\mathcal{T} = \mathbb{B}_{\varepsilon}(\phi)$ (all rooted blueprints)
  - $\mathcal{L} = \mathsf{Sub}(\phi)$ (labels for @)
  - $\mathcal{V} = \mathbb{B}(\phi) \times \mathbb{B}(\phi)$ (pairs of children of @)

- In our interpretation, four axioms are easy to check. The last one requires to prove that if $\Subset$ is a WQO on the subset $\mathbb{B}_\varepsilon(\phi)$ of *rooted* blueprints, then it is also a WQO on $\mathbb{B}(\phi)$.

- This part of the proof is the most esoteric and was the most painful to prove. Additionally, it requires the following theorem:



Theorem (Higman 1952) $\forall \mathcal{U}, \ll$

If $\ll$ is a WQO on $\mathcal{U}$,

then $\ll_{\mathbb{S}}$ is a WQO on $\mathbb{S}(\mathcal{U})$.

- the very last lemma is:

Key-Lemma. For all ϕ,

- $\Subset$ is a WQO on $\mathbb{B}(\phi)$,
- our algorithm terminates,
- the set of compact shadows labelled with subformulas of ϕ is a finite set, computable as a function of ϕ,

and our main result is...

Main result

(from the shadows to the light)

Theorem. Ticket Entailment is decidable.

Proof. ϕ is a theorem of $T_{\to}$.

$\Leftrightarrow$ ϕ is a inhabited in $\mathsf{BB'IW}$

$\Leftrightarrow$ ϕ is a inhabited by an HRM-term

$\Leftrightarrow$ there exists a compact inhabitant M of ϕ

$\Leftrightarrow$ there exists a compact shadow of same domain as M.

... and the shadow of M belongs to a finite set, computable as a function of ϕ.

$\square$

A never-ending quest? – the lost episode

December 2011, a few days before Christmas...

I was trying to relax, waiting for the next
(and hopefully the *last*) reports...

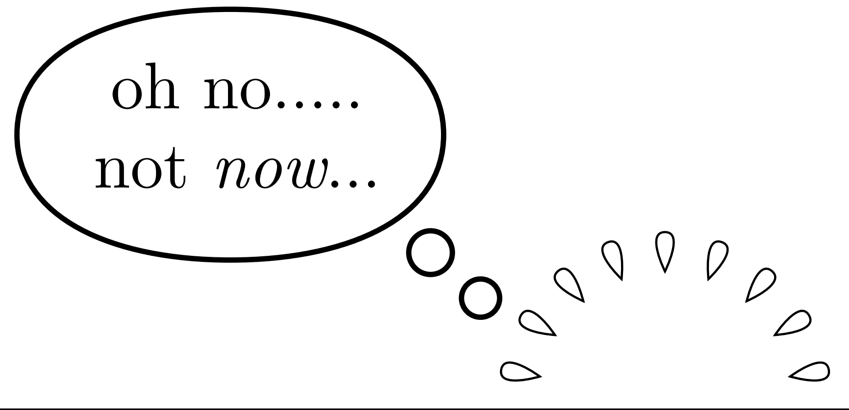
then...

”We show that the implicational fragment of the logic of ticket entailment is decidable *[...] Riche and Meyer say that:*

> ”Having been around since circa 1960, this is the most venerable problem in all of relevant logic.”

[...] We learned that a draft paper (Padovani 2010) etc.”

On the decidability of implicational Entailment
K. Bimbó and J.M. Dunn, JSL (accepted in 2012)



- The two proofs are now considered as independant.

- By the way, the full citation of Riche and Meyer is:

  "We note for the readers logical pleasure that he/she/it may achieve fame and fortune by solving the decision problem for $\mathsf{T}_{\rightarrow}$. Having been around since circa 1960, this is the most venerable problem in all of relevant logic."

"Das ist nicht Mathematik, das ist Theologie"

(footnote) Riche and Meyer, 1999

”We are not sportsmen aiming at record-breaking or something. We are workers trying to make progress and increase the global knowledge.”

Paweł Urzyczyn

THE CAST (2006-2013)

Paweł Urzyczyn	Lambda Buddha
Paul-André Melliès	Archmage of Kruskalian Black Magic
Pierre-Louis Curien	Grandmaster of the Holy Books
Antonio Bucciarelli	Stunt Auditor in Chief
(not so) Anonymous	Referee ♯ 1 ("good cop")
(simply) Anonymous	Referee ♮ 2 ("the neutral one")
(really) Anonymous	Referee ♭ 3 ("bad cop")
Daniele Varacca	Personal Showbiz Agent

All members of the PPS team

All the audience of Chocola

Many thanks
to all...

V. Padovani