Strong Normalizability as a Finiteness Structure via the Taylor Expansion of λ-terms ⋆

Michele Pagani[1], Christine Tasson[1], and Lionel Vaux[2]

[1] IRIF – Université Paris Diderot – Paris 7
{christine.tasson, michele.pagani}@pps.univ-paris-diderot.fr
[2] I2M – Université Aix-Marseille
lionel.vaux@univ-amu.fr

Abstract. In the folklore of linear logic, a common intuition is that the structure of finiteness spaces, introduced by Ehrhard, semantically reflects the strong normalization property of cut-elimination.
We make this intuition formal in the context of the non-deterministic λ-calculus by introducing a finiteness structure on resource terms, which is such that a λ-term is strongly normalizing iff the support of its Taylor expansion is finitary.
An application of our result is the existence of a normal form for the Taylor expansion of any strongly normalizable non-deterministic λ-term.

1 Introduction

It is well-known that sets and relations can be presented as a category of modules and linear functions over boolean semi-rings, giving one of the simplest semantics of linear logic. In [10] (see also [9]), it is shown how to generalize this construction to any complete[3] semi-ring $\mathcal{R}$ and yet obtain a model of linear logic. In particular, the composition of two matrices $\phi \in \mathcal{R}^{A\times B}$ and $\psi \in \mathcal{R}^{B\times C}$ is given by the usual matrix multiplication:

$$(\phi;\psi)_{a,c} := \sum_{b\in B} \phi_{a,b} \cdot \psi_{b,c} \tag{1}$$

The semi-ring $\mathcal{R}$ must be complete because the above sum might be infinite. This is an issue, because it prevents from considering standard vector spaces, which are usually constructed over "non-complete" fields like reals or complexes.

In order to overcome this problem, Ehrhard introduced the notion of finiteness space [3]. A *finiteness space* is a pair of a set A and a set $\mathfrak{A}$ (called *finiteness structure* in Definition 7) of subsets of A which is closed under a notion of duality. The point is that, for any field $\mathcal{K}$ (resp. any semi-ring), the set of vectors in $\mathcal{K}^A$ whose support[4] is in $\mathfrak{A}$ constitutes a vector space (resp. a module) over $\mathcal{K}$. Moreover, any two matrices $\phi \in \mathcal{K}^{A\times B}$ and $\psi \in \mathcal{K}^{B\times C}$ whose supports are

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[3] A semi-ring is complete if any sum, even infinite, is well-defined.
[4] The support of $v \in \mathcal{K}^A$ is the set of those $a \in A$ such that the scalar v_a is non-null.

in resp. $\mathfrak{A} \multimap \mathfrak{B}$ and $\mathfrak{B} \multimap \mathfrak{C}$ (the finiteness structures associated with the linear arrow) compose, because such finitary condition on the supports makes the terms in the sum of Equation (1) be zero almost everywhere. This gives rise to a category which is a model of linear logic and its differential extension.

The notion of finiteness space seems strictly related to the property of normalization. Already in [3], it is remarked that the coKleisli category of the exponential comonad is a model of simply typed λ-calculus, but it is not cpo-enriched and thus cannot interpret (at least in a standard way) fixed-point combinators, so neither PCF nor untyped λ-calculus. Moreover, in the setting of differential nets, Pagani showed that the property of having a finitary interpretation corresponds to an acyclicity criterion (called *visible acyclicity* [12]) which entails the normalization property of the cut-elimination procedure [11], while there are examples of visibly cyclic differential nets which do not normalize.

The goal of this paper is to shed further light on the link between finiteness spaces and normalization, this time considering the non-deterministic untyped λ-calculus. Since we deal with λ-terms and not with linear logic proofs (or differential nets), we will speak about formal power series rather than matrices at the semantical level. This corresponds to move from the morphisms of a linear category to those of its coKleisli construction. Moreover, following [7], we describe the monomials of these power series as *resource terms* in normal form. The benefit of this setting is that the interpretation of a λ-term M as a power series $[|M|]$ can be decomposed in two distinct steps: first, the term M is associated with a formal series M^* of resource terms possibly with redexes, called the *Taylor expansion* of M (see Table 1a); second, one reduces each resource term t appearing in the support $\mathcal{T}(M)$ of M^* into a normal form $\mathrm{NF}(t)$ and sum up all the results, that is (M^*_t denotes the coefficient of t in M^*):

$$[|M|] = \sum_{t \in \mathcal{T}(M)} M^*_t \cdot \mathrm{NF}(t) \tag{2}$$

The issue about the convergence of infinite sums appears in Equation (2) because there might be an infinite number of resource terms in $\mathcal{T}(M)$ reducing to the same normal form and thus possibly giving infinite coefficients to the formal series $[|M|]$. Ehrhard and Regnier have proven in [7] that this is not the case for deterministic λ-terms, however the situation gets worse in presence of non-deterministic primitives. If we allow sums $M + N$ representing potential reduction to M or N, then one can construct terms evaluating to a variable y an infinite number of times, like (where $\mathbf{\Theta}$ denotes Turing's fixed-point combinator):

$$(\mathbf{\Theta})\,\lambda x.(x+y) \to^*_\beta (\mathbf{\Theta})\,\lambda x.(x+y) + y \to^*_\beta (\mathbf{\Theta})\,\lambda x.(x+y) + y + y \to^*_\beta \dots \tag{3}$$

We postpone to Examples 1 and 4 a more detailed discussion of $(\mathbf{\Theta})\,\lambda x.(x+y)$, however we can already guess that this interplay between infinite reductions and non-determinism may produce infinite coefficients.

One can then wonder whether there are interesting classes of terms where the coefficients of the associated power series can be kept finite. Ehrhard proved in [4] that the terms typable by a non-deterministic variant of Girard's System

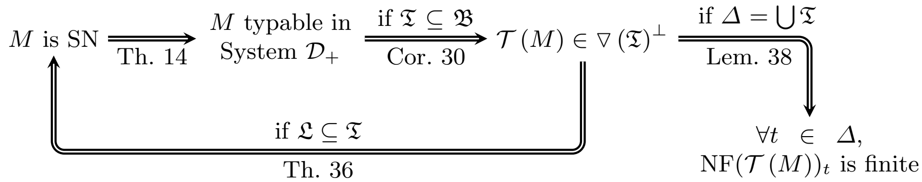

Fig. 1: main results of the paper

$\mathcal{F}$ have always finite coefficients. A by-product of our results is Corollary 39, which is a generalization of Ehrhard's result: every strongly normalizable non-deterministic λ-term can be interpreted by a power series with finite coefficients.

The main focus of our paper is however in the means used for obtaining this result rather than on the result itself. The proof in [4] is based on a finiteness structure $\mathfrak{S}$ over the set of resource terms Δ, such that any term M typable in System $\mathcal{F}$ has the support $\mathcal{T}(M)$ of its Taylor expansion in $\mathfrak{S}$. We show that this method can be both generalized and strengthened in order to characterize strong normalization via finiteness structures. Namely, we give sufficient conditions on a finiteness structure $\mathfrak{S}$ over Δ such that for every non-deterministic λ-term M: (i) if M is strongly normalizable, then $\mathcal{T}(M) \in \mathfrak{S}$ (Corollary 30); (ii) if $\mathcal{T}(M) \in \mathfrak{S}$, then M is strongly normalizable (Theorem 36).

Contents. Section 2 gives the preliminary definitions: the non-deterministic λ-calculus, its Taylor expansion into formal series of resource terms and the notion of finiteness structure. The proof of Item (i) splits into Sections 3 and 4, using an intersection type system (Table 2) for characterizing strong normalization. Section 5 gives the proof of Item (ii) and Section 6 concludes with Corollary 39 about the finiteness of the coefficients of the power series of strongly normalizable terms. Figure 1 sums up the main results of the paper.

2 Preliminaries

2.1 Non-deterministic λ-calculus Λ_+

The non-deterministic λ-calculus is defined by the following grammar[5]:

$$\lambda\text{-terms } \Lambda_+ : \qquad M := x \mid \lambda x.M \mid (M)\,M \mid M + M$$

subject to α-equivalence and to the following identities:

$$M + N = N + M, \qquad (M + N) + P = M + (N + P),$$

$$\lambda x.(M + N) = \lambda x.M + \lambda x.N, \qquad (M + N)\,P = (M)\,P + (N)\,P.$$

[5] We use Krivine's notation with the standard conventions, see [8] for reference.

The last two equalities state that abstraction is a linear operation (i.e. commutes with sums) while application is linear only in the function but not in the argument (i.e. $(P)\,(M+N) \neq (P)\,M + (P)\,N$). Notice also that the sum is not idempotent: $M + M \neq M$. This is a crucial feature for making a difference between terms reducing to a value once, twice, more times or an infinite number of times (see the discussion about Equation (3) in the Introduction).

Although this follows an old intuition from linear logic, the first extension of the λ-calculus with (a priori non idempotent) sums subject to these identities was, as far as we know, the differential λ-calculus of Ehrhard and Regnier [5]. This feature is now quite standard in the literature following this revival of quantitative semantics.

The (capture avoiding) substitution of a term for a variable is defined as usual and β-reduction $\to_\beta$ is defined as the context closure of:

$$(\lambda x.M)\,N \to_\beta M\,[N/x]\,.$$

We denote as $\to_\beta^*$ the reflexive-transitive closure of $\to_\beta$.

Example 1. Some λ-terms which will be used in the following are:

$$\mathbf{\Delta} := \lambda x.\,(x)\,x, \qquad \mathbf{\Omega} := (\mathbf{\Delta})\,\mathbf{\Delta}, \qquad \mathbf{\Delta_3} := \lambda x.\,(x)\,xx, \qquad \mathbf{\Omega_3} := (\mathbf{\Delta_3})\,\mathbf{\Delta_3},$$

$$\mathbf{\Theta} := (\lambda xy.\,(y)\,(x)\,xy)\,\lambda xy.\,(y)\,(x)\,xy.$$

The term $\mathbf{\Omega}$ is the prototypical diverging term, reducing to itself in one single β-step. $\mathbf{\Omega_3}$ is another example of diverging term, producing terms of greater and greater size: $\mathbf{\Omega_3} \to_\beta (\mathbf{\Omega_3})\,\mathbf{\Delta_3} \to_\beta (\mathbf{\Omega_3})\,\mathbf{\Delta_3}\mathbf{\Delta_3} \to_\beta \ldots$ It will be used in Remark 37 to prove the subtlety of characterizing strong normalization with finiteness spaces. The Turing fixed-point combinator $\mathbf{\Theta}$ has been used in the Introduction to construct $(\mathbf{\Theta})\,\lambda x.(x+y)$ as an example of non-deterministic λ-term morally reducing to normal forms with infinite coefficients (Equation (3)). Notice that, by abstraction linearity, $(\mathbf{\Theta})\,\lambda x.(x+y) = (\mathbf{\Theta})\,(\lambda x.x+\lambda x.y)$, however $(\mathbf{\Theta})\,(\lambda x.x+\lambda x.y) \neq (\mathbf{\Theta})\,\lambda x.x+(\mathbf{\Theta})\,\lambda x.y$, because application is not linear in the argument. This distinction is crucial: the latter term reduces to $((\mathbf{\Theta})\,\lambda x.x) + y$, with $(\mathbf{\Theta})\,\lambda x.x$ reducing to itself without producing any further occurence of y.

2.2 Resource calculus Δ and Taylor expansion

The syntax. Resource terms and *bags* are given by mutual induction:

$$\text{resource terms } \Delta: \quad s := x \mid \lambda x.s \mid \langle s\rangle\, \overline{s} \qquad \text{bags } \Delta^!: \quad \overline{s} := 1 \mid s \cdot \overline{s}$$

subject to both α-equivalence and commutativity of $(\cdot)$: we most often write $[s_1, \ldots, s_n]$ for $s_1 \cdot (\cdots (s_n \cdot 1) \cdots)$ and then $[s_1, \ldots, s_n] = [s_{\sigma(1)}, \ldots, s_{\sigma(n)}]$ for any permutation σ. In other words, bags are finite multisets of terms.

$x^* := x$	$\mathcal{T}(x) := \{x\}$
$(\lambda x.M)^* := \lambda x.M^*$	$\mathcal{T}(\lambda x.M) := \{\lambda x.s;\ s \in \mathcal{T}(M)\}$
$((M)\,N)^* := \sum_{n\in\mathbf{N}} \frac{1}{n!} \langle M^* \rangle\ (N^*)^n$	$\mathcal{T}((M)\,N) := \left\{\langle s \rangle\ \bar{t};\ s \in \mathcal{T}(M), \bar{t} \in \mathcal{T}(N)^!\right\}$
$(M+N)^* := M^* + N^*$	$\mathcal{T}(M+N) := \mathcal{T}(M) \cup \mathcal{T}(N)$
(a) expansion $(\ )^* : \Lambda_+ \mapsto \mathbb{Q}[\![\Delta]\!]$	(b) support $\mathcal{T}(\ ) : \Lambda_+ \mapsto \mathbb{B}[\![\Delta]\!]$

Table 1: Definition of the Taylor expansion $(\ )^*$ and of its support $\mathcal{T}(\ )$.

Linear extension. Let $\mathcal{R}$ be a rig (a.k.a. semi-ring). Of particular interest are the rigs $\mathbb{B} := (\{0,1\}, \max, \min)$ of booleans, $\mathbb{N} := (\mathbf{N}, +, \times)$ of non-negative integers and $\mathbb{Q} := (\mathbf{Q}, +, \times)$ of rational numbers. We denote as $\mathcal{R}[\![\Delta]\!]$ (resp. $\mathcal{R}[\![\Delta^!]\!]$) the set of all formal (finite or infinite) linear combinations of resource terms (resp. bags) with coefficients in $\mathcal{R}$. If $a \in \mathcal{R}[\![\Delta]\!]$ (resp. $\overline{a} \in \mathcal{R}[\![\Delta^!]\!]$) and $s \in \Delta$ (resp. $\overline{s} \in \Delta^!$), then $a_s \in \mathcal{R}$ (resp. $\overline{a}_{\overline{s}} \in \mathcal{R}$) denotes the coefficient of s in a (resp. $\overline{s}$ in $\overline{a}$). As is well-known, $\mathcal{R}[\![\Delta]\!]$ is endowed with a structure of $\mathcal{R}$-module, where addition and scalar multiplication are defined component-wise, i.e. for $a, b \in \mathcal{R}[\![\Delta]\!]$ and $\alpha \in \mathcal{R}$: $(a+b)_s := a_s + b_s$, and $(\alpha a)_s := \alpha a_s$. We will write $|a| \subseteq \Delta$ for the support of a: $|a| = \{s \in \Delta;\ a_s \neq 0\}$.

Moreover, each resource calculus constructor extends to $\mathcal{R}[\![\Delta]\!]$ component-wise, i.e. for any $a, a_1, \dots, a_n \in \mathcal{R}[\![\Delta]\!]$ and $\overline{a} \in \mathcal{R}[\![\Delta^!]\!]$, we set:

$$\lambda x.a := \textstyle\sum_{s\in\Delta} a_s \lambda x.s, \qquad \langle a \rangle\ \overline{a} := \textstyle\sum_{s\in\Delta,\, \overline{s}\in\Delta^!} a_s \overline{a}_{\overline{s}} \langle s \rangle\ \overline{s},$$

$$[a_1, \dots, a_n] := \textstyle\sum_{s_1,\dots,s_n\in\Delta} (a_1)_{s_1} \cdots (a_n)_{s_n} [s_1, \dots, s_n].$$

Notice that the last formula is coherent with the notation of the concatenation of bags as a product since it expresses a distributivity law. In particular, we denote $a^n := \underbrace{[a, \dots, a]}_{n}$ and if $\mathbb{Q} \subseteq \mathcal{R}$, $a^! := \sum_{n\in\mathbf{N}} \frac{1}{n!} a^n$.

In the case $\mathcal{R} = \mathbb{B}$, notice that $\mathbb{B}[\![\Delta]\!]$ is the power-set lattice $\mathfrak{P}(\Delta)$, so that we can use the set-theoretical notation: e.g. writing $s \in a$ instead of $a_s \neq 0$, or $a \cup b$ for $a + b$. Also, in that case the preceding formulas lead to: for $a, b \subseteq \Delta$, $t \in \Delta$, $\overline{a} \subseteq \Delta^!$: $\lambda x.a := \{\lambda x.s;\ s \in a\}$, $a^! := \{[s_1, \cdots, s_n];\ s_1, \cdots, s_n \in a\}$, $\langle t \rangle\ \overline{a} := \{\langle t \rangle\ \overline{s};\ \overline{s} \in \overline{a}\}$ and $\langle a \rangle\ \overline{a} := \bigcup \{\langle s \rangle\ \overline{a};\ s \in a\}$.

Taylor expansion. Ehrhard and Regnier have used in [7] the rig of rational numbers to express the λ-terms as formal combinations in $\mathbb{Q}[\![\Delta]\!]$. We refer to this translation $(\ )^* : \Lambda_+ \mapsto \mathbb{Q}[\![\Delta]\!]$ as the *Taylor expansion* and we recall it in Table 1a by structural induction. The supports of these expansions can be seen as a map $\mathcal{T}(\ ) : \Lambda_+ \mapsto \mathbb{B}[\![\Delta]\!]$ and directly defined by induction as in Table 1b.

Example 2. From the above definitions we have:

$$\mathbf{\Delta}^* = \sum_n \frac{1}{n!}\lambda x.\, \langle x\rangle\ x^n, \qquad \mathbf{\Delta_3}^* = \sum_{n,m} \frac{1}{n!m!}\lambda x.\, \langle\langle x\rangle\ x^n\rangle\ x^m.$$

Let us denote as δ_n (resp. $\delta_{n,m}$) the term $\lambda x.\, \langle x\rangle\ x^n$ (resp. $\lambda x.\, \langle\langle x\rangle\ x^n\rangle\ x^m$). We can then write:

$$\mathbf{\Omega}^* = \sum_k \frac{1}{k!} \sum_{n_0,\dots,n_k} \frac{1}{n_0!\cdots n_k!} \langle \delta_{n_0}\rangle\ [\delta_{n_1},\dots,\delta_{n_k}]\,, \tag{4}$$

$$\mathbf{\Omega_3}^* = \sum_k \frac{1}{k!} \sum_{\substack{n_0,\dots,n_k\\ m_0,\dots,m_k}} \frac{1}{n_0!m_0!\cdots n_k!m_k!} \langle \delta_{n_0,m_0}\rangle\ [\delta_{n_1,m_1},\dots,\delta_{n_k,m_k}]\,. \tag{5}$$

It is clear that the resource terms appearing with non-zero coefficients in M^* describe the structure of M taking an explicit number of times the argument of each application, and this recursively. The rôle of the rational coefficients will be clearer once defined the reduction rules over Δ (see Example 5).

Operational semantics. Let us write $\deg_x(t)$ for the number of free occurrences of a variable x in a resource term t. We define the *differential substitution* of a variable x with a bag $[s_1,\dots,s_n]$ in a resource term t, denoted $\partial_x t\cdot[s_1,\dots,s_n]$: it is a finite formal sum of resource terms, which is zero whenever $\deg_x(t)\neq n$; otherwise it is the sum of all possible terms obtained by linearly replacing each free occurrence of x with exactly one s_i, for $i=1,\dots,n$. Formally,

$$\partial_x t\cdot[s_1,\dots,s_n] := \begin{cases} \displaystyle\sum_{f\in\mathfrak{S}_n} t\left[s_{f(1)}/x_1,\dots,s_{f(n)}/x_n\right] & \text{if } \deg_x(t)=n,\\ 0 & \text{otherwise,}\end{cases} \tag{6}$$

where $\mathfrak{S}_n$ is the group of permutations over $n=\{1,\dots,n\}$ and $x_1,\dots,x_n$ is any enumeration of the free occurrences of x in t, so that $t\left[s_{f(i)}/x_i\right]$ denotes the term obtained from t by replacing the i-th free occurrence of x with $s_{f(i)}$. Then, we give a linear extension of the differential substitution: if $a\in\mathcal{R}[\![\Delta]\!]$ and $\overline{a}\in\mathcal{R}[\![\Delta^!]\!]$, we set: $\partial_x a\cdot\overline{a} = \sum_{t\in\Delta,\overline{s}\in\Delta^!} a_t\,\overline{a}_{\overline{s}}\partial_x t\cdot\overline{s}$.

The resource reduction $\to_r$ is then the smallest relation satisfying:

$$\langle\lambda x.t\rangle\ [s_1,\dots,s_n] \to_r \partial_x t\cdot[s_1,\dots,s_n]$$

and moreover linear and compatible with the resource calculus constructors. Spelling out these two last conditions: for any $t,u\in\Delta$, $\overline{s}\in\Delta^!$, $a,b\in\mathcal{R}[\![\Delta]\!]$, $\alpha\in\mathcal{R}\setminus\{0\}$, whenever $t\to_r a$, we have: (compatibility) $\lambda x.t\to_r\lambda x.a$, $\langle t\rangle\ \overline{s}\to_r \langle a\rangle\ \overline{s}$, $\langle u\rangle\ t\cdot\overline{s}\to_r\langle u\rangle\ a\cdot\overline{s}$, and (linearity) $\alpha t+b\to_r\alpha a+b$.

Proposition 3 ([7]). *Resource reduction $\to_r$ is confluent over the whole $\mathcal{R}[\![\Delta]\!]$ and, it is normalizing on the sums in $\mathcal{R}[\![\Delta]\!]$ having a finite support.*

Proposition 3 shows that any single resource term $t \in \Delta$ has a unique normal form that we can denote as $\mathrm{NF}(t)$. What about possibly infinite linear combinations in $\mathcal{R}[\![\Delta]\!]$? We would like to extend the normal form operator componentwise, as follows:

$$\mathrm{NF}(a) := \sum_{t \in \Delta} a_t \cdot \mathrm{NF}(t) \tag{7}$$

Example 4. The sum in Equation (7) can be undefined for general $a \in \mathcal{R}[\![\Delta]\!]$. Take $a = \lambda x.x + \langle \lambda x.x \rangle \, [\lambda x.x] + \langle \lambda x.x \rangle \, [\langle \lambda x.x \rangle \, [\lambda x.x]] + \dots$. Any single term reduces to $\lambda x.x$, hence $\mathrm{NF}(a)_{\lambda x.x}$ is infinite. Another example is given by the Taylor expansion of the term in Equation (3) discussed in the Introduction: one can check that $\mathrm{NF}\big(\mathcal{T}\left((\mathbf{\Theta})\, \lambda x.(x+y)\right)\big)_y$ is infinite because, for any $n \in \mathbb{N}$, there is a resource term in $\langle \mathcal{T}(\mathbf{\Theta}) \rangle \, [(\lambda x.x)^n, y]$ reducing to y.

In fact, Corollary 39 ensures that if a is the Taylor expansion of a strongly normalizing non-deterministic λ-term then Equation (7) is well-defined.

Example 5. Recall the expansions of Example 2 and consider $((\mathbf{\Delta})\, \lambda x.x)^* = \sum_{n,k} \frac{1}{n!k!} \langle \lambda x. \langle x \rangle \, [x^n] \rangle \, \left[(\lambda x.x)^k\right]$. The resource reduction applied to a term of this sum gives zero except for $k = n+1$; in this latter case we get $(n+1)! \, \langle \lambda x.x \rangle \, [(\lambda x.x)^n]$. Hence we have: $((\mathbf{\Delta})\, \lambda x.x)^* \to_r \sum_n \frac{1}{n!} \langle \lambda x.x \rangle \, [(\lambda x.x)^n]$, because the coefficient $(n+1)!$ generated by the reduction step is erased by the fraction $\frac{1}{k!}$ in the definition of Taylor expansion. Then, the term $\langle \lambda x.x \rangle \, [(\lambda x.x)^n]$ reduces to zero but for $n = 1$, in the latter case giving $\lambda x.x$. So we have: $\mathrm{NF}(((\mathbf{\Delta})\, \lambda x.x)^*) = \lambda x.x = (\mathrm{NF}((\mathbf{\Delta})\, \lambda x.x))^*$.

The commutation between computing normal forms and Taylor expansions has been proven in general for deterministic λ-terms [6][6] and witnesses the solidity of the definitions. The general case for Λ_+ is still an open issue.

Example 6. Recall the notation of Equation (4) from Example 2 expressing the sum $\mathbf{\Omega}^*$. All terms with $n_0 \neq k+1$ reduce to zero in one step. For $n_0 = k+1$, we have that a single term rewrites to $\sum_{f \in \mathfrak{S}_k} \left\langle \delta_{n_{f(1)}} \right\rangle \left[\delta_{n_{f(2)}}, \dots, \delta_{n_{f(k)}}\right]$, which is a sum of terms still in $\mathcal{T}(\mathbf{\Omega})$, but with smaller size w.r.t. the redex. Therefore, every term of $\mathbf{\Omega}^*$ eventually reduces to zero, after a reduction sequence whose length depends on the initial size of the term, and whose elements are sums with supports in $\mathcal{T}(\mathbf{\Omega})$. This is in some sense the way Taylor expansion models the unbounded resource consumption of the loop $\mathbf{\Omega} \to_\beta \mathbf{\Omega}$ in λ-calculus.

We postpone the discussion of the reduction of ${\mathbf{\Omega}_3}^*$ until Remark 37.

2.3 Finiteness structures induced by antireduction

Let us get back to Equation (7), and consider it pointwise: for all $s \in \Delta$ in normal form, we want to set $\mathrm{NF}(a)_s = \sum_{t \in \Delta} a_t \cdot \mathrm{NF}(t)_s$. Notice that this series can be obtained as the inner product between a and the vector $\uparrow s$ with $(\uparrow s)_t = \mathrm{NF}(t)_s$: one can think of s as a test, that a passes whenever the sum converges.

[6] The statement proven in [6] is actually more general, because it considers (possibly infinite) Böhm trees instead of the normal forms.

There is one very simple condition that one can impose on a formal series to ensure its convergence: just assume there are finitely many non-zero terms. This seemingly dull remark is in fact the starting point of the definition of finiteness spaces, introduced by Ehrhard [3] and discussed in the Introduction. The basic construction is that of a finiteness structure:

Definition 7 ([3]). *Let A be a fixed set. A* structure *on A is any set of subsets $\mathfrak{A} \subseteq \mathfrak{P}(A)$. For all subsets a and $a' \subseteq A$, we write $a \perp a'$ whenever $a \cap a'$ is finite. For all structure $\mathfrak{A} \subseteq \mathfrak{P}(A)$, we define its dual $\mathfrak{A}^{\perp} = \{a \subseteq A;\ \forall a' \in \mathfrak{A},\ a \perp a'\}$. A* finiteness structure *on A is any such $\mathfrak{A}^{\perp}$.*

Notice that: $\mathfrak{A} \subseteq \mathfrak{A}^{\perp\perp}$, also $\mathfrak{A} \subseteq \mathfrak{A}'$ entails $\mathfrak{A}'^{\perp} \subseteq \mathfrak{A}^{\perp}$, hence $\mathfrak{A}^{\perp} = \mathfrak{A}^{\perp\perp\perp}$.

Let $\mathfrak{C}_0 = \{|\uparrow s|\,;\ s \in \Delta, \text{ in normal form}\} \subseteq \mathfrak{P}(\Delta)$, we obtain that $|a| \in {\mathfrak{C}_0}^{\perp}$ iff Equation (7) involves only pointwise finite sums. So, one is led to focus on support sets only, leaving out coefficients entirely. Henceforth, unless specified otherwise, we will thus stick to the case of $\mathcal{R} = \mathbb{B}$, and use set-theoretical notations only. This approach of ensuring the normalization of Taylor expansion *via* a finiteness structure was first used by Ehrhard [4] for a non-deterministic variant of System $\mathcal{F}$. Our paper strengthens Ehrhard's result in several directions. In order to state them, we introduce a construction of finiteness structures on Δ induced by anticones for the reduction order defined as follows:

Definition 8. *For all $s, t \in \Delta$, we write $t \geq s$ whenever there exists a reduction $t \to_r^* a$ with $s \in a$.*

It should be clear that this defines a partial order relation (in particular we have antisymmetry because $\to_r$ terminates).

Definition 9. *If $a \subseteq \Delta$, $\uparrow a := \{t \in \Delta;\ \exists s \in a,\ t \geq s\}$ is the cone of antireduction over a.[7] For any structure $\mathfrak{T} \subseteq \mathfrak{P}(\Delta)$, we write $\triangledown(\mathfrak{T}) = \{\uparrow a\,;\ a \in \mathfrak{T}\}$.*

We can consider the elements of $\mathfrak{T}$ as tests, and say a set $a \subseteq \Delta$ passes a test $a' \in \mathfrak{T}$ iff $a \perp \uparrow a'$. The structure of sets that pass all tests is exactly $\triangledown(\mathfrak{T})^{\perp}$. Then Ehrhard's result can be rephrased as follows:

Theorem 10 ([4]). *If $M \in \Lambda_+$ is typable in System $\mathcal{F}$ then $\mathcal{T}(M) \in \triangledown(\mathfrak{S}_{sgl})^{\perp}$ where $\mathfrak{S}_{sgl} := \{\{s\}\,;\ s \in \Delta\}$.*

Notice that, in contrast with the definition of $\mathfrak{C}_0$, $\mathfrak{S}_{\text{sgl}}$ is in fact not restricted to normal forms. Our paper extends this theorem in three distinct directions: first, one can relax the condition that M is typed in System $\mathcal{F}$ and require only that M is strongly normalizable; second, the same result can be established for sets of "tests" larger (hence more demanding) than $\mathfrak{S}_{\text{sgl}}$; third, the implication can be reversed for a suitable set of tests $\mathfrak{T}$, i.e. M is strongly normalizable iff $\mathcal{T}(M) \in \triangledown(\mathfrak{T})^{\perp}$ (and we do need $\mathfrak{T}$ to provide more tests than just singletons: see Remark 37). In order to state our results precisely, we need to introduce a few more notions.

[7] Observe that $\uparrow\{s\} = \uparrow s$ (up to the identification of $\mathbb{B}[\![\Delta]\!]$ with $\mathfrak{P}(\Delta)$).

$$A, B := X \mid A \to B \mid A \cap B$$

(a) Grammar of types, X varying over a denumerable set of propositional variables

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$$\frac{\text{for } i \in \{1,2\}}{A_1 \cap A_2 \preceq A_i} \qquad \frac{A' \preceq A \quad B \preceq B'}{A \to B \preceq A' \to B'} \qquad \frac{}{(A \to B) \cap (A \to C) \preceq A \to (B \cap C)}$$

(b) Rules generating the subtyping relation $\preceq$

$$\frac{}{\Gamma, x : A \vdash x : A}\ (var) \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ (\to_i)$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash (M)\, N : B}\ (\to_e) \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}\ (\cap_i)$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}\ (+) \qquad \frac{\Gamma \vdash M : A \quad A \preceq A'}{\Gamma \vdash M : A'}\ (\preceq)$$

(c) Derivation rules

Table 2: the intersection type assignment system $\mathcal{D}_+$ for Λ_+

Definition 11. *We say that a resource term s is* linear *whenever each bag appearing in s has cardinality* 1*. A set a of resource terms is said* linear *whenever all its elements are* linear*. We say a* bounded *whenever there exists a number $n \in \mathbf{N}$ bounding the cardinality of all bags in all terms in a. We then write*

$$\mathfrak{L} := \{a \subseteq \Delta;\ a \textit{ linear}\} \qquad \textit{and} \qquad \mathfrak{B} := \{a \subseteq \Delta;\ a \textit{ bounded}\}\,.$$

We also denote as $\ell\,(M)$ the subset of the linear resource terms in $\mathcal{T}\,(M)$. Notice that $\ell\,(M)$ is always non-empty and can be directly defined by replacing the definition of $\mathcal{T}\,((M)\,N)$ in Table 1b with: $\ell\,((M)\,N) := \{\langle s\rangle\ [t];\ s \in \ell\,(M)\,, t \in \ell\,(N)\}$.

Notice that $\mathfrak{S}_{\mathrm{sgl}}, \mathfrak{L} \subseteq \mathfrak{B}$.

We can sum up our results Corollary 30 and Theorem 36 as follows:

Theorem 12. *Let $M \in \Lambda_+$:*

- *If M is strongly normalizing, then $\mathcal{T}\,(M) \in \nabla\,(\mathfrak{T})^\perp$ as soon as $\mathfrak{T} \subseteq \mathfrak{B}$.*
- *If $\mathcal{T}\,(M) \in \nabla\,(\mathfrak{T})^\perp$ with $\mathfrak{L} \subseteq \mathfrak{T}$, then M is strongly normalizing.*

3 Strongly Normalizing Terms Are $\mathcal{D}_+$ Typable

Intersection types are well known, as well as their relation with normalizability. We refer to [2] for the original system with subtyping characterizing the set of strongly normalizing λ-terms, and [1] and [8] for simpler systems. However, as far as we know, the literature about intersection types for non-deterministic λ-calculi is less well established and in fact we could find no previous characterization of strong normalization in a non-deterministic setting. Hence, we give in Table 2 a variant of Krivine's system $\mathcal{D}$ [8], characterizing the set of strongly normalizing terms in Λ_+. In this section, we only prove that strongly normalizing terms are typable (Theorem 14): the reverse implication follows from the rest of the paper (see Figure 1). These techniques are standard (see Appendix A).

Remark 13. Krivine's original System $\mathcal{D}$ does not have $(\preceq)$ and $(+)$, but it has the two usual elimination rules for intersection (here derivable). The rule $(+)$ is necessary to account for non-determinism, however adding just $(+)$ to System $\mathcal{D}$ is misbehaving. We can find terms M and N, and a context Γ such that $(M)\,N$ is typable in System $\mathcal{D}$ with $(+)$ under the context Γ but M is not: take $\Gamma = x : A \to B \cap B', y : A \to B \cap B'', z : A$, observe that $(x+y)\,z = (x)\,z + (y)\,z$, and thus $\Gamma \vdash (x+y)\,z : B$ but $x+y$ is not typable in Γ. This is the reason why we introduce subtyping.

Theorem 14. *For all $M \in \Lambda_+$, if M is strongly normalizable, then there exists a derivable judgement $\Gamma \vdash M : A$ in system $\mathcal{D}_+$.*

Proof (Sketch). For M a strongly normalizable term, let $\mathbf{l}\,(M)$ be the maximum length of a reduction from M, and $\mathbf{s}\,(M)$ the number of symbols occurring in M. By well-founded induction on the pair $(\mathbf{l}\,(M)\,, \mathbf{s}\,(M))$ we prove that there exists $n_M \in \mathbf{N}$ such that for all type B and all $n \geq n_M$, there is a context Γ and a sequence $(A_1, \ldots, A_n)$ of types such that $\Gamma \vdash M : A_1 \to \cdots \to A_n \to B$.

The proof splits depending on the structure of M. In case $M = M_1 + M_2$, we apply the induction hypothesis on both M_1 and M_2 and conclude by rule $(\preceq)$ and a contravariance property: $\Gamma \vdash M : A$ whenever $\Gamma' \vdash M : A$ and $\Gamma \preceq \Gamma'$.

In case of head-redexes, i.e. $M = ((\lambda x.N)\,P)\,M_1 \cdots M_q$, we apply the induction hypothesis on $M' = (N\,[P/x])\,M_1 \cdots M_q$ and on P. Then, we conclude via a subject expansion lemma stating that: $\Gamma \vdash (\lambda x.N)\,P\,M_1 \cdots M_n : A$, whenever $\Gamma \vdash (N\,[P/x])\,M_1 \cdots M_n : A$ and there exists B such that $\Gamma \vdash P : B$.

The other cases are similar to the first one. □

4 $\mathcal{D}_+$ Typable Terms Are Finitary

This section proves Corollary 30, giving sufficient conditions (to be dispersed, hereditary and expandable, see resp. Definition 23, 26 and 27) over a structure $\mathfrak{T}$ in order to have all cones $\uparrow a$ for $a \in \mathfrak{T}$ dual to the Taylor expansion of any strongly normalizing non-deterministic λ-term.

It is easy to check that these conditions are satisfied by the structures $\mathfrak{B}$ of bounded sets and $\mathfrak{L}$ of sets of linear terms (Definition 11). Moreover, as an immediate corollary one gets also that any subset $\mathfrak{T} \subseteq \mathfrak{B}$ is also such that $\mathcal{T}\,(M) \in \nabla\,(\mathfrak{T})^{\perp}$ for any strongly normalizable term $M \in \Lambda_+$, so getting the first Item of Theorem 12.

Thanks to previous Theorem 14 we can prove Corollary 30 by a realizability technique on the intersection type system $\mathcal{D}_+$. For a fixed structure $\mathfrak{S}$, we associate with any type A a realizer $\|A\|_{\mathfrak{S}}$ (Definition 18). In the case $\mathfrak{S}$ is adapted (Definition 17), we can prove that $\|A\|_{\mathfrak{S}}$ contains the Taylor expansion of any term of type A and that it is contained in $\mathfrak{S}$ (Theorem 21). These definitions and theorem are adapted from Krivine's proof for System $\mathcal{D}$ [8].

The crucial point is then to find structures $\mathfrak{S}$ which are *adapted*: here is our contribution. The structures that we study have the shape $\triangledown(\mathfrak{T})^{\perp}$, so that we are speaking of the interaction with cones of anti-reducts of tests in a structure $\mathfrak{T}$. Intuitively, $\mathfrak{T}$ is a set of tests that can be passed by any term typable in System $\mathcal{D}_+$ (hence by any strongly normalizing term). We prove (Lemma 29) that for a structure $\mathfrak{T}$, being dispersed, hereditary and expandable is sufficient to guarantee that the dual structure $\triangledown(\mathfrak{T})^{\perp}$ is adapted, then achieving Corollary 30.

Definition 15 (Functional). *Given two structures $\mathfrak{S}, \mathfrak{S}' \subseteq \mathfrak{P}(\Delta)$, we define the structure $\mathfrak{S} \to \mathfrak{S}' := \left\{f \subseteq \Delta;\ \forall a \in \mathfrak{S}, \langle f\rangle\ a^! \in \mathfrak{S}'\right\}$.*

Definition 16 (Saturation). *Let $\mathfrak{S}, \mathfrak{S}' \subseteq \mathfrak{P}(\Delta)$. We say $\mathfrak{S}'$ is* $\mathfrak{S}$-saturated *if $\forall e, f_0, \ldots, f_n \in \mathfrak{S}$, $\left\langle \partial_x e \cdot f_0{}^! \right\rangle f_1{}^! \ldots f_n{}^! \in \mathfrak{S}'$ implies $\langle \lambda x.e\rangle\ f_0{}^! f_1{}^! \ldots f_n{}^! \in \mathfrak{S}'$.*

Definition 17 (Adaptedness). *Let $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ and define the structure: $\mathfrak{S}_h := \{\{x\}\} \cup \left\{\langle x\rangle\ a_1{}^! \cdots a_n{}^!;\ x \in \mathcal{V},\ n > 0 \text{ and } \forall i,\ a_i \in \mathfrak{S}\right\}$. We call $\mathfrak{S}$* adapted *if:*

1. *$\mathfrak{S}$ is $\mathfrak{S}$-saturated;*
2. *$\mathfrak{S}_h \subset (\mathfrak{S} \to \mathfrak{S}_h) \subset (\mathfrak{S}_h \to \mathfrak{S}) \subset \mathfrak{S}$;*
3. *closed under finite unions: $\forall b, b' \in \mathfrak{S}$, $b \cup b' \in \mathfrak{S}$.*

Definition 18 (Realizers). *Let $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$. To each type A of System $\mathcal{D}_+$, we associate a structure $\|A\|_{\mathfrak{S}}$ defined inductively (X being a propositional variable):*

$$\|X\|_{\mathfrak{S}} := \mathfrak{S}, \quad \|A \to B\|_{\mathfrak{S}} := \|A\|_{\mathfrak{S}} \to \|B\|_{\mathfrak{S}}, \quad \|A \cap B\|_{\mathfrak{S}} := \|A\|_{\mathfrak{S}} \cap \|B\|_{\mathfrak{S}}.$$

Lemma 19. *Let $\mathfrak{S}$ be an adapted structure, then for every type A, $\|A\|_{\mathfrak{S}}$ is $\mathfrak{S}$-saturated, closed under finite unions and $\mathfrak{S}_h \subseteq \|A\|_{\mathfrak{S}} \subseteq \mathfrak{S}$.*

Lemma 20 (Adequacy). *If $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ is adapted, $x_1 : A_1, \ldots, x_n : A_n \vdash M : B$ and for all $1 \le i \le n$, $a_i \in \|A_i\|_{\mathfrak{S}}$, then $\partial_{x_1,\ldots,x_n} \mathcal{T}(M) \cdot a_1{}^!, \ldots, a_n{}^! \in \|B\|_{\mathfrak{S}}$.*

Proof (Sketch). By structural induction on the derivation of $x_1 : A_1, \ldots, x_n : A_n \vdash M : B$. The cases where the last rule is $(\to_i)$ or $(+)$ use respectively the facts that the realizers are saturated and closed by finite unions. The case where the last rule is $(\preceq)$ is an immediate consequence of the induction hypothesis and of a lemma stating that $A \preceq B$ implies $\|A\|_{\mathfrak{S}} \subseteq \|B\|_{\mathfrak{S}}$. □

Theorem 21. *If $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ is adapted and M is typable in System $\mathcal{D}_+$, then $\mathcal{T}(M) \in \mathfrak{S}$.*

Proof. Let $\Gamma \vdash M : B$. For any $x : A$ in Γ, $\{x\} \in \mathfrak{S}_h \subset \|A\|_{\mathfrak{S}}$ by Lemma 19 and Definition 17. Hence, $\mathcal{T}(M) = \partial_{x_1,\ldots,x_n} \mathcal{T}(M) \cdot \{x_1\}^!, \ldots, \{x_n\}^! \in \|B\|_{\mathfrak{S}}$ by Lemma 20. Again by Lemma 19, $\|B\|_{\mathfrak{S}} \subseteq \mathfrak{S}$, so $\mathcal{T}(M) \in \mathfrak{S}$. □

Now we look for conditions to ensure that a structure $\mathfrak{S}$ is *adapted*. These conditions (Definition 23, 26 and 27) are quite technical but they are easy to check, in particular the structures $\mathfrak{B}$ and $\mathfrak{L}$ enjoy them (Remark 28).

Definition 22. *The* height $\mathbf{h}(s)$ *of a resource term s is defined inductively:* $\mathbf{h}(x) := 1$, $\mathbf{h}(\lambda x.s) := 1 + \mathbf{h}(s)$ *and* $\mathbf{h}(\langle s_0 \rangle\ [s_1, \ldots, s_n]) := 1 + \max_i(\mathbf{h}(s_i))$.

Definition 23 (Dispersed). *A set $a \subseteq \Delta$ is* dispersed *whenever for all $n \in \mathbf{N}$, and all finite set V of variables, the set $\{s \in a;\ \mathbf{h}(s) \leq n \text{ and } \mathsf{fv}(s) \subseteq V\}$ is finite. A structure $\mathfrak{S}$ is dispersed whenever $\forall a \in \mathfrak{S}$, a is dispersed.*

Definition 24. *Let $s \in \Delta$ and $x \notin \mathsf{fv}(s)$. We define* immediate subterm projections $\pi_x s \in \mathfrak{P}(\Delta)$, $\pi_0 s \in \mathfrak{P}(\Delta)$, $\overline{\pi}_1 s \in \mathfrak{P}\left(\Delta^!\right)$ *and* $\pi_1 s \in \mathfrak{P}(\Delta)$ *as follows:*

- *if $s = \lambda x.t$ then $\pi_x s = \{t\}$; otherwise $\pi_x s = \emptyset$;*
- *if $s = \langle t \rangle\ \overline{u}$ then $\pi_0 s = \{t\}$, $\overline{\pi}_1 s = \{\overline{u}\}$, $\pi_1 s = |\overline{u}|$; otherwise $\pi_0 s = \overline{\pi}_1 s = \pi_1 s = \emptyset$.*

Observe that up to α-conversion and the hypothesis $x \notin \mathsf{fv}(s)$, the abstraction case is exhaustive. These functions obviously extend to sets of terms, up to some care about free variables. If $V \subseteq \mathcal{V}$ is a set of variables, we write Δ^V for the set of resource λ-terms with free variables in V.

Definition 25. *For all $V \subseteq \mathcal{V}$ and $a \subseteq \Delta^V$, let*

$$\pi_0 a := \bigcup_{s \in a} \pi_0 s \subseteq \Delta^V, \qquad \overline{\pi}_1 a := \bigcup_{s \in a} \overline{\pi}_1 s \subseteq \Delta^{V^!}, \qquad \pi_1 a := \bigcup_{s \in a} \pi_1 s \subseteq \Delta^V,$$

and if moreover $x \notin V$, then let $\pi_x a := \bigcup_{s \in a} \pi_x s \subseteq \Delta^{V \cup \{x\}}$.

Definition 26 (Hereditary). *A structure $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ is said to be* hereditary *if, $\mathfrak{S}$ is downwards closed, and for all $a \in \mathfrak{S}$, $\pi_0 a \in \mathfrak{S}$, $\pi_1 a \in \mathfrak{S}$ and for all $x \in \mathcal{V} \setminus \mathsf{fv}(a)$, $\pi_x a \in \mathfrak{S}$.*

Definition 27 (Expandable). *A structure $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ is said to be* expandable *if, for all $x \in \mathcal{V}$ and all $a \in \mathfrak{S}$, we have $\{\langle s \rangle\ [x];\ s \in a\} \in \mathfrak{S}$.*

Remark 28. The structures $\mathfrak{S}_{\text{sgl}}$ (Theorem 10) and $\mathfrak{L}$, $\mathfrak{B}$ (Definition 11) are dispersed and expandable. The last two are also hereditary, while $\mathfrak{S}_{\text{sgl}}$ is not.

Lemma 29. *For any structure $\mathfrak{T}$ which is dispersed, hereditary and expandable, we have that $\triangledown(\mathfrak{T})^{\perp}$ is adapted.*

Proof (Sketch). One has to prove the three conditions of Definition 17. Hereditariness and dispersion are used to obtain saturation (Condition 1). The inclusions of Condition 2 need all hypotheses and an auxiliary lemma proving that ${\triangledown(\mathfrak{T})^{\perp}}_h \subseteq \triangledown(\mathfrak{T})^{\perp}$, for which dispersion is crucial. Finally, the closure under finite unions (Condition 3) is an easy property for any dual structure. See Appendix B.2 for more details. $\square$

Corollary 30. *Let $\mathfrak{T} \subseteq \mathfrak{P}(\Delta)$ be dispersed, hereditary and expandable. For every strongly normalizable term M, we have $\mathcal{T}(M) \in \triangledown(\mathfrak{T})^{\perp}$. Hence, this holds for $\mathfrak{T} \in \{\mathfrak{L}, \mathfrak{B}\}$ and for any of their subsets, such as $\mathfrak{S}_{sgl} \subset \mathfrak{B}$.*

Proof. The general statement follows from Theorems 14, 21 and Lemma 29. Remark 28 implies $\mathcal{T}(M) \in \triangledown(\mathfrak{T})^{\perp}$ for $\mathfrak{T} \in \{\mathfrak{B}, \mathfrak{L}\}$. The rest of the statement follows because, for any structures $\mathfrak{S}, \mathfrak{S}'$, $\mathfrak{S} \subseteq \mathfrak{S}'$ implies ${\mathfrak{S}'}^{\perp} \subseteq \mathfrak{S}^{\perp}$. □

5 Finitary Terms Are Strongly Normalizing

In this section we prove Theorem 36, giving a sufficient condition for a structure $\mathfrak{T}$ to be able to test strong normalization. The condition is that $\mathfrak{T}$ includes at least $\mathfrak{L}$, i.e. the set of all sets of linear terms[8].

The proof is by contraposition, suppose that M is divergent, then $\mathcal{T}(M)$ is not dual to some cone $\uparrow a$, with $a \in \mathfrak{L}$. The proof enlightens two kinds of divergence in λ-calculus: the one generated by looping terms: $\mathbf{\Omega} \to_\beta \mathbf{\Omega} \to_\beta \dots$ and the other generated by infinite reduction sequences $(M_i)_{i\in\mathbf{N}}$ with an infinite number of different terms: $\mathbf{\Omega_3} \to_\beta (\mathbf{\Omega_3})\,\mathbf{\Delta_3} \to_\beta ((\mathbf{\Omega_3})\,\mathbf{\Delta_3})\,\mathbf{\Delta_3} \to_\beta \dots$ (see Example 1).

In the first case, the cone $\big\uparrow \ell(\mathbf{\Omega})$ of the linear expansion (Definition 11) of the looping term $\mathbf{\Omega}$ suffices to show up the divergence, since $\mathcal{T}(\mathbf{\Omega}) \cap \big\uparrow \ell(\mathbf{\Omega})$ is infinite. Indeed the Taylor expansion of a looping term, say $\mathcal{T}(\mathbf{\Omega})$, is a kind of "contractible space", where any resource term reduces to a smaller term within the same Taylor expansion or vanishes (see Example 6). In particular, there are unboundedly large resource terms reducing to the linear expansion $\ell(\mathbf{\Omega})$.

In the case of an infinite reduction sequence of different terms, one should take, basically, the cone of all linear expansions of the terms occurring in the sequence: the linear expansion of a single term (or of a finite set of terms) might not suffice to test this kind of divergence. For example, $\mathcal{T}(\mathbf{\Omega_3}) \cap \big\uparrow \ell(\mathbf{\Omega_3})$ is finite, while $\mathcal{T}(\mathbf{\Omega_3}) \cap \big\uparrow \{\ell(\mathbf{\Omega_3}), \ell((\mathbf{\Omega_3})\,\mathbf{\Delta_3}), \dots\}$ is infinite, so $\mathcal{T}(\mathbf{\Omega_3}) \notin \triangledown(\mathfrak{L})^{\perp}$.

In the presence of the non-deterministic sum $+$, we have a third kind of divergence, which is given by infinite reduction sequences of terms $(M_i)_{i\in\mathbf{N}}$ which are pairwise different but whose Taylor expansion support repeats infinitely many times as the reducts of $(\mathbf{\Theta})\,(\lambda x.x + y)$ (see Equation (3)). We prove that this kind of divergence is much more similar to a loop rather than to a sequence of different λ-terms. In particular, there is a single linear resource term (depending on the reduction sequence) whose cone is able to show up the divergence. Indeed, most of the effort in the proof of Theorem 36 is devoted to deal with this kind of "looping Taylor expansion". Namely, Definition 31 gives a notion of non-deterministic reduction $\rightharpoonup$ allowing Lemma 35, which is the key statement used in the proof of Theorem 36 for dealing with both the divergence of looping terms (like $\mathbf{\Omega}$) and that of looping Taylor expansions (like $(\mathbf{\Theta})\,(\lambda x.x + y)$).

[8] This condition can be slightly weakened replacing $\mathfrak{L}$ with: $\{a \subseteq \Delta;\ a \text{ linear and } \mathsf{fv}(a) \text{ finite}\}$. However, we prefer to stick to the more intuitive definition of $\mathfrak{L}$.

The omitted or sketched proofs are detailed in Appendix C.

We introduce a reduction rule $\rightharpoonup$ on Λ_+ which corresponds to one step of β-reduction and a potential loss of some addenda in a term. For that, we need an order $\trianglerighteq$ on Λ_+ expressing this loss. For instance, $(\mathbf{\Theta})\,(\lambda x.x+y)+y \trianglerighteq (\mathbf{\Theta})\,\lambda x.x$, thus $(\mathbf{\Theta})\,(\lambda x.x+y) \rightharpoonup^* (\mathbf{\Theta})\,\lambda x.x$, and similarly, $(\mathbf{\Theta})\,(\lambda x.x+y) \rightharpoonup^* y$.

Definition 31 (Partial reduction). *We write $M \rightharpoonup N$ if there exists P such that $M \rightarrow_\beta P$ and $P \trianglerighteq N$, where the partial order $\trianglelefteq$ over Λ_+ is defined as the least order such that $M \trianglelefteq M+N$; $N \trianglelefteq M+N$ and if $M \trianglelefteq N$ then: $M+P \trianglelefteq N+P$, $\lambda x.M \trianglelefteq \lambda x.N$, $(M)\,P \trianglelefteq (N)\,P$, and $(P)\,M \trianglelefteq (P)\,N$.*

A reduction $M \rightharpoonup N$ is at top level *if $M = (\lambda x.M')\,M'' \rightarrow M'[M''/x] \trianglerighteq N$.*

Write $s > t$ whenever $s \geq t$ (Definition 8) and $s \neq t$: this is a strict partial order relation.

Lemma 32. *If $M \rightharpoonup N$ and $t \in \mathcal{T}(N)$, then there exists $s \in \mathcal{T}(M)$ such that $s \geq t$. If moreover, $M \rightharpoonup N$ is at top level, then $s > t$.*

Lemma 33. *Let $M \rightharpoonup N$ and $u \in \Delta$. If $\mathcal{T}(N) \cap {\uparrow}u$ is infinite, then $\mathcal{T}(M) \cap {\uparrow}u$ is also infinite.*

Definition 34. *The* height $\mathbf{h}(M)$ *of a term $M \in \Lambda_+$ is defined inductively as follows: $\mathbf{h}(x) := 1$, $\mathbf{h}(\lambda x.M) := 1+\mathbf{h}(M)$, $\mathbf{h}((M)\,N) := 1+\max(\mathbf{h}(M),\mathbf{h}(N))$ and $\mathbf{h}(M+N) := \max(\mathbf{h}(M),\mathbf{h}(N))$.*

Lemma 35. *Let $(M_i)_{i\in\mathbf{N}}$ be a sequence. If $\forall i \in \mathbf{N}$, $M_i \rightharpoonup M_{i+1}$ and $(\mathbf{h}(M_i))_{i\in\mathbf{N}}$ is bounded, then there exists a linear term t such that $\mathcal{T}(M_0) \cap {\uparrow}t$ is infinite.*

Proof (Sketch). First, it is sufficient to address the case of a sequence $(S_i)_{i\in\mathbf{N}}$ of simple terms (i.e. without $+$ as the top-level constructor) such that $S_i \rightharpoonup S_{i+1}$ for all $i \in \mathbf{N}$ and $(\mathbf{h}(S_i))_{i\in\mathbf{N}}$ is bounded. Besides, by Lemma 33, it is sufficient to have $\mathcal{T}(S_{i_0}) \cap {\uparrow}t$ infinite for some $i_0 \in \mathbf{N}$.

Then, by induction on $h = \max\{\mathbf{h}(S_i);\ i \in \mathbf{N}\}$, we show that there exists $i_0 \in \mathbf{N}$ and a sequence $(s_j)_{j\in\mathbf{N}} \in \mathcal{T}(S_{i_0})^{\mathbf{N}}$ such that s_0 is linear and, for all $j \in \mathbf{N}$, $s_{j+1} > s_j$. Since $>$ is a strict order relation, this implies that the set $\{s_j;\ j \in \mathbf{N}\} \subseteq \mathcal{T}(S_{i_0}) \cap {\uparrow}s_0$ is infinite.

First assume that there are infinitely many top level reductions. Observe that, since $\mathbf{h}(S_i) \leq h$ and $\mathsf{fv}(S_i) \subseteq \mathsf{fv}(S_0)$ for all i, the set $\{\mathcal{T}(S_i);\ i \in \mathbf{N}\}$ is finite. Hence there exists an index $i_0 \in \mathbf{N}$ such that $\{i \in \mathbf{N};\ \mathcal{T}(S_i) = \mathcal{T}(S_{i_0})\}$ is infinite. As there are infinitely many top level reductions, there are i_1 and i_2 such that $i_0 < i_1 < i_2$, the reduction i_1 is at top level and $\mathcal{T}(S_{i_2}) = \mathcal{T}(S_{i_0})$. We inductively define the required sequence by choosing arbitrary $s_0 \in \ell(S_{i_0}) \subseteq \mathcal{T}(S_{i_0})$, and by iterating Lemma 32: for $s_j \in \mathcal{T}(S_{i_0}) = \mathcal{T}(S_{i_2})$, we obtain $s_{j+1} \in \mathcal{T}(S_{i_0})$ with $s_{j+1} > s_j$ since the reduction i_1 is at top level.

Now assume that there are only finitely many top level reductions. Let i_1 be such that no reduction $S_i \rightharpoonup S_{i+1}$ with $i \geq i_1$ is at top level. Either for all $i \geq i_1$, $S_i = \lambda x.M'_i$ with $M'_i \rightharpoonup M'_{i+1}$, and we conclude by applying the induction hypothesis to the sequence $(M'_{i+i_1})_{i\in\mathbf{N}}$; or for all $i \geq i_1$, $S_i = (M'_i)\, N'_i$ so that $(M'_i)_{i\geq i_1}$ and $(N'_i)_{i\geq i_1}$ are partial reduction sequences, with infinitely many different terms for at least one of them. In this case, assume for instance that $(M'_i)_{i\geq i_1}$ produces infinitely many different terms and apply the induction hypothesis to the extracted subsequence $(M'_{\phi(i)})_{i\in\mathbf{N}}$ of different terms. It provides i'_0 and a sequence $(s'_j) \in \mathcal{T}(M'_{\phi(i'_0)})^N$ with s'_0 linear and, $s'_{j+1} > s'_j$. Fix $t \in \ell(N'_{\phi(i'_0)})$ arbitrarily. So we set $i_0 = \phi(i'_0)$ and $s_j = \langle s'_j \rangle\, [t]$ for all $j \in \mathbf{N}$. $\square$

Theorem 36. *Let $\mathfrak{T}$ be a structure such that $\mathfrak{L} \subseteq \mathfrak{T}$. If $\mathcal{T}(M) \in \nabla(\mathfrak{T})^{\perp}$ then M is strongly normalizable. In particular, this holds for $\mathfrak{T} \in \{\mathfrak{L}, \mathfrak{B}\}$.*

Proof. Assume that $(M_i)_{i\in\mathbf{N}}$ is such that $M = M_0$ and for all i, $M_i \to_\beta M_{i+1}$. We prove that $\mathcal{T}(M) \notin \nabla(\mathfrak{T})^{\perp}$ by exhibiting $a \in \mathfrak{T}$ such that $\mathcal{T}(M) \not\perp \uparrow a$.

If $(\mathbf{h}(M_i))_{i\in\mathbf{N}}$ is bounded, then fix $a = \{t\}$ with t given by Lemma 35.

Otherwise, $\forall i \in \mathbf{N}$, fix $t_i \in \ell(M_i)$ such that $\mathbf{h}(t_i) = \mathbf{h}(M_i)$. Lemma 32 implies that there is $s_i \in \mathcal{T}(M)$ such that $s_i \geq t_i$. Denote by $\mathbf{s}(s)$ the number of symbols occurring in s. Since there is no duplication in reduction $\to_r$ it should be clear that if $s \geq t$ then $\mathbf{s}(s) \geq \mathbf{s}(t)$. Besides, $\mathbf{s}(s) \geq \mathbf{h}(s)$. Therefore, since $\{\mathbf{h}(M_i)\,;\ i \in \mathbf{N}\}$ is unbounded, $\{\mathbf{h}(t_i)\,;\ i \in \mathbf{N}\}$, $\{\mathbf{s}(t_i)\,;\ i \in \mathbf{N}\}$ and $\{\mathbf{s}(s_i)\,;\ i \in \mathbf{N}\}$ are unbounded. Fix $a = \bigcup_{i\in\mathbf{N}} \ell(M_i) \in \mathfrak{T}$, we have proved that $\mathcal{T}(M) \cap \uparrow a$ is infinite. $\square$

Notice that the structure $\mathfrak{S}_{\mathrm{sgl}}$ of singletons used in [4] does not enjoy the hypothesis of Theorem 36 ($\mathfrak{L} \not\subseteq \mathfrak{S}_{\mathrm{sgl}}$). In fact:

Remark 37. We prove that $\mathcal{T}(\mathbf{\Omega_3}) \in \nabla(\mathfrak{S}_{\mathrm{sgl}})^{\perp}$, although $\mathbf{\Omega_3}$ is not normalizing. Recall from Example 2, that the support of the Taylor expansion of $\mathbf{\Omega_3}$ is made of terms of the form $\langle \delta_{n_0,m_0} \rangle\, [\delta_{n_1,m_1}, \dots, \delta_{n_k,m_k}]$ (for $k, n_i, m_i \in \mathbf{N}$). Write $\Delta_h = \{\langle \delta_{-,-} \rangle\, [\dots \delta_{-,-} \dots] \cdots [\dots \delta_{-,-} \dots]$ with h bags$\}$: in particular $\mathcal{T}(\mathbf{\Omega_3}) = \Delta_1$. One can easily check that if $s \in \Delta_h$ and $s \geq s'$, then $s' \in \Delta_{h'}$ with $h \leq h'$. A careful inspection of such reductions shows that they are moreover reversible: for all $s' \in \Delta_{h'}$ and all $h \leq h'$ there is exactly one $s \in \Delta_h$ such that $s \geq s'$. It follows that $\Delta_1 \cap \uparrow s$ is either empty or a singleton. Therefore $\mathcal{T}(\mathbf{\Omega_3}) \in \nabla(\mathfrak{S}_{\mathrm{sgl}})^{\perp}$.

6 Conclusion

We achieved all implications of Figure 1, but the rightmost one, concerning the finiteness of the coefficients in the normal form of the Taylor expansion of a strongly normalizing λ-term (recall Equation (2) in the Introduction).

Thanks to the definition of cones (Definition 9) we immediately have the following lemma, which is the last step to Corollary 39.

Lemma 38. *Let $\mathfrak{T}$ be a structure. If $\mathcal{T}(M) \in \nabla(\mathfrak{T})^{\perp}$, then $\forall t \in \bigcup \mathfrak{T}$, $\mathrm{NF}(\mathcal{T}(M))_t$ is finite.*

Applying Corollary 30 and Lemma 38 to a structure like $\mathfrak{B}$ or $\mathfrak{S}_{\mathrm{sgl}}$, we get:

Corollary 39. *Given a non-deterministic λ-term M, if M is strongly normalizable, then $\mathrm{NF}(\mathcal{T}(M))_t$ is finite for all $t \in \Delta$.*

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A Proof of Theorem 14

Let us first introduce notations. A typing context Γ is a finite variable-indexed family of types: we write $|\Gamma|$ for the index set of this family, i.e. the set of variables declared in Γ. As is usual, we will abuse notation and write contexts as ordered lists of typing declarations: $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ where $x_1, \ldots, x_n$ are the pairwise distinct variables of $|\Gamma|$ and $A_n = \Gamma(x_n)$.

We then need a series of technical results.

Definition 40. *We distinguish the class of* simple *(i.e. non-sum) terms as follows:*

- *any variable x is a simple term;*
- *$\lambda x.M$ is simple iff M is simple;*
- *$(M)\,N$ is simple iff M is simple.*

Definition 41. *A type is said to be prime if it is not of the form $A \cap B$. For all type A, we define the set $\mathsf{p}\,(A)$ of prime factors of A as follows:*

$$\begin{aligned} \mathsf{p}\,(X) &= \{X\} \\ \mathsf{p}\,(A \to B) &= \{A \to B\} \\ \mathsf{p}\,(A \cap B) &= \mathsf{p}\,(A) \cup \mathsf{p}\,(B) \end{aligned}$$

Lemma 42. *For all type A and all $B \in \mathsf{p}\,(A)$, if $\Gamma \vdash M : A$ then $\Gamma \vdash M : B$.*

Proof. By a simple induction on type A, we prove that $A \preceq B$ and then we use rule $(\preceq)$.

Lemma 43. *For all type A, if $\Gamma \vdash M : B$ for all $B \in \mathsf{p}\,(A)$, then $\Gamma \vdash M : A$.*

Proof. By a simple induction on A.

Lemma 44. *Let A and B be types such that $\mathsf{p}\,(A) = \mathsf{p}\,(B)$. Then $\Gamma \vdash M : A$ iff $\Gamma \vdash M : B$.*

Proof. By the previous two lemmas.

Definition 45. *We write $\Gamma' \preceq \Gamma$ if $|\Gamma| \subseteq |\Gamma'|$ and, for all $x \in |\Gamma|$, $\Gamma'(x) \preceq \Gamma(x)$.*

Lemma 46. *If $\Gamma' \vdash M : A$ and $\Gamma \preceq \Gamma'$ then $\Gamma \vdash M : A$.*

Proof. Easy induction on the typing derivation of $\Gamma \vdash M : A$.

Lemma 47. *If $\Gamma, x : A \vdash M : B$ and $x \notin \mathsf{fv}\,(M)$ then $\Gamma \vdash M : B$.*

Proof. By a "λ-calculus 101" induction on typing derivations.

Lemma 48. *Fix a context Γ and a term M. Suppose $A' \preceq A$ and all the prime factors of A' are of the form $B' \to C'$ such that $\Gamma, x : B' \vdash M : C'$. Then all the prime factors of A are of the form $B \to C$ with $\Gamma, x : B \vdash M : C$.*

Proof. Fix Γ and M, and for all type A write $\Psi(A)$ for the statement: all the prime factors of A are of the form $B \to C$ such that $\Gamma, x : B \vdash M : C$. Consider the binary relation $\leq_\Psi$ on types defined by:

$$A' \leq_\Psi A \text{ iff } \Psi(A') \text{ implies } \Psi(A).$$

We are to prove that this contains the subtyping relation $\preceq$. Since $\leq_\Psi$ is clearly reflexive and transitive, it is sufficient to check the conditions of Table 2b for $\leq_\Psi$:

- assume $A' = A'_0 \cap A'_1$ and $A = A'_i$ for $i \in \{0, 1\}$: then $\mathsf{p}(A) \subseteq \mathsf{p}(A')$ so the result is obvious;
- assume $A' = (B' \to C'_0) \cap (B' \to C'_1)$ and $A = B' \to (C'_0 \cap C'_1)$: then, setting $B = B'$ and $C = C'_0 \cap C'_1$, we have $\mathsf{p}(A) = \{B \to C\}$, $\Gamma, x : B' \vdash M : C'_0$ and $\Gamma, x : B' \vdash M : C'_1$, hence $\Gamma, x : B \vdash M : C$ by rule $(\cap_i)$;
- assume $A' = B' \to C'$ and $A = B \to C$ with $B \preceq B'$ and $C' \preceq C$: then $\mathsf{p}(A) = \{B \to C\}$, and $\Gamma, x : B' \vdash M : C'$, hence $\Gamma, x : B \vdash M : C'$ by Lemma 46, and then $\Gamma, x : B \vdash M : C$ by rule $(\preceq)$.

Lemma 49. *If $\Gamma \vdash \lambda x.M : A$ then for all prime type $A_0 \in \mathsf{p}(A)$, there are types B and C such that $\Gamma, x : B \vdash M : C$ and $A_0 = B \to C$.*

Proof. The proof is by induction on the derivation δ of $\Gamma \vdash \lambda x.M : A$.

The last rule of δ cannot be any of (var) nor $(\to_e)$ because $\lambda x.M$ is not of the appropriate shape.

If the last rule of δ is $(\to_i)$, we directly obtain B and C such that $\Gamma, x : B \vdash M : C$, $A = B \to C$ and $\mathsf{p}(A) = \{B \to C\}$.

If the last rule is $(+)$, then we have subderivations δ' and δ'' with respective conclusions $\Gamma \vdash \lambda x.M' : A$ and $\Gamma \vdash \lambda x.M'' : A$, with $M = M' + M''$. Let $A_0 \in \mathsf{p}(A)$. Applying the induction hypothesis to δ' and δ'', we obtain B', C', B'' and C'' such that $\Gamma, x : B' \vdash M' : C'$, $\Gamma, x : B'' \vdash M'' : C''$ and $A_0 = B' \to C' = B'' \to C''$. It is then sufficient to set $B = B'(= B'')$ and $C = C'(= C'')$.

If the last rule is $(\cap_i)$, then we have subderivations δ' and δ'' with respective conclusions $\Gamma \vdash \lambda x.M : A'$ and $\Gamma \vdash \lambda x.M : A''$, with $A = A' \cap A''$. Since $\mathsf{p}(A) = \mathsf{p}(A') \cup \mathsf{p}(A'')$ it is sufficient to apply the induction hypothesis to δ' or δ''.

If the last rule is $(\preceq)$, then the immediate subderivation δ' of δ has conclusion $\Gamma \vdash \lambda x.M : A'$ with $A' \preceq A$. By induction hypothesis applied to δ', all the prime factors of A' are of the form $B' \to C'$ with $\Gamma, x : B' \vdash M : C'$. We conclude by applying Lemma 48.

Lemma 50. *If $\Gamma \vdash M : A \to B$ and $\Gamma \vdash M : A' \to B'$ then $\Gamma \vdash M : (A \cap A') \to (B \cap B')$*

Proof. Write $A'' = A \cap A'$, then $A'' \preceq A$ and $A'' \preceq A'$. We can build the proof:

$$\dfrac{\dfrac{\Gamma \vdash M : A \to B \quad A \to B \preceq A'' \to B}{\Gamma \vdash M : A'' \to B}\,(\preceq) \quad \dfrac{\Gamma \vdash M : A' \to B' \quad A' \to B' \preceq A'' \to B'}{\Gamma \vdash M : A'' \to B'}\,(\preceq)}{\Gamma \vdash M : (A'' \to B) \cap (A'' \to B')}\,(\cap_i)$$

then conclude by rule $(\preceq)$ since $(A'' \to B) \cap (A'' \to B') \preceq A'' \to (B \cap B')$.

Lemma 51. *If $\Gamma \vdash (M)\,N : A$ then there is a type B such that $\Gamma \vdash M : B \to A$ and $\Gamma \vdash N : B$.*

Proof. The proof is by induction on the derivation δ of $\Gamma \vdash (M)\,N : A$.

The last rule of δ can not be any of (var) nor $(\to_i)$ because $(M)\,N$ is not of the appropriate shape.

If the last rule of δ is $(\to_e)$, we directly obtain B such that $\Gamma \vdash M : B \to A$ and $\Gamma \vdash N : B$.

If the last rule is $(+)$, then we have subderivations δ' and δ'' with respective conclusions $\Gamma \vdash (M')\,N : A$ and $\Gamma \vdash (M'')\,N : A$, with $M = M' + M''$. Applying the induction hypothesis to δ' and δ'', we obtain B' and B'' such that $\Gamma \vdash M' : B' \to A$, $\Gamma \vdash N : B'$, $\Gamma \vdash M'' : B'' \to A$ and $\Gamma \vdash N : B''$. Setting $B = B' \cap B''$, we have $\Gamma \vdash N : B$ by $(\cap_i)$, and we can build the following derivation:

$$\dfrac{\dfrac{\Gamma \vdash M' : B' \to A \quad B' \to A \preceq B \to A}{\Gamma \vdash M' : B \to A}\,(\preceq) \quad \dfrac{\Gamma \vdash M'' : B'' \to A \quad B'' \to A \preceq B \to A}{\Gamma \vdash M'' : B \to A}\,(\preceq)}{\Gamma \vdash M : B \to A}\,(+)$$

If the last rule is $(\preceq)$, then the immediate subderivation δ' of δ has conclusion $\Gamma \vdash (M)\,N : A'$ with $A' \preceq A$. The induction hypothesis applied to δ' gives B such that $\Gamma \vdash N : B$ and $\Gamma \vdash M : B \to A'$, and we obtain $\Gamma \vdash M : B \to A$ by applying rule $(\preceq)$, observing that $B \to A' \preceq B \to A$.

If the last rule is $(\cap_i)$, then we have subderivations δ' and δ'' with respective conclusions $\Gamma \vdash (M)\,N : A'$ and $\Gamma \vdash (M)\,N : A''$, with $A = A' \cap A''$. Applying the induction hypothesis to δ' and δ'', we obtain B' and B'' such that $\Gamma \vdash M : B' \to A'$, $\Gamma \vdash N : B'$, $\Gamma \vdash M : B'' \to A''$ and $\Gamma \vdash N : B''$. Setting $B = B' \cap B''$, we have $\Gamma \vdash N : B$ by rule $(\cap_i)$ and we conclude by Lemma 50.

Lemma 52. *If $\Gamma \vdash M + M' : A$ then $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$.*

Proof. By induction on the derivation δ of $\Gamma \vdash M + M' : A$. The last rule of δ can not be (var) because $M + M'$ is not a variable.

If the last rule of δ is $(\to_i)$, there are terms P and P' and types B and C such that $A = B \to C$, $M = \lambda x.P$, $M' = \lambda x.P'$ and $\Gamma, x : B \vdash P + P' : C$. By induction hypothesis, we obtain $\Gamma, x : B \vdash P : C$ and $\Gamma, x : B \vdash P' : C$, hence $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$ by rule $(\to_i)$.

If the last rule of δ is $(\to_e)$, there are terms P, P' and Q, and a type B such that $M = (P)\,Q$, $M' = (P')\,Q$, $\Gamma \vdash P + P' : B \to A$ and $\Gamma \vdash Q : B$. By induction hypothesis, we obtain $\Gamma \vdash P : B \to A$ and $\Gamma \vdash P' : B \to A$, hence $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$ by rule $(\to_e)$.

If the last rule is $(+)$, then there are terms such that P, P', Q and Q' such that $M = P+Q$, $M' = P'+Q'$, $\Gamma \vdash P+P' : A$ and $\Gamma \vdash Q+Q' : A$. By induction hypothesis, we obtain $\Gamma \vdash P : A$, $\Gamma \vdash P' : A$, $\Gamma \vdash Q : A$ and $\Gamma \vdash Q' : A$, hence $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$ by rule $(+)$.

If the last rule is $(\preceq)$, then there is a type $A' \preceq A$ such that $\Gamma \vdash M + M' : A'$. By induction hypothesis, we obtain $\Gamma \vdash M : A'$ and $\Gamma \vdash M' : A'$, hence $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$ by rule $(\preceq)$.

If the last rule is $(\cap_i)$, then there are types B and C such that $A = B \cap C$, $\Gamma \vdash M + M' : B$ and $\Gamma \vdash M + M' : C$. By induction hypothesis, we obtain $\Gamma \vdash M : B$, $\Gamma \vdash M' : B$, $\Gamma \vdash M : C$ and $\Gamma \vdash M' : C$, hence $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$ by rule $(\cap_i)$.

Definition 53. *Let Γ and Δ be two contexts. We write $\Gamma \cap \Delta$ for the context such that $|\Gamma \cap \Delta| = |\Gamma| \cup |\Delta|$ and, for all $x \in |\Gamma \cap \Delta|$,*

$$(\Gamma \cap \Delta)(x) = \begin{cases} \Gamma(x) \cap \Delta(x) & \textit{if } x \in |\Gamma| \cap |\Delta| \\ \Gamma(x) & \textit{if } x \in |\Gamma| \setminus |\Delta| \\ \Delta(x) & \textit{if } x \in |\Delta| \setminus |\Gamma| \end{cases}$$

Lemma 54. *For all contexts Γ and Δ, $\Gamma \cap \Delta \preceq \Gamma$ and $\Gamma \cap \Delta \preceq \Delta$.*

Proof. By definition.

Lemma 55. *Let Γ be a context, $x \in \mathcal{V} \setminus |\Gamma|$, $M, N \in \Lambda_+$ and A, B be types such that $\Gamma \vdash N : B$ and $\Gamma \vdash M\,[N/x] : A$. Then there is a type B' such that $\Gamma \vdash N : B'$ and $\Gamma, x : B' \vdash M : A$.*

Proof. Write $M' = M\,[N/x]$. The proof is by induction on M. We first assume that M is a simple term.

If $M \in \mathcal{V}$ then one of the following holds:

- $M = x$, then $M' = N$, hence $\Gamma \vdash N : A$ and we can set $B' = A$;
- $M \neq x$, then $M' = M$, hence $\Gamma \vdash M : A$ and we can set $B' = B$.

If $M = \lambda y.P$ then $M' = \lambda y.P'$ with $P' = P\,[N/x]$. Write $\mathsf{p}\,(A) = \{A_1, \ldots, A_n\}$. By Lemma 49, for $1 \leq i \leq n$ there are types C_i and D_i such that $A_i = C_i \to D_i$ and $\Gamma, y : C_i \vdash P' : D_i$. By the induction hypothesis applied to P (using the fact that $\Gamma, y : C_i \vdash N : B$ by Lemma 46), there is a type B_i such that $\Gamma, x : B_i, y : C_i \vdash P : D_i$ and $\Gamma, y : C_i \vdash N : B_i$, hence $\Gamma \vdash N : B_i$ by Lemma 47. Then fix $B' = B_1 \cap \cdots \cap B_n$ so that $\Gamma \vdash N : B'$ by repeated application of rule $(\cap_i)$. By Lemma 46 we also have $\Gamma, x : B', y : C_i \vdash P : D_i$ for all i, hence $\Gamma, x : B' \vdash M : C_i \to D_i$ by rule $(\to_i)$ and then $\Gamma, x : B' \vdash M : A$ by Lemma 43.

If $M = (P)\,Q$ then $M' = (P')\,Q'$ with $P' = P\,[N/x]$ and $Q' = Q\,[N/x]$. By Lemma 51, there is a type C such that $\Gamma \vdash P : C \to A$ and $\Gamma \vdash Q : C$. By the induction hypothesis applied to P and Q, we obtain types B'' and B''' such that $\Gamma, x : B'' \vdash P : C \to A$, $\Gamma, x : B''' \vdash Q : C$, $\Gamma \vdash N : B''$ and $\Gamma \vdash N : B'''$. Set $B' = B'' \cap B'''$ so that $\Gamma \vdash N : B'$ by rule $(\cap_i)$. By Lemma 46 we also have $\Gamma, x : B' \vdash P : C \to A$ and $\Gamma, x : B' \vdash Q : C$, and then $\Gamma, x : B' \vdash M : A$ by rule $(\to_e)$.

Now assume that M is a sum of simple terms $M = S_1 + \cdots + S_n$. We then have $M' = S'_1 + \cdots + S'_n$ with $S'_i = S_i\,[N/x]$ for $1 \leq i \leq n$. Lemma 52 and $\Gamma \vdash M' : A$ implies $\Gamma \vdash S'_i : A$ for $1 \leq i \leq n$. Since we have already covered all possible cases for simple terms, we obtain types B_i such that $\Gamma \vdash N : A$ and $\Gamma, x : B_i \vdash S_i : A$ for $1 \leq i \leq n$. We can then fix $B' = B_1 \cap \cdots \cap B_n$ so that $\Gamma \vdash N : B'$ by repeated application of rule $(\cap_i)$, $\Gamma, x : B' \vdash S_i : A$ by Lemma 46 for $1 \leq i \leq n$, and then $\Gamma, x : B' \vdash M : A$ by repeated application of rule $(+)$.

Remark 56. The previous lemma fails if we include a neutral term 0 in the syntax, such that $(0)\,M = 0$. This is why we only consider a non-deterministic λ-calculus rather than an algebraic one.

Lemma 57. *If $\Gamma \vdash (M\,[N/x])\,N_1 \cdots N_n : A$ and there exists B such that $\Gamma \vdash N : B$ then $\Gamma \vdash (\lambda x.M)\,N\,N_1 \cdots N_n : A$.*

Proof. The proof is by induction on n.

If $n = 0$, then $\Gamma \vdash M\,[N/x] : A$. By Lemma 55, there exists C such that $\Gamma \vdash N : C$ and $\Gamma, x : C \vdash M : A$, and we conclude that $\Gamma \vdash (\lambda x.M)\,N : A$. by rules $(\rightarrow_i)$ and $(\rightarrow_e)$.

Assume the result holds up to n and consider a judgement

$$\Gamma \vdash ((M\,[N/x])\,N_1 \cdots N_n)\,N_{n+1} : A.$$

By Lemma 51, there exists a type C such that $\Gamma \vdash (M\,[N/x])\,N_1 \cdots N_n : C \rightarrow A$ and $\Gamma \vdash N_{n+1} : C$. By induction hypothesis, $\Gamma \vdash (\lambda x.M)\,N\,N_1 \cdots N_n : C \rightarrow A$ and we conclude by rule $(\rightarrow_e)$.

Proof (of Theorem 14).

For M a strongly normalizable term, write $\mathbf{l}\,(M)$ for the maximum length of a reduction from M.

We prove by well founded induction on the pair $(\mathbf{l}\,(M), \mathbf{s}\,(M))$ that there exists $n_M \in \mathbf{N}$ such that for all type B, there is a context Γ and a sequence $(A_1, \ldots, A_{n_M})$ of types such that $\Gamma \vdash M : A_1 \rightarrow \cdots \rightarrow A_{n_M} \rightarrow B$.

First observe that in this case, we more generally have: for all type B and all $n \geq n_M$, there is a context Γ and a sequence $(A_1, \ldots, A_n)$ of types such that $\Gamma \vdash M : A_1 \rightarrow \cdots \rightarrow A_n \rightarrow B$.

If $M = M_1 + M_2$ then $\mathbf{l}\,(M_i) \leq \mathbf{l}\,(M)$ and $\mathbf{s}\,(M_i) < \mathbf{s}\,(M)$ for $i \in \{1, 2\}$. Then the induction hypothesis holds for terms M_1 and M_2. Set $n_M = \max(n_{M_1}, n_{M_2})$. Then, for all type B there are contexts Σ and Θ and sequences $(C_1, \ldots, C_{n_M})$ and $(D_1, \ldots, D_{n_M})$ of types such that $\Sigma \vdash M_1 : C_1 \rightarrow \cdots \rightarrow C_{n_M} \rightarrow B$ and $\Theta \vdash M_2 : D_1 \rightarrow \cdots \rightarrow D_{n_M} \rightarrow B$. Set $\Gamma = \Sigma \cap \Theta$ and $A_i = C_i \cap D_i$ for $1 \leq i \leq n_M$. By rule $(\preceq)$ and Lemma 46, we obtain $\Gamma \vdash M_1 : A_1 \rightarrow \cdots \rightarrow A_{n_M} \rightarrow B$ and $\Gamma \vdash M_2 : A_1 \rightarrow \cdots \rightarrow A_{n_M} \rightarrow B$, hence $\Gamma \vdash M : A_1 \rightarrow \cdots \rightarrow A_{n_M} \rightarrow B$ by rule $(+)$.

If $M = \lambda x.N$ then $\mathbf{l}\,(N) = \mathbf{l}\,(M)$ and $\mathbf{s}\,(N) < \mathbf{s}\,(M)$: we then apply the induction hypothesis to N and set $n_M = n_N + 1$. For all type B, we obtain $\Gamma' \vdash N : A'_1 \rightarrow \cdots \rightarrow A'_{n_N} \rightarrow B$. Let $\Gamma = \Gamma' \setminus \{x\}$, $A_1 = \Gamma'(x)$ (or an arbitrary value if $x \notin |\Gamma'|$) and $A_{i+1} = A'_i$ for $1 \leq i \leq n_N$: we obtain (possibly using Lemma 46 if $x \notin |\Gamma'|$) $\Gamma, x : A_1 \vdash N : A_2 \to \cdots \to A_{n_M} \to B$, and then $\Gamma \vdash M : A_1 \to A_2 \to \cdots \to A_{n_M} \to B$ by rule $(\to_i)$.

Now assume that M neither a sum nor an abstraction. In this case we can write $M = (M_0)\, M_1 \cdots M_q$, with $q \geq 0$, so that one of the following holds.

Either M_0 is a variable: $M_0 = x \in \mathcal{V}$. We then set $n_M = 0$. Observe that $\mathbf{l}\,(M_i) \leq \mathbf{l}\,(M)$ and $\mathbf{s}\,(M_i) < \mathbf{s}\,(M)$ for $0 \leq i \leq n$, so the induction hypothesis applies to each M_i. In particular, for all i, there is a context Γ_i and a type B_i such that $\Gamma_i \vdash M_i : B_i$. For all type B, consider $\Gamma = \Gamma_1 \cap \cdots \cap \Gamma_q \cap (x : B_1 \to \cdots \to B_q \to B)$. By Lemma 46, we have $\Gamma \vdash x : B_1 \to \cdots \to B_q \to B$ and $\Gamma \vdash M_i : B_i$ for all i, hence $\Gamma \vdash M : B$ by iterating rule $(\to_e)$.

Or M_0 is a redex: $M_0 = (\lambda x.N)\, P$. Write $M' = (N\,[P/x])\, M_1 \cdots M_q$. Since $\mathbf{l}\,(M') < \mathbf{l}\,(M)$, so we can apply the induction hypothesis to M': for all type B, we obtain $\Gamma \vdash M' : A_1 \to \cdots \to A_{n_M} \to B$. Since moreover $\mathbf{l}\,(P) \leq \mathbf{l}\,(M)$ and $\mathbf{s}\,(P) < \mathbf{s}\,(M)$, we can apply the induction hypothesis to P and in particular P is typeable. By Lemma 57, we obtain $\Gamma \vdash M : A_1 \to \cdots \to A_{n_M} \to B$. □

B Proofs of Section 4

B.1 The realizability technique (Theorem 21)

This Appendix details the proof of realizability theorem (Theorem 21) . The proofs are completely standard, we slightly adapt the ones given in [8].

Notice that if $\mathcal{R} = \mathbb{B}$, the differential substitution (6) leads to $\partial_x a \cdot \overline{a} := \bigcup \{\partial_x s \cdot \overline{t};\ s \in a,\ \overline{t} \in \overline{a}\}$.

Proof (of Lemma 19). By structural induction on A. The case X is a variable is immediate from the hypothesis on $\mathfrak{S}$. The case of an intersection follows immediately from the induction hypothesis. The only delicate case is the implication.

Suppose $A = B \to C$. First let us prove the $\mathfrak{S}$-saturation. Let $e, f_1, \ldots, f_n \in \mathfrak{S}$ and assume $\left\langle \partial_x e \cdot f_0{}^! \right\rangle f_1{}^! \ldots f_n{}^! \in \|B \to C\|_{\mathfrak{S}}$. Let $b \in \|B\|_{\mathfrak{S}}$: we deduce $\left\langle \partial_x e \cdot f_0{}^! \right\rangle f_1{}^! \ldots f_n{}^! \, b^! \in \|C\|_{\mathfrak{S}}$. Since we also have $b \in \mathfrak{S}$, and $\|C\|_{\mathfrak{S}}$ is $\mathfrak{S}$-saturated, we obtain $\langle \lambda x.e \rangle\ f_0{}^! f_1{}^! \ldots f_n{}^! b^! \in \|C\|_{\mathfrak{S}}$. We have just proved that $\langle \lambda x.e \rangle\ f_0{}^! f_1{}^! \ldots f_n{}^! \in \|B \to C\|_{\mathfrak{S}}$.

Using the induction hypothesis $\mathfrak{S}_h \subseteq \|B\|_{\mathfrak{S}}, \|C\|_{\mathfrak{S}} \subseteq \mathfrak{S}$ and the adaptedness of $\mathfrak{S}$ we get the following inclusions:

$$\mathfrak{S}_h \subseteq \mathfrak{S} \to \mathfrak{S}_h \subset \|B \to C\|_{\mathfrak{S}} \subset \mathfrak{S}_h \to \mathfrak{S} \subset \mathfrak{S}.$$

Finally, for the closure under finite unions, take $f, f' \in \|B \to C\|_{\mathfrak{S}}$, $b \in \|B\|_{\mathfrak{S}}$. We have, applying the induction hypothesis on C, $\langle f \cup f' \rangle\ b^! = \langle f \rangle\ b^! \cup \langle f' \rangle\ b^! \in \|C\|_{\mathfrak{S}}$. We conclude that $f \cup f' \in \|B \to C\|_{\mathfrak{S}}$. □

Lemma 58. *If $A \preceq B$ then $\|A\|_{\mathfrak{S}} \subseteq \|B\|_{\mathfrak{S}}$ for all $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$.*

Proof. By the definition of realizers, one can easily check that the statement holds for the rules generating the subtyping relation $\preceq$ (Table 2b). □

Proof (of Lemma 20). The proof is by induction on the derivation δ of $x_1 : A_1, \ldots, x_n : A_n \vdash M : B$. Let us write $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ and $b = \partial_{x_1,\ldots,x_n} \mathcal{T}(M) \cdot a_1{}^!, \ldots, a_n{}^!$.

If δ is an instance of rule (var), $M = x_i$, $B = A_i$, then $b = a_i \in \|B\|_{\mathfrak{S}}$.

If the last rule of δ is $(\to_i)$, then the subderivation of δ is δ' with conclusion $\Gamma, x_{n+1} : A_{n+1} \vdash M' : B'$ with $M = \lambda x_{n+1}.M'$ and $B = A_{n+1} \to B'$. Then $b = \lambda x_{n+1}.b'$, with $b' = \partial_{x_1,\ldots,x_n} \mathcal{T}(M') \cdot a_1{}^!, \ldots, a_n{}^!$. By the definition of $\|A_{n+1} \to B'\|_{\mathfrak{S}}$, it is sufficient to prove that, for all $a_{n+1} \in \|A_{n+1}\|_{\mathfrak{S}}$, $\langle b \rangle\ a_{n+1}{}^! \in \|B'\|_{\mathfrak{S}}$. Since $\|B'\|_{\mathfrak{S}}$ is $\mathfrak{S}$-saturated (Lemma 19), it is sufficient to prove that $\partial_{x_{n+1}} b' \cdot a_{n+1}{}^! \in \|B'\|_{\mathfrak{S}}$. Since $\partial_{x_{n+1}} b' \cdot a_{n+1}{}^! = \partial_{x_1,\ldots,x_{n+1}} \mathcal{T}(M') \cdot a_1{}^!, \ldots, a_{n+1}{}^!$, we conclude by the induction hypothesis applied to δ'.

If the last rule of δ is $(\to_e)$, then the subderivations of δ are δ' with conclusion $\Gamma \vdash N : A \to B$ and δ'' with conclusion $\Gamma \vdash P : A$, such that $M = (N)\,P$. Observe that $b = \langle f \rangle\ a^!$ with $f = \partial_{x_1,\ldots,x_n} \mathcal{T}(N) \cdot a_1{}^!, \ldots, a_n{}^!$ and $a = \partial_{x_1,\ldots,x_n} \mathcal{T}(P) \cdot a_1{}^!, \ldots, a_n{}^!$. The induction hypothesis for δ' implies that

$f \in \|A \to B\|_{\mathfrak{S}}$ and that for δ'' implies that and $a \in \|A\|_{\mathfrak{S}}$. We conclude by definition of $\|A \to B\|_{\mathfrak{S}}$.

If the last rule of δ is $(\cap_i)$, then the subderivations of δ are δ' with conclusion $\Gamma \vdash M : B'$ and δ'' with conclusion $\Gamma \vdash M : B''$, such that $B = B' \cap B''$. The induction hypothesis for δ' implies that $b \in \|B'\|_{\mathfrak{S}}$ and that for δ'' implies that and $b \in \|B''\|_{\mathfrak{S}}$. We conclude since $\|B\|_{\mathfrak{S}} = \|B'\|_{\mathfrak{S}} \cap \|B''\|_{\mathfrak{S}}$.

If the last rule of δ is $(\preceq)$, then the subderivation of δ is δ' with conclusion $\Gamma \vdash M : B'$ such that $B' \preceq B$. The induction hypothesis for δ' implies that $b \in \|B'\|_{\mathfrak{S}}$ and we conclude by Lemma 58.

If the last rule of δ is $(+)$, then the subderivations of δ are δ' with conclusion $\Gamma \vdash M' : B$ and δ'' with conclusion $\Gamma \vdash M'' : B$, such that $M = M' + M''$. Observe that $b = b' \cup b''$ with $b' = \partial_{x_1,\ldots,x_n} \mathcal{T}(M') \cdot {a_1}^!, \ldots, {a_n}^!$ and $b'' = \partial_{x_1,\ldots,x_n} \mathcal{T}(M'') \cdot {a_1}^!, \ldots, {a_n}^!$. The induction hypothesis for δ' implies that $b' \in \|B\|_{\mathfrak{S}}$ and that for δ'' implies that and $b \in \|B\|_{\mathfrak{S}}$. We conclude since $\|B\|_{\mathfrak{S}}$ is closed under finite unions (Lemma 19).

B.2 Sufficient conditions to be adapted to the realizability technique (Lemma 29)

This appendix is devoted to prove Lemma 29, stating sufficient conditions for a structure $\mathfrak{T}$ to have the dual of its cone $\triangledown(\mathfrak{T})^\perp$ adapted and hence allowing to apply the Realizability Theorem 21. The conditions are already stated in Definitions 23, 26 and 27.

Let us decompose the definition of hereditariness (recall Definitions 24 and 25):

Definition 59. *We say that a structure $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ is*

- application hereditary *if, for all $a \in \mathfrak{S}$, $\pi_0 a \in \mathfrak{S}$, $\pi_1 a \in \mathfrak{S}$;*
- abstraction hereditary *if, for all $a \in \mathfrak{S}$, $\mathsf{fv}(a) \neq \mathcal{V}$, and for all $x \notin \mathsf{fv}(a)$, $\pi_x a \in \mathfrak{S}$;*
- hereditary *if it is downwards closed and both application and abstraction hereditary (as it is defined in Defintion 26).*

Write $s \succ t$ if there is $a \subseteq \Delta$ such that $s \to_r a$ and $t \in a$: the reduction order $\geq$ used to define cones is just the reflexive and transitive closure of $\succ$. We generalize $\geq$ over bags as: $\bar{s} \geq \bar{t}$ whenever there is n such that $\bar{s}$ and $\bar{t}$ can be respectively written as $[s_1, \ldots, s_n]$ and $[t_1, \ldots, t_n]$ with for all i, $s_i \geq t_i$.

Lemma 60. *Let $t \succ t'$. Then $\mathsf{fv}(t') = \mathsf{fv}(t)$, and $\mathbf{s}(t') + 2 \leq \mathbf{s}(t) \leq 2\mathbf{s}(t') + 2$.*

Proof. This result is the immediate contextual extension of the following:

Let $t = \langle \lambda x.s_0 \rangle [s_1, \ldots, s_n]$ and $t' \in \partial_x s_0 \cdot [s_1, \ldots, s_n]$. Then $\mathsf{fv}(t') = \mathsf{fv}(t)$, and $\mathbf{s}(t') + 2 \leq \mathbf{s}(t) \leq 2\mathbf{s}(t') + 2$.

The conservation of free variables is well known. If $\partial_x s_0 \cdot [s_1, \ldots, s_n]$ is empty, then the statement is trivial, otherwise we know that the number of free occurrences of x in s_0 must be n. By induction on s_0, we prove that $n \leq \mathbf{s}(t')$ and $\mathbf{s}(t') = \mathbf{s}(t) - n - 2$. Thus, $\mathbf{s}(t') + 2 \leq \mathbf{s}(t)$ and $\mathbf{s}(t) \leq 2\mathbf{s}(t') + 2$.

Lemma 61. *For all $t' \in \Delta$, the set $\{t \in \Delta;\ t \succ t'\}$ is finite.*

Proof. By the previous Lemma, this is a subset of the finite set:

$$\{t \in \Delta;\ \mathsf{s}(t) \leq 2\mathsf{s}(t') + 2 \text{ and } \mathsf{fv}(t) = \mathsf{fv}(t')\}.$$

Lemma 62. *If $s \geq s'$, $\bar{t} \geq \bar{t}'$ and $v' \in \partial_x s' \cdot \bar{t}'$ then there exists $v \in \partial_x s \cdot t$ such that $v \geq v'$.*

Proof. It is sufficient to prove the result when:

– $s \succ s'$ and $\bar{t} = \bar{t}'$: this is done by induction on the reduction $s \succ s'$;
– $s = s'$ and $\bar{t} \succ \bar{t}'$: this is done by induction on s;

(and in fact, in both cases, we can further require that $v \succ v'$).

The following property is a simple consequence of the fact that the reduction of a head-redex can be always done at first. It will be used for Lemma 66.

Lemma 63. *Let $s = \langle \lambda x.t \rangle\, \overline{u_0}\, \overline{u_1} \ldots \overline{u_n}$ and assume that $s \geq s'$. Then one of the following holds:*

(i) $s' = \langle \lambda x.t' \rangle\, \overline{u'_0}\, \overline{u'_1} \ldots \overline{u'_n}$, with $t \geq t'$ and $\overline{u_i} \geq \overline{u'_i}$ for all $i \in \{0, \ldots, n\}$;
(ii) or there exists $v \in \langle \partial_x t \cdot \overline{u_1} \rangle\, \overline{u_2} \ldots \overline{u_n}$ such that $v \geq s'$.

Proof. By induction on the length of the reduction $s \geq s'$.

If $s = s'$, the first case immediately holds. Otherwise, we have $s \geq s'' \succ s'$ and the induction hypothesis applies to the reduction from s to s''.

Case (ii) for the latter reduction extends to the former directly, because $v \geq s''$ implies $v \geq s'$.

Assume (i) holds: $s'' = \langle \lambda x.t'' \rangle\, \overline{u''_0}\, \overline{u''_1} \ldots \overline{u''_n}$, with $t \geq t''$ and $\overline{u_i} \geq \overline{u''_i}$ for all $i \in \{0, \ldots, n\}$.

Then there are two possible cases for the reduction $s'' \succ s'$:

– $s' = \langle \lambda x.t' \rangle\, \overline{u'_0}\, \overline{u'_1} \ldots \overline{u'_n}$, with $t'' \geq t'$ and $\overline{u''_i} \geq \overline{u'_i}$ for all $i \in \{0, \ldots, n\}$ (in fact, there is only one actual reduction among these) and then we conclude that (i) also holds for the reduction $s \geq s'$;
– or $s' = \langle w' \rangle\, \overline{u''_1} \ldots \overline{u''_n}$ with $w' \in \left\langle \partial_x t'' \cdot \overline{u''_0} \right\rangle$.

In that latter case, the previous lemma gives $w \in \partial_x t \cdot \overline{u}_0$ such that $w \geq w'$. We then obtain (ii), setting $v = \langle w \rangle\, \overline{u_1} \ldots \overline{u_n}$.

Lemma 64. *Assume $\mathfrak{H} \subseteq \mathfrak{P}(\Delta)$ is abstraction hereditary and downwards closed. If $a \in \nabla(\mathfrak{H})^{\perp}$, then $\lambda x.a \in \nabla(\mathfrak{H})^{\perp}$.*

Proof. Let $b' \in \mathfrak{H}$, write $b = (\lambda x.a) \cap \upharpoonleft b'$: we prove that b is finite. We know $x \notin \mathsf{fv}(b)$. Write $b'' = b' \cap \mathfrak{P}\left(\Delta^{\mathsf{fv}(b)}\right)$ so that, in particular, $x \notin \mathsf{fv}(b'')$ and $b'' \in \mathfrak{H}$ because $\mathfrak{H}$ is downwards closed, hence $\pi_x b'' \in \mathfrak{H}$ because $\mathfrak{H}$ is abstraction hereditary.

For all $s \in b$, there is $s' \in b'$ such that $s \geq s'$, hence $s' \in b''$. Moreover there is $t \in a$ such that $s = \lambda x.t$; then we can write $s' = \lambda x.t'$ with $t \geq t'$. Since $s' \in b''$, we have $t' \in \pi_x b''$, hence $t \in a \cap \upharpoonleft \pi_x b''$. We deduce that $b \subseteq \lambda x.\left(a \cap \upharpoonleft \pi_x b''\right)$, which is finite because $a \in \nabla(\mathfrak{H})^{\perp}$ and $\pi_x b'' \in \mathfrak{H}$.

The following is a technical lemma that will be used only twice in the following: its sole purpose is to factor out otherwise redundant proofs.

Lemma 65. *Consider* $n \in \mathbf{N}$ *and two families* $(s_k)_{k\in K}$ *and* $(s'_k)_{k\in K}$ *of resource terms such that, for all* $k \in K$*:*

- $s_k = \langle t_{k,0}\rangle\, \bar{t}_{k,1} \cdots \bar{t}_{k,n}$;
- $s'_k = \left\langle t'_{k,0}\right\rangle\, \bar{t}'_{k,1} \cdots \bar{t}'_{k,n}$;
- $t_{k,0} \geq t'_{k,0}$ *and for all* $i \in \{1, \dots, n\}$, $\bar{t}_{k,i} \geq \bar{t}'_{k,i}$.

Let $b = \{s_k;\ k \in K\}$, $b' = \{s'_k;\ k \in K\}$, $b_0 = \{t_{k,0};\ k \in K\}$ *and, for* $1 \leq i \leq n$, $b_i = \bigcup_{k\in K} \left|\bar{t}_{k,i}\right|$. *Assume there is an application hereditary structure* $\mathfrak{H}$ *which is dispersed and such that* $b' \in \mathfrak{H}$*, and* $b_i \in \nabla(\mathfrak{H})^{\perp}$ *for* $0 \leq i \leq n$*. Then* b *is finite.*

Proof. By the definition of reduction on resource terms, for all $k \in K$ and $i \in \{1, \dots, n\}$, we can write $\bar{t}_{k,i} = \left[t_{k,i,1}, \dots, t_{k,i,p_{k,i}}\right]$ and $\bar{t}'_{k,i} = \left[t'_{k,i,1}, \dots, t'_{k,i,p_{k,i}}\right]$, and assume w.l.o.g. that $t_{k,i,j} \geq t'_{k,i,j}$ for $1 \leq j \leq p_{k,i}$. To prove that b is finite, it is sufficient to prove that each b_i is finite and the $p_{k,i}$'s are bounded.

Write $b'_0 = \left\{t'_{k,0};\ k \in K\right\}$ and $b'_i = \bigcup_{k\in K} \left|\bar{t}'_{k,i}\right|$ for $1 \leq i \leq n$. Notice that $b'_0 = \pi_0^n b'$ and $b'_i = \pi_1 \pi_0^{n-i} b'$: since $\mathfrak{H}$ is hereditary and $b' \in \mathfrak{H}$, $b'_i \in \mathfrak{H}$ for $0 \leq i \leq n$. Since $b_i \subset \uparrow b'_i$ and $b_i \in \nabla(\mathfrak{H})^{\perp}$ we deduce that b_i is finite.

By the properties of the reduction of resource terms (namely, Lemma 60), any term has only finitely many reducts, hence each b'_i is also finite. In particular, each of the sets $\left\{\mathbf{h}\left(t'_{k,0}\right);\ k \in K\right\}$ and $\left\{\mathbf{h}\left(t'_{k,i,j}\right);\ k \in K \wedge 1 \leq j \leq p_{k,i}\right\}$ for $1 \leq i \leq n$ is finite. Write d for a common upper bound. We obtain that, for all $k \in K$, $\mathbf{h}(s'_k) \leq d + n$. Moreover observe that $\mathsf{fv}(b') \subseteq \mathsf{fv}(b) \subseteq \bigcup_{0\leq i\leq n} \mathsf{fv}(b_i)$, which is finite because $\bigcup_{0\leq i\leq n} b_i$ is finite. Since $b' \in \mathfrak{H}$ and $\mathfrak{H}$ is dispersed, b' is finite. This entails that the set $\{p_{k,i};\ k \in K \wedge 1 \leq i \leq n\}$ is finite, which concludes the proof.

Lemma 66 (Saturation). *For all* $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ *which is hereditary and dispersed, the structure* $\nabla(\mathfrak{S})^{\perp}$ *is* $\nabla(\mathfrak{S})^{\perp}$*-saturated.*

Proof. We have to prove that for all $e, f_0, \dots, f_n \in \nabla(\mathfrak{S})^{\perp}$,

$$\left\langle \partial_x(e, {f_0}^!)\right\rangle {f_1}^! \dots {f_n}^! \in \nabla(\mathfrak{S})^{\perp}, \text{ we also have } \langle \lambda x.e\rangle\ {f_0}^! {f_1}^! \dots {f_n}^! \in \nabla(\mathfrak{S})^{\perp}.$$

It is sufficient to prove that for all $a' \in \mathfrak{S}$, for all family $(s_k)_{k\in K}$ of elements of $\left(\langle \lambda x.e\rangle\ {f_0}^! {f_1}^! \dots {f_n}^!\right) \cap \uparrow a'$, the set $b = \{s_k;\ k \in K\}$ is finite. For all $k \in K$, we write $s_k = \langle \lambda x.t_k\rangle\, \bar{u}_{k,0} \cdots \bar{u}_{k,n}$ and we fix $s'_k \in a'$ such that $s_k \geq s'_k$. Then for all $k \in K$, thanks to Lemma 63 at least one of the following holds:

(i) we can write $s'_k = \langle \lambda x.t'_k\rangle\, \bar{u}'_{k,0} \cdots \bar{u}'_{k,n}$, with $t_k \geq t'_k$ and $\bar{u}_{k,i} \geq \bar{u}'_{k,i}$ for all $i \in \{0, \dots, n\}$;

(ii) there exists $s''_k \in \langle \partial_x(t_k, \bar{u}_{k,0})\rangle\, \bar{u}_{k,1} \cdots \bar{u}_{k,n}$ such that $s''_k \geq s'_k$.

Write K' (resp. K'') for the set of all $k \in K$ such that (i) (resp.(ii)) holds. If we set $b' = \{s_k;\ k \in K'\}$ and $b'' = \{s''_k;\ k \in K''\}$, we have that $b \subseteq b' \cup \{s \in \Delta;\ \exists s'' \in b'',\ s \succ s''\}$. By Lemma 61, it is sufficient to prove that b' and b'' are finite.

Notice that $\{s'_k;\ k \in K'\} \subseteq a' \in \mathfrak{S}$. We also have $\{\lambda x.t_k;\ k \in K'\} \subseteq \lambda x.e \in \nabla(\mathfrak{S})^\perp$ (Lemma 64) and for all $i \in \{0, \dots, n\}$, $\bigcup_{k \in K'} |\overline{u}_{k,i}| \subseteq f_i \in \nabla(\mathfrak{S})^\perp$. As a consequence of Lemma 65, b' is finite.

It remains to prove that b'' is finite, which follows from the fact that, for all $k \in K''$, $s''_k \in \left\langle \partial_x(e, {f_0}^!) \right\rangle {f_1}^! \dots {f_n}^! \in \nabla(\mathfrak{H})^\perp$ and $s''_k \geq s'_k \in a' \in \mathfrak{H}$. $\square$

Lemma 67. *For all structure $\mathfrak{H}$ which is application hereditary and dispersed,* ${\nabla(\mathfrak{H})^\perp}_h \subseteq \nabla(\mathfrak{H})^\perp$.

Proof. Assume that $a \in {\nabla(\mathfrak{H})^\perp}_h$: we can write $a = \langle x \rangle\ {a_1}^! \cdots {a_n}^!$ with $a_1, \dots, a_n \in \nabla(\mathfrak{H})^\perp$. To show that $a \in \nabla(\mathfrak{H})^\perp$ amounts to prove that $a \perp \uparrow a'$ for all $a' \in \mathfrak{H}$. Let $a' \in \mathfrak{H}$ and $(s_k)_{k \in K}$ be an arbitrary family of elements of $a \cap \uparrow a'$: we prove that $b = \{s_k;\ k \in K\}$ is finite.

Fix $k \in K$. Since $s_k \in a$, there are bags $\overline{t}_{k,i} \in {a_i}^!$ for $1 \leq i \leq n$ such that $s_k = \langle x \rangle\ \overline{t}_{k,1} \cdots \overline{t}_{k,n}$. Since $s_k \in \uparrow a'$, there exists $s'_k \in a'$ such that $s_k \geq s'_k$. By the definition of reduction on resource terms, s'_k is necessarily of the form $s'_k = \langle x \rangle\ \overline{t}'_{k,1} \cdots \overline{t}'_{k,n}$ with $\overline{t}_{k,i} \geq \overline{t}'_{k,i}$ for $1 \leq i \leq n$.

Setting $t_{k,0} = t'_{k,0}$ for all k, the hypotheses of Lemma 65 are directly satisfied and we conclude that b is finite.

Lemma 68. *Let $\mathfrak{S}$ be dispersed, hereditary and expandable, we show that:*

$$\nabla(\mathfrak{S})^\perp{}_h \subset \left(\nabla(\mathfrak{S})^\perp \to \nabla(\mathfrak{S})^\perp{}_h\right) \subset \left(\nabla(\mathfrak{S})^\perp{}_h \to \nabla(\mathfrak{S})^\perp\right) \subset \nabla(\mathfrak{S})^\perp.$$

Proof. The first inclusion is immediate from the definition of ${\nabla(\mathfrak{S})^\perp}_h$. The second inclusion is a consequence of the fact that ${\nabla(\mathfrak{S})^\perp}_h \subset \nabla(\mathfrak{S})^\perp$ (Lemma 67). It remains to prove that $\left({\nabla(\mathfrak{S})^\perp}_h \to \nabla(\mathfrak{S})^\perp\right) \subset \nabla(\mathfrak{S})^\perp$.

Let $f \in {\nabla(\mathfrak{S})^\perp}_h \to \nabla(\mathfrak{S})^\perp$, fix $h \in \mathfrak{S}$ and consider any family $(s_k)_{k \in K}$ such that $s_k \in f \cap \uparrow h$ for all $k \in K$. Let $x \in \mathcal{V}$ and $h' = \{\langle s \rangle\ [x];\ s \in h\} \in \mathfrak{S}$ (since $\mathfrak{S}$ is supposed to be expandable): we have $\{\langle s_k \rangle\ [x];\ k \in K\} \subseteq \left(\uparrow h'\right) \cap \langle f \rangle\ \{x\}^!$. Since $\{x\} \in {\nabla(\mathfrak{S})^\perp}_h$ and $f \in {\nabla(\mathfrak{S})^\perp}_h \to \nabla(\mathfrak{S})^\perp$, we have $\langle f \rangle\ \{x\}^! \in \nabla(\mathfrak{S})^\perp$. We deduce that $\{\langle s_k \rangle\ [x];\ k \in K\}$ is finite, hence so is $\{s_k;\ k \in K\}$. We have just proved that $(\uparrow h) \cap f$ is finite for all $h \in \mathfrak{S}$, hence $f \in \nabla(\mathfrak{S})^\perp$. $\square$

Proof (of Lemma 29). Conditions 1 and 2 of the definition of adaptedness (Definition 17) are given respectively by Lemma 66 and 68. The closure under finite unions (Condition 3) is an easy property to check for any dual structure (like $\nabla(\mathfrak{S})^\perp$). $\square$

B.3 A Variant in the Definition of the Realizers

In this Appendix, we present realizers as finiteness spaces $\mathcal{A}$ that are given by a web $|\mathcal{A}|$ and a finiteness structure $\mathfrak{Fin}(\mathcal{A})$ on it (see Definition 7). Although it is not necessary for our purpose, it is noticable that this structures enjoy the finiteness condition.

Definition 69 (Realizers). *Let $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$. To each type A of system D_+, we associate a finiteness space $\|A\|_{\mathfrak{S}}$ defined inductively as follows (X varying on propositional variables):*

$$\begin{aligned}
|\|X\|_{\mathfrak{S}}| &:= \Delta, \\
\mathfrak{Fin}(\|X\|_{\mathfrak{S}}) &:= \nabla(\mathfrak{S})^{\perp}, \\
|\|A \to B\|_{\mathfrak{S}}| &:= \left\{s \in \Delta;\ \forall a \in \|A\|_{\mathfrak{S}}, \langle s\rangle\ a^! \in \|B\|_{\mathfrak{S}}\right\}, \\
\mathfrak{Fin}(\|A \to B\|_{\mathfrak{S}}) &:= \left\{f \subseteq |\|A \to B\|_{\mathfrak{S}}|;\ \forall a \in \|A\|_{\mathfrak{S}}, \langle f\rangle\ a^! \in \|B\|_{\mathfrak{S}}\right\}, \\
|\|A \cap B\|_{\mathfrak{S}}| &:= |\|A\|_{\mathfrak{S}}| \cap |\|B\|_{\mathfrak{S}}|, \\
\mathfrak{Fin}(\|A \cap B\|_{\mathfrak{S}}) &:= \mathfrak{Fin}(\|A\|_{\mathfrak{S}}) \cap \mathfrak{Fin}(\|B\|_{\mathfrak{S}}).
\end{aligned}$$

We then write $|A|_{\mathfrak{S}} = |\|A\|_{\mathfrak{S}}|$ and $\mathfrak{Fin}_{\mathfrak{S}}(A) = \mathfrak{Fin}(\|A\|_{\mathfrak{S}})$.

Proposition 70. *For any $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ and type A, $\|A\|_{\mathfrak{S}}$ is a finiteness space.*

Proof. The atomic case $\|X\|_{\mathfrak{S}}$ is a finiteness space by construction. The intersection case follows because finiteness spaces are closed under intersection. The fact that $\mathfrak{Fin}_{\mathfrak{S}}(A \to B)$ is indeed a finiteness structure (i.e. closed by biorthogonality) is a consequence of the following result:

(∗) For all $a, b \subseteq \Delta$, write $a \bullet b = \left\{s \in \Delta;\ \langle s\rangle\ a^! \cap b \neq \emptyset\right\}$. Then

$$\mathfrak{Fin}_{\mathfrak{S}}(A \to B) = \left\{a \bullet b';\ a \in \mathfrak{Fin}_{\mathfrak{S}}(A) \text{ and } b' \in \mathfrak{Fin}_{\mathfrak{S}}(B)^{\perp_{|B|_{\mathfrak{S}}}}\right\}^{\perp_{|A\to B|_{\mathfrak{S}}}}$$

Let $f \subseteq |A \to B|_{\mathfrak{S}}$. We prove that $f \in \mathfrak{Fin}_{\mathfrak{S}}(A \to B)$ iff for all $a \in \mathfrak{Fin}_{\mathfrak{S}}(A)$, and all $b' \in \mathfrak{Fin}_{\mathfrak{S}}(B)^{\perp_{|B|_{\mathfrak{S}}}}$, $f \cap (a \bullet b')$ is finite.

First assume that $f \in \mathfrak{Fin}_{\mathfrak{S}}(A \to B)$, $a \in \mathfrak{Fin}_{\mathfrak{S}}(A)$ and $b' \in \mathfrak{Fin}_{\mathfrak{S}}(B)^{\perp_{|B|_{\mathfrak{S}}}}$. Then $\langle f\rangle\ a^! \in \mathfrak{Fin}_{\mathfrak{S}}(B)$ hence $b' \cap \langle f\rangle\ a^!$ is finite. Take $s \in f \cap (a \bullet b')$. Since $s \in a \bullet b'$, there exists $\bar{t} \in a^!$ such that $\langle s\rangle\ \bar{t} \in b'$. Since $s \in f$ and $\bar{t} \in a^!$, we also have $\langle s\rangle\ \bar{t} \in \langle f\rangle\ a^!$. We deduce that $f \cap (a \bullet b') \subseteq \pi_0\left(b' \cap \langle f\rangle\ a^!\right)$, which is finite.

Now assume that for all $a \in \mathfrak{Fin}_{\mathfrak{S}}(A)$, and all $b' \in \mathfrak{Fin}_{\mathfrak{S}}(B)^{\perp_{|B|_{\mathfrak{S}}}}$, the set $f'' = f \cap (a \bullet b')$ is finite. We prove that $b'' = b' \cap \langle f\rangle\ a^!$ is also finite. Take $u \in b''$. Since $u \in \langle f\rangle\ a^!$, there are $s \in f$ and $\bar{t} \in a^!$ such that $u = \langle s\rangle\ \bar{t}$. Since $u \in b'$, we moreover have $s \in a \bullet b'$, hence $s \in f''$. We then have $u \in \bigcup\left\{\langle s\rangle\ a^!; s \in f''\right\}$. Observe that, for all $s \in f''$, since $s \in |A \to B|_{\mathfrak{S}}$ and $a \in \mathfrak{Fin}_{\mathfrak{S}}(A)$, $\langle s\rangle\ a^! \in \mathfrak{Fin}_{\mathfrak{S}}(B)$. Since $\mathfrak{Fin}_{\mathfrak{S}}(B)$ is closed under finite unions, we obtain $\bigcup\left\{\langle s\rangle\ a^!; s \in f''\right\} \in \mathfrak{Fin}_{\mathfrak{S}}(B)$. Finally, $b'' \subseteq b' \cap \bigcup\left\{\langle s\rangle\ a^!; s \in f''\right\}$, which is finite because $b' \in \mathfrak{Fin}_{\mathfrak{S}}(B)^{\perp_{|B|_{\mathfrak{S}}}}$.

Remark 71. In the previous proof, the fact that $f \subseteq |A \to B|_{\mathfrak{S}}$ is crucial: if we defined $|A \to B|_{\mathfrak{S}} = \varDelta$, Proposition 70 would fail. Consider the term $s = \lambda x. \langle x \rangle \, [I]$. The singleton $\{s\}$ is in $\left\{a \bullet b';\ a \in \mathfrak{Fin}_{\mathfrak{S}}(A) \text{ and } b' \in \mathfrak{Fin}_{\mathfrak{S}}\left(B^{\perp}\right)\right\}^{\perp}$, for any A and B, because finite. However, taking $A = B = X$ and $\mathfrak{S} = \mathfrak{P}_f(\varDelta)$, we argue that $\{s\}$ does not enjoy the condition of the definition of $\|X \to X\|_{\mathfrak{P}_f(\varDelta)}$. Take $a = \{t_n;\ n \in \mathbf{N}\}$ where $t_n = \lambda y. \langle \cdots \langle y \rangle \, [I] \cdots \rangle \, [I]$ (n times). Notice that $a \in \nabla(\mathfrak{P}_f(\varDelta))^{\perp}$, however $\langle s \rangle \, a^{!} \notin \nabla(\mathfrak{P}_f(\varDelta))^{\perp}$. In fact, for all $n \in \mathbf{N}$, $\langle s \rangle \, [t_n] \to_{\beta}^{*} I$, hence $\{\langle s \rangle \, [t_n];\ n \in \mathbf{N}\} \subset \langle s \rangle \, a^{!}$ is infinite and included in $\uparrow I \in \nabla(\mathfrak{P}_f(\varDelta))^{\perp}$.

C Proofs of Section 5

Definition 72 (Partial Productive sequences). *Let $(M_i)_{i \in \mathbf{N}}$ be a sequence of terms. We say it is a* partial reduction sequence *if, for all $i \in \mathbf{N}$, we have $M_i \rightharpoonup M_{i+1}$ or $M_i \trianglerighteq M_{i+1}$. We say that it is* productive *if moreover the set $\{i \in \mathbf{N};\ M_i \rightharpoonup M_{i+1}\}$ is infinite. In this case, we call the only monotone bijection $\mathbf{N} \to \{i \in \mathbf{N};\ M_i \rightharpoonup M_{i+1}\}$ the* squash-indexing *of the sequence.*

Lemma 73. *If $M \trianglelefteq N$ then $\mathbf{s}(M) \leq \mathbf{s}(N)$, $\mathbf{h}(M) \leq \mathbf{h}(N)$, $\mathsf{fv}(M) \subseteq \mathsf{fv}(N)$ and $\mathcal{T}(M) \subseteq \mathcal{T}(N)$.*

Proof. Easy by induction on the definition of $\trianglelefteq$.

Lemma 74. *If $s \in \mathcal{T}(M[N/x])$ then there exist $t \in \mathcal{T}(M)$ and $\overline{u} \in \mathcal{T}(N)^!$ such that $s \in \partial_x t \cdot \overline{u}$.*

Proof. Write $M' = M[N/x]$. The proof is by induction on M.

If M is x then $M' = N$ and $s \in \mathcal{T}(N)$: we can set $t = x$ and $\overline{u} = [s]$.

If M is $y \in \mathcal{V} \setminus \{x\}$ then $M' = y$: we can set $t = y$ and $\overline{u} = []$.

If M is $\lambda y.M_0$ with $y \neq x$ and $y \notin \mathsf{fv}(N)$, then $M' = \lambda y.M_0'$ with $M_0' = M_0[N/x]$. Necessarily $s = \lambda y.s_0$ with $s_0 \in \mathcal{T}(M_0')$. By induction hypothesis, we obtain $t_0 \in \mathcal{T}(M_0)$ and $\overline{u} \in \mathcal{T}(N)^!$ such that $s_0 \in \partial_x t_0 \cdot \overline{u}$. Then we can set $t = \lambda x.t_0$.

If M is $(M_0)\,M_1$ then $M' = (M_0')\,M_1'$ with $M_i' = M_i[N/x]$ for $i \in \{0,1\}$. Necessarily, $s = \langle s_0\rangle\,[s_1,\ldots,s_k]$ with $s_0 \in \mathcal{T}(M_0')$ and $s_i \in \mathcal{T}(M_1')$ for $1 \leq i \leq n$. By induction hypothesis, we obtain $t_0 \in \mathcal{T}(M_0)$, $\overline{u}_0 \in \mathcal{T}(N)^!$, and $t_i \in \mathcal{T}(M_1)$ and $\overline{u}_i \in \mathcal{T}(N)^!$ for $1 \leq i \leq n$, such that $s_0 \in \partial_x t_0 \cdot \overline{u}_0$ and $s_i \in \partial_x t_i \cdot \overline{u}_i$ for $1 \leq i \leq n$. Then we can set $t = \langle t_0\rangle\,[t_1,\ldots t_n]$ and $\overline{u} = \overline{u}_0 + \cdots + \overline{u}_n$.

If M is $M_0 + M_1$ then $M' = M_0' + M_1'$ with $M_i' = M_i[N/x]$ for $i \in \{0,1\}$. Necessarily, $s \in \mathcal{T}(M_i')$ for some i. By induction hypothesis, we obtain $t \in \mathcal{T}(M_i)$, and $\overline{u} \in \mathcal{T}(N)^!$ such that $s \in \partial_x t \cdot \overline{u}$. We conclude since $\mathcal{T}(M_i) \subseteq \mathcal{T}(M)$. □

Proof (of Lemma 32). We begin by proving the top level case. Assume that $M \rightharpoonup N$ is at top level, that is $M = (\lambda x.M')\,M'' \to M'[M''/x] \trianglerighteq N$. Let $s \in \mathcal{T}(N)$, by Lemma 73, $s \in \mathcal{T}(M'[M''/x])$. Now, by Lemma 74, there are $t \in \mathcal{T}(M)$ and $\overline{u} \in \mathcal{T}(N)^!$ such that $s \in \partial_x t \cdot \overline{u}$, so that $\langle \lambda x.t\rangle\,\overline{u} > s$ and $\langle \lambda x.t\rangle\,\overline{u} \in \mathcal{T}(M)$.

Now, for proving the general case, by Lemma 73, it is sufficient to consider the case of $M \to_\beta N$. The proof is by induction on the definition of $\to_\beta$.

If $M = (\lambda x.S)\,P$ and $N = S[P/x]$ then we conclude by the top level case.

If $M = \lambda x.M_0$ and $N = \lambda x.N_0$ with $M_0 \to_\beta N_0$ then $t = \lambda x.t_0$ with $t_0 \in \mathcal{T}(N_0)$ and the induction hypothesis provides $s_0 \in \mathcal{T}(M_0)$ such that $s_0 \geq t_0$. We set $s = \lambda x.s_0$.

If $M = (M_0)\,M_1$ and $M = (N_0)\,N_1$ with $M_0 \to_\beta N_0$ or $M_1 \to_\beta N_1$, then $t = \langle t_0\rangle\,[t_1,\ldots,t_n]$ with $t_0 \in \mathcal{T}(N_0)$ and $t_i \in \mathcal{T}(N_1)$ for $1 \leq i \leq n$. The induction hypothesis provides:

- either $s_0 \in \mathcal{T}(M_0)$ such that $s_0 \geq t_0$, and we set $s = \langle s_0 \rangle\ [t_1, \ldots, t_n]$;
- or $s_1, \ldots, s_n \in \mathcal{T}(N_1)$ such that $s_i \geq t_i$ for $1 \leq i \leq n$, and in this case we set $s = \langle t_0 \rangle\ [s_1, \ldots, s_n]$.

If $M = M_0 + M_1$ and $N = N_0 + M_1$ with $M_0 \to_\beta N_0$. If $t \in \mathcal{T}(N)_0$, then the induction hypothesis directly provides $s \in \mathcal{T}(M_0) \subseteq \mathcal{T}(M)$ such that $s \geq t$. Otherwise, $t \in \mathcal{T}(M_1)$ and we set $s = t$. ☐

Proof (of Lemma 33). Write $a = \mathcal{T}(M) \cap \uparrow u$ and $b = \mathcal{T}(N) \cap \uparrow u$. Since b is infinite, and $\mathsf{fv}(b) \subseteq \mathsf{fv}(N)$ is finite, the set $\{\mathbf{s}(t)\,;\ t \in b\}$ is unbounded. Moreover, for all $t \in b$, by Lemma 32, there exists $s \in \mathcal{T}(M)$ such that $s \geq t$: by Lemma 60, this implies that the set $\{\mathbf{s}(s)\,;\ s \in a\}$ is also unbounded, hence a is infinite.

Lemma 75. *If $M = P + Q$ then:*

- *if $M \to_\beta M'$ then $M' = P' + Q'$ with either $P \to_\beta P'$ and $Q = Q'$ or $P = P'$ and $Q \to_\beta Q'$;*
- *if $M \trianglerighteq M'$ then either $P \trianglerighteq M'$ or $Q \trianglerighteq M'$ or $M' = P' + Q'$ with $P \trianglerighteq P'$ and $Q \trianglerighteq Q'$.*

Proof. Each result is proved by an easy induction on the definition of the relation.

Lemma 76. *Let $(M_i)_{i \in \mathbf{N}}$ be a productive partial reduction sequence, and ϕ be its squash-indexing. Then $M_{\phi(i)} \rightharpoonup M_{\phi(i+1)}$ for all $i \in \mathbf{N}$.*

Proof. Observe that, since for $\phi(i) < j < \phi(i+1)$ $M_j \trianglerighteq M_{j+1}$ and since $\trianglelefteq$ is reflexive and transitive, we have $M_{\phi(i)+1} \trianglerighteq M_{\phi(i+1)}$ hence $M_{\phi(i)} \rightharpoonup M_{\phi(i+1)}$.

Lemma 77. *Let $(M_i)_{i \in \mathbf{N}}$ be a sequence of terms such that $M_i \rightharpoonup M_{i+1}$ for all $i \in \mathbf{N}$. Assume moreover that $M_0 = P_0 + Q_0$. Then there exists a sequence $(N_i)_{i \in \mathbf{N}}$ and a stricly increasing $\phi : \mathbf{N} \to \mathbf{N}$ such that either $P_0 \trianglerighteq N_0$ or $Q_0 \trianglerighteq N_0$, and $N_i \rightharpoonup N_{i+1}$ and $N_i \trianglelefteq M_{\phi(i)}$ for all $i \in \mathbf{N}$.*

Proof. Assume there is no such sequence: we construct one, which establishes a contradiction.[9] More precisely, by induction on $i \in \mathbf{N}$, we define a pair (P_i, Q_i) of terms such that $M_i = P_i + Q_i$ and, if $i > 0$, either $P_{i-1} \rightharpoonup P_i$ and $Q_{i-1} \trianglerighteq Q_i$ or $P_{i-1} \trianglerighteq P_i$ and $Q_{i-1} \rightharpoonup Q_i$: then either $(P_i)_{i \in \mathbf{N}}$ or $(Q_i)_{i \in \mathbf{N}}$ is productive, and we obtain the desired $(N_i)_{i \in \mathbf{N}}$ by the previous lemma.

Assume P_i and Q_i are defined. Consider the reduction $M_i \rightharpoonup M_{i+1}$. By definition there exists M'_i such that $M_i \to_\beta M'_i \trianglerighteq M_{i+1}$. Since $M_i = P_i + Q_i$, Lemma 75 gives $M'_i = P'_i + Q'_i$ with either $P_i \to_\beta P'_i$ and $Q_i = Q'_i$ or $P_i = P'_i$ and $Q_i \to_\beta Q'_i$. Since $P'_i + Q'_i \trianglerighteq M_{i+1}$, using Lemma 75 again, we obtain one of the following:

- either $P'_i \trianglerighteq M_{i+1}$;

[9] We use *reductio ad absurdum* to keep the proof short, but one could perfectly construct $(N_i)_{i \in \mathbf{N}}$ directly, either by a coinductive argument or by establishing a more contorted statement by induction on i.

– or $Q'_i \trianglerighteq M_{i+1}$;
– or $M_{i+1} = P_{i+1} + Q_{i+1}$ with $P'_i \trianglerighteq P_{i+1}$ and $Q'_i \trianglerighteq Q_{i+1}$.

The first two cases directly lead to a contradiction: assume, e.g., that $P'_i \trianglerighteq M_{i+1}$; then, setting $P_j = M_j$ for all $j > i$, we obtain a productive partial reduction sequence such that $P_j \trianglelefteq M_j$ for $j \in \mathbf{N}$, we obtain a suitable $(N_i)_{i\in\mathbf{N}}$ by the previous lemma. Hence, the third case holds, and we have constructed the desired pair (P_{i+1}, Q_{i+1}).

We call a Λ^+ term *simple* if its top-level constructor is not a sum (i.e. it is a variable, or an abstraction or an application).

Lemma 78. *Let $(M_i)_{i\in\mathbf{N}}$ be a sequence such that $M_i \rightharpoonup M_{i+1}$ for all $i \in \mathbf{N}$. Then there exists a sequence $(S_i)_{i\in\mathbf{N}}$ of simple terms (i.e. terms without sum as top-level constructor) and a strictly increasing $\phi : \mathbf{N} \to \mathbf{N}$ such that, for all $i \in \mathbf{N}$, $S_i \rightharpoonup S_{i+1}$ and such that $S_i \trianglelefteq M_{\phi(i)}$.*

Proof. By induction on $i \in \mathbf{N}$, we construct a sequence $(N_{i,j})_{j\in\mathbf{N}}$ together with a strictly increasing $\phi_i : \mathbf{N} \to \mathbf{N}$, a natural number $\phi(i)$ and a simple term S_i, such that: for all $i \in \mathbf{N}$, $S_i \trianglelefteq M_{\phi(i)}$, $S_i \rightharpoonup S_{i+1}$, $\phi(i+1) > \phi(i)$ and for all $j \in \mathbf{N}$, $N_{i,j} \trianglelefteq M_{\phi_i(j)}$ and $N_{i,j} \rightharpoonup N_{i,j+1}$.

We first let $(N_{0,j})_{j\in\mathbf{N}} = (M_j)_{j\in\mathbf{N}}$ and $\phi_0(j) = j$.

Suppose $(N_{i,j})_{j\in\mathbf{N}}$ and ϕ_i are defined. We define S_i, $\phi(i)$, ϕ_{i+1} and $(N_{i+1,j})_{j\in\mathbf{N}}$ as follows: by iterating Lemma 77 starting from $(N_{i,j})_{j\in\mathbf{N}}$, we obtain $(N'_{i,j})_{j\in\mathbf{N}}$ and ϕ'_i such that $N'_{i,j} \trianglelefteq N_{i,\phi'_i(j)} \trianglelefteq M_{\phi_i(\phi'_i(j))}$ and $N'_{i,j} \rightharpoonup N'_{i,j+1}$ for all $j \in \mathbf{N}$, with $N'_{i,0}$ simple. We set $S_i = N'_{i,0}$, $\phi(i) = \phi_i(\phi'_i(0))$, $N_{i+1,j} = N'_{i,j+1}$ and $\phi_{i+1}(j) = \phi_i(\phi'_i(j+1))$.

Proof (of Lemma 35). Observe that by Lemma 33 it is sufficient to have $\mathcal{T}(M_{i_0}) \not\subseteq {\uparrow}t$ for some $i_0 \in \mathbf{N}$.

Then, by induction on $h = \max\{\mathbf{h}(M_i);\ i \in \mathbf{N}\}$, we show there exists $i_0 \in \mathbf{N}$ and a sequence $(s_j)_{j\in\mathbf{N}} \in \mathcal{T}(M_{i_0})^{\mathbf{N}}$ such that s_0 is linear and, for all $j \in \mathbf{N}$ $s_{j+1} \succ^+ s_j$. By Lemma 60, this implies that the set $\{s_j;\ j \in \mathbf{N}\} \subseteq \mathcal{T}(M_{i_0}) \cap {\uparrow}s_0$ is infinite.

Also observe that by Lemmas 78 and 73, it is sufficient to address the case of a sequence of simple terms (i.e. terms without sum as top-level constructor) $(S_i)_{i\in\mathbf{N}}$ such that $S_i \rightharpoonup S_{i+1}$ for all $i \in \mathbf{N}$ and $(\mathbf{h}(S_i))_{i\in\mathbf{N}}$ is bounded.

We say the reduction $S_i \rightharpoonup S_{i+1}$ is at top level whenever $S_i = (\lambda x.M_i)\, T_i \to M_i[T_i/x] \trianglerighteq S_{i+1}$.

First assume that there are infinitely many top level reductions. Observe that, since $\mathbf{h}(S_i) \leq h$ and $\mathsf{fv}(S_i) \subseteq \mathsf{fv}(S_0)$ for all i, the set $\{\mathcal{T}(S_i);\ i \in \mathbf{N}\}$ is finite. Hence there exists an index $i_0 \in \mathbf{N}$ such that $\{i \in \mathbf{N};\ \mathcal{T}(S_i) = \mathcal{T}(S_{i_0})\}$ is infinite. Since there also are infinitely many top level reductions, we can find i_1 and i_2 such that $i_0 < i_1 < i_2$, the reduction i_1 is at top level and $\mathcal{T}(S_{i_2}) = \mathcal{T}(S_{i_0})$.

We can then define the required sequence $(s_j)_{j\in\mathbf{N}}$. Let s_0 be an arbitrary element of $\ell(S_{i_0}) \subseteq \mathcal{T}(S_{i_0})$. We construct s_{j+1} from s_j as follows: we observe that $s_j \in \mathcal{T}(S_{i_2}) = \mathcal{T}(S_{i_0})$; by iterating Lemma 32, we obtain $s'_j \in \mathcal{T}(S_{i_1+1})$ such that $s'_j \geq s_j$; by Lemma the top level case of 74, we obtain $t'_j \in \mathcal{T}(S_{i_1})$ such that $t_j \succ s'_j$; by Lemma 32 again we obtain $s_{j+1} \in \mathcal{T}(S_{i_0})$ such that $s_{j+1} \geq t_j$, hence $s_{j+1} \succ^+ s_j$.

Now assume there are only finitely many top level reductions. Let i_1 be such that no reduction $S_i \rightharpoonup S_{i+1}$ with $i \geq i_1$ is at top level. One of the following necessarily holds:

1. for all $i \geq i_1$, $S_i = \lambda x.M'_i$ with $M'_i \rightharpoonup M'_{i+1}$;
2. for all $i \geq i_1$, $S_i = (M'_i)\, N'_i$ so that $(M'_i)_{i \geq i_1}$ and $(N'_i)_{i \geq i_1}$ are partial reduction sequences, at least one of them being productive.

In case 1, we apply the induction hypothesis to the sequence $(M'_{i+i_1})_{i \in \mathbf{N}}$: this provides i'_0 and a sequence $(s'_j) \in \mathcal{T}\left(M'_{i'_0+i_1}\right)^{\mathbf{N}}$ with s'_0 linear and, for all $j \in \mathbf{N}$ $s'_{j+1} \succ^+ s'_j$. Then we set $i_0 = i'_0 + i_1$ and $s_j = \lambda x.s'_j$ for all $j \in \mathbf{N}$.

In case 2, first assume that $(M'_i)_{i \geq i_1}$ is productive and let ϕ be its squashing index. We can then apply the induction hypothesis to $(M'_{\phi(i)})_{i \in \mathbf{N}}$ which provides i'_0 and a sequence $(s'_j) \in \mathcal{T}\left(M'_{\phi(i'_0)}\right)^{\mathbf{N}}$ with s'_0 linear and, for all $j \in \mathbf{N}$ and $s'_{j+1} \succ^+ s'_j$. Fix $t \in \ell\left(N'_{\phi(i'_0)}\right)$ arbitrarily. Then we set $i_0 = \phi(i'_0)$ and $s_j = \left\langle s'_j \right\rangle [t]$ for all $j \in \mathbf{N}$.

Now assume that $(N'_i)_{i \geq i_1}$ is productive and let ϕ be its squashing index. We apply the induction hypothesis to the sequence $(N'_{\phi(i)})_{i \in \mathbf{N}}$: this provides i'_0 and a sequence $(s'_j) \in \mathcal{T}\left(N'_{\phi i'_0}\right)^{\mathbf{N}}$ with s'_0 linear and, for all $j \in \mathbf{N}$ and $s'_{j+1} \succ^+ s'_j$. Fix $t \in \ell\left(M'_{\phi i'_0}\right)$. Then we set $i_0 = \phi(i'_0)$, and $s_j = \langle t \rangle \left[s'_j\right]$ for all $j \in \mathbf{N}$.

D Proofs of Section 6

Proof (Proof of Lemma 38). If $NF(\mathcal{T}(M))_t$ is infinite, then there is an infinite set $S \in \mathcal{T}(M) \cap \uparrow t$. Let $a \in \mathfrak{H}$ such that $t \in a$ (which exists since $t \in \bigcup \mathfrak{H}$), we conclude $\mathcal{T}(M) \cap \uparrow a$ is infinite, hence $\mathcal{T}(M) \notin \triangledown(\mathfrak{H})^{\perp}$.

Proof (Proof of Corollary 39). By Theorem 14, M is typable in System $\mathcal{D}_+$. Hence, choosing for example $\mathfrak{S} = \mathfrak{B}$, we have $\mathcal{T}(M) \in (\uparrow\mathfrak{S})^{\perp}$ (Theorem 21), and thus $NF(\mathcal{T}(M))_t$ is finite by Lemma 38 and the fact that $\bigcup \mathfrak{B} = \Delta$. □