Parallel Reduction in Resource Lambda-Calculus*

Michele Pagani[1] and Paolo Tranquilli[2]

[1] Dipartimento di Informatica – Università di Torino
pagani@di.unito.it
[2] Laboratoire PPS – Université Paris Diderot
ptranqui@pps.jussieu.fr

Abstract. We study the resource calculus – the non-lazy version of Boudol's λ-calculus with resources. In such a calculus arguments may be finitely available and mixed, giving rise to nondeterminism, modelled by a formal sum. We define parallel reduction in resource calculus and we apply, in such a nondeterministic setting, the technique by Tait and Martin-Löf to achieve confluence. Then, slightly generalizing a technique by Takahashi, we obtain a standardization result.

1 Introduction

In the '90s Boudol introduced resource calculus [1] – an extension of λ-calculus where arguments may come in limited availability and mixed together. Boudol's main motivation was studying a finer observational equivalence, arriving in particular to the one given by π-calculus via Milner's translation [2].

The main difference with ordinary λ-calculus is the renewal of the application of a function to an argument along two directions: on the one hand by introducing depletable arguments that must be used exactly once, on the other by letting the arguments come in multisets. Resource calculus is similar to Ehrhard and Regnier's differential λ-calculus [3]: the application of a function f to a linear argument corresponds, in the terminology of [3], to applying the derivative of f in 0 (which is a linear map) to that argument. Indeed, the second author shows in [4] that resource calculus corresponds to the intuitionistic minimal fragment of differential nets with promotion [5], exactly as λ-calculus corresponds to the intuitionistic minimal fragment of linear logic proof-nets [6]. This translation is therefore built on top of the *proofs-as-programs* correspondence, thus linking a language for nondeterministic programs with a new kind of nondeterministic proofs, the differential nets of differential linear logic.

Let us give a sample of resource calculus by means of an example. Let

$$\mathbf{I} := \lambda z.z \quad D := \lambda dz.z[d^!][d^!] \quad B := \lambda xy.\mathbf{I}[x^!, y^!] \quad M := \lambda b.b[(b[d^!][D[a]])^!][c^!],$$

where we follow the definition of the syntax as given in Figure 1(a). $\mathbf{I}$ is the standard λ-calculus identity. D is a standard λ-term too: it is $\lambda dz.zdd$. The

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slight difference is only in the notation: we write the two arguments of z with a bang as superscript, emphasizing the fact that they are infinitely available arguments, and provide them as two distinct multiset singletons, delimited by brackets and called *bags*. This way of writing the application comes from Girard's linear logic [6]: indeed !-marked arguments (called *perpetual*) correspond exactly to exponential boxes (see [4]), the synchronized areas of proofs viable for non-linear operations (duplication and erasing). Along the same lines, the multiset bag constructor is semantically justified by several denotational models of linear logic, by their interpretation of the exponential modality.

Let us resume our example. The term B shows nondeterministic application: $\mathbf{I}$ is applied to a bag of two (infinitely available) terms, x and y. The term M is very like to the λ-term $\lambda b.b(bd(Da))c$, a nesting of two `if_then_else` with arguments d, Da and c (if b is fed with a boolean). All bags contain exactly one element, modelling deterministic λ-calculus application. However the bags $[D[a]]$ and, inside it, $[a]$ contain an element with no ! superscript, which sets the term apart from ordinary λ-calculus. This means that the argument $D[a]$ (resp. a) *must* be used exactly once by the function which is applied to $[D[a]]$ (resp. $[a]$). Let us evaluate $M[B^!]$ following the reduction of Definition 10.

$$\begin{aligned}
&(1) \quad M[B^!] \to B[(B[d^!][D[a]])^!][c^!] \to \big(\lambda y.\mathbf{I}[(B[d^!][D[a]])^!, y^!]\big)[c^!] \\
&\qquad\qquad \to \mathbf{I}[(B[d^!][D[a]])^!, c^!] \\
&(2) \quad \to B[d^!][D[a]] + c \to \big(\lambda y.\mathbf{I}[d^!, y^!]\big)[D[a]] + c \to \mathbf{I}[d^!, D[a]] + c \\
&(3) \quad \to D[a] + c \to \lambda z.z[a]1 + \lambda z.z1[a] + c.
\end{aligned}$$

The steps in line (1) of the example are akin to ordinary λ-calculus ones: we have a λ-abstraction fed with a bag containing exactly one infinitely available element. The step from line (1) to line (2) is a nondeterministic one, the argument of $\mathbf{I}$ being a bag with two elements, whence we have a sum of the two possible results. Sums intuitively correspond to a version of nondeterminism where the actual choice operation is left outside the calculus: the result of a term reduction will in general be a large formal sum of terms. The next steps in line (2) are again standard λ-calculus ones. The last term of line (2) has the nondeterministic redex $\mathbf{I}[d^!, D[a]]$. One could be tempted to contract the redex into $d+D[a]$, analogously to the previous nondeterministic step, but in this case the element $D[a]$ occurs linearly in the bag, hence only the choices using $D[a]$ exactly once are allowed. Specifically $\mathbf{I}[d^!, D[a]] \to D[a]$. Finally the last step has also a nondeterministic feature, this time due to a concurrency effect. Indeed in the redex $D[a]$, the function D encodes a pair where both the left and right components ask for the abstracted variable d. However the redex has only one linear occurrence of a available, for which the left and right components are in concurrency for fetching it. We thus have two possible outcomes, depending on which component takes linearly a while forcing the other to collapse to 1 i.e. the empty multiset.

In this paper we prove two basic properties of resource calculus — confluence (Theorem 20) and standardization (Theorem 27). Confluence does not contradict nondeterminism because the result of a nondeterministic reduction is a sum of

Λ:	M, N, L	$::= x \mid \lambda x.M \mid (MP)$	terms
Λ^{arg}:	$M^{(!)}, N^{(!)}$	$::= M \mid M^!$	arguments
Λ^b:	P, Q, R	$::= [M_1^{(!)}, \ldots, M_n^{(!)}]$	bags
$\Lambda^{(b)}$:	A, B	$::= M \mid P$	expressions
$\mu, \nu \in \mathbb{N}\langle\Lambda\rangle$	$\pi, \rho \in \mathbb{N}\langle\Lambda^b\rangle$	$\alpha, \beta, \gamma \in \mathbb{N}\langle\Lambda^{(b)}\rangle := \mathbb{N}\langle\Lambda\rangle \cup \mathbb{N}\langle\Lambda^b\rangle$	sums

(a) Grammar of terms, bags, expressions, sums.

$$\lambda x.(\textstyle\sum_i M_i) := \sum_i \lambda x.M_i \qquad [(\sum_i M_i)]\cdot P := \sum_i [M]_i \cdot P$$

$$(\textstyle\sum_i M_i)P := \sum_i M_i P \qquad [(\sum_i M_i)^!]\cdot P := [M_1^!, \ldots, M_k^!]\cdot P$$

$$M(\textstyle\sum_i P_i) := \sum_i MP_i.$$

(b) notation on $\mathbb{N}\langle\Lambda^{(b)}\rangle$.

Fig. 1: Syntax of resource calculus.

terms. It remains meaningful, as it states that nondeterminism is really internal, and not caused by what an evaluator chooses to reduce. We achieve Theorem 20 by adapting the technique by Tait and Martin-Löf, using a suitable notion of parallel reduction (Definition 14). A similar result is in [3] where confluence of differential λ-calculus is proven. However our proof is somewhat simpler, using a notion of *development* (Definition 18) as defined by Takahashi [7] for λ-calculus. The result is at the same time proved for the *outer* reduction, which is meaningful for the standardization theorem.

A reduction step is *inner* if the redex to be contracted is under the scope of a bang, otherwise it is *outer* (Definition 11). Standardization states that every reduction chain can be split into a concatenation of outer steps followed by inner ones (Theorem 27). In λ-calculus such a result turned out to be fundamental for designing abstract machines computing (weak, head) normal forms, thus giving the theoretical justification of actual evaluators of functional languages. Although in our setting standardization does not give immediately a deterministic normalizing strategy (Example 28), it will help in implementing abstract machines for resource calculus, which can in turn also help in analyzing resource usage by ordinary λ-calculus programs [8]. Our proof of standardization adapts the one by Takahashi for λ-calculus, based on parallel reduction and inner parallel reduction [7]. Actually our notion of inner parallel reduction (Definition 21) is quite peculiar, possibly yielding a slight generalization of such technique.

We conclude the paper by discussing another, more atomic reduction of resource terms, called *baby-step* reduction in [4] (here Definition 31). Although confluence of baby-step reduction is an easy consequence of Theorem 20 (Theorem 34), we show how baby-step standardization fails in general, though it holds for normal and head normal forms (Theorem 36).

2 Syntax and Reduction

We will now introduce resource calculus. Though the "protagonists" are terms, for the ease of proofs it is best to present also other types of syntactic entities. We

$$y\langle N/x\rangle := \begin{cases} N & \text{if } y = x, \\ 0 & \text{otherwise,}\end{cases} \qquad (\lambda y.M)\langle N/x\rangle := \lambda y.(M\langle N/x\rangle), \quad y \notin \mathrm{FV}(N)\cup\{x\},$$
$$(MP)\langle N/x\rangle := M\langle N/x\rangle P + M(P\langle N/x\rangle),$$
$$[M]\langle N/x\rangle := [M\langle N/x\rangle], \qquad 1\langle N/x\rangle := 0,$$
$$[M^!]\langle N/x\rangle := [M\langle N/x\rangle, M^!], \qquad (P\cdot R)\langle N/x\rangle := P\langle N/x\rangle\cdot R + P\cdot R\langle N/x\rangle.$$

(a) Linear substitution.

$$A\langle\!\langle N^{(!)}/x\rangle\!\rangle := \begin{cases} A\langle N/x\rangle & \text{if } N^{(!)} = N, \\ A\,\{x+N/x\} & \text{if } N^{(!)} = N^!,\end{cases}$$
$$A\langle\!\langle [N_1^{(!)},\dots,N_k^{(!)}]/x\rangle\!\rangle := A\langle\!\langle N_1^{(!)}/x\rangle\!\rangle\cdots\langle\!\langle N_k^{(!)}/x\rangle\!\rangle, \quad x\notin \bigcup_{i=1}^{k}\mathrm{FV}(N_i^{(!)}).$$

(b) Argument and bag substitutions.

Fig. 2: Linear, argument and bag substitutions. Notice that the condition on bag substitution can always be achieved by renaming.

thus introduce the calculus as a many-sorted one: the grammars for generating *terms* Λ and *bags* Λ^b (which are in fact multisets of *arguments* Λ^{arg}) is presented in Figure 1(a) together with their typical metavariables. $\Lambda^{(b)}$ (*expressions*) denotes either terms or bags. As we already mentioned, we also have formal sums, denoted by the $\mathbb{N}\langle\,.\,\rangle$ notation (as formal sums are the freely generated modules over natural numbers). However in $\mathbb{N}\langle\Lambda^{(b)}\rangle$, rather than taking freely generated sums, we allow only objects of the same sort to be summed.

Bags are multisets presented in multiplicative notation, so that $P{\cdot}Q$ is multiset union, and $1 = [\,]$ is the empty bag. It must be noted though that we will never omit the dot $\cdot$, to avoid confusion with application.

The grammar for terms and bags does not include sums in any point, so that in a sense they may arise only on the "surface". However as an inductive notation (and *not* in the actual syntax) we extend all the constructors to sums as shown in Figure 1(b). In fact all constructors but the $(\cdot)^!$ are, as expected, linear. Notice the similarity between the equation $[(M+N)^!] = [M^!]\cdot[N^!]$ and $\mathrm{e}^{x+y} = \mathrm{e}^x\cdot\mathrm{e}^y$: this is not a coincidence, as Taylor expansion and semantics show well [9], and can be traced back to linear logic's *exponential isomorphism* $!A\otimes !B \cong !(A \,\&\, B)$.

There is no technical difficulty in defining α-equivalence and the set $\mathrm{FV}(\alpha)$ of free variables as in ordinary λ-calculus.

Definition 1 (Substitutions). *We define the following substitution operators.*

1. $A\,\{N/x\}$ *is the usual capture free substitution of* N *for* x*. It is extended to sums as in* $\alpha\,\{\beta/x\}$ *by linearity*[3] *in* α *and using the notations of Figure 1(b) for* β*. The form* $A\,\{x+N/x\}$ *is called* partial substitution.
2. $A\langle N/x\rangle$ *is the* linear substitution *defined inductively in Figure 2(a). It is extended to* $\alpha\langle\beta/x\rangle$ *by bilinearity in both* α *and* β.

[3] $F(A)$ (resp. $F(A,B)$) is extended by linearity (resp. bilinearity) by setting $F\big(\sum_i A_i\big) = \sum_i F(A_i)$ (resp. $F\big(\sum_i A_i, \sum_j B_j\big) = \sum_{i,j} F(A_i,B_j)$).

3. Argument substitution $A\langle\!\langle N^{(!)}/x\rangle\!\rangle$ *and its iteration* $A\langle\!\langle P/x\rangle\!\rangle$*, the* bag substitution*, are shown in Figure 2(b). Notice that* $A\langle\!\langle 1/x\rangle\!\rangle = A$*. Bag substitution is further generalized to* $\alpha\langle\!\langle \pi/x\rangle\!\rangle$ *by bilinearity in both* α *and* π*.*

As examples we show (supposing x not free in M, N):

$$\begin{aligned} x[x^!]\,\{M+N/x\} &= (M+N)[(M+N)^!] = M[M^!,N^!] + N[M^!,N^!],\\ x[x^!]\langle M+N/x\rangle &= x[x^!]\langle M/x\rangle + x[x^!]\langle N/x\rangle \\ &= M[x^!] + x[M,x^!] + x[N,x^!] + N[x^!],\\ x[x^!]\langle\!\langle [M,N^!]/x\rangle\!\rangle &= M[x^!]\,\{x+N/x\} + x[M,x^!]\,\{x+N/x\}\\ &= M[N^!,x^!] + x[M,N^!,x^!] + N[M,N^!,x^!]. \end{aligned}$$

The definition of the linear substitution on a product of bags is clearly well defined regardless of the decomposition of the bag. On the other hand in order for the bag substitution to be well defined, we need to know that argument substitutions can be freely commuted. Commutation of linear substitutions is obtained from the so-called Schwartz lemma, a name due to linear substitution corresponding to partial derivation[4]. Both of the following lemmas are proved by structural induction (for details we refer to [3]).

Lemma 2 (Schwartz). *For* α *a sum of expressions,* μ,ν *sums of terms and* x, y *variables such that* $y \notin \mathrm{FV}(\mu)$*, we have*

$$\big(\alpha\langle\nu/y\rangle\big)\langle\mu/x\rangle = \big(\alpha\langle\mu/x\rangle\big)\langle\nu/y\rangle + \alpha\langle\nu\langle\mu/x\rangle/y\rangle.$$

In particular if $x \notin \mathrm{FV}(\nu)$ *then the second addend is* 0 *and the two substitutions commute.*

Lemma 3. *For* α,μ,ν,x,y *as in the above lemma, and moreover* $y \notin \mathrm{FV}(\nu)$*, we have*

$$\big(\alpha\,\{y+\nu/y\}\big)\langle\mu/x\rangle = \big(\alpha\langle\mu/x\rangle\big)\,\{y+\nu/y\} + \alpha\langle\nu\langle\mu/x\rangle/y\rangle\,\{y+\nu/y\}\,.$$

In particular if $x \notin \mathrm{FV}(\nu)$ *then the two commute.*

Furthermore we have, if $x \notin \mathrm{FV}(\mu)\cup\mathrm{FV}(\nu)$,

$$(\alpha\,\{x+\mu/x\})\,\{x+\nu/x\} = \alpha\,\{x+\mu+\nu/x\} = (\alpha\,\{x+\nu/x\})\,\{x+\mu/x\}\,.$$

Combined together, all the above implies that bag substitution is well defined, given its condition on the variable. We give another result we will need later.

Lemma 4. *If* $y \notin \mathrm{FV}(\mu)\cup\mathrm{FV}(\pi)$ *and* $x \neq y$*, then*

- $\alpha\langle\!\langle\pi/y\rangle\!\rangle\langle\mu/x\rangle = \alpha\langle\mu/x\rangle\langle\!\langle\pi/y\rangle\!\rangle + \alpha\langle\!\langle\pi\langle\mu/x\rangle/y\rangle\!\rangle$*, and*
- $\alpha\,\{0/y\}\,\langle\mu/x\rangle = \alpha\langle\mu/x\rangle\,\{0/y\}$*.*

[4] Indeed, notice the parallel between $\frac{\partial \mathrm{e}^y}{\partial x} = \frac{\partial y}{\partial x}\mathrm{e}^y$ and $[M^!]\langle N/x\rangle = [M\langle N/x\rangle]\cdot[M^!]$.

$$\frac{M \mathrel{\mathtt{R}} \mu}{\lambda x.M \mathrel{\mathtt{R}} \lambda x.\mu}\ \lambda \qquad \frac{M \mathrel{\mathtt{R}} \mu}{MP \mathrel{\mathtt{R}} \mu P}\ \mathtt{@l} \qquad \frac{P \mathrel{\mathtt{R}} \pi}{MP \mathrel{\mathtt{R}} M\pi}\ \mathtt{@r}$$

$$\frac{M \mathrel{\mathtt{R}} \mu}{[M]\cdot P \mathrel{\mathtt{R}} [\mu]\cdot P}\ \mathtt{bag}\ell \qquad \frac{M \mathrel{\mathtt{R}} \mu}{[M^!]\cdot P \mathrel{\mathtt{R}} [\mu^!]\cdot P}\ \mathtt{bag!} \qquad \frac{A \mathrel{\mathtt{R}} \alpha}{A+\beta \mathrel{\mathtt{R}} \alpha+\beta}\ \mathtt{sum}$$

Fig. 3: Rules defining the passing to the context of a relation R. For linear context, one just drops the bag! rule.

Proof. Sums pose no problems. Let us therefore reason, for the first point, by induction on $\pi = P$. For $P = 1$ it amounts to seeing $\alpha\langle\mu/x\rangle = \alpha\langle\mu/x\rangle + \alpha\langle\langle 0/y\rangle\rangle$. For $P = P'\cdot[L]$ we have by Schwartz lemma and inductive hypothesis:

$$\begin{aligned}
\alpha\langle\langle P'/y\rangle\rangle\langle L/y\rangle\langle\mu/x\rangle &= \alpha\langle\langle P'/y\rangle\rangle\langle\mu/x\rangle\langle L/y\rangle + \alpha\langle\langle P'/y\rangle\rangle\langle L\langle\mu/x\rangle/y\rangle \\
&= \alpha\langle\mu/x\rangle\langle\langle P'\cdot[L]/y\rangle\rangle + \alpha\langle\langle(P'\langle\mu/x\rangle)\cdot[L]/y\rangle\rangle + \alpha\langle\langle P'\cdot([L]\langle\mu/x\rangle)/y\rangle\rangle \\
&= \alpha\langle\mu/x\rangle\langle\langle P/y\rangle\rangle + \alpha\langle\langle\big((P'\langle\mu/x\rangle)\cdot[L] + P'\cdot([L]\langle\mu/x\rangle)\big)/y\rangle\rangle \\
&= \alpha\langle\mu/x\rangle\langle\langle P/y\rangle\rangle + \alpha\langle\langle P\langle\mu/x\rangle/y\rangle\rangle.
\end{aligned}$$

For $P = P'\cdot[L^!]$ we have by Lemma 3 and inductive hypothesis:

$$\begin{aligned}
&\alpha\langle\langle P'/y\rangle\rangle\,\{y+L/y\}\,\langle\mu/x\rangle \\
&\quad= \alpha\langle\langle P'/y\rangle\rangle\langle\mu/x\rangle\langle\langle L^!/y\rangle\rangle + \alpha\langle\langle P'/y\rangle\rangle\langle L\langle\mu/x\rangle/y\rangle\langle\langle L^!/y\rangle\rangle \\
&\quad= \alpha\langle\mu/x\rangle\langle\langle P'\cdot[L^!]/y\rangle\rangle + \alpha\langle\langle P'\langle\mu/x\rangle\cdot[L^!]/y\rangle\rangle + \alpha\langle\langle P'\cdot[L\langle\mu/x\rangle, L^!]/y\rangle\rangle \\
&\quad= \alpha\langle\mu/x\rangle\langle\langle P/y\rangle\rangle + \alpha\langle\langle(P'\cdot[L^!])\langle\mu/x\rangle/y\rangle\rangle = \alpha\langle\mu/x\rangle\langle\langle P/y\rangle\rangle + \alpha\langle\langle P\langle\mu/x\rangle/y\rangle\rangle.
\end{aligned}$$

The second point is a straightforward induction on α. □

2.1 Relations

We will now introduce the relations defining reductions in resource calculus. Such relations will be in general defined by *rules* with premises and a conclusion. Such rules then *generate* the relation R, meaning that R is the least relation satisfying them, or equivalently is defined by *inferences*, i.e. trees made of such rules. A relation T *satisfies* the rules generating a relation R if such rules with T substituted for R are valid: then clearly $\mathtt{R} \subseteq \mathtt{T}$. We will use this to avoid repeating identical steps in proofs by induction on the size of an inference of R. We denote composition of relations by juxtaposition, so that $a \mathrel{\mathtt{RT}} b$ iff $\exists c$ s.t. $a \mathrel{\mathtt{R}} c$ and $c \mathrel{\mathtt{T}} b$.

Definition 5 (Passing to the context). *A binary relation* R *on* $\mathbb{N}\langle\Lambda^{(b)}\rangle$ passes to the context *(resp. to the* linear context*) whenever it satisfies all the rules of Figure 3 (resp. all the rules but the* bag! *rule).*

Definition 6 (Compatibility). *We take a binary relation* R *on* $\mathbb{N}\langle\Lambda^{(b)}\rangle$ *to be* compatible *if it commutes with all constructors of* $\mathbb{N}\langle\Lambda^{(b)}\rangle$*, i.e. it satisfies all the*

$$\frac{}{x \mathrel{\mathtt{R}} x}\,\mathtt{var} \qquad \frac{M \mathrel{\mathtt{R}} \mu \qquad P \mathrel{\mathtt{R}} \pi}{MP \mathrel{\mathtt{R}} \mu\pi}\,\overline{@} \qquad \frac{M \mathrel{\mathtt{R}} \mu}{\lambda x.M \mathrel{\mathtt{R}} \lambda x.\mu}\,\overline{\lambda}$$

$$\frac{}{1 \mathrel{\mathtt{R}} 1}\,\mathtt{bag1} \qquad \frac{M \mathrel{\mathtt{R}} \mu \qquad P \mathrel{\mathtt{R}} \pi}{[M]\cdot P \mathrel{\mathtt{R}} [\mu]\cdot\pi}\,\overline{\mathtt{bag}\ell} \qquad \frac{M \mathrel{\mathtt{R}} \mu \qquad P \mathrel{\mathtt{R}} \pi}{[M^!]\cdot P \mathrel{\mathtt{R}} [\mu^!]\cdot\pi}\,\overline{\mathtt{bag!}}$$

$$\frac{A_i \mathrel{\mathtt{R}} \alpha_i, \qquad \text{for } 1 \leq i \leq k}{\sum_{i=1}^{k} A_i \mathrel{\mathtt{R}} \sum_{i=1}^{k} \alpha_i}\,\overline{\mathtt{sum}}$$

Fig. 4: Rules defining the compatibility for a relation R. In $\overline{\mathtt{sum}}$, $0 \leq k \neq 1$.

rules of Figure 4. We write of linear compatibility *when commutation is with all constructs but the* $(\cdot)^!$ *one: formally,* R *is linearly compatible if it satisfies all rules for compatibility but the* $\overline{\mathtt{bag!}}$ *one, which is replaced by*

$$\frac{P \mathrel{\mathtt{R}} \pi}{[M^!]\cdot P \mathrel{\mathtt{R}} [M^!]\cdot\pi}\,\overline{\mathtt{bag!}=}$$

Lemma 7. *A (linearly) compatible relation* R *is necessarily reflexive and passing to (linear) context.*

Proof. Reflexivity is evident as soon as one sees that equality is precisely the relation generated by the rules for both linear and regular compatibility. One then sees that all rules for passing to (linear) context are admissible under the rules for (linear) compatibility, by using reflexivity. □

We write that a relation is *sum-independent* if $\sum_i A_i \mathrel{\mathtt{R}} \alpha$ implies that $\alpha = \sum_i \beta_i$ with $A_i \mathrel{\mathtt{R}^=} \beta_i$ for all i, where $\mathtt{R}^=$ is the reflexive closure of R. All the relations we study here are sum-independent, a notion capturing the fact that no interaction is possible between different addends of a sum. If the only rules introducing a sum on the left are among the two for passing to context (sum) or compatibility ($\overline{\mathtt{sum}}$), the generated relation is clearly sum-independent.

Further, we speak of a *generalized rule* meaning a rule where all expressions in it are replaced by sums (using the notations of Figure 1(b) in the conclusion). A relation *strongly satisfies* a rule if it satisfies its generalized version.

Lemma 8. *The reflexive transitive closure* $\mathtt{R}^*$ *of a sum-independent relation* R *passing to (linear) context is sum-independent and (linearly) compatible.*

Proof. (sketch) Sum-independence is immediate. Then, by going through all passing to (linear) context rules, one sees that each one is strongly satisfied, which enables to easily check that also compatibility rules are. All single passages are carried out by inductions on the reduction length.

Lemma 9. *A (linearly) compatible sum-independent relation* R *strongly satisfies the rules for (linear) compatibility.*

Proof. (sketch) Straightforward check of all the rules.

Definition 10 (β-Reduction). *The β-reduction $\to$ is given by the rules for passing to the context (Figure 3) plus the following one:*

$$\frac{}{(\lambda x.M)P \to M\langle\!\langle P/x\rangle\!\rangle\,\{0/x\}}\ \mathtt{g}$$

For an example of reduction, see the one given in the introduction. In [4,10] this reduction is called the *giant-step* one (hence the name of the rule) to distinguish it from the baby-step one we will discuss in Section 5.

Definition 11 (Outer, Inner Reduction). *The* outer reduction *is the relation $\xrightarrow{\mathtt{o}}$ generated by the rule* g *of Definition 10 and the rules of passing the linear context (Figure 3 but the* bag! *rule). The* inner reduction *is the relation $\xrightarrow{\mathtt{i}}$ generated by the rules of passing the context and the following rule*

$$\frac{M \xrightarrow{\mathtt{o}} \mu}{[M^!]\cdot P \xrightarrow{\mathtt{i}} [\mu^!]\cdot P}\ \mathtt{in}$$

Informally, outer reduction is the one reducing linear redexes not inside a $(\cdot)^!$, inner is the rest. The rules we have provided for the inner reduction allow for more neat proofs. Notice the difference between $\xrightarrow{\mathtt{o}}$ and the λ-calculus head reduction: we have $\big(\lambda x.(\lambda y.y)[N^!]\big)[L^!] \xrightarrow{\mathtt{o}} (\lambda x.N)[L^!]$, which is false for head reduction. We will see in Example 28 how usual head redexes are not sufficient for reaching head normal forms, and linear arguments are to be taken into account as well. At this point we decided, mainly for the sake of elegance, to extend the notion to all *linear* redexes, even if under the scope of another linear redex.

Fact 12. We have that $\to\, = \xrightarrow{\mathtt{o}} \cup \xrightarrow{\mathtt{i}}$ as is expected: $\xrightarrow{\mathtt{o}} \cup \xrightarrow{\mathtt{i}}$ satisfies the rules of $\to$, and $\to$ those of both $\xrightarrow{\mathtt{o}}$ and $\xrightarrow{\mathtt{i}}$.

Fact 13. Using Lemma 9 one can also easily check that the relations $\xrightarrow{*}$, $\xrightarrow{\mathtt{i}*}$ and $\xrightarrow{\mathtt{o}*}$ are sum-independent and strongly satisfying all of the rules for (linear) compatibility.

3 Confluence

Definition 14 (Parallel reduction). *The* parallel reduction $\Rightarrow$ *(resp. the* parallel outer reduction $\overset{\mathtt{o}}{\Rightarrow}$*) is generated by the compatibility rules (resp. the linear compatibility rules) plus the following one:*

$$\frac{M \Rightarrow \mu \qquad P \Rightarrow \pi}{(\lambda x.M)P \Rightarrow \mu\langle\!\langle \pi/x\rangle\!\rangle\,\{0/x\}}\ \bar{\mathtt{g}}$$

Fact 15. $\Rightarrow$ and $\overset{\mathtt{o}}{\Rightarrow}$ are sum-independent and strongly satisfying all of their rules: by Lemma 9 only the new rule must be checked, which is immediate by multilinearity of the substitution operator.

We will thus be liberal when saying we apply one of the rules for parallel reduction, by allowing them with sums of expressions in the premises.

Lemma 16 (Closures coincide). *We have that* $\to \subseteq \Rightarrow \subseteq \xrightarrow{*}$*. In particular* $\xrightarrow{*} = \overset{*}{\Rightarrow}$ *The same holds for* $\xrightarrow{\circ}$ *and* $\overset{\circ}{\Rightarrow}$*.*

Proof. We show both inclusions by seeing that the right end satisfies the rules of the left one. For the first inclusion, by Lemma 7 just the $\mathtt{g}$ rule needs to be checked. This is straightforward by the $\overline{\mathtt{g}}$ rule and reflexivity of $\Rightarrow$ (Lemma 7). For the second inclusion, by Lemma 8 only the $\overline{\mathtt{g}}$ rule must be checked. Suppose therefore that $M \xrightarrow{*} \mu$ and $P \xrightarrow{*} \pi$. By compatibility of $\xrightarrow{*}$ (Lemma 8) we have $(\lambda x.M)P \xrightarrow{*} (\lambda x.\mu)\pi \xrightarrow{*} \mu\langle\!\langle \pi/x\rangle\!\rangle \{0/x\}$, where the last reduction (given by $\mathtt{g}$) is by compatibility with sum. The distinction between $\to$ and $\xrightarrow{\circ}$ is left to Lemma 8. ☐

Lemma 17 (Substitution for $\Rightarrow$). *For* $\alpha \Rightarrow \beta$ *and* $\pi \Rightarrow \sigma$ *we have* $\alpha\langle\!\langle \pi/x\rangle\!\rangle \Rightarrow \beta\langle\!\langle \sigma/x\rangle\!\rangle$ *and* $\alpha\{0/x\} \Rightarrow \beta\{0/x\}$*. The same holds for* $\overset{\circ}{\Rightarrow}$*.*

Proof. For the first result we reason by a primary induction on the size of π. We proceed by splitting over the last rule used to infer $\pi \Rightarrow \sigma$. The proof for $\overset{\circ}{\Rightarrow}$ proceeds almost identically, and we will highlight only its differences.
Case I ($\mathtt{bag1}, \pi = 1 = \sigma$). As $\alpha\langle\!\langle 1/x\rangle\!\rangle = \alpha \Rightarrow \beta = \beta\langle\!\langle 1/x\rangle\!\rangle$ we are done.
Case II ($\overline{\mathtt{bag!}}$, $\pi = [N^!]\cdot Q$). We have $\sigma = [\nu^!]\cdot\tau$ with $N \Rightarrow \nu$ ($\nu = N$ for $\overset{\circ}{\Rightarrow}$) and $Q \Rightarrow \tau$. Once we show that $\alpha\{x + N/x\} \Rightarrow \beta\{x+\nu/x\} = \beta\langle\!\langle [\nu^!]/x\rangle\!\rangle$ we would be done, as by inductive hypothesis on Q we would get

$$\alpha\langle\!\langle \pi/x\rangle\!\rangle = \alpha\{x + N/x\}\langle\!\langle Q/x\rangle\!\rangle \Rightarrow \beta\langle\!\langle [\nu^!]/x\rangle\!\rangle\langle\!\langle \tau/x\rangle\!\rangle = \beta\langle\!\langle \sigma/x\rangle\!\rangle.$$

We show it by induction on α. All but the base step for α a variable is trivial, as the substitution commutes with all the constructors, and $\Rightarrow$ is strongly compatible with them by Fact 15. For $\overset{\circ}{\Rightarrow}$, in the case $\alpha = [M^!]$, we have $\beta = \alpha$ and there is nothing to prove. If $\alpha = y = \beta$ we have

$$y\{x + N/x\} = y + \delta_{x,y}N \Rightarrow y + \delta_{x,y}\nu = y\langle\!\langle [\nu^!]/x\rangle\!\rangle.$$

Case III ($\overline{\mathtt{bag}\ell}$, $\pi = [N]\cdot Q$). As in the above case, we just need to show that $\alpha\langle N/x\rangle \Rightarrow \beta\langle \nu/x\rangle$ when $N \Rightarrow \nu$, as then the rest follows by inductive hypothesis on Q. Again we reason by a secondary induction on α, splitting on which rule was last used to infer $\alpha \Rightarrow \beta$. Apart the base cases $\mathtt{var}, \mathtt{bag1}$, the other cases uses secondary induction hypothesis and the strong compatibility of $\Rightarrow$ (resp. strong linear compatibility of $\overset{\circ}{\Rightarrow}$).
Subcase III.a ($\mathtt{var}$). We have $\alpha = y = \beta$, and $y\langle N/x\rangle = \delta_{x,y}N \Rightarrow \delta_{x,y}\nu = y\langle\!\langle [\nu]/x\rangle\!\rangle$.
Subcase III.b ($\overline{@}$). We have $\alpha = MR$ with $M \Rightarrow \mu$, $R \Rightarrow \rho$ and $\beta = \mu\rho$. Then

$$\alpha\langle N/x\rangle \;=\; M\langle N/x\rangle R + MR\langle N/x\rangle \;\Rightarrow\; \mu\langle \nu/x\rangle\rho + \mu\rho\langle \nu/x\rangle \;=\; (\mu\rho)\langle \nu/x\rangle.$$

$$
\begin{array}{rlr}
x^* &:= x, & 1^* := 1,\\
(\lambda x.M)^* &:= \lambda x.M^*, & [N]^* := [N^*],\\
(MP)^* &:= M^*P^* \quad \text{if } M \text{ is not an abstraction}, & [N^!]^* := [(N^*)^!],\\
((\lambda x.M)P)^* &:= M^*\langle\!\langle P^*/x\rangle\!\rangle\,\{0/x\}\,, & (P\cdot Q)^* := (P^*\cdot Q^*).
\end{array}
$$

Fig. 5: Inductive definition of developments.

Subcase III.c ($\overline{\mathtt{g}}$). $\alpha = (\lambda y.M)R$, with $M \Rightarrow \mu$, $R \Rightarrow \rho$ and $\beta = \mu\langle\!\langle\rho/y\rangle\!\rangle\,\{0/y\}$, and by inductive hypothesis $M\langle N/x\rangle \Rightarrow \mu\langle\nu/x\rangle$ and $R\langle N/x\rangle \Rightarrow \rho\langle\nu/x\rangle$. Now, supposing $y \neq x$ and $y \notin \mathrm{FV}(N) \supseteq \mathrm{FV}(\nu)$,

$$
\begin{aligned}
\alpha\langle N/x\rangle = (\lambda y.M\langle N/x\rangle)R + (\lambda y.M)(R\langle N/x\rangle)&\\
\Rightarrow \mu\langle\nu/x\rangle\langle\!\langle\rho/y\rangle\!\rangle\,\{0/y\} + \mu\langle\!\langle\rho\langle\nu/x\rangle/y\rangle\!\rangle\,\{0/y\}&\\
= \mu\langle\!\langle\rho/y\rangle\!\rangle\langle\nu/x\rangle\,\{0/y\} = \mu\langle\!\langle\rho/y\rangle\!\rangle\,\{0/y\}\,\langle\nu/x\rangle&,
\end{aligned}
$$

where apart the inductive hypothesis we used Lemma 4.
Subcase III.d (otherwise). The other inductive steps are either trivial or easily carried over by using arguments like the above.
Case IV ($\overline{\mathtt{sum}}$, $\pi = P_1 + \cdots + P_k$). We have $\alpha\langle\!\langle\pi/x\rangle\!\rangle = \sum_i \alpha\langle\!\langle P_i/x\rangle\!\rangle$ and $\sigma = \sum_i \rho_i$ with $P_i \Rightarrow \rho_i$. We can apply inductive hypothesis k times and the $\overline{\mathtt{sum}}$ rule to get

$$\alpha\langle\!\langle\pi/x\rangle\!\rangle = \textstyle\sum_i \alpha\langle\!\langle P_i/x\rangle\!\rangle \Rightarrow \sum_i \beta\langle\!\langle\rho_i/x\rangle\!\rangle = \beta\langle\!\langle\sigma/x\rangle\!\rangle.$$

The result for $\alpha\,\{0/x\}$ is an easy induction on the derivation $\alpha \Rightarrow \beta$. □

Definition 18 (Developments α^* and $\alpha^{\circledast}$). *Given an expression A its* development $A^* \in \mathbb{N}\langle\Lambda^{(b)}\rangle$ *is defined inductively in Figure 5. The definition is extended to sums by linearity. The* linear development $\alpha^{\circledast}$ *is defined by the same inductive rules (just replace $*$ with $\circledast$), but for $[N^!]$ where $[N^!]^{\circledast} := [N^!]$.*

The name is due to the fact that it is a direct definition of the unique normal form one would get in proving the finite development theorem.

Lemma 19 (Main Lemma). *For any β such that $\alpha \Rightarrow \beta$ (resp. $\alpha \overset{\circ}{\Rightarrow} \beta$), we have $\beta \Rightarrow \alpha^*$ (resp. $\beta \overset{\circ}{\Rightarrow} \alpha^{\circledast}$).*

Proof. By induction on α, splitting on the last rule used for $\alpha \Rightarrow \beta$ (resp. $\alpha \overset{\circ}{\Rightarrow} \beta$). Again, we use only $\Rightarrow$, and we mark only where the proof differs for $\overset{\circ}{\Rightarrow}$.
Case I ($\mathtt{var}$, $\alpha = x$). As $\alpha^* = x = \beta$ and we are done.
Case II ($\overline{@}$, $\alpha = NP$). We have $\beta = \nu\pi$ with $N \Rightarrow \nu$ and $P \Rightarrow \pi$. By inductive hypothesis $\nu \Rightarrow N^*$ and $\pi \Rightarrow P^*$. We have two subcases.
Subcase II.a (N not an abstraction). We directly have $\nu\pi \Rightarrow N^*P^* = (NP)^*$ by a generalized $\overline{@}$ rule (Fact 15).
Subcase II.b ($N = \lambda x.L$). We have then that $\nu = \lambda x.\delta$ with $L \Rightarrow \delta \Rightarrow L^*$ by inductive hypothesis. Then by a generalized $\overline{\mathtt{g}}$ rule (Fact 15) we have $(\lambda x.\delta)\pi \Rightarrow$

$L^*\langle\!\langle P^*/x\rangle\!\rangle\{0/x\} = \alpha^*$.
Case III ($\overline{\mathtt{g}}$, $\alpha = (\lambda x.L)P$). Again we have $L \Rightarrow \delta \Rightarrow L^*$ and $P \Rightarrow \pi \Rightarrow P^*$ by inductive hypothesis, where $\beta = \delta\langle\!\langle \pi/x\rangle\!\rangle\{0/x\}$. Then by Lemma 17 we have $\beta = \delta\langle\!\langle\pi/x\rangle\!\rangle\{0/x\} \Rightarrow L^*\langle\!\langle P^*/x\rangle\!\rangle\{0/x\} = \alpha^*$.
Case IV (Otherwise). The cases for $\overline{\mathtt{sum}}$ and $\mathtt{bag1}$ are trivial, while the ones for $\overline{\mathtt{bag}\ell}$ and $\overline{\mathtt{bag!}}$ (resp. $\overline{\mathtt{bag!=}}$ for $\overset{\mathtt{o}}{\Rightarrow}$) are analogous to the non-redex application (Subcase II.a). □

Theorem 20 (Confluence). *Both the β-reduction and the outer reductions are confluent.*

Proof. Lemma 19 gives strong confluence of $\Rightarrow$, which in turn gives strong confluence of $\overset{*}{\Rightarrow} = \overset{*}{\to}$ (Lemma 16), another way to say that $\to$ is confluent. The reasoning for $\overset{\mathtt{o}}{\to}$ is identical. □

We could similarly prove the same for $\overset{\mathtt{i}}{\to}$, though we restrain from doing so just because the proof would not have the same complete similarity as do the two for $\to$ and $\overset{\mathtt{o}}{\to}$.

4 Standardization

Definition 21 (Inner Parallel Reduction). *The* inner parallel reduction *is the relation $\overset{\mathtt{i}}{\Rightarrow}$ generated by the rule*

$$\frac{M \overset{\mathtt{o}*}{\longrightarrow} \nu \qquad \nu \overset{\mathtt{i}}{\Rightarrow} \mu \qquad P \overset{\mathtt{i}}{\Rightarrow} \pi}{[M^!]\cdot P \overset{\mathtt{i}}{\Rightarrow} [\mu^!]\cdot \pi}\ \overline{\mathtt{in}}$$

and those for compatibility (Figure 4) but the $\overline{\mathtt{bag!}}$ rule.

We excluded the $\overline{\mathtt{bag!}}$ as it is derivable from $\overline{\mathtt{in}}$, so that $\overset{\mathtt{i}}{\Rightarrow}$ is compatible anyway. Notice that $\overset{\mathtt{i}}{\Rightarrow} \not\subseteq \Rightarrow$, as the outer reduction in the premise of $\overline{\mathtt{in}}$ can go out of it. In fact it is an inductive definition of a "huge" relation: once the standardization theorem will be proved, but only then, it will turn out that $\overset{\mathtt{i}}{\Rightarrow} = \overset{\mathtt{i}*}{\longrightarrow}$.

Fact 22. Using Lemma 9, one sees that $\overset{\mathtt{i}}{\Rightarrow}$ is sum-independent and strongly satisfying all of its rules.

Lemma 23. *We have that $\overset{\mathtt{i}}{\to} \subseteq \overset{\mathtt{i}}{\Rightarrow} \subseteq \overset{\mathtt{i}*}{\longrightarrow}$. In particular $\overset{\mathtt{i}*}{\longrightarrow} = \overset{\mathtt{i}*}{\Rightarrow}$.*

Proof. By $\overline{\mathtt{in}}$ and the reflexivity of $\overset{\mathtt{i}}{\Rightarrow}$ (Lemma 7), $\overset{\mathtt{i}}{\Rightarrow}$ satisfies the $\mathtt{in}$ rule. Moreover $\overset{\mathtt{i}}{\Rightarrow}$ passes to the context (still Lemma 7), so $\overset{\mathtt{i}}{\Rightarrow}$ satisfies all rules generating $\overset{\mathtt{i}}{\to}$. We conclude $\overset{\mathtt{i}}{\to} \subseteq \overset{\mathtt{i}}{\Rightarrow}$.

Let us prove $\overset{\mathtt{i}}{\Rightarrow} \subseteq \overset{\mathtt{i}*}{\longrightarrow}$ by showing that $\overset{\mathtt{i}*}{\longrightarrow}$ enjoys the rules generating $\overset{\mathtt{i}}{\Rightarrow}$. Lemma 8 proves that $\overset{\mathtt{i}*}{\longrightarrow}$ is compatible. As for $\overline{\mathtt{in}}$, suppose $M \overset{\mathtt{o}*}{\longrightarrow} \nu$, $\nu \overset{\mathtt{i}*}{\longrightarrow} \mu$ and $P \overset{\mathtt{i}*}{\longrightarrow} \pi$, we must prove $[M^!]\cdot P \overset{\mathtt{i}*}{\longrightarrow} [\mu^!]\cdot\pi$. By an easy induction on the length of $M \overset{\mathtt{o}*}{\longrightarrow} \nu$ one has $[M^!] \overset{\mathtt{i}*}{\longrightarrow} [\nu^!]$, and by the compatibility of $\overset{\mathtt{i}*}{\longrightarrow}$ we have $[M^!]\cdot P \overset{\mathtt{i}*}{\longrightarrow} [\nu^!]\cdot P \overset{\mathtt{i}*}{\longrightarrow} [\mu^!]\cdot\pi$. □

Lemma 24 (Substitution for $\xrightarrow{\mathtt{o}*}$). *For $\alpha \xrightarrow{\mathtt{o}*} \beta$ and $\pi \xrightarrow{\mathtt{o}*} \rho$ we have that $\alpha\langle\!\langle\pi/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \beta\langle\!\langle\rho/x\rangle\!\rangle$ and $\alpha\{0/x\} \xrightarrow{\mathtt{o}*} \beta\{0/x\}$.*

Proof. By Lemma 16, we have $\xrightarrow{\mathtt{o}*} = \overset{\mathtt{o}*}{\Rrightarrow}$, so we can reason with $\overset{\mathtt{o}}{\Rrightarrow}$ only. Then a direct iteration of Lemma 17 (together with reflexivity of $\overset{\mathtt{o}}{\Rrightarrow}$) yields $\alpha\langle\!\langle\pi/x\rangle\!\rangle \overset{\mathtt{o}*}{\Rrightarrow} \beta\langle\!\langle\pi/x\rangle\!\rangle \overset{\mathtt{o}*}{\Rrightarrow} \beta\langle\!\langle\sigma/x\rangle\!\rangle$, together with $\alpha\{0/x\} \overset{\mathtt{o}*}{\Rrightarrow} \beta\{0/x\}$. □

Substitution on inner reductions is subtler: in general $\alpha \xrightarrow{\mathtt{i}*} \beta$ and $\pi \xrightarrow{\mathtt{i}*} \rho$ do not entail $\alpha\langle\pi/x\rangle \xrightarrow{\mathtt{i}*} \beta\langle\rho/x\rangle$. For example take $M = y[(\mathbf{I}[x])^!]$ and $N = y[x^!]$, we have $M \xrightarrow{\mathtt{i}*} N$ but $M\langle z/x\rangle \equiv y[\mathbf{I}[z], (\mathbf{I}[x])^!] \not\xrightarrow{\mathtt{i}*} N\langle z/x\rangle \equiv y[z, x^!]$. However what suffices for standardization is the following lemma.

Lemma 25 (Substitution for $\overset{\mathtt{i}}{\Rrightarrow}$). *Suppose $\alpha \overset{\mathtt{i}}{\Rrightarrow} \beta$ and $\pi \overset{\mathtt{i}}{\Rrightarrow} \rho$, then there is $\beta_o \in \mathbb{N}\langle\Lambda^{(b)}\rangle$ such that $\alpha\langle\!\langle\pi/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \beta_o \overset{\mathtt{i}}{\Rrightarrow} \beta\langle\!\langle\rho/x\rangle\!\rangle$. Moreover $\alpha\{0/x\} \overset{\mathtt{i}}{\Rrightarrow} \beta\{0/x\}$.*

Proof. The proof of $\alpha\{0/x\} \overset{\mathtt{i}}{\Rrightarrow} \beta\{0/x\}$ is a straightforward induction on the derivation of $\alpha \overset{\mathtt{i}}{\Rrightarrow} \beta$, using Fact 22 and, for the $\overline{\mathtt{in}}$ rule case, Lemma 24. As for $\alpha\langle\!\langle\pi/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \beta_o \overset{\mathtt{i}}{\Rrightarrow} \beta\langle\!\langle\rho/x\rangle\!\rangle$, we do induction on the derivation of $\pi \overset{\mathtt{i}}{\Rrightarrow} \rho$. As usual, we will use Fact 22 implicitly. We split in cases, depending on the last rule inferring $\pi \overset{\mathtt{i}}{\Rrightarrow} \rho$. The case `bag1` is trivial, and the case $\overline{\mathtt{sum}}$ is an easy consequence of the linearity in π and the induction hypothesis.

Case I ($\overline{\mathtt{in}}$). We have $\pi = [N^!]\cdot Q$, $\rho = [\nu^!]\cdot\tau$ and $N \xrightarrow{\mathtt{o}*} \nu_o \overset{\mathtt{i}}{\Rrightarrow} \nu$, $Q \overset{\mathtt{i}}{\Rrightarrow} \tau$. Once we have proved that $\alpha\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \beta_o \overset{\mathtt{i}}{\Rrightarrow} \beta\langle\!\langle[\nu^!]/x\rangle\!\rangle$, we would be done, as by Lemma 24 and inductive hypothesis on Q we would have

$$\alpha\langle\!\langle N^!/x\rangle\!\rangle\langle\!\langle Q/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \beta_o\langle\!\langle Q/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \beta_{oo} \overset{\mathtt{i}}{\Rrightarrow} \beta\langle\!\langle[\nu^!]/x\rangle\!\rangle\langle\!\langle Q/x\rangle\!\rangle = \beta\langle\!\langle\rho/x\rangle\!\rangle.$$

The proof of $\alpha\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \beta_o \overset{\mathtt{i}}{\Rrightarrow} \beta\langle\!\langle[\nu^!]/x\rangle\!\rangle$ is by induction on the derivation $\alpha \overset{\mathtt{i}}{\Rrightarrow} \beta$.

Subcase I.a (`var`). If $\alpha = \beta = y$, then by compatibility $\alpha\{x + N/x\} = y + \delta_{x,y}N \xrightarrow{\mathtt{o}*} y + \delta_{x,y}\nu_o \overset{\mathtt{i}}{\Rrightarrow} y + \delta_{x,y}\nu = \beta\langle\!\langle[\nu^!]/x\rangle\!\rangle$.

Subcase I.b ($\overline{\mathtt{in}}$). We have $\alpha = [M^!]\cdot R$, $\beta = [\mu^!]\cdot\rho'$ and $M \xrightarrow{\mathtt{o}*} \mu_o \overset{\mathtt{i}}{\Rrightarrow} \mu$, $R \overset{\mathtt{i}}{\Rrightarrow} \rho'$. By Lemma 24 and inductive hypothesis on $\mu_o \overset{\mathtt{i}}{\Rrightarrow} \mu$ and $R \overset{\mathtt{i}}{\Rrightarrow} \rho'$, we have $M\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \mu_o\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \mu_{oo} \overset{\mathtt{i}}{\Rrightarrow} \mu\langle\!\langle[\nu^!]/x\rangle\!\rangle$ and $R\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \rho_o \overset{\mathtt{i}}{\Rrightarrow} \rho'\langle\!\langle[\nu^!]/x\rangle\!\rangle$. By compatibility of $\xrightarrow{\mathtt{o}*}$ and a generalized $\overline{\mathtt{in}}$ rule we have

$$[(M\langle\!\langle N^!/x\rangle\!\rangle)^!]\cdot R\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} [(M\langle\!\langle N^!/x\rangle\!\rangle)^!]\cdot\rho_o \overset{\mathtt{i}}{\Rrightarrow} [(\mu\langle\!\langle[\nu^!]/x\rangle\!\rangle)^!]\cdot\rho'\langle\!\langle[\nu^!]/x\rangle\!\rangle.$$

Subcase I.c ($\overline{@}$). We have $\alpha = MR$, $\beta = \mu\rho'$ and $M \overset{\mathtt{i}}{\Rrightarrow} \mu$, $R \overset{\mathtt{i}}{\Rrightarrow} \rho'$. By induction hypothesis we have $M\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \mu_o \overset{\mathtt{i}}{\Rrightarrow} \mu\langle\!\langle[\nu^!]/x\rangle\!\rangle$ and $R\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \rho'_o \overset{\mathtt{i}}{\Rrightarrow} \rho'\langle\!\langle[\nu^!]/x\rangle\!\rangle$. We conclude by strong compatibility $M\langle\!\langle N^!/x\rangle\!\rangle R\langle\!\langle N^!/x\rangle\!\rangle \xrightarrow{\mathtt{o}*} \mu_o\rho'_o \overset{\mathtt{i}}{\Rrightarrow} \mu\langle\!\langle[\nu^!]/x\rangle\!\rangle\rho'\langle\!\langle[\nu^!]/x\rangle\!\rangle$.

Subcase I.d (Otherwise). The case $\overline{\mathtt{bag}\ell}$ is similar to the previous $\overline{@}$ case; $\mathtt{bag1}$ is trivial and $\overline{\lambda}$, $\overline{\mathtt{sum}}$ are easy consequences of the induction hypothesis.

Case II ($\overline{\mathtt{bag}\ell}$). We have $\pi = [N]\cdot Q$, $\rho = \nu\cdot\tau$, with $N \overset{\mathtt{i}}{\Rightarrow} \nu$ and $Q \overset{\mathtt{i}}{\Rightarrow} \tau$. As in the previous case, once we prove that $\alpha\langle N/x\rangle \overset{\mathtt{o}*}{\longrightarrow} \beta_o \overset{\mathtt{i}}{\Rightarrow} \beta\langle\langle[\nu]/x\rangle\rangle$ we would have concluded, as by Lemma 24 and inductive hypothesis on $Q \overset{\mathtt{i}}{\Rightarrow} \tau$, we have $\alpha\langle N/x\rangle\langle\langle Q/x\rangle\rangle \overset{\mathtt{o}*}{\longrightarrow} \beta_o\langle\langle Q/x\rangle\rangle \overset{\mathtt{o}*}{\longrightarrow} \beta_{oo} \overset{\mathtt{i}}{\Rightarrow} \beta\langle\langle[\nu]/x\rangle\rangle\langle\langle\tau/x\rangle\rangle = \beta\langle\langle\rho/x\rangle\rangle$. We do induction on the derivation of $\alpha \overset{\mathtt{i}}{\Rightarrow} \beta$.

Subcase II.a ($\mathtt{var}$). If $\alpha = \beta = y$, then by compatibility $\alpha\langle N/x\rangle = \delta_{x,y}N \overset{\mathtt{i}}{\Rightarrow} \delta_{x,y}\nu = \beta\langle\nu/x\rangle$.

Subcase II.b ($\overline{\mathtt{in}}$). If $\alpha = [M^!]\cdot Q$, $\beta = [\mu^!]\cdot\tau$ and $M \overset{\mathtt{o}*}{\longrightarrow} \mu_o$, $\mu_o \overset{\mathtt{i}}{\Rightarrow} \mu$, $Q \overset{\mathtt{i}}{\Rightarrow} \tau$, then Lemma 24 gives $M\langle N/x\rangle \overset{\mathtt{o}*}{\longrightarrow} \mu_o\langle N/x\rangle$, and induction hypothesis yields $\mu_o\langle N/x\rangle \overset{\mathtt{o}*}{\longrightarrow} \mu_{oo} \overset{\mathtt{i}}{\Rightarrow} \mu\langle\nu/x\rangle$ and $Q\langle N/x\rangle \overset{\mathtt{o}*}{\longrightarrow} \tau_o \overset{\mathtt{i}}{\Rightarrow} \tau\langle\nu/x\rangle$. By strong compatibility and $\overline{\mathtt{in}}$, we have

$$\begin{aligned}\alpha\langle N/x\rangle = [M\langle N/x\rangle, M^!]\cdot Q + [M^!]\cdot Q\langle N/x\rangle &\overset{\mathtt{o}*}{\longrightarrow} [\mu_{oo}, M^!]\cdot Q + [M^!]\cdot\tau_o \\ &\overset{\mathtt{i}}{\Rightarrow} [\mu\langle\nu/x\rangle, \mu^!]\cdot\tau + [\mu^!]\cdot\tau\langle\nu/x\rangle = \beta\langle\nu/x\rangle.\end{aligned}$$

Subcase II.c ($\overline{\mathtt{bag}\ell}$). If $\alpha = [M]\cdot Q$, $\beta = [\mu]\cdot\tau$ and $M \overset{\mathtt{i}}{\Rightarrow} \mu$, $Q \overset{\mathtt{i}}{\Rightarrow} \tau$, then induction hypothesis yields $M\langle N/x\rangle \overset{\mathtt{o}*}{\longrightarrow} \mu_o \overset{\mathtt{i}}{\Rightarrow} \mu\langle\nu/x\rangle$ and $Q\langle N/x\rangle \overset{\mathtt{o}*}{\longrightarrow} \tau_o \overset{\mathtt{i}}{\Rightarrow} \tau\langle\nu/x\rangle$. Strong compatibility of $\overset{\mathtt{o}*}{\longrightarrow}$ yields $[M\langle N/x\rangle]\cdot Q \overset{\mathtt{o}*}{\longrightarrow} [\mu_o]\cdot Q$ and $[M]\cdot[Q\langle N/x\rangle] \overset{\mathtt{o}*}{\longrightarrow} [M]\cdot\tau_o$. Strong compatibility of $\overset{\mathtt{i}}{\Rightarrow}$ gives $[\mu_o]\cdot Q \overset{\mathtt{i}}{\Rightarrow} [\mu\langle\nu/x\rangle]\cdot\tau$ and $[M]\cdot\tau_o \overset{\mathtt{i}}{\Rightarrow} [\mu]\cdot\tau\langle\nu/x\rangle$. Finally we conclude by the $\overline{\mathtt{sum}}$ rule:

$$\begin{aligned}\alpha\langle N/x\rangle = [M\langle N/x\rangle]\cdot Q + [M]\cdot Q\langle N/x\rangle &\overset{\mathtt{o}*}{\longrightarrow} [\mu_o]\cdot Q + [M]\cdot\tau_o \\ &\overset{\mathtt{i}}{\Rightarrow} [\mu\langle\nu/x\rangle, \mu]\cdot\tau + [\mu]\cdot\tau\langle\nu/x\rangle = \beta\langle\nu/x\rangle.\end{aligned}$$

Subcase II.d (Otherwise). The rule $\overline{@}$ is handled similarly to the case $\overline{\mathtt{bag}\ell}$; the cases $\overline{\lambda}$ and $\overline{\mathtt{sum}}$ are easy consequences of the induction hypothesis; the rule $\mathtt{bag1}$ is immediate. □

Lemma 26 (Postponement). *We have* $\overset{\mathtt{i}}{\Rightarrow}\overset{\mathtt{o}}{\to} \subseteq \overset{\mathtt{o}*}{\longrightarrow}\overset{\mathtt{i}}{\Rightarrow}$.

Proof. Let $\alpha \overset{\mathtt{i}}{\Rightarrow} \alpha' \overset{\mathtt{o}}{\to} \beta$, we prove there is $\gamma \in \mathbb{N}\langle\Lambda^{(b)}\rangle$ such that $\alpha \overset{\mathtt{o}*}{\longrightarrow} \gamma \overset{\mathtt{i}}{\Rightarrow} \beta$. Suppose that $\alpha' \overset{\mathtt{o}}{\to} \beta$ is inferred by rule $\mathtt{g}$, all other cases are easy variants and omitted. So, let $\alpha' = (\lambda x.M')P'$ and $\beta = M'\langle\langle P'/x\rangle\rangle\,\{0/x\}$. Under these hypothesis $\alpha \overset{\mathtt{i}}{\Rightarrow} \alpha'$ can be obtained only by means of a $\overline{@}$ rule, therefore $\alpha = (\lambda x.M)P$ with $M \overset{\mathtt{i}}{\Rightarrow} M'$ and $P \overset{\mathtt{i}}{\Rightarrow} P'$. By Lemma 25 there is a sum $\nu \in \mathbb{N}\langle\Lambda\rangle$ such that $M\langle\langle P/x\rangle\rangle\,\{0/x\} \overset{\mathtt{o}*}{\longrightarrow} \nu \overset{\mathtt{i}}{\Rightarrow} M'\langle\langle P'/x\rangle\rangle\,\{0/x\}$. Hence $\alpha \overset{\mathtt{o}}{\to} M\langle\langle P/x\rangle\rangle\,\{0/x\} \overset{\mathtt{o}*}{\longrightarrow} \nu \overset{\mathtt{i}}{\Rightarrow} M'\langle\langle P'/x\rangle\rangle\,\{0/x\} = \beta$. □

Theorem 27 (Standardization). *We have* $\overset{*}{\to} = \overset{\mathtt{o}*}{\longrightarrow}\overset{\mathtt{i}*}{\longrightarrow}$.

Proof. Fact 12 gives $\xrightarrow{*} = (\xrightarrow{\mathtt{o}} \cup \xrightarrow{\mathtt{i}})^*$, and Lemma 23 $\xrightarrow{\mathtt{o}} \cup \xrightarrow{\mathtt{i}} \subseteq \xrightarrow{\mathtt{o}} \cup \xRightarrow{\mathtt{i}}$. Then $\alpha \xrightarrow{*} \beta$ entails $\alpha \xrightarrow{\mathtt{o}*}\xRightarrow{\mathtt{i}} \cdots \xrightarrow{\mathtt{o}*}\xRightarrow{\mathtt{i}} \beta$. By iterating Lemma 26 we have $\alpha \xrightarrow{\mathtt{o}*}\xRightarrow{\mathtt{i}*} \beta$. So Lemma 23 allows us to conclude $\alpha \xrightarrow{\mathtt{o}*}\xrightarrow{\mathtt{i}*} \beta$. □

In λ-calculus we have a notion of *strong* standardization stating that there is a deterministic *history-free* strategy leading to a normal form (resp. head normal form), e.g. left reduction. By history-free, we mean that the redex is chosen by just looking at the term, regardless of the previous steps. In contrast, we argue that resource calculus has no history-free effective strategy assuring a normal form (resp. a head normal form) whenever it exists.

Example 28. Let us consider $\mathbf{I}[\mathbf{I}^!, \big(x[\mathbf{\Omega}, \mathbf{I}1]\big)^!]$, where $\mathbf{\Omega} = (\lambda x.x[x^!])\lambda x.x[x^!]$ is the typical diverging term. We have $\mathbf{I}[\mathbf{I}^!, (x[\mathbf{\Omega}, \mathbf{I}1])^!] \xrightarrow{\mathtt{o}} \mathbf{I} + x[\mathbf{\Omega}, \mathbf{I}1] \xrightarrow{\mathtt{o}} \mathbf{I}$. In the second term, we have two choices among the two linear arguments of the bag. Choosing the first loops, while the second normalizes. However in general making the right decision should be akin to solving the halting problem.

What could probably be done, though it is outside the scope of this work, is devising a kind of *fair* strategy, in the sense of concurrent programming. By craftily marking the redexes, one could probably make sure that, though sequentially, all parallel subterms get a chance to be reduced, so that if there is a reduction to 0 it would be found.

4.1 An application.

In a forthcoming paper the first author and Ronchi della Rocca characterize different notions of resource calculus solvability by means of the following definition of *may-head* and *must-head normalizability*:

Definition 29 (Head Normal Form). *We define simultaneously the class of terms and that of bags in* head normal form*,* hnf *for short:*

- $\lambda x.M$ *is a* hnf *iff* M *is a* hnf*;*
- $yP_1 \dots P_n$ *is a* hnf *iff each* P_i *is a* hnf*;*
- $P = [M_1^{(!)}, \dots, M_m^{(!)}]$ *is a* hnf *iff for each* i*,* $M_i^{(!)} = M_i$ *entails* M_i *is a* hnf*.*

In case of a sum $\sum_{i=1}^m A_i$ *of expressions, we have two different notions of head normal form:*

- $\sum_{i=1}^m A_i$ *is a* may-head normal form*,* mhnf *for short, iff there is a* $i \leq m$ *such that* A_i *is a head normal form;*
- $\sum_{i=1}^m A_i$ *is a* must-head normal form*,* Mhnf *for short, iff* $m \neq 0$ *and for every* $i \leq m$*,* A_i *is a head normal form.*

An expression A *is* may-head normalizable *(resp.* must-head normalizable*) if it is reducible to a* mhnf *(resp.* Mhnf*).*

Corollary 30 (Head Normalization). *Whenever* α *is may-head (resp. must-head) normalizable, there is a* mhnf *(resp.* Mhnf*)* β *such that* $\alpha \xrightarrow{\mathtt{o}*} \beta$*.*

Proof. Immediate from Theorem 27 and the fact that whenever $\beta' \xrightarrow{\mathtt{i}*} \beta$ we have β' mhnf (resp. Mhnf) iff β mhnf (resp. Mhnf). □

5 Baby-Step Reduction

This section is devoted to presenting another, more atomic reduction of resource terms.

Definition 31. *The* baby-step reduction $\xrightarrow{\mathtt{b}}$ *(resp.* outer baby-step reduction $\xrightarrow{\mathtt{ob}}$*) is the relation generated by the rules for passing to context (resp. to linear context) of Figure 3 plus the following two:*

$$\frac{}{(\lambda x.M)[N^{(!)}]\cdot P \xrightarrow{\mathtt{b}} (\lambda x.M\langle\!\langle N^{(!)}/x\rangle\!\rangle)P}\ \mathtt{b} \qquad \frac{}{(\lambda x.M)1 \xrightarrow{\mathtt{b}} M\,\{0/x\}}\ \mathtt{b1}$$

The inner baby-step reduction $\xrightarrow{\mathtt{ib}}$ *is* $\xrightarrow{\mathtt{b}} \setminus \xrightarrow{\mathtt{ob}}$*.*

Lemma 32. *We have* $(\lambda x.\mu)P \xrightarrow{\mathtt{b}*} (\lambda x.\mu\langle\!\langle P/x\rangle\!\rangle)1$*. In particular* $\rightarrow \subseteq \xrightarrow{\mathtt{b}*}$*. The same holds for the outer and inner versions.*

Proof. We can proceed by induction on P: if $P = 1$ then the reduction chain is empty and we are done. If $P = [L^{(!)}]\cdot Q$ then (supposing $\mu = \sum_i M_i$)

$$\begin{aligned}(\lambda x.\mu)P = \textstyle\sum_i(\lambda x.M_i)[L^{(!)}]\cdot Q &\xrightarrow{\mathtt{b}*} \textstyle\sum_i(\lambda x.M_i\langle\!\langle L^{(!)}/x\rangle\!\rangle)Q \\ &\xrightarrow{\mathtt{b}*} \textstyle\sum_i(\lambda x.M_i\langle\!\langle L^{(!)}/x\rangle\!\rangle\langle\!\langle Q/x\rangle\!\rangle)1 = (\lambda x.\mu\langle\!\langle P/x\rangle\!\rangle)1,\end{aligned}$$

where in the first reduction we used compatibility with sum, while in the second both it and inductive hypothesis. Applying final baby steps to all addends in $(\lambda x.M)P \xrightarrow{*} (\lambda x.M\langle\!\langle P/x\rangle\!\rangle)1$ and closing by (linear) context gives the result on $\rightarrow$ (resp. $\xrightarrow{\mathtt{o}}$). The result for $\xrightarrow{\mathtt{ib}}$ and $\xrightarrow{\mathtt{i}}$ follows. □

Lemma 33. *We have that* $\xrightarrow{\mathtt{b}} \subseteq \equiv_\beta$*, where the* β*-equivalence* $\equiv_\beta$ *is as usual* $(\leftarrow \cup \rightarrow)^*$*. The same holds for the outer versions of the two reductions.*

Proof. We check that $\rightarrow\xleftarrow{*}$ satisfies the rules of $\xrightarrow{\mathtt{b}}$, concluding $\xrightarrow{\mathtt{b}} \subseteq \rightarrow\xleftarrow{*} \subseteq \equiv_\beta$. Notice first of all that $\xleftarrow{*}$ is compatible as $\xrightarrow{*}$ is (Lemma 8). Then $\rightarrow\xleftarrow{*}$ passes to context, and we need to check just the two new rules. Of the two `b1` is a special case of `g`, while for the `b` rule we have

$$(\lambda x.M)[L^{(!)}]\cdot P \rightarrow M\langle\!\langle [L^{(!)}]\cdot P/x\rangle\!\rangle\,\{0/x\} \xleftarrow{*} (\lambda x.M\langle\!\langle [L^{(!)}]/x\rangle\!\rangle)P,$$

where there is the need of the transitive reflexive closure as a sum may have arisen. Notice that Lemma 8 assures that the result is valid for $\xrightarrow{\mathtt{ob}}$ also. □

In particular by the above lemmas we have that the equational theory of the two reductions is the same.

Theorem 34 (Confluence of baby-steps). *The baby-step reduction and the outer baby-step reduction is confluent.*

Proof. By Lemma 33, confluence of $\to$ (which as known entails $\equiv_\beta = \xrightarrow{*}\xleftarrow{*}$) and Lemma 32, we have $\xleftarrow{\mathtt{b}*}\xrightarrow{\mathtt{b}*} \subseteq \equiv_\beta = \xrightarrow{*}\xleftarrow{*} \subseteq \xrightarrow{\mathtt{b}*}\xleftarrow{\mathtt{b}*}$. The same passages on $\xrightarrow{\mathtt{ob}}$ give the result for the outer reduction. □

The next question is whether the standardization result as presented by Theorem 27 is valid also for $\xrightarrow{\mathtt{b}}$, but the answer is negative.

Example 35. Take $M = (\lambda d.I)[(I[x^!, y^!])^!]$. It is easily shown that $I[x^!, y^!] \xrightarrow{\mathtt{ob}*} x + y$, which in turn gives rise to the following chain:

$$M \xrightarrow{\mathtt{i}*} (\lambda d.I)[(x+y)^!] = (\lambda d.I)[x^!, y^!] \xrightarrow{\mathtt{ob}} (\lambda d.I)[y^!].$$

However the only outer redex in M gives rise in a single step to $M \xrightarrow{\mathtt{ob}} (\lambda d.I)1$, so that y is lost as soon as we make an outer reduction, which makes $(\lambda d.I)[y^!]$ unreachable.

The catch is that the baby-step reduction is somewhat too atomic: as inner sums may split the elements of a bag, an inner reduction may change outer redexes in way to which only the baby reduction is sensitive to. However, as normal forms of the two reduction coincide, we can still get a weaker result.

Theorem 36. *Whenever* $\alpha \xrightarrow{\mathtt{b}*} \beta$ *and* β *is normal (resp. a* mhnf *or a* Mhnf*), then* $\alpha \xrightarrow{\mathtt{ob}*}\xrightarrow{\mathtt{ib}*} \beta$.

Proof. By Lemma 33 and confluence (thus uniqueness of normal form) we get $\alpha \xrightarrow{*} \beta$. By Theorem 27 $\alpha \xrightarrow{\mathtt{o}*}\xrightarrow{\mathtt{i}*} \beta$, which by Lemma 32 entails the result.

The part about head normalization is a direct consequence of Corollary 30 and the above fact. □

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