Fixed points of functors

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Abstract

We summarize general categorical results underlying the existence of inductively defined datatypes in denotational semantics.

1 From datatypes to initial algebras

1.1 Categories of diagrams

Familiar datatypes first appear as sets of expressions inductively defined by grammars, as for example :

- Integers, by `N` $=$ `0` $\mid$ `sN`;
- Lists of elements of type `A`, by `L` $=$ `nil` $\mid$ `cAL`;
- Planar (binary, unlabeled) rooted trees, by $T =$ `*` $\mid$ `wTT`.

In any categorical interpretation, types become objects and various constructors become arrows. Suppose in addition that our category has finite products and coproducts—hence also a terminal object 1 and an initial object 0— then the term constructors in the above examples lead to diagrams:

$$1 \xrightarrow{0} N \xrightarrow{s} N$$

$$1 \xrightarrow{nil} L \xleftarrow{c} A \times L$$

$$1 \xrightarrow{*} T \xleftarrow{w} T \times T$$

But we still need to express that in each case we define the *smallest* set containing some given terms and closed by the constructors: in categorical terms, this means that the above diagrams are *initial* among diagrams of the same shape.

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Let us look in detail at the case of integers: given a category $\mathbf{C}$, we may build a new category $\mathbf{C}^D$ having as objects diagrams in $\mathbf{C}$ of the shape:

$$1 \xrightarrow{u} X \xrightarrow{v} X \tag{1}$$

and as morphisms between to such diagrams arrows g of $\mathbf{C}$ making the following diagram commutative:

$$\begin{tikzcd} 1 \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "g"'] \\ 1 \arrow[r, "u'"'] & X' \arrow[r, "v'"'] & X' \end{tikzcd}$$

Now an *object of integers* is simply a diagram

$$1 \xrightarrow{0} N \xrightarrow{s} N$$

which is initial in $\mathbf{C}^D$, in other words, for each diagram of the shape (1), there is a *unique* arrow $f : N \to X$ such that the following diagram commutes

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "\mathrm{id}"'] & N \arrow[r, "s"] \arrow[d, "f"'] & N \arrow[d, "f"'] \\ 1 \arrow[r, "u"'] & X \arrow[r, "v"'] & X \end{tikzcd}$$

1.2 Initial algebras

The previous example may be seen from a slightly different point of view: an object of $\mathbf{C}^D$ amounts to a pair of arrows $u : 1 \to X$, $v : X \to X$ or equivalently to a single arrow $[u, v] : 1 + X \to X$. Let F be the functor $X \mapsto 1 + X$, we may define yet another category $\mathbf{C}^F$ having as objects all arrows of the form $x : FX \to X$ and as morphisms from x to $x' : FX' \to X'$ arrows $g : x \to x'$ such that the following diagram commutes:

$$\begin{tikzcd} FX \arrow[r, "Fg"] \arrow[d, "x"'] & FX' \arrow[d, "x'"] \\ X \arrow[r, "g"'] & X' \end{tikzcd}$$

We easily remark that $\mathbf{C}^F$ and $\mathbf{C}^D$ are equivalent categories, thus an initial object in $\mathbf{C}^D$ amounts to an initial object in $\mathbf{C}^F$.

Definition 1 *An* initial algebra *of F is an initial object in $\mathbf{C}^F$.*

In the case of lists of elements of type A, the functor would be

$$X \mapsto 1 + A \times X$$

and in the case of trees

$$X \mapsto 1 + X \times X$$

The general case leads to *polynomial functors*:

$$F : X \mapsto A_0 + A_1 \times X + \cdots + A_n \times X^n$$

Before investigating existence conditions for initial algebras, we recall a simple but important fact:

Lemma 1 *If $a : FA \to A$ is an initial algebra for F, then a is an isomorphism.*

Proof. Because a is initial, there is a unique $b : A \to FA$ such that the following diagram commutes:

$$\begin{tikzcd}
FA \arrow[r, "Fb"] \arrow[d, "a"'] & FFA \arrow[d, "Fa"] \\
A \arrow[r, "b"'] & FA
\end{tikzcd}$$

If we define $v = a \circ b$, then $Fv = F(a \circ b) = Fa \circ Fb = b \circ a$, and the above diagram may be completed as follows:

$$\begin{tikzcd}
FA \arrow[r, "Fb"] \arrow[d, "a"'] \arrow[dr, "Fv"] & FFA \arrow[d, "Fa"] \\
A \arrow[r, "b"'] \arrow[dr, "v"'] & FA \arrow[d, "a"] \\
 & A
\end{tikzcd}$$

which is still commutative. Now $v : A \to A$ satisfies $v \circ a = a \circ Fv$ hence it is the identity on A, because a is initial. Thus a, b are mutual inverses and a is an isomorphism. $\square$

Because of lemma 1, we may consider A as a *fixed point* of F.

2 Basic lemmas

2.1 Diagrams, cones and (co)limits

Let us first give a very brief review of limits and colimits in categories, together with a few notational conventions we shall adopt.

Let $\mathbf{I}$ be a category, a *diagram of type* $\mathbf{I}$ in $\mathbf{C}$ will be simply a functor

$$\Delta : \mathbf{I} \to \mathbf{C}$$

Often, the so-called *index*-category $\mathbf{I}$ will not be given as a category, but rather as a graph, in which case it is the free category generated by this graph that we have in mind. For example, a diagram of type

$$\bullet \qquad \bullet \tag{2}$$

is nothing but a pair $\langle X, Y\rangle$ of objects of $\mathbf{C}$. Diagrams of fixed type $\mathbf{I}$ are objects of a category $\mathbf{C}^I$ whose morphisms are the *natural transformations* between functors. Now each object X of $\mathbf{C}$ gives rise to a constant diagram $\mathbf{I} \to \mathbf{C}$, which sends all objects of $\mathbf{I}$ to X and all arrows to id_X. Whenever there is no ambiguity on $\mathbf{I}$, we still denote this diagram by X. Moreover any morphism $f : X \to X'$ in $\mathbf{C}$ induces an obvious natural transformation between the corresponding constant diagrams. Notice that, conversely, a natural transformation from X to X' seen as constant diagrams necessarily comes from an arrow $f : X \to X'$ of $\mathbf{C}$.

Definition 2 *Let Δ be a diagram of type* $\mathbf{I}$*, a* projective cone *to the base Δ is a natural transformation $X \to \Delta$, where X is an object of* $\mathbf{C}$*. Likewise, an* inductive cone *on the base Δ is a natural transformation $\Delta \to X$.*

Let us emphasize the point that cones are special cases of morphisms in $\mathbf{C}^I$. Now projective (resp. inductive) cones to (resp. from) a given base Δ are also objects of a new category $\mathbf{Cone}_\Delta$ (resp. $\mathbf{Cone}^\Delta$). In the projective case a morphism from $\xi : X \to \Delta$ to $\xi' : X' \to \Delta$ is an arrow $f : X \to X'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\xi"'] & X' \arrow[d, "\xi'"] \\ & \Delta \end{tikzcd}$$

As for inductive cones, just reverse the arrows.

Definition 3 *A* limiting cone *to the base Δ is a terminal object in* $\mathbf{Cone}_\Delta$*. Likewise, a* colimiting cone *from the base Δ is an initial object in* $\mathbf{Cone}^\Delta$*.*

If $\gamma : L \to \Delta$ is a limiting cone, we refer to L as "the" *limit* of the diagram Δ: it is in fact unique, up to isomorphism. Likewise, we will speak of the *colimit* of a diagram. Let us insist on the fact that sentences like

> " Let L be the limit of a diagram Δ"

always suppose, implicitely, that a certain cone is given from L to Δ.

Let us take for example a diagram Δ of type (2), that is a pair $\langle A, B\rangle$ of objects of $\mathbf{C}$. A projective cone to the base Δ amounts to an object X of $\mathbf{C}$ together with a pair $\langle f, g\rangle$ of arrows $f : X \to A$, $g : X \to B$. A cone (P, p, q) is now limiting if, for each (X, f, g), there is a unique $u : X \to P$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "u"] \arrow[dr, "g"] & \\ A & P \arrow[l, "p"] \arrow[r, "q"'] & B \end{tikzcd}$$

Of course this is the definition of a cartesian product $A \times B$ in $\mathbf{C}$. Likewise, a colimiting cone from the same diagrams amounts to a coproduct (or sum), $A + B$ in $\mathbf{C}$.

2.2 Countable chains

Another important example is given by taking the ordered set of integers $\langle \omega, \leq \rangle$ as the index category. Precisely, the objects of ω are the integers $\{0, 1, \ldots\}$ and there is exactly one morphism from m to n whenever $m \leq n$, and no morphism at all otherwise. Equivalently, ω is the free category on the following graph:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

Hence a diagram of type ω in $\mathbf{C}$ amounts to a sequence of objects and arrows in $\mathbf{C}$ of the shape:

$$X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} \cdots$$

We call such a diagram a *countable chain* in $\mathbf{C}$.

Definition 4 *An* ω-category *is a category where all countable chains have colimits.*

Let us finally define a useful class of functors:

Definition 5 *A functor* $\mathbf{A} \to \mathbf{B}$ *is* ω-continuous *if it transforms any colimiting cone of a countable chain in* $\mathbf{A}$ *into a colimiting cone in* $\mathbf{B}$*.*

Here should be noticed that such functors are *not* required to preserve arbitrary finite colimits, not even sums.

2.3 Variable indices

If we allow variable index categories, we get yet another category having as objects diagrams $\Delta : I \to \mathbf{C}$, and as arrows from $\Delta : I \to \mathbf{C}$ to $\Delta' : I' \to \mathbf{C}$ all functors $f : I \to I'$ such that the following diagram commutes:

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[dr, "\Delta"'] & I' \arrow[d, "\Delta'"] \\ & \mathbf{C} \end{tikzcd} \tag{3}$$

This situation will be denoted by $f^* : \Delta \to \Delta'$. We may refer to such an arrow as a *change of base*.

2.4 Remark on diagrams

In many situations, three different kinds of arrows occur simultaneously: arrows in **C**—viewed as natural transformations between constant functors—, cones and arrows between diagrams as defined in section 2.3.

It will be convenient to see these arrows as particular cases of arrows in a bigger, *single category* $\widehat{\mathbf{C}}$, depending on **C** and on a fixed class of index categories.

- objects are all diagrams $\Delta : I \to \mathbf{C}$ (with variable I);
- morphisms from $\Delta : I \to \mathbf{C}$ to $\Delta' : I' \to \mathbf{C}$ are *pairs* $\langle f, \tau \rangle$ where f is a functor $I \to I'$ and τ is a natural transformation from Δ to $\Delta' \circ f$;
- arrows $\langle f, \tau \rangle : \Delta \to \Delta'$ and $\langle f', \tau' \rangle : \Delta' \to \Delta''$ compose according to

$$\langle f', \tau' \rangle \circ \langle f, \tau \rangle = \langle f' \circ f, (\tau' f) \circ \tau \rangle$$

The case of natural transformations on a fixed I is obtained by taking $f = \mathrm{id}$. Changes of base are obtained by taking $\tau = \mathrm{id}$.

2.5 Main construction

If F is an endofunctor of C, and 0 is initial in C, there is a unique arrow $i : 0 \to F0$, and we may define a countable chain (see section 2.2) by

$$0 \xrightarrow{i} F0 \xrightarrow{Fi} FF0 \xrightarrow{FFi} \cdots \tag{4}$$

The main idea is to build an initial algebra for F by taking the colimit of (4), if it exists.

Theorem 1 *Let* **C** *be an ω-category with an initial object, and F an ω-continuous endofunctor of* **C**. *Then F has an initial algebra.*

Proof. Because **C** has an initial object 0, (4) defines a diagram $\Delta : \omega \to \mathbf{C}$. It has a colimiting cone $\delta : \Delta \to A$ by hypothesis on **C**. Now F is ω-continuous, hence $F\delta : F\Delta \to FA$ is also colimiting.

But $F\Delta$ is also $\Delta \circ m$ where m is the endofunctor on ω induced by $n \mapsto n+1$. Hence we get a change of base

$$m^* : F\Delta \to \Delta$$

But $\kappa = \delta m$ is then a natural transformation from $\Delta \circ m$ to $A \circ m$; because A is constant, $A \circ m = A$, hence an inductive cone:

$$\kappa : F\Delta \to A$$

Now $F\delta$ is colimiting, and there is a unique arrow $a : FA \to A$ such that

$$a \circ F\delta = \kappa \tag{5}$$

It remains to check that a is an initial algebra for F. Let X be an object of $\mathbf{C}$ and $x : FX \to X$ an arrow. There is a unique arrow $j : 0 \to X$, whence a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r, "i"] \arrow[d, "j"'] & F0 \arrow[d, "Fj"] \\
X & FX \arrow[l, "x"]
\end{tikzcd}$$

By successively applying the functor F we get a sequence of commutative diagrams

$$\begin{tikzcd}
F^k 0 \arrow[r, "F^k i"] \arrow[d, "F^k j"'] & F^{k+1} 0 \arrow[d, "F^{k+1} j"] \\
F^k X & F^{k+1} X \arrow[l, "F^k x"]
\end{tikzcd}$$

If we put all the previous diagrams side by side, like this:

$$\begin{tikzcd}
0 \arrow[r, "i"] \arrow[d, "j"'] & F0 \arrow[r, "Fi"] \arrow[d, "Fj"'] & FF0 \arrow[r, "FFi"] \arrow[d, "FFj"'] & \cdots \\
X & FX \arrow[l, "x"] & FFX \arrow[l, "Fx"] & \cdots \arrow[l, "FFx"]
\end{tikzcd} \tag{6}$$

we get, for each $k \geq 1$, an arrow $\xi_k = x \circ \cdots \circ F^{k-1}x \circ F^k j$ from $F^k 0$ to X, plus $\xi_k = j$ for $k = 0$. The commutativity of (6) ensures that the ξ_k's define an inductive cone $\xi : \Delta \to X$. δ being colimiting, there is a unique $f : A \to X$ such that

$$f \circ \delta = \xi \tag{7}$$

The whole construction may be summarized in the following diagram which we draw *in* $\widehat{\mathbf{C}}$

$$\begin{tikzcd}
F\Delta \arrow[dr, "{\langle \mathrm{id}, F\delta \rangle}"] \arrow[drr, "{\langle \mathrm{id}, F\xi \rangle}"] \arrow[dd, "{\langle m, \mathrm{id} \rangle}"'] & & \\
& FA \arrow[r, "{\langle \mathrm{id}, Ff \rangle}"] \arrow[dd, "{\langle m, a \rangle}"] & FX \arrow[dd, "{\langle m, x \rangle}"] \\
\Delta \arrow[dr, "{\langle \mathrm{id}, \delta \rangle}"'] \arrow[drr, "{\langle \mathrm{id}, \xi \rangle}"] & & \\
& A \arrow[r, "{\langle \mathrm{id}, f \rangle}"'] & X
\end{tikzcd} \tag{8}$$

First of all, the lower and upper horizontal triangles in (8) commute: because the first components are identities, this boils down to (7) and its immediate consequence

$$Ff \circ F\delta = F\xi$$

Also, by (5), the left vertical square $(F\Delta, FA, A, \Delta)$ commutes, and, by definition of ξ, the back vertical square $(F\Delta, FX, X, \Delta)$ commutes. This implies

$$\langle \mathrm{id}, f\rangle \circ \langle m, a\rangle \circ \langle \mathrm{id}, F\delta\rangle = \langle m, x\rangle \circ \langle \mathrm{id}, Ff\rangle \circ \langle \mathrm{id}, F\delta\rangle \tag{9}$$

By computing the second component on each side of (9) we get

$$(fm \circ a) \circ F\delta = (x \circ Ff) \circ F\delta$$

But $F\delta$ is colimiting, hence

$$fm \circ a = x \circ Ff$$

Because A and X are constant functors, $A \circ m = A$, $X \circ m = X$ and $fm = f$. Back to the category $\mathbf{C}$, we get a commutative diagram

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "a"'] & FX \arrow[d, "x"] \\ A \arrow[r, "f"'] & X \end{tikzcd} \tag{10}$$

Finally, there is only one f making (10) commutative: given such an f, we may consider again the diagram (8). This time, the three vertical squares are supposed to be commutative. As a consequence

$$\langle \mathrm{id}, f\rangle \circ \langle \mathrm{id}, \delta\rangle \circ \langle m, \mathrm{id}\rangle = \langle \mathrm{id}, \xi\rangle \circ \langle m, \mathrm{id}\rangle$$

so by keeping only the second components on each side:

$$(f \circ \delta)m = \xi m \tag{11}$$

But (11) means that $f \circ \delta$ and ξ take the same value on all indices $n > 0$, and because 0 is initial, they also coincide on 0, hence

$$f \circ \delta = \xi \tag{12}$$

and there is only one f such that (12) holds. $\square$

3 Polynomial functors

3.1 Functor categories

We first remark that if $\mathbf{C}$ is an ω-category, and has an initial object, then the same holds for $\mathbf{C}^n$, $n > 1$ (that is the category with ojects n-tuples of objects in $\mathbf{C}$ and morphisms n-tuples of morphisms of $\mathbf{C}$).

This is an easy consequence of the fact that for all categories $\mathbf{I}$, $\mathbf{C}$, $\mathbf{D}$, if $\mathbf{C}$ has all (co)limits of type $\mathbf{I}$, then the functor category $\mathbf{C}^D$ also has all (co)limits of type $\mathbf{I}$. Moreover these (co)limits may be computed "point by point". Now $\mathbf{C}^n$ is equivalent to the functor category $\mathbf{C}^D$ where $\mathbf{D}$ is the discrete category with n objects.

3.2 Constants, projections and coproducts

Let us prove a few easy lemmas.

Lemma 2 *The composition of two ω-continuous functors is ω-continuous.*

Immediate consequence of the definitions.

Lemma 3 *Let $\mathbf{C}$ be any category. Constant endofunctors of $\mathbf{C}$ are ω-continuous.*

This is immediate. Observe however that constant functors do *not* preserve general limits and colimits.

Lemma 4 *Let $\mathbf{C}$ be any category. Projections $\pi_i : \langle X_1, \ldots, X_n \rangle \mapsto X_i$ from $\mathbf{C}^n$ to $\mathbf{C}$ are ω-continuous.*

Proof. Without any special hypotheses on $\mathbf{C}$, it can be proved directly from the definitions that projections preserve all limits and colimits, essentially because a diagram in $\mathbf{C}^n$ is nothing but n independent diagrams in $\mathbf{C}$. Another proof holds in the case where $\mathbf{C}$ has a terminal object 1: then the i-th projection

$$\pi_i : \langle X_1, \ldots, X_n \rangle \mapsto X_i$$

has a right adjoint, wich is the insertion of Y as the i-th component in an n-tuple:

$$\rho_i : Y \mapsto \langle 1, \ldots, Y, \ldots, 1 \rangle$$

Hence π_i preserves all colimits, in particular ω-colimits. $\square$

Lemma 5 *If $F : \mathbf{C} \to \mathbf{A}$ and $G : \mathbf{C} \to \mathbf{B}$ are ω-continuous functors, then $\langle F, G \rangle : \mathbf{C} \to \mathbf{A} \times \mathbf{B}$ is ω-continuous.*

Proof. The result holds in fact for any type of limits or colimits. Tedious but essentially obvious verification. A more conceptual argument is to look at $\mathbf{A} \times \mathbf{B}$ as a subcategory of *functors* from $\mathbf{2}$ (two objects 0 and 1, trivial arrows) to $\mathbf{A} + \mathbf{B}$: an object $\langle a, b \rangle$ becomes a functor $0 \mapsto a$, $1 \mapsto b$. Now the (co)limits are computed point by point, which gives the result. $\square$

Lemma 6 *A binary coproduct $+ : \mathbf{C}^2 \to \mathbf{C}$ is ω-continuous.*

Proof. Under the hypotheses of the lemma, the functor $+$ has a right adjoint, which is $X \mapsto \langle X, X \rangle$, therefore it preserves all colimits. $\square$

3.3 Products

We now look for conditions ensuring that

$$\langle X, Y\rangle \mapsto X \times Y$$

is an ω-continuous functor from $\mathbf{C}^2$ to $\mathbf{C}$. The first remark is that it suffices to check ω-continuity on each variable separately. Precisely,

Lemma 7 *Suppose that $X \mapsto X \times B$ and $Y \mapsto A \times Y$ are ω-continuous for each choice of objects A, B in $\mathbf{C}$, then the product functor $\times : \mathbf{C}^2 \to \mathbf{C}$ is ω-continuous.*

Proof. We only sketch the proof, and leave details to the reader. Suppose the hypotheses of the lemma are satisfied, and consider an ω-diagram $\langle X_i, Y_i\rangle$ in $\mathbf{C}^2$, with colimit $\langle L, M\rangle$. The projections give arrows $x_{ij} : X_i \to X_j$ and $y_{kl} : Y_k \to Y_l$ whenever $i \leq j$ and $k \leq l$. Hence a commutative diagram of type $\omega \times \omega$ in $\mathbf{C}^2$:

$$\begin{tikzcd}
\langle X_0, Y_0\rangle \arrow[r] \arrow[d] & \langle X_0, Y_1\rangle \arrow[r] \arrow[d] & \cdots \\
\langle X_1, Y_0\rangle \arrow[r] \arrow[d] & \langle X_1, Y_1\rangle \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots &
\end{tikzcd}$$

By applying the functor $\times$ to the above diagrams, we get a commutative diagram Δ of type ω^2 in $\mathbf{C}$. Now for each k, the k-th column is an ω-diagram with colimit $L \times Y_k$, by ω-continuity of $X \mapsto X \times Y_k$. From the $L \times Y_k$'s, we get another ω-diagram:

$$L \times Y_0 \longrightarrow L \times Y_1 \longrightarrow \cdots$$

with colimit $L \times M$, by ω-continuity of $Y \mapsto L \times Y$. Finally, we easily verify that $L \times M$ is also colimit of the diagonal of Δ, which is the image of our initial diagram under $\times$. Hence the result. $\Box$

The following proposition is an immediate consequence of lemma 7.

Lemma 8 *If $\mathbf{C}$ is cartesian closed, the functor $\times : \mathbf{C}^2 \to \mathbf{C}$ is ω-continuous.*

Proof. If $\mathbf{C}$ is cartesian closed, the partial product functors $X \mapsto X \times B$ and $Y \mapsto A \times Y$ have a right adjoint, hence commute to all colimits, in particular those of type ω. By lemma 7, the product is ω-continuous. $\Box$

Let us point out, however, that the (total) product does *not* have a right adjoint in general: for example, in **Sets**, it does not commute to the coproduct, while partial products do.

3.4 Theorem

Let us summarize our discussion by the following statement.

Theorem 2 *Suppose* $\mathbf{C}$ *is an* ω*-category with binary products and coproducts, an initial element, and cartesian closed. Then polynomial endofunctors of* $\mathbf{C}$ *have initial algebras.*

Proof. In the hypotheses of the theorem, all constructions used to build polynomial functors preserve ω-colimits, as shown by lemmas 2 to 8. $\square$

4 Comments and further reading

Theorem 2 is merely a starting point in the study of least fixpoints of functors: many interesting categories do not satisfy all its hypotheses. Fortunately, there are many ways to relax the existence conditions, by a simple inspection of the proofs we have given. There are essentially two distinct issues:

- As far as the existence of a fixpoint for a *given* polynomial functor F is concerned, remark that we *never* used the fact that "+" and "×" are the actual coproduct and product in $\mathbf{C}$, only that they are bifunctors $\oplus$ and $\otimes$, with certain commutation properties with colimits. Familiar example are categories $\mathbf{Cpo}$ of complete, pointed, partial ordered sets with a least element and continuous maps, and $\mathbf{Cpo}_\perp$, where we require in addition that maps preserve the least element: the first one is cartesian closed, but has no coproduct, and the second one has a coproduct, but is not cartesian closed anymore. Nevertheless, $\mathbf{Cpo}$ has a bifunctor $\oplus$ making the interpretation of polynomial functors possible. As for $\mathbf{Cpo}_\perp$, even though it is not cartesian closed, the product is still ω-continuous;
- much more delicate is the issue related to the semantical interpretation of grammar-like definitions in a categorical setting. The previous point immediately leads to the following question: what does it mean to interpret the solution of $\mathtt{L} = \mathtt{nil} \mid \mathtt{cAL}$ for example, as a fixpoint of $X \mapsto 1 \oplus (A \otimes X)$ if $\oplus$ and $\otimes$ are not the categorical sum and product anymore?

The classic source for general category theory is still [3]. A main source for the particular issues discussed here is [4]. Much of recent work on the subject concentrates on the relationship between initial algebras and the dual notion of terminal coalgebras, see for example [1] and [2].

References

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