Syntax vs. semantics: a polarized approach

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Abstract

We present a notion of sliced proof-nets for the polarized fragment of Linear Logic and a corresponding game model. We show that the connection between them is very strong through an equivalence of categories (this contains soundness, full completeness and faithful completeness).

An important topic in the recent developments of denotational semantics has been the quest for stronger and stronger connections between the syntactical systems and the denotational models. Work towards bringing the two notions closer has come from both sides, and can be seen as an attempt to solve the general question "*what is a proof?*".

Full abstraction and full completeness (see [1, 8]) results have been initiated with game semantics [1, 2, 15] and come with models containing only elements definable by the syntax. These results have been mainly obtained in the last ten years for fragments of linear logic (for example MLL with and without MIX [1, 14, 25, 5, 6], MALL [4], ILL [19], LLP [21], ...) and for extensions of PCF (for example PCF [2, 15], μPCF [18], Idealized Algol [3], ...). This full completeness property can be considered as a measurement of the precision of the semantics (whatever the syntax might be).

On the other side, the syntactical settings for logical systems have evolved progressively: sequent calculus, natural deduction, proof-nets. Although natural deduction is satisfactory for intuitionistic logic with $\rightarrow$, $\wedge$ and $\forall$, proof-nets permit intrinsic syntactical presentations of richer systems. Much work has been done since the original version [7] to remove sequentiality (boxes) and to make them more canonical. A way to evaluate the precision of a given syntax is to compare it with another one and to show that it realizes a quotient. Another approach is to use semantical means to see the identifications realized by a given model and that are not present in the syntax. In the spirit of a tight connection between syntax and semantics, the perfect case would correspond to an injective interpretation of the syntax in the model. The main results in this direction are due to L. Tortora de Falco [30] for fragments of linear logic with respect to coherent semantics. Work on full completeness led also to such faithfulness results (see [1, 2, 15]) but correspond to quite particular cases: MLL (and MLL with MIX) or λ-calculus. The extension of faithfulness results to the additive connectives was a very open question. Proof-nets for MALL recently given by D. Hughes and R. van Glabbeek [13] are very likely to lead to faithfulness results with respect to various models such as coherence spaces [7] or game semantics [4]. A solution for a polarized setting has also been given with proof-nets [24] but with a restriction on the use of the exponential connectives. These faithfulness results may be considered as a property of the syntax comparable to Böhm's theorem and more generally separation theorems. These syntactical theorems are about separation of terms by contexts. Here the separation is based on semantics but these two points of view sometimes coincide in particular in realizability models where terms are precisely interpreted by a set of accepting contexts (see for example J.-L. Krivine's classical realizability [17] or J.-Y. Girard's ludics [11]).

In the spirit of J.-Y. Girard's program [10] to remove the distinction between syntax and semantics, this paper describes a strict correspondence between the polarized propositional fragment of linear logic $\mathsf{LL}_{\mathrm{pol}}$ [20] (with all the connectives and the units) and a polarized game model. Combining work from the syntactical side (sliced proof-nets) and work from the semantical side (game semantics), we prove that we arrive to a meeting point for the polarized framework. The notion of proof-nets we use is mainly the one described in [24] and our game model is a simplified version of [21] for $\mathsf{LL}_{\mathrm{pol}}$, also used in [22], which is enriched with variables. The polarization constraint we deal with here is not too strong since our setting is expressive enough to encode propositional classical logic with all connectives [23, 20] (using a variation on Girard's embedding of intuitionistic logic into linear logic).

The results we prove in this paper can be summarized in categorical terms through an equivalence of categories between the syntax and the game model:

- Each object of the model is isomorphic to the interpretation of a formula.
- Each morphism of the model is the interpretation of a proof.
- The interpretations of two proofs are the same *iff* these proofs are $\beta\eta$-equal (with a canonical representative in each class of $\beta\eta$-equivalence given by cut-free sliced proof-nets).

We do not claim that this paper contains completely new ideas and structures. It is mainly a nice combination of (almost) known objects in order to get a precise comparison between them. The two main really new ingredients are the extension of the game model to variables with full completeness and the proof of the faithfulness result.

1 Polarized linear logic and proof-nets

The syntactical objects we are interested in are polarized sequent calculus proofs and polarized proof-nets. We first present the sequent calculus $\mathsf{LL}_{\mathrm{pol}}$ based on a restriction of LL to polarized formulas. Using the properties coming from polarization, we are then able to introduce the notion of sliced proof-nets.

1.1 $\mathsf{LL}_{\mathrm{pol}}$

The system $\mathsf{LL}_{\mathrm{pol}}$ is the fragment of LL obtained by the following restriction on formulas:

$$\begin{array}{lcccccccccccc} P & ::= & !X & \mid & 1 & \mid & 0 & \mid & P \otimes P & \mid & P \oplus P & \mid & !N \\ N & ::= & ?X^{\perp} & \mid & \perp & \mid & \top & \mid & N \parr N & \mid & N \mathbin{\&} N & \mid & ?P \end{array}$$

The rules of the system LL_{pol} are just the usual LL rules restricted to these polarized formulas:

$$\frac{}{\vdash N, N^{\perp}}\ ax \qquad \frac{\vdash \Gamma, N \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ cut$$

$$\frac{\vdash \Gamma, N, M}{\vdash \Gamma, N \parr M}\ \parr \qquad \frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\ \otimes$$

$$\frac{\vdash \Gamma, N \qquad \vdash \Gamma, M}{\vdash \Gamma, N \,\&\, M}\ \& \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\ \oplus_1 \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\ \oplus_2$$

$$\frac{\vdash ?\Gamma, N}{\vdash ?\Gamma, !N}\ ! \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\ ?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c$$

$$\frac{}{\vdash \Gamma, \top}\ \top \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \frac{}{\vdash 1}\ 1$$

where $?A$ is any negative formula starting with a $?$ symbol and where the context Γ of the $\top$-rule contains at most one positive formula (which is the only difference with LL).

Lemma 1 (Positive formula)
If $\vdash \Gamma$ is provable in LL_{pol}, Γ contains at most one positive formula.

Lemma 2 (Negative structural rules)
If N is a negative formula, the rules $\dfrac{\vdash \Gamma}{\vdash \Gamma, N}$ and $\dfrac{\vdash \Gamma, N, N}{\vdash \Gamma, N}$ are derivable in LL_{pol}.

1.2 Sliced polarized proof-nets

Proof-nets permit the definition of a more parallel syntax than sequent calculus which is less sensible to the order of rules and thus represent proofs up to certain commutations of rules. In our polarized setting, we are able to introduce a sliced notion of proof-nets. This gives an independent representation of the two premises of the &-rule which is the key ingredient in order to quotient some "additive" commutations of rules ($\&/\&$ or $\&/\parr$ for example).

Examples of the various objects and notions described in this section are given in appendix B.

Definition 1 (Flat proof-structure)
A *flat proof-structure* is a directed graph with edges labeled by types, where a *type* can be:

- either a polarized formula (P or N);
- or a $\flat$-formula $\flat P$ where P is a positive formula;
- or a $\flat$-formula $\flat X^{\perp}$ where X is a variable;
- or a variable X or $X^{\perp}$.

Edges typed with negative formulas, $\flat$-formulas or variables X are called *negative* and edges typed with positive formulas or variables $X^{\perp}$ are called *positive*.

Nodes are given with constraints on the typing of their edges, on the number of incoming edges, called the *premises*, and on the number of outgoing edges, called the *conclusions*, according to the following rules:

(ax) An *ax*-node has no premise and two conclusions typed with X and $X^\perp$ for some variable X.

(cut) A *cut*-node has two premises typed with dual types P and $P^\perp$ or X and $X^\perp$.

($\otimes$) A $\otimes$-node has two premises typed with positive formulas P and Q and one conclusion of type $P \otimes Q$.

($\parr$) A $\parr$-node has two premises typed with negative formulas N and M and one conclusion of type $N \parr M$.

($\oplus_1$) A $\oplus_1$-node has one premise typed with a positive formula P and one conclusion of type $P \oplus Q$.

($\oplus_2$) A $\oplus_2$-node has one premise typed with a positive formula Q and one conclusion of type $P \oplus Q$.

($\&_1$) A $\&_1$-node has one premise typed with a negative formula N and one conclusion of type $N \& M$.

($\&_2$) A $\&_2$-node has one premise typed with a negative formula M and one conclusion of type $N \& M$.

($!$) A $!$-node has no premise, one conclusion of type $!N$ or $!X$ and any number of other conclusions typed with $\flat$-formulas.

($\flat$) A $\flat$-node has one premise typed with a positive formula P or a variable $X^\perp$ and one conclusion typed with the corresponding $\flat$-formula $\flat P$ or $\flat X^\perp$.

($?$) A $?$-node has any number of premises (possibly 0) with the same type $\flat P$ or $\flat X^\perp$ and one conclusion of the corresponding type $?P$ or $?X^\perp$.

(1) A 1-node has no premise and one conclusion typed with the positive formula 1.

($\perp$) A $\perp$-node has no premise and one conclusion typed with the negative formula $\perp$.

The nodes with only positive edges, that is $\otimes$, $\oplus_1$, $\oplus_2$ and 1, are called *positive*. The nodes with only negative edges, that is $\parr$, $\&_1$, $\&_2$, $\perp$ and ? are called *negative*.

Any edge must have a source but we allow edges without target and these edges are called the *conclusions* of the flat proof-structure.

Remark: The main connective of a type specifies the nodes of which it can be a conclusion with three particular cases: $P \oplus Q$ can be conclusion of a $\oplus_1$ or $\oplus_2$-node, $N \& M$ can be conclusion of a $\&_1$ or $\&_2$-node, and $\flat A$ can be conclusion of a $\flat$-node or of a !-node. For the other connectives, there is only one possible kind of node.

Definition 2 (Sliced proof-structure)
Sliced proof-structures and *slices* with conclusions $\Gamma, \flat\Delta$ are defined inductively by:

- A flat proof-structure without !-node and with conclusions $\Gamma, \flat\Delta$ is a *slice* s with conclusions $\Gamma, \flat\Delta$. Its nodes are said to have *depth* 0 in s.
- A flat proof-structure with conclusions $\Gamma, \flat\Delta$ and with, for each !-node n with conclusions $!N, \flat\Xi$, an associated sliced proof-structure $\mathcal{S}_n$ with conclusions $N, \flat\Xi$ (called the box of n) is a *slice* s with conclusions $\Gamma, \flat\Delta$. If a node has depth d in $\mathcal{S}_n$, it has depth $d+1$ in the slice s and if it is in the flat proof-structure, it has depth 0 in s.

- A finite set (possibly empty) of k slices s_i with conclusions $\Gamma, \flat\Delta_i$ $(1 \leq i \leq k)$ is a *sliced proof-structure* $\mathcal{S}$ with conclusions $\Gamma, \flat\Delta$ if $\Delta = \Delta_1, \ldots, \Delta_k$. If a node has depth d in s_i, it has the same depth in the sliced proof-structure $\mathcal{S}$.

The *depth* of a sliced proof-structure is the maximal depth of its nodes.

Definition 3 (Correction graph)
The *correction graph* of a flat proof-structure is the directed graph obtained by orienting each positive edge upwardly and each negative edge downwardly.

Definition 4 (Acceptable proof-structure)
A sliced proof-structure $\mathcal{S}$ is *acceptable* if:

- it is *acyclic*: the correction graphs of all the flat proof-structures of $\mathcal{S}$ are acyclic.

- it is *connected*: all the flat proof-structures of $\mathcal{S}$ contain exactly one $\flat$-node or one positive conclusion.

- it is *type completed*: all its conclusions are typed with polarized formulas (no $\flat$-formula, no X, no $X^\perp$).

Definition 5 (Cut elimination)
The cut elimination procedure for acceptable sliced proof-structures is defined as usual for proof-nets [28] except that we work independently in each slice. The only particular case is a cut between a $\&_1$-node and a $\oplus_2$-node (or a $\&_2$-node and a $\oplus_1$-node) which is reduced by erasing the slice [24].

The main properties of sliced proof-nets are proved in [24, 20]. We just recall here the definitions and statements.

Proposition 1 (Confluence)
If $\mathcal{R} \to^* \mathcal{R}_1$ and $\mathcal{R} \to^* \mathcal{R}_2$, there exists $\mathcal{R}_0$ such that $\mathcal{R}_1 \to^* \mathcal{R}_0$ and $\mathcal{R}_2 \to^* \mathcal{R}_0$.

Proposition 2 (Strong normalization)
If $\mathcal{R}$ is an acceptable sliced proof-structure, there is no infinite sequence of reductions starting from $\mathcal{R}$.

Definition 6 (Translation of sequent calculus)
The translation of a sequent calculus proof π of $\vdash \Gamma$ as a sliced proof-structure $\mathcal{R}_\pi$ with conclusions Γ is defined for an η-expanded proof (that is with axioms introducing only $\vdash ?X^\perp, !X$ sequents) by induction on the structure of this proof. If π is not η-expanded, we first expand all its axioms (see appendix A).

(ax) The sliced proof-structure $\mathcal{R}_\pi$ contains one flat proof-structure reduced to a !-node with conclusions $!X$ and $\flat X^\perp$ and a unary ?-node under this $\flat X^\perp$ conclusion introducing $?X^\perp$. The sliced proof-structure associated with the !-node contains one flat proof-structure with an *ax*-node with conclusions X and $X^\perp$ and one $\flat$-node under this $X^\perp$ introducing $\flat X^\perp$.

(cut) If the sliced proof-structures associated with the two premises of this rule are $\mathcal{R}_{\pi_1}$ and $\mathcal{R}_{\pi_2}$, the slices of $\mathcal{R}_\pi$ are obtained for each slice $s_1 \in \mathcal{R}_{\pi_1}$ and each slice $s_2 \in \mathcal{R}_{\pi_2}$ by putting a *cut*-node between the conclusions of s_1 and s_2 cut in π.

($\otimes$) If the sliced proof-structures associated with the two premises of this rule are $\mathcal{R}_{\pi_1}$ and $\mathcal{R}_{\pi_2}$, the slices of $\mathcal{R}_\pi$ are obtained for each slice $s_1 \in \mathcal{R}_{\pi_1}$ and each slice $s_2 \in \mathcal{R}_{\pi_2}$ by putting a $\otimes$-node between the conclusions of s_1 and s_2 corresponding to the active formulas.

($\parr$) If the sliced proof-structure associated with the premise of this rule is $\mathcal{R}_{\pi_1}$, the slices of $\mathcal{R}_\pi$ are obtained by adding a $\parr$-node in each slice of $\mathcal{R}_{\pi_1}$.

($\oplus_1$) If the sliced proof-structure associated with the premise of this rule is $\mathcal{R}_{\pi_1}$, the slices of $\mathcal{R}_\pi$ are obtained by adding a $\oplus_1$-node in each slice of $\mathcal{R}_{\pi_1}$.

($\oplus_2$) If the sliced proof-structure associated with the premise of this rule is $\mathcal{R}_{\pi_1}$, the slices of $\mathcal{R}_\pi$ are obtained by adding a $\oplus_2$-node in each slice of $\mathcal{R}_{\pi_1}$.

(&) If the sliced proof-structures associated with the two premises of this rule are $\mathcal{R}_{\pi_1}$ and $\mathcal{R}_{\pi_2}$, the slices of $\mathcal{R}_\pi$ are obtained by adding a $\&_1$-node to each slice of $\mathcal{R}_{\pi_1}$ and by adding a $\&_2$-node to each slice of $\mathcal{R}_{\pi_2}$.

(!) If the sliced proof-structure associated with the premise of this rule is $\mathcal{R}_{\pi_1}$ with conclusions $?\Gamma$ and N, we remove in each slice s_1 of $\mathcal{R}_{\pi_1}$ the concluding ?-nodes and we obtain a sliced proof-structure $\mathcal{R}'_{\pi_1}$ with the conclusion N and many $\flat$-conclusions $\flat\Gamma'$. The sliced proof-structure $\mathcal{R}_\pi$ contains a unique flat proof-structure with one !-node with conclusions $!N$ and $\flat\Gamma'$ (with associated sliced proof-structure $\mathcal{R}'_{\pi_1}$) and the required ?-nodes corresponding to the erased ones of $\mathcal{R}'_{\pi_1}$, so that we obtain $\mathcal{R}_\pi$ with conclusions $?\Gamma$ and $!N$.

($?d$) If the sliced proof-structure associated with the premise of this rule is $\mathcal{R}_{\pi_1}$, the slices of $\mathcal{R}_\pi$ are obtained by adding a $\flat$-node and a unary ?-node under it in each slice of $\mathcal{R}_{\pi_1}$.

($?w$) If the sliced proof-structure associated with the premise of this rule is $\mathcal{R}_{\pi_1}$, the slices of $\mathcal{R}_\pi$ are obtained by adding a 0-ary ?-node in each slice of $\mathcal{R}_{\pi_1}$.

($?c$) If the sliced proof-structure associated with the premise of this rule is $\mathcal{R}_{\pi_1}$, the slices of $\mathcal{R}_\pi$ are obtained by merging the two ?-nodes above the contracted conclusions in each slice of $\mathcal{R}_{\pi_1}$.

($\top$) The sliced proof-structure $\mathcal{R}_\pi$ associated with a $\top$-rule introducing $\vdash \Gamma, \top$ is the empty set of slices with conclusions Γ and $\top$.

($\perp$) If the sliced proof-structure associated with the premise of this rule is $\mathcal{R}_{\pi_1}$, the slices of $\mathcal{R}_\pi$ are obtained by adding a $\perp$-node in each slice of $\mathcal{R}_{\pi_1}$.

(1) The sliced proof-structure $\mathcal{R}_\pi$ associated with a 1-rule contains a unique flat proof-structure reduced to a unique 1-node.

Proposition 3 (Correctness of the translation)
The sliced proof-structure $\mathcal{R}_\pi$ associated with the proof π is acceptable.

Proposition 4 (Simulation)
If π_0 is a normal form of the proof π, the normal form of $\mathcal{R}_\pi$ is $\mathcal{R}_{\pi_0}$.

To get a sequentialization theorem we need to add some requirements to be sure that enough slices are present in the proof-structure and that no contradiction appears between slices. In order to give the appropriate definitions, we have to restrict ourselves to the case of cut-free proof-structures.

Definition 7 (Equivalence of negative nodes)
We define a partial equivalence relation $\equiv$ on the 0-depth negative nodes of a sliced cut-free proof-structure $\mathcal{S}$:

- if n_1 and n_2 are two conclusion negative nodes of two slices of $\mathcal{S}$ with the same conclusion, they are equivalent: $n_1 \equiv n_2$;
- if n_1 and n_2 are two negative nodes of $\mathcal{S}$ above the same premise (left or right for binary nodes) of two nodes m_1 and m_2 (that must be negative) of the same kind such that $m_1 \equiv m_2$, then $n_1 \equiv n_2$.

Two equivalent nodes are of the same kind except if we have a $\&_1$-node and a $\&_2$-node but in this case the nodes above their premise are not equivalent. The equivalence classes of nodes for $\equiv$ correspond to occurrences of connectives in the types of the conclusions of the sliced proof-structure.

To compare the use of the &-nodes in different slices we use Girard's notion of additive weights [9].

Definition 8 (Weights)
Let $\mathcal{S}$ be a sliced cut-free proof-structure, we denote by $\&^i$ the equivalence classes of the &-nodes of $\mathcal{S}$ with respect to $\equiv$. With each such equivalence class $\&^i$, we associate a boolean variable p_i and we denote its negation by $\overline{p_i}$.

- the weight of a slice s is:
$$w(s) = \prod_{\&_1^i \in s} p_i \prod_{\&_2^i \in s} \overline{p_i}$$
- the weight of a sliced proof-structure $\mathcal{S}$ is:
$$w(\mathcal{S}) = \sum_{s \in \mathcal{S}} w(s)$$
- if the types of the conclusions of a sliced proof-structure $\mathcal{S}$ are $A_1, \ldots, A_n$, the variables p_i are associated with occurrences of the & connective in these types and we define the $\top$-weight of a sub-type of A_i by:
$$\begin{aligned}
w_\top(A) &= 0 && \text{if } A \text{ is positive}\\
w_\top(\flat A) &= 0\\
w_\top(?A) &= 0\\
w_\top(\bot) &= 0\\
w_\top(\top) &= 1\\
w_\top(N \parr M) &= w_\top(N) + w_\top(M)\\
w_\top(N \,\&\, M) &= p w_\top(N) + \overline{p} w_\top(M) && \text{where } p \text{ is the associated variable of the } \&
\end{aligned}$$

  and the $\top$-weight of $\mathcal{S}$ is $w_\top(\mathcal{S}) = w_\top(A_1) + \cdots + w_\top(A_n)$.

Definition 9 (Sliced cut-free proof-net)
A sliced *cut-free* proof-structure $\mathcal{S}$ is *correct* or is a *sliced proof-net* if it is *acceptable* and moreover:

- it is *full*: $w(\mathcal{S}) + w_\top(\mathcal{S}) = 1$ and for each sliced proof-structure $\mathcal{S}'$ associated with a !-node $w(\mathcal{S}') + w_\top(\mathcal{S}') = 1$.
- it is *compatible*: for any two slices $s \neq t$ of $\mathcal{S}$, $w(s)w(t) = 0$ and for each sliced proof-structure $\mathcal{S}'$ associated with a !-node if $s' \neq t' \in \mathcal{S}'$, $w(s')w(t') = 0$.

Proposition 5 (Correctness of the cut-free translation)
The sliced cut-free proof-structure $\mathcal{R}_\pi$ associated with the cut-free proof π is a sliced cut-free proof-net.

Proposition 6 (Sequentialization)
If $\mathcal{R}$ is a sliced cut-free proof-net*, there exists a proof π in $\mathsf{LL}_{\mathrm{pol}}$ such that $\mathcal{R} = \mathcal{R}_\pi$.*

2 Game semantics

2.1 Polarized games

The game model we are interested in can be seen both as a simplification for $\mathsf{LL}_{\mathrm{pol}}$ of the polarized model described in [21] and as a generalization of Laird's model [18]. We extend the usual Hyland-Ong/Nickau games [15, 27] (HON) with explicit polarities on arenas and with new constructions $\times$ and $\updownarrow$. The explicit polarization leads to some variations in the interpretations of contraction, weakening, …

For more explanations on this kind of HON games, more details and more intuitions, the reader may look at [12].

Definition 10 (Forest)
A *forest* is a partial order $(E, \leq)$ such that for any $x \in E$, $(\{y \in E \mid y \leq x\}, \leq)$ is a finite total order.

The *nodes* of a forest $(E, \leq)$ are the elements of E and the forest is *finite* if E is finite. The *roots* are the minimal elements. The set of the roots of a forest is denoted by E^r. The *leaves* are the maximal elements and a *strict leaf* is a leaf which is not a root. If x is the maximal element under y, we say that y is a *son* of x.

Definition 11 (Arena)
A *polarized arena* $(A, \leq_A, \pi_A, \mathcal{V}_A)$ is a finite forest $(A, \leq_A)$ whose nodes are called *moves* with a polarity π_A which is P or O (also denoted by $+$ or $-$) and a given set $\mathcal{V}_A$ of strict leaves.

A *labeled polarized arena* is a polarized arena with a function from $\mathcal{V}_A$ to variables X, Y, …

The *polarity of a move* m is π_A (resp. $\overline{\pi_A}$) if the length of the path (*i.e.* its number of edges) going from a root of A to m is even (resp. odd).

A move m of A is *initial*, denoted by $\vdash_A m$, if it is a root of A. If m is a son of n in A, we say that n enables m, denoted by $n \vdash_A m$. We will usually describe arenas by means of $\vdash_A$ instead of $\leq_A$.

Definition 12 (Constructions of arenas)
We consider the following constructions on arenas:

Dual. If A is an arena, its dual is obtained by changing its polarity with the same set $\mathcal{V}_{A^\perp} = \mathcal{V}_A$.

Empty. There are two empty polarized arenas: the positive one $(\emptyset, \emptyset, P, \emptyset)$ and the negative one $(\emptyset, \emptyset, O, \emptyset)$.

Unit. The two unit arenas are the forests reduced to one node: $(\{\star\}, \emptyset, P, \emptyset)$ and $(\{\star\}, \emptyset, O, \emptyset)$.

Sum. If A and B are two arenas of the same polarity, $A + B$ is the union of the two forests:

- the underlying set of $A + B$ is the disjoint union of A and B;
- if $a \in A$, $a' \in A$ and $a \vdash_A a'$ then $a \vdash_{A+B} a'$;
- if $b \in B$, $b' \in B$ and $b \vdash_B b'$ then $b \vdash_{A+B} b'$;
- $\pi_{A+B} = \pi_A = \pi_B$;
- $\mathcal{V}_{A+B} = \mathcal{V}_A + \mathcal{V}_B$.

Product. If A and B are two arenas of the same polarity, the trees of $A \times B$ are obtained by taking a tree in A and a tree in B and by identifying their roots. More formally:

- the underlying set of $A \times B$ is $(A^r \times B^r) + ((A \setminus A^r) \times B^r) + (A^r \times (B \setminus B^r))$;
- if $(a_0, b_0) \in A^r \times B^r$, $(a, b_0) \in (A \setminus A^r) \times B^r$, and $a_0 \vdash_A a$ then $(a_0, b_0) \vdash_{A \times B} (a, b_0)$;
- if $(a_0, b_0) \in A^r \times B^r$, $(a_0, b) \in A^r \times (B \setminus B^r)$, and $b_0 \vdash_B b$ then $(a_0, b_0) \vdash_{A \times B} (a_0, b)$;
- if $(a, b_0) \in (A \setminus A^r) \times B^r$, $(a', b_0) \in (A \setminus A^r) \times B^r$, and $a \vdash_A a'$ then $(a, b_0) \vdash_{A \times B} (a', b_0)$;
- if $(a_0, b) \in A^r \times (B \setminus B^r)$, $(a_0, b') \in A^r \times (B \setminus B^r)$, and $b \vdash_B b'$ then $(a_0, b) \vdash_{A \times B} (a_0, b')$;
- $\pi_{A \times B} = \pi_A = \pi_B$;
- $\mathcal{V}_{A \times B} = \mathcal{V}_A \times B^r + A^r \times \mathcal{V}_B$.

Lift. If A is an arena, $\updownarrow A$ is obtained by adding a new root $\star$ under all the trees of A:

- the underlying set of $\updownarrow A$ is the disjoint union of A and $\{\star\}$;
- if $a \vdash_A a'$ then $a \vdash_{\updownarrow A} a'$;
- if $\vdash_A a$ then $\star \vdash_{\updownarrow A} a$;
- $\vdash_{\updownarrow A} \star$;
- $\pi_{\updownarrow A} = \overline{\pi_A}$;
- $\mathcal{V}_{\updownarrow A} = \mathcal{V}_A$.

In the spirit of categories, we will use the notation $A \to B$ for the arena $\updownarrow A^\perp \times B$.

Definition 13 (Justified sequence)
Let A be an arena, a *justified sequence* s on A is a sequence of moves of A with, for each non-initial move b of s, a pointer to an earlier occurrence of move a of s, called the *justifier* of b, such that $a \vdash_A b$.

Definition 14 (Projections)
If s is a justified sequence on $A + B$, the *projection* $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the justified sequence containing only the moves of s in A (resp. in B).

If s is a justified sequence on $A \times B$, the *projection* $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is the justified sequence containing only the moves a (resp. b) such that (a, b_0) (resp. (a_0, b)) is a move of s for some initial move b_0 (resp. a_0). In this spirit, we will say that a move of the shape (a, b_0) in $A \times B$ with a non-initial and b_0 initial (resp. (a_0, b) with a_0 initial and b non-initial) is a move in A (resp. in B).

If s is a justified sequence on $\updownarrow A$, the *projection* $s \upharpoonright_A$ is the justified sequence containing only the moves of s in A.

In these three cases, $s \upharpoonright_A$ (resp. $s \upharpoonright_B$) is a justified sequence on A (resp. on B).

Definition 15 (Play)
Let A be an arena, a *play* s on A is a justified sequence on A with moves of *alternated polarity.*

The set of plays of A is denoted by $\mathcal{P}_A$. We use the notation $t \leq^P s$ if t is a prefix of s ending with a P-move. We say that t is a *P-prefix* of s.

In the sequel we will use the following notations:

- A, B, C, ... for arenas and formulas;
- $\mathcal{A}$, ... for arenas (when confusion with the corresponding formula is possible);
- a, b, c, ..., m, n, ... for moves;
- s, t, ... for justified sequences and plays and also u, v, ... but mainly for interaction sequences;
- σ, τ, ... for strategies;
- φ for view functions.

Definition 16 (View)
Let A be an arena and s be a play on A, the *view* $\ulcorner s \urcorner$ of s is the sub-play of s defined by:

- $\ulcorner sa \urcorner = a$ if a is an initial move;
- $\ulcorner sa \urcorner = \ulcorner s \urcorner a$ if a is a non-initial P-move;
- $\ulcorner satb \urcorner = \ulcorner s \urcorner ab$ if b is an O-move justified by a.

A play s on A is called a *view* if $s = \ulcorner s \urcorner$.

Definition 17 (Strategy)
A *strategy* σ on a *negative* arena A, denoted by $\sigma : A$, is a non-empty P-prefix-closed set of even length plays of A such that:

- *determinism*: if $sab \in \sigma$ and $sac \in \sigma$, then $sab = sac$.
- *visibility*: if $sab \in \sigma$, the justifier of b is in $\ulcorner sa \urcorner$.
- *innocence*: if $sab \in \sigma$, $t \in \sigma$, $ta \in \mathcal{P}_A$ and $\ulcorner sa \urcorner = \ulcorner ta \urcorner$ then $tab \in \sigma$.

Definition 18 (View function)
Let A be a negative arena, a *view function* φ on A is a non-empty P-prefix-closed set of even length views of A which is deterministic: if $sab \in \varphi$ and $sac \in \varphi$ then $sab = sac$ (this can also be seen as a partial function from odd length views to P-moves with a pointer). If $\sigma : A$ is a strategy, its *view function* is $\varphi_\sigma = \{\ulcorner s \urcorner \mid s \in \sigma\}$.

According to the following lemma, a strategy can be described as before or by its view function.

Lemma 3 (Innocence and view function)
If φ is a view function, there exists a unique strategy $\sigma : A$ such that $\varphi_\sigma = \varphi$.

Definition 19 (Linear strategy)
A strategy $\sigma : A \to B = \ddagger A^\perp \times B$ is *linear*, denoted by $\sigma : \ddagger A^\perp \times B$, if in any play s of σ, each initial move $(\star, b)$ is immediately followed by a move (a, b) in $A^r \times B^r$ justified by $(\star, b)$ and no other move of s in $A^r \times B^r$ is justified by $(\star, b)$.

Definition 20 (Total strategy)
Let $\sigma : A$ be a strategy, σ is *total* if whenever $s \in \sigma$ and $sa \in \mathcal{P}_A$, there exists some b such that $sab \in \sigma$.

Definition 21 (Finite strategy)
The *size* of a strategy σ is the sum of the lengths of the views of its view function φ_σ. A strategy is *finite* if its size is finite.

Definition 22 (Balanced strategy)
A strategy $\sigma : A$ is *balanced* if, for any play sab of σ, a is in $\mathcal{V}$ if and only if b is in $\mathcal{V}$. If A is labeled, σ is *label-balanced* if it is balanced and moreover the variables associated with these pairs of moves are the same.

This definition is very similar to Murawski's *token-reflecting strategies* [26].

Definition 23 (Identity)
Let A be a negative arena, the *identity* strategy id_A is $id_A = \{s \in \mathcal{P}_{A_1 \to A_2} \mid \forall t \leq^P s, t \upharpoonright_{A_1} = t \upharpoonright_{A_2}\} : A \to A$ (the indexes are only used to distinguish occurrences).

Definition 24 (Composition)
Let A, B and C be three negative arenas, an *interaction sequence* u on A, B and C is a justified sequence on $(A \to B) \to C$ such that $u \upharpoonright_{A \to B} \in \mathcal{P}_{A \to B}$, $u \upharpoonright_{B \to C} \in \mathcal{P}_{B \to C}$ and $u \upharpoonright_{A \to C} \in \mathcal{P}_{A \to C}$. A move of u in A pointing to a move in B is an initial move of A and its justifier is an initial move of B. The play $u \upharpoonright_{A \to C}$ is obtained by choosing as a pointer for these initial moves of A the justifier of their justifier which is an initial move of C. The set of the interaction sequences on A, B and C is denoted by $\mathrm{int}(A, B, C)$.

Let $\sigma : A \to B$ and $\tau : B \to C$ be two strategies, the *composition* of σ and τ is the strategy $\sigma;\tau = \{u \upharpoonright_{A \to C} \mid u \in \mathrm{int}(A, B, C) \wedge u \upharpoonright_{A \to B} \in \sigma \wedge u \upharpoonright_{B \to C} \in \tau\} : A \to C$.

Remark: If A is the empty arena, $(A \to B) \to C$ is the same as $B \to C$, so that we can generalize the previous definition to the notion of interaction sequences on B and C (denoted by $\mathrm{int}(B, C)$) and to composition of a strategy $\sigma : B$ with a strategy $\tau : B \to C$ which gives the strategy $\sigma;\tau$ on C.

Lemma 4 (Composition of balancing)
If $\sigma : A \to B$ and $\tau : B \to C$ are two balanced (resp. label-balanced) strategies, then $\sigma;\tau : A \to C$ is a balanced (resp. label-balanced) strategy.

PROOF: Let smn be a play in $\sigma;\tau$ and u be an interaction sequence such that $u \upharpoonright_{A \to C} = smn$, $u \upharpoonright_{A \to B} \in \sigma$ and $u \upharpoonright_{B \to C} \in \tau$, we decompose u into $u'mu''nu'''$ and we prove by induction on the length of u'' that if m (resp. n) is in $\mathcal{V}$ then all the moves of u'' and n (resp. m) are in $\mathcal{V}$:

- If u'' is empty, m and n are both in A (or both in C which is similar) and, by balancing of σ, m is in $\mathcal{V}$ if and only if n is in $\mathcal{V}$.
- If $u'' = va$, if m is in $\mathcal{V}$, by induction hypothesis, all the moves of v are in $\mathcal{V}$ and if a is a P-move (resp. O-move), by balancing of τ (resp. σ), a is in $\mathcal{V}$ and then by balancing of σ (resp. τ), n is in $\mathcal{V}$. In the other direction, we decompose u'' into bw and in the same way, if n is in $\mathcal{V}$, both w, b, and m are in $\mathcal{V}$.

The label-balanced case is proved exactly like the balanced case. □

2.2 The game model of LL_{pol}

The game model of LL_{pol} is given by an interpretation of polarized formulas by polarized arenas of the same polarity and of proofs by strategies.

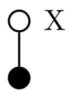
The interpretation of the polarized formula A is the labeled polarized arena $A^\star$ defined by:

$$
\begin{array}{rcc@{\qquad\qquad}rcc}
(!X)^\star & = & (\{q,\checkmark\},\{(q,\checkmark)\},P,\{\checkmark\}) & (?X^\perp)^\star & = & (\{q,\checkmark\},\{(q,\checkmark)\},O,\{\checkmark\}) \\
 & & \text{with } \checkmark \mapsto X & & & \text{with } \checkmark \mapsto X \\
0^\star & = & (\emptyset,\emptyset,P,\emptyset) & \top^\star & = & (\emptyset,\emptyset,O,\emptyset) \\
1^\star & = & (\{\star_1\},\emptyset,P,\emptyset) & \perp^\star & = & (\{\star_\perp\},\emptyset,O,\emptyset) \\
(P \oplus Q)^\star & = & P^\star + Q^\star & (N \mathbin{\&} M)^\star & = & N^\star + M^\star \\
(P \otimes Q)^\star & = & P^\star \times Q^\star & (N \mathbin{⅋} M)^\star & = & N^\star \times M^\star \\
(!N)^\star & = & \updownarrow N^\star & (?P)^\star & = & \updownarrow P^\star
\end{array}
$$

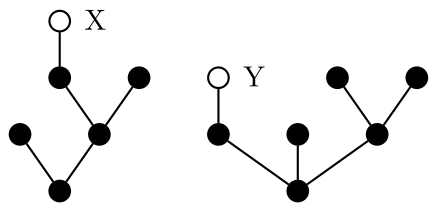
with the property $A^{\star\perp} = A^{\perp\star}$.

For the non-labeled case, we just forget the variable informations.

Example 1
The polarized arenas associated with $?X^\perp$ and $?(1 \oplus !(?X^\perp \mathbin{\&} \perp)) \mathbin{\&} (?(!Y \oplus 1) \mathbin{⅋} ?!(\perp \mathbin{\&} \perp))$ are:



and



with polarity O and $\mathcal{V}$ is denoted with ○ nodes.

For another example, see the end of appendix B

Lemma 5 (Product of strategies)
If A and B are two positive arenas, C and D are two negative arenas, $\sigma : \ddagger A \times C$ and $\tau : \ddagger B \times D$ are two linear *strategies, the set $\sigma \times \tau = \{s \in \mathcal{P}_{\ddagger(A\times B)\times C\times D} \mid s \upharpoonright_{\ddagger A\times C} \in \sigma \wedge s \upharpoonright_{\ddagger B\times D} \in \tau\}$ is a linear strategy on $\ddagger(A \times B) \times C \times D$. Moreover, if σ and τ are label-balanced, $\sigma \times \tau$ is label-balanced.*

Proof: We first prove by induction on the length of s that if $sab \in \sigma \times \tau$ then $a \in \updownarrow A \times C \iff b \in \updownarrow A \times C$. If s is empty, a is an initial move thus in $C \times D$ and by linearity of σ and τ, b corresponds to an initial move in $A \times B$. If s is not empty, if $a \in \updownarrow A \times C$ then $b \in \updownarrow A \times C$ otherwise $sab \upharpoonright_{\ddagger A\times C}$ ends with an O-move and $sab \upharpoonright_{\ddagger A\times C} \in \sigma$ is impossible; if $b \in \updownarrow A \times C$ then $a \in \updownarrow A \times C$ otherwise $sab \upharpoonright_{\ddagger A\times C}$ is not alternated.

We can now easily verify that $\sigma \times \tau$ is a non-empty P-prefix-closed set of even length plays of $\updownarrow(A \times B) \times C \times D$.

If $sab \in \sigma \times \tau$ and $sac \in \sigma \times \tau$, if $a = (a', a'') \in C \times D$ then we must have $b = (b', b'') \in A \times B$ (resp. $c = (c', c'') \in A \times B$) and $a'b' \in \sigma$ (resp. $a'c' \in \sigma$) and $a''b'' \in \tau$ (resp. $a''c'' \in \tau$) so that $b' = c'$ and $b'' = c''$. If $a \in \updownarrow A \times C$ (the case $a \in \updownarrow B \times D$ is very similar), by our preliminary result, $b \in \updownarrow A \times C$ and $c \in \updownarrow A \times C$ so that $sab \upharpoonright_{\updownarrow A \times C} = s \upharpoonright_{\updownarrow A \times C} ab \in \sigma$ and $sac \upharpoonright_{\updownarrow A \times C} = s \upharpoonright_{\updownarrow A \times C} ac \in \sigma$ which entails $b = c$ by determinism of σ.

We now show a second intermediary result by induction on the length of s: if $sab \in \sigma \times \tau$ and $a \in \updownarrow A \times C$ then $\ulcorner sa \urcorner \subset sa \upharpoonright_{\updownarrow A \times C}$. If s is empty or s is not empty and a is initial, then $a \in C \times D$. If s is not empty and a is not initial, we decompose sa into $s'bta$ where b is the justifier of a. We have $\ulcorner sa \urcorner = \ulcorner s' \urcorner ba$. If $b \in \updownarrow A \times C$ then the last move of s' is in $\updownarrow A \times C$ and by induction hypothesis $\ulcorner s' \urcorner \subset s' \upharpoonright_{\updownarrow A \times C}$ thus $\ulcorner sa \urcorner \subset sa \upharpoonright_{\updownarrow A \times C}$. If $b \notin \updownarrow A \times C$ then $b \in A \times B$ and $\ulcorner s' \urcorner$ is reduced to one initial move in $C \times D$ so that $\ulcorner sa \urcorner \subset sa \upharpoonright_{\updownarrow A \times C}$.

If $sab \in \sigma \times \tau$, we consider the case where $b \in \updownarrow A \times C$ which entails $a \in \updownarrow A \times C$ thus $\ulcorner sa \urcorner \subset sa \upharpoonright_{\updownarrow A \times C}$ and by visibility of σ, b points in its view in $sa \upharpoonright_{\updownarrow A \times C}$ thus it points in its view in sab.

If $sab \in \sigma \times \tau$, $t \in \sigma \times \tau$, $ta \in \mathcal{P}_{\updownarrow(A \times B) \times C \times D}$ and $\ulcorner sa \urcorner = \ulcorner ta \urcorner$, we assume $b \in \updownarrow A \times C$ thus $a \in \updownarrow A \times C$ and $\ulcorner sa \upharpoonright_{\updownarrow A \times C} \urcorner = \ulcorner sa \urcorner \upharpoonright_{\updownarrow A \times C} = \ulcorner ta \urcorner \upharpoonright_{\updownarrow A \times C} = \ulcorner ta \upharpoonright_{\updownarrow A \times C} \urcorner$ and by innocence of σ we have $tab \upharpoonright_{\updownarrow A \times C} \in \sigma$. Moreover we have $tab \upharpoonright_{\updownarrow B \times D} = t \upharpoonright_{\updownarrow B \times D} \in \tau$ so that $tab \in \sigma \times \tau$. □

If $\sigma : \ddagger A \times B$ is a linear strategy, we use the notation $\sigma \times C$ for $\sigma \times id_C : \ddagger(A \times C^{\perp}) \times B \times C$.

Lemma 6 (Exponential of a strategy)
If A and $\updownarrow B$ are two negative arenas and $\sigma : A \times \updownarrow B$ is a strategy, we define $\varphi = \{\varepsilon, (\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})\} \cup \{(\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})(a, \star_{\updownarrow B})s \mid (a, \star_{\updownarrow B})s \in \varphi_\sigma\}$ where $\star$ is the initial move of $\updownarrow\updownarrow A$, $\star_{\updownarrow A}$ is the initial move of $\updownarrow A$ and $\star_{\updownarrow B}$ is the initial move of $\updownarrow B$.

The set φ is a view function on $\updownarrow\updownarrow A \times \updownarrow B$ and we denote by $!\sigma : \ddagger\updownarrow A \times \updownarrow B$ the associated linear *label-balanced strategy.*

PROOF: If $(a, \star_{\updownarrow B})s \in \varphi_\sigma$ then $(\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})(a, \star_{\updownarrow B})s$ is an even length view, thus φ is a non-empty set of even length views, and we easily see that it is P-prefix-closed. If $sab \in \varphi$ and $sac \in \varphi$, either $s = \varepsilon$ and $ab = ac = (\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})$ or we conclude by determinism of φ_σ. □

Lemma 7 (Contraction)
If $\updownarrow A$ is a negative arena and s is a play in $\updownarrow(\updownarrow A_1 \times \updownarrow A_2)^{\perp} \times \updownarrow A_0$ (indexes are for occurrences), we define s_i ($i = 1, 2$) to be the sub-sequence of s defined by:

- *the initial moves $(\star, \star)$ are in s_i;*
- *the moves in $\updownarrow A_i$ are in s_i;*
- *a P-move in $\updownarrow A_0$ following a move of s_i is in s_i;*
- *an O-move in $\updownarrow A_0$ pointing to a move of s_i is in s_i.*

The set $c_{\updownarrow A} = \{s \in \mathcal{P}_{\updownarrow(\updownarrow A_1 \times \updownarrow A_2)^{\perp} \times \updownarrow A_0} \mid \forall t \leq^P s, t_1 \in id_{\updownarrow A} \wedge t_2 \in id_{\updownarrow A}\}$ is a linear label-balanced strategy on $\ddagger(\updownarrow A \times \updownarrow A)^{\perp} \times \updownarrow A$.

According to lemma 1, a proof π in $\mathsf{LL}_{\mathrm{pol}}$ has a conclusion $\vdash \mathcal{N}, \Pi$ where $\mathcal{N} = N_1, \ldots, N_k$ contains only negative formulas and Π is either empty or contains one positive formula. If Π is empty, π is interpreted as a label-balanced strategy $\pi^\star$ on $\mathcal{N}^\star = N_1^\star \times \cdots \times N_k^\star$. If $\Pi = P$, π is interpreted as a *linear* label-balanced strategy $\pi^\star$ on $\ddagger P^\star \times \mathcal{N}^\star$. In the particular case where $\mathcal{N}$ is empty we have a linear strategy on $\ddagger P^\star$.

In order to simplify the following definitions, we introduce the notation $\sigma : \ddagger\Pi^\star \times \mathcal{N}^\star$ which means that either Π is empty and σ is a strategy on $\mathcal{N}^\star$ or $\Pi = P$ and σ is a *linear* strategy on $\ddagger P^\star \times \mathcal{N}^\star$. The strategy $\pi^\star$ is defined by induction on the structure of π by:

- (ax) The strategy $\pi^\star$ is the identity strategy on $\ddagger N^{\perp\star} \times N^\star$.
- (cut) If $\sigma_1 : \ddagger\Pi^\star \times N^\star \times \Gamma^\star$ and $\sigma_2 : \ddagger N^{\perp\star} \times \Delta^\star$ are the interpretations of the two premises, the strategy $\pi^\star$ is $\sigma_1; (\sigma_2 \times \Gamma^\star) : \ddagger\Pi^\star \times \Delta^\star \times \Gamma^\star$.
- ($\otimes$) If $\sigma_1 : \ddagger P^\star \times \Gamma^\star$ and $\sigma_2 : \ddagger Q^\star \times \Delta^\star$ are the interpretations of the two premises, the strategy $\pi^\star$ is $\sigma_1 \times \sigma_2 : \ddagger(P \otimes Q)^\star \times \Gamma^\star \times \Delta^\star$.
- ($\parr$) If $\sigma : \ddagger\Pi^\star \times \Gamma^\star \times N^\star \times M^\star$ is the interpretation of the premise, $\pi^\star = \sigma : \ddagger\Pi^\star \times \Gamma^\star \times (N \parr M)^\star$.
- ($\oplus_1$) If $\sigma : \ddagger P^\star \times \Gamma^\star$ is the interpretation of the premise, $\pi^\star = \sigma : \ddagger(P \oplus Q)^\star \times \Gamma^\star$.
- ($\oplus_2$) If $\sigma : \ddagger Q^\star \times \Gamma^\star$ is the interpretation of the premise, $\pi^\star = \sigma : \ddagger(P \oplus Q)^\star \times \Gamma^\star$.
- (&) If $\sigma_1 : \ddagger\Pi^\star \times \Gamma^\star \times N^\star$ and $\sigma_2 : \ddagger\Pi^\star \times \Gamma^\star \times M^\star$ are the interpretations of the two premises, the strategy $\pi^\star$ is $\sigma_1 + \sigma_2 : \ddagger\Pi^\star \times \Gamma^\star \times (N \,\&\, M)^\star$ where $\sigma_1 + \sigma_2$ is the strategy such that $\varphi_{\sigma_1+\sigma_2} = \varphi_{\sigma_1} \cup \varphi_{\sigma_2}$.
- (!) If $\sigma : (?\Gamma)^\star \times N^\star$ is the interpretation of the premise, $\pi^\star$ is the linear strategy $!\sigma : \ddagger(!N)^\star \times (?\Gamma)^\star$.
- ($?d$) If $\sigma : \ddagger P^\star \times \Gamma^\star$ is the interpretation of the premise, $\pi^\star = \sigma : (?P)^\star \times \Gamma^\star$.
- ($?w$) If $\sigma : \ddagger\Pi^\star \times \Gamma^\star$, the views of the strategy $\pi^\star : \ddagger\Pi^\star \times \Gamma^\star \times (?A)^\star$ are ε and $\{(m, \star)(s, \star) \mid ms \in \varphi_\sigma\}$ where $(s, \star)$ is obtained by replacing each move n of s by $(n, \star)$.
- ($?c$) If $\sigma : \ddagger\Pi^\star \times \Gamma^\star \times (?A)^\star \times (?A)^\star$, the strategy $\pi^\star$ is $\sigma; (\Gamma^\star \times c_{(?A)^\star}) : \ddagger\Pi^\star \times \Gamma^\star \times (?A)^\star$.
- ($\top$) The strategy $\pi^\star$ is the strategy $\{\varepsilon\} : \ddagger\Pi^\star \times \Gamma^\star \times \top^\star$.
- ($\perp$) If $\sigma : \ddagger\Pi^\star \times \Gamma^\star$, the views of the strategy $\pi^\star : \ddagger\Pi^\star \times \Gamma^\star \times \perp^\star$ are ε and $\{(m, \star)(s, \star) \mid ms \in \varphi_\sigma\}$, this is a particular case of ($?w$) with $A = 0$.
- (1) The views of the linear strategy $\pi^\star : \ddagger 1^\star$ are ε and $\star\star_1$.

Proposition 7 (Soundness for η-expansion)
Given a formula A, the strategies associated with the proof $\dfrac{}{\vdash A, A^\perp}\, ax$ *and with its η-expansion (see appendix A) are the same.*

Lemma 8 (Model of proof-nets)
If π_1 and π_2 are two proofs such that $\mathcal{R}_{\pi_1} = \mathcal{R}_{\pi_2}$, then $\pi_1^\star = \pi_2^\star$. As a consequence if $\mathcal{R}$ is a proof-net and π is a proof such that $\mathcal{R}_\pi = \mathcal{R}$, we define $\mathcal{R}^\star = \pi^\star$.

Moreover it is easily possible to directly define the strategy $\mathcal{R}^\star$ associated with any proof-net $\mathcal{R}$ in the same spirit as for sequent calculus proofs, in such a way that $(\mathcal{R}_\pi)^\star = \pi^\star$.

3 Between syntax and semantics

We have defined our two main objects: sliced polarized proof-nets and polarized HON games. The goal of this section is to describe the very tight connection between them. We are going to show that the two objects are almost "isomorphic" in the following way: the interpretation of proof-nets by strategies is compatible with cut elimination (soundness), any strategy is the interpretation of a proof-net (completeness) and if two proof-nets have the same interpretation they are equal up to cut elimination (faithfulness). This gives a bijection between cut-free sliced proof-nets and strategies.

3.1 Soundness

Lemma 9 (Binoidal product)
The product of strategies is bifunctorial for linear strategies:

- *If A and B are two negative arenas, $id_A \times id_B = id_{A\times B}$.*
- *If $\sigma : \ddagger A^\perp \times C$ and $\tau : \ddagger B^\perp \times D$ are two strategies, $\sigma \times \tau = (\sigma \times B); (C \times \tau) = (A \times \tau); (\sigma \times D)$.*

Lemma 10 (Projection of sum)
If $\sigma_1 : A$, $\sigma_2 : B$ and $\tau : \ddagger A^\perp \times C$ are three strategies, we can also see τ as a strategy on $\ddagger(A^\perp + B^\perp)\times C$ and we have $(\sigma_1 + \sigma_2); \tau = \sigma_1; \tau : C$.

Lemma 11 (Duplication of exponential)
If $\sigma : A \times \updownarrow B$ is a strategy, $c_{\updownarrow A^\perp}; !\sigma = (!\sigma \times !\sigma); c_{\updownarrow B} : \ddagger(\updownarrow A \times \updownarrow A) \times \updownarrow B$.

Lemma 12 (Bang lemma)
If $\sigma : \ddagger\updownarrow A \times \updownarrow B$ is a strategy, we have $\sigma = !(id_A; (\sigma \times A))$.

PROOF: We first prove that $\varphi_{id_A;(\sigma\times A)} = \{\varepsilon\} \cup \{(\star_{\updownarrow B}, a)s \mid (\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})(a, \star_{\updownarrow B})s \in \varphi_\sigma\}$. If $(\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})(a, \star_{\updownarrow B})s \in \varphi_\sigma$, we consider the interaction sequence u on $\updownarrow A_1^\perp \times A_2$ and $\updownarrow B \times A_0$ (we use the indexes only to distinguish the occurrences of A) such that $u \restriction_{A_0} = u \restriction_{A_1^\perp} = u \restriction_{A_2}$ and $u \restriction_{\ddagger\updownarrow A_1 \times \updownarrow B} = (\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})(a, \star_{\updownarrow B})s$, we have $u \restriction_{\updownarrow A_1^\perp \times A_2} \in id_A$, $u \restriction_{A_2 \to A_0} \in id_A$ thus $u \restriction_{\updownarrow B \times A_0} = (\star_{\updownarrow B}, a)s \in id_A; (\sigma \times A)$. Conversely, if $(\star_{\updownarrow B}, a)s \in \varphi_{id_A;(\sigma\times A)}$, there exists an interaction sequence u such that $u \restriction_{\updownarrow A_1^\perp \times A_2} \in id_A$, $u \restriction_{\updownarrow A_1^\perp \times A_2 \to \updownarrow B \times A_0} \in \sigma \times A$ and $u \restriction_{\updownarrow B \times A_0} = (\star_{\updownarrow B}, a)s$ this entails $u \restriction_{\ddagger\updownarrow A_1 \times \updownarrow B} \in \sigma$ and $u \restriction_{A_2 \to A_0} \in id_A$ so that $u \restriction_{\ddagger\updownarrow A_1 \times \updownarrow B} = (\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})(a, \star_{\updownarrow B})s$ and finally $(\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})(a, \star_{\updownarrow B})s \in \sigma$.

By definition of the exponential of a strategy, we have $\varphi_{!(id_A;(\sigma\times A))} = \{\varepsilon, (\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})\} \cup \{(\star, \star_{\updownarrow B})(\star_{\updownarrow A}, \star_{\updownarrow B})(a, \star_{\updownarrow B})s \mid (\star_{\updownarrow B}, a)s \in \varphi_{id_A;(\sigma\times A)}\}$ and we immediately conclude $\varphi_{!(id_A;(\sigma\times A))} = \varphi_\sigma$ thus $\sigma = !(id_A; (\sigma \times A))$. □

Theorem 1 (Soundness)
If $\pi \to \pi'$ then $\pi^\star = \pi'^\star$.

PROOF: According to lemma 8, it suffices to prove the preservation of the semantics for the cut-elimination steps that correspond to a cut-elimination step in proof-nets. We do not have to prove all the commutative steps ignored by proof-nets.

We use the notations of section 2.2 for the interpretation of proofs.

- (ax) If the cut formula is the negative formula N in the ax-rule, and if π_1 is the proof of the other premise of the cut-rule, the interpretation of π is $id_{N^\star};\pi_1^\star = \pi_1^\star = \pi'^\star$. If the cut formula is the positive formula P in the ax-rule, the interpretation of π is $\pi_1^\star;(id_{P^{\perp\star}} \times \Gamma^\star) = \pi_1^\star; id_{P^{\perp\star}\times\Gamma^\star} = \pi_1^\star = \pi'^\star$ (with lemma 9).
- ($\otimes - \parr$) If π_1 and π_2 are the premises of the $\otimes$-rule and π_0 is the premise of the $\parr$-rule, the interpretation of π is $\pi_0^\star;(\pi_1^\star \times \pi_2^\star \times \Gamma^\star) = \pi_0^\star;(\pi_1^\star \times M^\star \times \Gamma^\star);(\Delta^\star \times \pi_2^\star \times \Gamma^\star) = \pi'^\star$ (with lemma 9).
- ($\oplus - \&$) We consider the case of a $\oplus_1$-rule with premise π_0. If π_1 and π_2 are the premises of the $\&$-rule, the interpretation of π is $(\pi_1^\star + \pi_2^\star);(\pi_0^\star \times \Gamma^\star) = \pi_1^\star;(\pi_0^\star \times \Gamma^\star) = \pi'^\star$ (with lemma 10).
- ($! - ?d$) If π_1 is the premise of the $?d$-rule and π_2 is the premise of the !-rule, the interpretation of π is $\pi_1^\star;(!\pi_2^\star \times \Gamma^\star)$. A play in $\pi^\star$ comes from an interaction sequence $u \in \mathrm{int}(\updownarrow N^{\perp\star} \times \Gamma^\star, (?\Delta)^\star \times \Gamma^\star)$ such that $u \upharpoonright_{\updownarrow N^{\perp\star}\times\Gamma^\star} \in \pi_1^\star$ and $u \upharpoonright_{\updownarrow(\updownarrow N^{\perp\star}\times\Gamma^\star)^\perp\times(?\Delta)^\star\times\Gamma^\star} \in !\pi_2^\star \times \Gamma^\star$. We build an interaction sequence $v \in \mathrm{int}((?\Delta)^\star \times N^\star, (?\Delta)^\star \times \Gamma^\star)$ by induction on the length of u:
  - If u is empty, v is empty.
  - If $u = u'(\star, m)$ where $(\star, m)$ is an initial move in $(?\Delta)^\star \times \Gamma^\star$, and v' is the interaction sequence obtained from u', we define $v = v'(\star, m)$.
  - If $u = u'(\star, m)$ where $(\star, m)$ is an initial move in $\updownarrow N^{\perp\star} \times \Gamma^\star$, and v' is the interaction sequence obtained from u', we define $v = v'$.
  - If $u = u'n$ where n is an initial move in $N^{\perp\star}$, we define $v = v'(\star, n)$.
  - If $u = u'n$ where n is a non-initial move in $N^{\perp\star}$, we define $v = v'n$.
  - If $u = u'm$ where m is a move in the rightmost $\Gamma^\star$, we define $v = v'm$.
  - If $u = u'm$ where m is a move in the leftmost $\Gamma^\star$, we define $v = v'$.
  - If $u = u'm$ where m is a move in $(?\Delta)^\star$, we define $v = v'm_1m_2$ if m is an O-move and $v = v'm_2m_1$ if m is a P-move, where m_1 (resp. m_2) is an occurrence of m in the rightmost (resp. leftmost) $(?\Delta)^\star$.

  We can verify that $v \upharpoonright_{(?\Delta)^\star\times N^\star} \in \pi_2^\star$, $v \upharpoonright_{\updownarrow((?\Delta)^\star\times N^\star)^\perp\times(?\Delta)^\star\times\Gamma^\star} \in (?\Delta)^\star\times\pi_1^\star$ and $v \upharpoonright_{(?\Delta)^\star\times\Gamma^\star} = u \upharpoonright_{(?\Delta)^\star\times\Gamma^\star}$ thus $\pi^\star \subset \pi_2^\star;((?\Delta)^\star \times \pi_1^\star) = \pi'^\star$. The other direction is proved in a similar way.
- ($! - ?c$) If π_1 is the premise of the $?c$-rule and π_2 is the premise of the !-rule, the interpretation of π is $\pi_1^\star;(\Gamma^\star \times c_{(?P)^\star});(\Gamma^\star \times !\pi_2^\star) = \pi_1^\star;(\Gamma^\star \times (c_{(?P)^\star};!\pi_2^\star)) = \pi_1^\star;(\Gamma^\star \times ((!\pi_2^\star \times !\pi_2^\star);c_{(?\Delta)^\star})) = \pi_1^\star;(\Gamma^\star \times !\pi_2^\star \times (?P)^\star);(\Gamma^\star \times (?\Delta)^\star \times !\pi_2^\star);(\Gamma^\star \times c_{(?\Delta)^\star}) = \pi'^\star$ (with lemma 11).
- ($! - ?w$) If π_1 is the premise of the $?w$-rule and π_2 is the premise of the !-rule, a play in the interpretation of π comes from an interaction sequence $u \in \mathrm{int}(\Pi^{\perp\star}, \Gamma^\star \times (?A)^\star, \Gamma^\star \times (?\Delta)^\star)$ such that $u \upharpoonright_{\updownarrow\Pi^\star\times\Gamma^\star\times(?A)^\star}$ is a play in $\pi_1^\star$ if we replace every initial moves $(\star, m, \star)$ by $(\star, m)$ and $u \upharpoonright_{\updownarrow(\Gamma^\star\times(?A)^\star)^\perp\times\Gamma^\star\times(?\Delta)^\star} \in \Gamma^\star \times !\pi_2^\star$ so that the only moves in $(?A)^\star$ in u are initial moves $\star$. The play $u \upharpoonright_{\updownarrow\Pi^\star\times\Gamma^\star\times(?\Delta)^\star}$ is $u \upharpoonright_{\updownarrow\Pi^\star\times\Gamma^\star\times(?A)^\star}$ if we identify the initial move $\star$ of $(?A)^\star$ with the initial move $\star$ of $(?\Delta)^\star$ and thus belongs to $\pi'^\star$. The converse is similar.
- ($! - !$) If π_1 is the premise of the !-rule which does not introduce the cut !-formula and π_2 is

the premise of the !-rule which introduces the cut !-formula, the interpretation of π is:

$$\begin{aligned} !\pi_1^\star;((?\Gamma)^\star \times !\pi_2^\star \times N^\star) &= !(id_{N^\star};(\pi_1^\star \times N^\star);((?\Gamma)^\star \times !\pi_2^\star \times N^\star)) && \text{lemma 12} \\ &= !(\pi_1^\star;((?\Gamma)^\star \times (!M)^\star \times id_{N^\star});((?\Gamma)^\star \times !\pi_2^\star \times N^\star)) && (! - ?d) \text{ case} \\ &= !(\pi_1^\star;((?\Gamma)^\star \times !\pi_2^\star \times N^\star)) \\ &= \pi'^\star \end{aligned}$$

$(1 - \bot)$ This can be seen as a particular case of the $(! - ?w)$ reduction step.

$(\top)$ If π_1 is the premise of the *cut*-rule which is not the $\top$-rule, the interpretation of π is $\{\varepsilon\};(\pi_1^\star \times \Gamma^\star) = \{\varepsilon\} = \pi'^\star$. □

Corollary 1.1 (Soundness for proof-nets)
If $\mathcal{R} \to \mathcal{R}'$ then $\mathcal{R}^\star = \mathcal{R}'^\star$.

Using the game model, it is possible to extract from the strategy interpreting a proof π, the minimal identification of variables in π really required for π to be a correct proof.

Proposition 8 (Generalization of variables)
Let π be a proof of $\vdash \Gamma$ in LL_{pol}, there exists a proof π' of $\vdash \Gamma'$ and a substitution θ of variables by variables such that $\Gamma = \Gamma'\theta$, $\pi = \pi'\theta$ and moreover (π', Γ', θ) is the most general such triple.

Proof: The strategy $\pi^\star$ is a label-balanced strategy on $\Gamma^\star$. Let $\mathcal{A}$ be the non-labeled arena obtained from $\Gamma^\star$ by removing the variable names, $\pi^\star$ is a balanced strategy on $\mathcal{A}$. We can derive from it a (finest) partition p of the elements of $\mathcal{V}_\mathcal{A}$ such that if $\mathcal{A}'$ is the labeled arena obtained from $\mathcal{A}$ by associating the same variable to the elements of each class of p, $\pi^\star$ is a label-balanced strategy on $\mathcal{A}'$. Let θ be the substitution such that $\mathcal{A}'\theta = \Gamma^\star$, let Γ' be the sequent such that $\Gamma'\theta = \Gamma$ and $\Gamma'^\star = \mathcal{A}'$, and let π' be the proof of $\vdash \Gamma'$ such that $\pi'\theta = \pi$ (and thus $\pi'^\star = \pi^\star$), we can show that (π', Γ', θ) is the most general triple such that $\pi'\theta = \pi$ and $\Gamma'\theta = \Gamma$: for any other triple $(\pi'', \Gamma'', \theta'')$, we must have $\pi''^\star = \pi^\star = \pi'^\star$ and $\Gamma''^\star$ is a labeling of $\mathcal{A}$ so that $\mathcal{A}'\theta' = \Gamma''^\star$ for some substitution θ' and we have $\theta = \theta'' \circ \theta'$ and $\pi'' = \pi'\theta'$ and $\Gamma'' = \Gamma'\theta'$. □

In a model of non-labeled arenas and balanced strategies, two such proofs π and π' are identified.

3.2 Full completeness

Proposition 9 (Essentially surjective interpretation of formulas)
If $\mathcal{A}$ is a polarized arena, there exists a formula A such that $A^\star$ is isomorphic to $\mathcal{A}$.

Proof: By induction on the size of $\mathcal{A}$, we define A and an order-preserving bijection f from $A^\star$ to $\mathcal{A}$. We only consider the case $\mathcal{A}$ positive, the negative case is then easy to derive.

- If $\mathcal{A}$ is empty, we have $A = 0$ and f is the empty function.
- If $\mathcal{A}$ contains at least two trees, let $\mathcal{A}'$ and $\mathcal{A}''$ be two non-empty sub-forests of $\mathcal{A}$. By induction hypothesis, there exist two formulas A' and A'' and two functions f' and f'' such that f' (resp. f'') is an order-preserving bijection from $A'^\star$ (resp. $A''^\star$) to $\mathcal{A}'$ (resp. $\mathcal{A}''$). We choose $A = A' \oplus A''$ and f is the union of f' and f''.

- If $\mathcal{A}$ is a tree reduced to one node, we choose $A = 1$ and f associates the unique node of $\mathcal{A}$ with the unique node of $1^\star$.
- If $\mathcal{A}$ is a tree with exactly one leaf above the root which is in $\mathcal{V}$, let X be the label of this leaf, we choose $A = {!X}$ and f associates the root of $\mathcal{A}$ with the root of $(!X)^\star$ and the leaf of $\mathcal{A}$ with the leaf of $(!X)^\star$.
- If $\mathcal{A}$ is a tree with exactly one node above the root which is not in $\mathcal{V}$, let $\mathcal{A}'$ be the tree above the unique root of $\mathcal{A}$. By induction hypothesis, there exists a formula A' and a function f' such that f' is an order-preserving bijection from $A'^\star$ to $\mathcal{A}'^\perp$. We choose $A = {!A'^\perp}$ and f is defined by extending f' with the root of $\mathcal{A}$ as image of the root of $A^\star$.
- If $\mathcal{A}$ is a tree with more than one node above the root, let $\mathcal{A}'$ and $\mathcal{A}''$ be two non-empty sub-forests of the forest above the root (so that $\mathcal{A} \approx \updownarrow\mathcal{A}' \times \updownarrow\mathcal{A}''$). By induction hypothesis, there exist two formulas A' and A'' and two functions f' and f'' such that f' (resp. f'') is an order-preserving bijection from $A'^\star$ (resp. $A''^\star$) to $\updownarrow\mathcal{A}'$ (resp. $\updownarrow\mathcal{A}''$). We choose $A = A' \otimes A''$ and f is defined by associating $f'(x)$ (resp. $f''(x)$) with each node x of $A'^\star$ (resp. $A''^\star$) which is not the root and by associating the root of $\mathcal{A}$ with the root of $A^\star$.

The strategy $\sigma_f = \{s \in \mathcal{P}_{A^\star \to \mathcal{A}} \mid \forall t \leq^P s, t \upharpoonright_{\mathcal{A}} = f(t \upharpoonright_{A^\star})\} : A^\star \to \mathcal{A}$ is an isomorphism between $A^\star$ and $\mathcal{A}$. □

According to the convention of section 2.2, a strategy σ on $\Gamma^\star$ is required to be linear if Γ contains a positive formula and we apply this convention for the following statement.

Theorem 2 (Full completeness)
Let σ be a finite total label-balanced strategy on the arena $\Gamma^\star$, there exists a proof π of $\vdash \Gamma$ in LL_{pol} such that $\pi^\star = \sigma$.

In section 2.2, we have described how to "apply an LL_{pol} rule" to strategies on the arenas corresponding to the premises of the rule to get a strategy on the arena associated with its conclusion. Using this, we will be able to consider, for example, the strategy:

$$\dfrac{\overset{\sigma}{\vdash \Gamma, A} \qquad \dfrac{}{\vdash A^\perp, A}\,ax}{\vdash \Gamma, A}\,cut$$

if σ is a strategy on the arena associated with $\vdash \Gamma, A$, by applying the appropriate constructions to σ. This allows us to describe constructions on strategies with sequent calculus rules.

Proof: By induction on the pair (size of σ, size of Γ). Let $\Gamma = A_1, \dots, A_n$, we look at the main connectives of the A_is:

($\parr$) Let assume that $A_1 = B_1 \parr B_2$, σ is also a strategy on the arena corresponding to $\vdash B_1, B_2, A_2, \dots, A_n$. By induction hypothesis we get a proof of $\vdash B_1, B_2, A_2, \dots, A_n$ and by adding a $\parr$-rule, we obtain a proof of $\vdash \Gamma$.

We can also proceed as for the next cases by using the reversibility of $\parr$. Let σ' be the strategy given by:

$$\begin{prooftree}\AxiomC{σ}\noLine\UnaryInfC{$\vdash B_1 \parr B_2, A_2, \ldots, A_n$}\AxiomC{}\RightLabel{$ax$}\UnaryInfC{$\vdash B_1^\perp, B_1$}\AxiomC{}\RightLabel{$ax$}\UnaryInfC{$\vdash B_2^\perp, B_2$}\RightLabel{$\otimes$}\BinaryInfC{$\vdash B_1^\perp \otimes B_2^\perp, B_1, B_2$}\RightLabel{$\parr$}\UnaryInfC{$\vdash B_1^\perp \otimes B_2^\perp, B_1 \parr B_2$}\RightLabel{$cut$}\BinaryInfC{$\vdash B_1 \parr B_2, A_2, \ldots, A_n$}\end{prooftree}$$

By proposition 7 and theorem 1, we have $\sigma' = \sigma$. Moreover, by theorem 1, this strategy is also the same as:

$$\begin{prooftree}\AxiomC{σ}\noLine\UnaryInfC{$\vdash B_1 \parr B_2, A_2, \ldots, A_n$}\AxiomC{}\RightLabel{$ax$}\UnaryInfC{$\vdash B_1^\perp, B_1$}\AxiomC{}\RightLabel{$ax$}\UnaryInfC{$\vdash B_2^\perp, B_2$}\RightLabel{$\otimes$}\BinaryInfC{$\vdash B_1^\perp \otimes B_2^\perp, B_1, B_2$}\RightLabel{$cut$}\BinaryInfC{$\vdash B_1, B_2, A_2, \ldots, A_n$}\RightLabel{$\parr$}\UnaryInfC{$\vdash B_1 \parr B_2, A_2, \ldots, A_n$}\end{prooftree}$$

Let σ'' be the strategy obtained without the last rule, by induction hypothesis, we obtain a proof π'' of $\vdash B_1, B_2, A_2, \ldots, A_n$ such that $\sigma'' = \pi''^\star$. The strategy $\sigma' = \sigma$ is then the interpretation of:

$$\begin{prooftree}\AxiomC{π''}\noLine\UnaryInfC{$\vdash B_1, B_2, A_2, \ldots, A_n$}\RightLabel{$\parr$}\UnaryInfC{$\vdash B_1 \parr B_2, A_2, \ldots, A_n$}\end{prooftree}$$

($\perp$) If $\Gamma = \perp, A_2, \ldots, A_n$, let σ' be the strategy given by:

$$\begin{prooftree}\AxiomC{σ}\noLine\UnaryInfC{$\vdash \perp, A_2, \ldots, A_n$}\AxiomC{}\RightLabel{$1$}\UnaryInfC{$\vdash 1$}\RightLabel{$\perp$}\UnaryInfC{$\vdash 1, \perp$}\RightLabel{$cut$}\BinaryInfC{$\vdash \perp, A_2, \ldots, A_n$}\end{prooftree}$$

By proposition 7 and theorem 1, we have $\sigma' = \sigma$. Moreover, by theorem 1, this strategy is also the same as:

$$\begin{prooftree}\AxiomC{σ}\noLine\UnaryInfC{$\vdash \perp, A_2, \ldots, A_n$}\AxiomC{}\RightLabel{$1$}\UnaryInfC{$\vdash 1$}\RightLabel{$cut$}\BinaryInfC{$\vdash A_2, \ldots, A_n$}\RightLabel{$\perp$}\UnaryInfC{$\vdash \perp, A_2, \ldots, A_n$}\end{prooftree}$$

Let σ'' be the strategy obtained without the last rule, by induction hypothesis, we obtain a proof π'' of $\vdash A_2, \ldots, A_n$ such that $\sigma'' = \pi''^\star$. The strategy $\sigma' = \sigma$ is then the interpretation of:

$$\begin{prooftree}\AxiomC{π''}\noLine\UnaryInfC{$\vdash A_2, \ldots, A_n$}\RightLabel{$\perp$}\UnaryInfC{$\vdash \perp, A_2, \ldots, A_n$}\end{prooftree}$$

(&) Let assume that $A_1 = B_1 \,\&\, B_2$, let σ' be the strategy given by:

$$\begin{prooftree}\AxiomC{σ}\noLine\UnaryInfC{$\vdash B_1 \,\&\, B_2, A_2, \ldots, A_n$}\AxiomC{}\RightLabel{$ax$}\UnaryInfC{$\vdash B_1^\perp, B_1$}\RightLabel{$\oplus_1$}\UnaryInfC{$\vdash B_1^\perp \oplus B_2^\perp, B_1$}\AxiomC{}\RightLabel{$ax$}\UnaryInfC{$\vdash B_2^\perp, B_2$}\RightLabel{$\oplus_2$}\UnaryInfC{$\vdash B_1^\perp \oplus B_2^\perp, B_2$}\RightLabel{$\&$}\BinaryInfC{$\vdash B_1^\perp \oplus B_2^\perp, B_1 \,\&\, B_2$}\RightLabel{$cut$}\BinaryInfC{$\vdash B_1 \,\&\, B_2, A_2, \ldots, A_n$}\end{prooftree}$$

By proposition 7 and theorem 1, we have $\sigma' = \sigma$. Moreover, by theorem 1, this strategy is also the same as:

$$\dfrac{\dfrac{\sigma''_1}{\vdash B_1, A_2, \dots, A_n} \qquad \dfrac{\sigma''_2}{\vdash B_2, A_2, \dots, A_n}}{\vdash B_1 \mathbin{\&} B_2, A_2, \dots, A_n}\,\&$$

with $\sigma''_1 = \dfrac{\dfrac{\sigma}{\vdash B_1 \mathbin{\&} B_2, A_2, \dots, A_n} \qquad \dfrac{\dfrac{}{\vdash B_1^\perp, B_1}\,ax}{\vdash B_1^\perp \oplus B_2^\perp, B_1}\,\oplus_1}{\vdash B_1, A_2, \dots, A_n}\,cut$

and $\sigma''_2 = \dfrac{\dfrac{\sigma}{\vdash B_1 \mathbin{\&} B_2, A_2, \dots, A_n} \qquad \dfrac{\dfrac{}{\vdash B_2^\perp, B_2}\,ax}{\vdash B_1^\perp \oplus B_2^\perp, B_2}\,\oplus_2}{\vdash B_2, A_2, \dots, A_n}\,cut$

By induction hypothesis, we obtain a proof π''_1 of $\vdash B_1, A_2, \dots, A_n$ and a proof π''_2 of $\vdash B_2, A_2, \dots, A_n$ such that $\sigma''_1 = \pi''^{\star}_1$ and $\sigma''_2 = \pi''^{\star}_2$. The strategy $\sigma' = \sigma$ is then the interpretation of:

$$\dfrac{\dfrac{\pi''_1}{\vdash B_1, A_2, \dots, A_n} \qquad \dfrac{\pi''_2}{\vdash B_2, A_2, \dots, A_n}}{\vdash B_1 \mathbin{\&} B_2, A_2, \dots, A_n}\,\&$$

($\top$) If $\Gamma = \top, A_2, \dots, A_n$, $\sigma = \{\varepsilon\}$ is the interpretation of the proof:

$$\dfrac{}{\vdash \top, A_2, \dots, A_n}\,\top$$

We now assume that none of these connectives is a main one in Γ, so that we have $\Gamma = \Pi, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$. If Π is not empty, we look at its main connective:

(0) A strategy on $\vdash 0, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$ cannot be linear.

(1) A maximal view of a linear strategy on $\vdash 1, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$ is $(\star, ?, \dots, ?, q, \dots, q)\star_1$ (where $?$ is the unique initial move of $(?P_i)^\star$, q is the unique initial move of $(?X_j^\perp)^\star$ and $\star$ and $\star_1$ are the two moves of $\updownarrow 1^\star$), and this corresponds to the interpretation of the proof:

$$\dfrac{\dfrac{}{\vdash 1}\,1}{\vdash 1, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp}\,?w$$

($\oplus$) Let assume that $\Pi = Q_1 \oplus Q_2$, by linearity and innocence, there is exactly one move played by σ in $Q_1 \oplus Q_2$ and justified by the unique first move of $?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$. This implies that all the moves of σ in $Q_1 \oplus Q_2$ are in one of the Q_is depending on this particular move. If this move is in Q_i, then σ is also a strategy on $(Q_i, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp)^\star$ and by induction hypothesis we get a proof of $\vdash Q_i, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$ that can be completed into a proof of $\vdash Q_1 \oplus Q_2, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$ by adding a $\oplus_i$-rule and its interpretation is σ.

($\otimes$) If $\Pi = Q_1 \otimes Q_2$, let ρ_i be the strategy associated with the following proof:

$$\dfrac{\dfrac{\dfrac{}{\vdash Q_i^\perp, Q_i}\,ax}{\vdash Q_1^\perp, Q_2^\perp, Q_i}\,\text{lemma 2}}{\vdash Q_1^\perp \parr Q_2^\perp, Q_i}\,\parr$$

By induction hypothesis we get a proof π_1 of $\vdash Q_1, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$ from $\rho_1;\sigma$ and a proof π_2 of $\vdash Q_2, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$ from $\rho_2;\sigma$, and if we add a $\otimes$-rule between π_1 and π_2 and $n+k$ $?c$-rules, we obtain a proof of $\vdash \Gamma$ whose interpretation is $((\rho_1;\sigma) \times (\rho_2;\sigma)); (c_{(?P_1)^\star} \times ?P_2 \times ?P_2 \times \cdots \times ?P_n \times ?P_n \times ?X_1^\perp \times ?X_1^\perp \times \cdots \times ?X_k^\perp \times ?X_k^\perp); \cdots; (?P_1 \times \cdots \times ?P_n \times ?X_1^\perp \times \cdots \times ?X_{k-1}^\perp \times c_{(?X_k^\perp)^\star}) = \sigma$.

Remark: this could be made more similar to the &-case, by studying the proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash Q_1^\perp, Q_1}\,ax}{\vdash Q_1^\perp, Q_2^\perp, Q_1}\,\text{lemma 2}}{\vdash Q_1^\perp \parr Q_2^\perp, Q_1}\,\parr \qquad \dfrac{\dfrac{\dfrac{}{\vdash Q_2^\perp, Q_2}\,ax}{\vdash Q_1^\perp, Q_2^\perp, Q_2}\,\text{lemma 2}}{\vdash Q_1^\perp \parr Q_2^\perp, Q_2}\,\parr}{\vdash Q_1^\perp \parr Q_2^\perp, Q_1^\perp \parr Q_2^\perp, Q_1 \otimes Q_2}\,\otimes}{\vdash Q_1^\perp \parr Q_2^\perp, Q_1 \otimes Q_2}\,\text{lemma 2}$$

and by showing that its interpretation is the identity.

$(!N)$ If $\Pi = !N$, let σ' be the strategy given by:

$$\dfrac{\begin{matrix}\sigma \\ \vdash !N, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp\end{matrix} \qquad \dfrac{\dfrac{\dfrac{}{\vdash N^\perp, N}\,ax}{\vdash ?N^\perp, N}\,?d}{\vdash ?N^\perp, !N}\,!}{\vdash !N, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp}\,cut$$

By proposition 7 and theorem 1, we have $\sigma' = \sigma$. Moreover, by theorem 1, this strategy is also the same as:

$$\dfrac{\dfrac{\begin{matrix}\sigma \\ \vdash !N, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp\end{matrix} \qquad \dfrac{\dfrac{}{\vdash N^\perp, N}\,ax}{\vdash ?N^\perp, N}\,?d}{\vdash N, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp}\,cut}{\vdash !N, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp}\,!$$

Let σ'' be the strategy obtained without the last rule, by induction hypothesis, we obtain a proof π'' of $\vdash N, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$ such that $\sigma'' = \pi''^\star$. The strategy $\sigma' = \sigma$ is then the interpretation of:

$$\dfrac{\begin{matrix}\pi'' \\ \vdash N, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp\end{matrix}}{\vdash !N, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp}\,!$$

($!X$) If $\Pi = !X$, by linearity, a (long enough) view in σ starts like this:

$$\begin{array}{cccccccc}
\updownarrow !X & ?P_1 & \dots & ?P_n & ?X_1^\perp & \dots & ?X_k^\perp \\
\star & ? & \dots & ? & q & \dots & q \\
q & & & & & & \\
\checkmark_X & & & & & &
\end{array}$$

and then contains a move m that must be a $\checkmark_{X_j}$ in one of the $?X_j^\perp$s (which is an occurrence of $?X^\perp$ thus $X_j = X$) by the label-balancing condition. A view cannot continue after that. This means that σ is the interpretation of the proof:

$$\dfrac{\dfrac{}{\vdash !X, ?X^\perp}\,ax}{\vdash !X, ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp}\,?w$$

We now arrive to the case of a sequent of the shape $\vdash ?P_1, \dots, ?P_n, ?X_1^\perp, \dots, ?X_k^\perp$. If a view of σ contains two moves justified by the first one in the same formula, we linearize this formula in the sequent to obtain a strategy σ' on the arena associated with $\vdash \cdots ?P_1^i \cdots, \dots, \cdots ?P_n^i \cdots, ?X_1^\perp, \dots, ?X_k^\perp$ (we do not need to linearize the $?X_j^\perp$ formulas since a view cannot contain two moves justified by the first one) such that σ' composed with the corresponding contraction strategies is σ (this corresponds to $?c$-rules). By determinism, the first P-move is always in the same formula, and by linearization there is at most one move justified by the first one in this formula in a view of σ'. We have two possible cases for this formula:

($?P_{i_0}$) σ' can be transformed into a strategy on $\vdash \cdots ?P_1^i \cdots, \dots, P_{i_0}, \dots, \cdots ?P_n^i \cdots, ?X_1^\perp, \dots, ?X_k^\perp$ and by induction hypothesis we obtain a proof such that the interpretation of this proof followed by a $?d$-rule is σ';

($?X_{j_0}^\perp$) this is impossible because this P-move in $?X_{j_0}^\perp$ must be after a move in $\mathcal{V}$ according to the balancing condition. □

Corollary 2.1 (Full completeness for proof-nets)
Let σ be a finite total label-balanced strategy on an arena $\Gamma^\star$, there exists a sliced proof-net $\mathcal{R}$ with conclusions Γ such that $\mathcal{R}^\star = \sigma$.

3.3 Faithful completeness

Theorem 3 (Faithful completeness)
If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two cut-free sliced proof-nets such that $\mathcal{R}_1^\star = \mathcal{R}_2^\star$ then $\mathcal{R}_1 = \mathcal{R}_2$.

PROOF: First, if one of the conclusions of $\mathcal{R}_1$ and $\mathcal{R}_2$ is positive, we can add a $\flat$-node and a ?-node in both $\mathcal{R}_1$ and $\mathcal{R}_2$ to get purely negative conclusions. Let σ be the strategy $\mathcal{R}_1^\star = \mathcal{R}_2^\star$, we prove the result by induction on the size of $\mathcal{R}_1$.

- If one of the conclusions of $\mathcal{R}_1$ and $\mathcal{R}_2$ has a $⅋$ (resp. $\perp$) as main connective both $\mathcal{R}_1$ and $\mathcal{R}_2$ must have a $⅋$ (resp. $\perp$) node above it. We can remove it and we obtain two proof-nets $\mathcal{R}_1'$ and $\mathcal{R}_2'$ with the same interpretation so that, by induction hypothesis, $\mathcal{R}_1' = \mathcal{R}_2'$ and finally $\mathcal{R}_1 = \mathcal{R}_2$.

- If one of the conclusions of $\mathcal{R}_1$ and $\mathcal{R}_2$ has a $\&$ as main connective we denote by $\mathcal{R}_1'$ (resp. $\mathcal{R}_1''$) the set of the slices of $\mathcal{R}_1$ containing the corresponding $\&_1$ (resp. $\&_2$) node in which we remove this node, and the same for $\mathcal{R}_2'$ and $\mathcal{R}_2''$. Since both $\mathcal{R}_1'$ and $\mathcal{R}_2'$ are obtained from $\mathcal{R}_1$ and $\mathcal{R}_2$ by eliminating cut with the translation of the proof:

$$\dfrac{\dfrac{}{\vdash N^\perp, N}\,ax}{\vdash N^\perp \oplus M^\perp, N}\,\oplus_1$$

  this entails $\mathcal{R}_1'^\star = \mathcal{R}_2'^\star$ (and in the same way $\mathcal{R}_1''^\star = \mathcal{R}_2''^\star$), thus by induction hypothesis $\mathcal{R}_1' = \mathcal{R}_2'$ (and $\mathcal{R}_1'' = \mathcal{R}_2''$) so that $\mathcal{R}_1 = \mathcal{R}_2$.
- If one of the conclusions of $\mathcal{R}_1$ and $\mathcal{R}_2$ has a $\top$ main connective, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are empty.
- If all the conclusions of $\mathcal{R}_1$ and $\mathcal{R}_2$ are ?-formulas: $?A_1, \ldots, ?A_k$, both $\mathcal{R}_1$ and $\mathcal{R}_2$ contain a ?-node for each $?A_i$. We look at the answer m_2 of σ to the unique initial move m_1 of the corresponding arena. The move m_2 is in one of the A_is and the corresponding ?-node must have a $\flat$-node above it in both $\mathcal{R}_1$ and $\mathcal{R}_2$. Moreover we can show that m_2 describes the structure of the "positive tree" $\mathcal{T}$ above A_i. That is the sub-graph containing positive nodes and !-nodes and with conclusion A_i. We do it by induction on the positive type A_i:
  - if its main connective is a $\otimes$, $\mathcal{T}$ must end with a $\otimes$-node;
  - if its main connective is a $\oplus$, the move m_2 is in an arena of the shape $P + Q$ and if m_2 belongs to P, $\mathcal{T}$ ends with a $\oplus_1$-node and if m_2 belongs to Q, $\mathcal{T}$ ends with a $\oplus_2$-node;
  - if its main connective is a 1, $\mathcal{T}$ is reduced to a 1-node;
  - if its main connective is a !, $\mathcal{T}$ is reduced to a !-node.

  Since we have found the unique $\flat$-node at this depth, all the other $\flat$-formulas (of the shape $\flat A_i$) are conclusions of !-nodes and these !-nodes are leaves of $\mathcal{T}$. Given such a !-node n, we consider the proof-net $\mathcal{R}_1^n$, without $\flat$-conclusions, obtained by adding to the proof-net associated with n, k ?-nodes (corresponding to those with conclusions $?A_1, \ldots, ?A_k$) in such a way that two conclusions of n are premises of the same ?-node in $\mathcal{R}_1$ if and only if they are premises of the same ?-node in $\mathcal{R}_1^n$.

  If the premise of n has type X, the proof-net associated with n must be the following: above this X we can only have an *ax*-node and the other conclusion of this node (of type $X^\perp$) must be the premise of a $\flat$-node. Using the correctness criterion and the fact that the conclusions of this proof-net which are not X are $\flat$-formulas, we can see that no other node appears in this proof-net and we easily have $\mathcal{R}_1^n = \mathcal{R}_2^n$.

  Otherwise, let m_3 be the move corresponding to the premise of n, we consider the strategy σ' with views $(m_1, m_3)s$ where $m_1 m_2 m_3 s$ is a view of σ. We can show that $\sigma' = \mathcal{R}_1^{n\star}$ (and in the same way $\sigma' = \mathcal{R}_2^{n\star}$) so that $\mathcal{R}_1^{n\star} = \mathcal{R}_2^{n\star}$ and by induction hypothesis $\mathcal{R}_1^n = \mathcal{R}_2^n$.

We have shown that the 0-depth parts of $\mathcal{R}_1$ and $\mathcal{R}_2$ are the same, except maybe for the $\flat$-typed edges. Moreover for each !-node n, the associated proof-nets coming from $\mathcal{R}_1^n$ and $\mathcal{R}_2^n$ are the same. But, by definition of $\mathcal{R}_1^n$, we can rebuild the 0-depth $\flat$-typed edges of $\mathcal{R}_1$ since a $\flat$-conclusion of the node n has an edge to a given ?-node if it has an edge to the corresponding ?-node in $\mathcal{R}_1^n$, and the same for $\mathcal{R}_2$ and $\mathcal{R}_2^n$. We can conclude $\mathcal{R}_1 = \mathcal{R}_2$. $\square$

3.4 Categorical interpretation

To explain the relation between our results and the terminology coming from categories we reformulate them in (control) categorical terms. This leads to an equivalence of categories.

In order to build categories, we have to break the symmetry between positive and negative objects. Both choices are possible and we focus on the negative case since this corresponds to what we have already done with strategies.

Definition 25 (Syntactical category)
Our *syntactical category* of sliced proof-nets is given by:

- *objects*: objects are negative formulas.
- *morphisms*: a morphism from N to M is a cut-free sliced proof-net with conclusions $?N^\perp$ and M.
- *identity*: the identity morphism from N to N is the proof-net associated with the proof:

$$\dfrac{\dfrac{}{\vdash N^\perp, N}\,ax}{\vdash ?N^\perp, N}\,?d$$

- *composition*: if $\mathcal{R}_1$ is a proof-net with conclusions $?N^\perp$ and M and $\mathcal{R}_2$ is a proof-net with conclusions $?M^\perp$ and L, the composition of $\mathcal{R}_1$ and $\mathcal{R}_2$ is obtained by the normalization of $\mathcal{R}_1$ in a !-box cut on $!M$ with the conclusion $?M^\perp$ of $\mathcal{R}_2$ which gives a cut-free proof-net with conclusions $?N^\perp$ and L.

Definition 26 (Game category)
The *game category* is given from game semantics by:

- *objects*: objects are negative labeled arenas;
- *morphisms*: a morphism from A to B is a finite total label-balanced strategy on $\updownarrow A^\perp \times B$.
- *identity*: the identity morphism from A to A is the identity strategy id_A on $\updownarrow A^\perp \times A$.
- *composition*: if $\sigma : \updownarrow A^\perp \times B$ and $\tau : \updownarrow B^\perp \times C$ are two strategies, the composition of these strategies gives a strategy $\sigma;\tau$ on $\updownarrow A^\perp \times C$.

Theorem 4 (Equivalence completeness)
There exists an equivalence of categories between the syntactical category of sliced proof-nets and the game category.

Proof: Corollary 1.1 allows to show that $\mathcal{R} \mapsto \mathcal{R}^\star$ defines a functor from the syntactical category to the game category and by proposition 9, corollary 2.1 and theorem 3, it is an equivalence of categories. □

Corollary 4.1 (Isomorphisms of types)
If A and B are two polarized formulas and if there exists an isomorphism between $A^\star$ and $B^\star$ in the game category, then A and B are isomorphic in the syntactical category.

Remark: We have shown in [22] how it is possible to use the game model described here to characterize the isomorphisms of types of classical logic. This requires a model that contains exactly the isomorphisms of the logic (not more) and in which these isomorphisms are possible to compute. In [22] we obtained the first point in a indirect way by computing isomorphisms and by verifying that all of them are valid in the syntax. We have here a direct proof that syntactical isomorphisms and game isomorphisms are the same.

The equivalence result can also be expressed in the particular setting of Selinger's control categories [29].

Proposition 10 (Control categories)
Both the category of sliced proof-nets and the category of games are control categories.

Lemma 13
If $\sigma : \ddagger A \times B$ is a linear strategy, it is a central morphism.

PROOF: By theorems 2 and 3, σ is the interpretation of a proof-net which ends with a unary ?-node and we can verify that such a proof-net is central in the syntactical category thus σ is central in the game category by theorem 1. □

The converse is also true and proved in [20].

Corollary 4.2 (Equivalence of control categories)
There exists an equivalence of control categories between the syntactical category of sliced proof-nets and the game category.

PROOF: According to Selinger's definition of equivalence of control categories [29], we just have to remark that in the proof of proposition 9 the isomorphism σ_f is linear thus central by lemma 13. □

The main direction to extend the previous results is the introduction of second order quantification which is not very problematic on the syntactical side but more tricky for the game model.

Whereas proof-nets seem to give an almost ultimate solution for the analysis of syntax, we could try to find some other semantical presentations of this polarized logic, that is some other equivalent categories.

Acknowledgments. I would like to thank V. Danos for his suggestions on the polarized game model that have led to this work and the referee for his useful comments.

A Expansion of axioms (η-rules)

$$\dfrac{}{\vdash 1, \bot}\, ax \quad \rightsquigarrow \quad \dfrac{\dfrac{}{\vdash 1}\, 1}{\vdash 1, \bot}\, \bot$$

$$\dfrac{}{\vdash 0, \top}\, ax \quad \rightsquigarrow \quad \dfrac{}{\vdash 0, \top}\, \top$$

$$\dfrac{}{\vdash N^\perp \otimes M^\perp, N \parr M}\, ax \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{\vdash N^\perp, N}\, ax \quad \dfrac{}{\vdash M^\perp, M}\, ax}{\vdash N^\perp \otimes M^\perp, N, M}\, \otimes}{\vdash N^\perp \otimes M^\perp, N \parr M}\, \parr$$

$$\dfrac{}{\vdash N^\perp \oplus M^\perp, N \,\&\, M}\, ax \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{\vdash N^\perp, N}\, ax}{\vdash N^\perp \oplus M^\perp, N}\, \oplus_1 \quad \dfrac{\dfrac{}{\vdash M^\perp, M}\, ax}{\vdash N^\perp \oplus M^\perp, M}\, \oplus_2}{\vdash N^\perp \oplus M^\perp, N \,\&\, M}\, \&$$

$$\dfrac{}{\vdash !N, ?N^\perp}\, ax \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{\vdash N, N^\perp}\, ax}{\vdash N, ?N^\perp}\, ?d}{\vdash !N, ?N^\perp}\, !$$

B A sliced proof-net

Flat proof-structures. The five graphs of figure 1 are *flat proof-structures.*

Slices and sliced proof-structures. Starting from these five flat proof-structures, we define the following *slices* s_i and *sliced proof-structures* $\mathcal{S}_i$:

- $\mathcal{S}_1 = \{s_1\}$ with $s_1 = \mathcal{F}_1$ which has no !-node.
  It has conclusions X, $\flat X^\perp$.
- $\mathcal{S}_2 = \{s_2\}$ where s_2 is obtained by associating $\mathcal{S}_1$ with the unique !-node of $\mathcal{F}_2$.
  It has conclusions $\flat(1 \otimes (!(\bot \,\&\, (\top \parr N)) \oplus !X))$, $?X^\perp$.
- $\mathcal{S}_3 = \{s_3\}$ where s_3 is obtained by associating $\mathcal{S}_2$ with the unique !-node of $\mathcal{F}_3$.
  It has conclusions $\bot \,\&\, (\top \parr N)$, $\flat(1 \otimes (!(\bot \,\&\, (\top \parr N)) \oplus !X))$, $\flat !?X^\perp$.
- $\mathcal{S}_4 = \{s_4\}$ where s_4 is obtained by associating $\mathcal{S}_3$ with the unique !-node of $\mathcal{F}_4$.
  It has conclusions $?(1 \otimes 1) \parr ?(1 \otimes (!(\bot \,\&\, (\top \parr N)) \oplus !X))$, $(\bot \parr \bot) \,\&\, ?!?X^\perp$.
- $\mathcal{S}_5 = \{s_5\}$ where $s_5 = \mathcal{F}_5$ which has no !-node.
  It has conclusions $?(1 \otimes 1) \parr ?(1 \otimes (!(\bot \,\&\, (\top \parr N)) \oplus !X))$, $(\bot \parr \bot) \,\&\, ?!?X^\perp$.
- $\mathcal{S} = \{s_4, s_5\}$.
  It has conclusions $?(1 \otimes 1) \parr ?(1 \otimes (!(\bot \,\&\, (\top \parr N)) \oplus !X))$, $(\bot \parr \bot) \,\&\, ?!?X^\perp$.

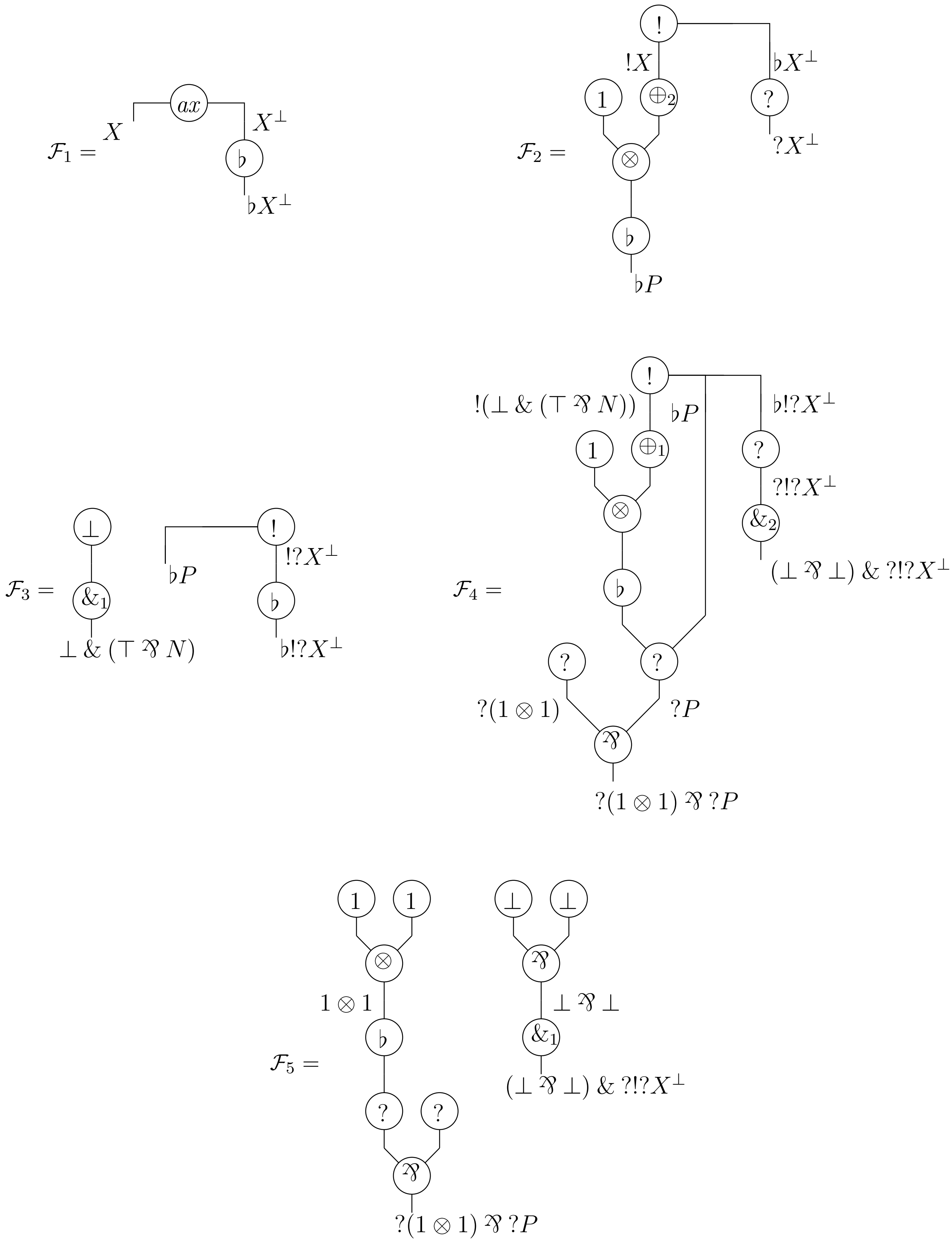


Figure 1: Some flat proof-structures (with $P = 1 \otimes (!(\bot \,\&\, (\top \parr N)) \oplus !X)$ and with some omitted types)

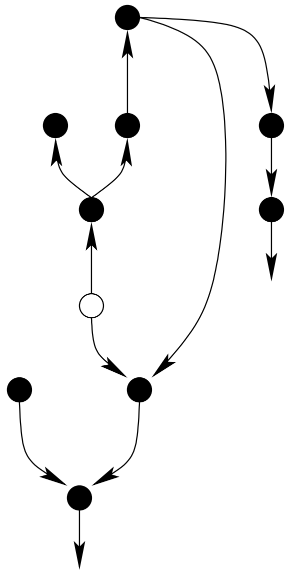


Figure 2: A correction graph

Acceptable proof-structures. $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}$ are *acceptable*. For example, the correction graph of $\mathcal{F}_4$ is given in figure 2. It is acyclic and it contains exactly one $\flat$-node.

Correct proof-structures. There is two occurrences of $\&_i$-nodes at depth 0 in $\mathcal{S}$ and they are equivalent with respect to $\equiv$. We associate the boolean variable p with this equivalence class. There is also a $\&_1$-node at depth 0 in $\mathcal{S}_3$ with which we associate the variable q. We have:

$$\begin{aligned}
w(s_3) &= q\\
w(s_4) &= \overline{p}\\
w(s_5) &= p\\
w(\mathcal{S}_3) &= w(s_3) = q\\
w(\mathcal{S}) &= w(s_4) + w(s_5) = 1\\
w_\top(\mathcal{S}_3) &= w_\top(\bot \mathbin{\&} (\top \parr N)) + w_\top(?(1 \otimes (!(\bot \mathbin{\&} (\top \parr N)) \oplus !X))) + w_\top(?!?X^\bot)\\
&= w_\top(\bot \mathbin{\&} (\top \parr N)) = q w_\top(\bot) + \overline{q} w_\top(\top \parr N) = \overline{q} w_\top(\top) + \overline{q} w_\top(N)\\
&= \overline{q}\\
w_\top(\mathcal{S}) &= w_\top(?(1 \otimes 1) \parr ?(1 \otimes (!(\bot \mathbin{\&} (\top \parr N)) \oplus !X))) + w_\top((\bot \parr \bot) \mathbin{\&} ?!?X^\bot)\\
&= w_\top(?(1 \otimes 1)) + w_\top(?(1 \otimes (!(\bot \mathbin{\&} (\top \parr N)) \oplus !X))) + p w_\top(\bot \parr \bot) + \overline{p} w_\top(?!?X^\bot)\\
&= p w_\top(\bot) + p w_\top(\bot)\\
&= 0
\end{aligned}$$

Thus $w(\mathcal{S}_3) + w_\top(\mathcal{S}_3) = q + \overline{q} = 1$, $w(\mathcal{S}) + w_\top(\mathcal{S}) = 1$ and $w(s_4)w(s_5) = \overline{p}p = 0$ (the other sliced proof-structures $\mathcal{S}_1$ and $\mathcal{S}_2$ are immediately full and compatible). So that $\mathcal{S}$ is a *proof-net*.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\vdash 1$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\vdash 1$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash {!X}, ?X^\perp$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash {!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X}, ?X^\perp$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash 1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X}), ?X^\perp$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?X^\perp$}
\RightLabel{$!$}
\UnaryInfC{$\vdash ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), {!?X^\perp}$}
\RightLabel{$\perp$}
\UnaryInfC{$\vdash \perp, ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), {!?X^\perp}$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash \perp, ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\AxiomC{}
\RightLabel{$\top$}
\UnaryInfC{$\vdash \top, N, ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash \top \parr N, ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash \perp \mathbin{\&} (\top \parr N), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash ?(1 \otimes 1), \perp \mathbin{\&} (\top \parr N), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$!$}
\UnaryInfC{$\vdash ?(1 \otimes 1), {!(\perp \mathbin{\&} (\top \parr N))}, ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash ?(1 \otimes 1), {!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X}, ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash ?(1 \otimes 1), 1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X}), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(1 \otimes 1), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$?c$}
\UnaryInfC{$\vdash ?(1 \otimes 1), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\noLine
\UnaryInfC{$\vdots$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash 1 \otimes 1, \perp \parr \perp$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash ?(1 \otimes 1), 1 \otimes 1, \perp \parr \perp$}
\RightLabel{$?d$}
\UnaryInfC{$\vdash ?(1 \otimes 1), ?(1 \otimes 1), \perp \parr \perp$}
\RightLabel{$?c$}
\UnaryInfC{$\vdash ?(1 \otimes 1), \perp \parr \perp$}
\RightLabel{$?w$}
\UnaryInfC{$\vdash ?(1 \otimes 1), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), \perp \parr \perp$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\vdash ?(1 \otimes 1), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), ?{!?X^\perp}$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash ?(1 \otimes 1), ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), (\perp \parr \perp) \mathbin{\&} ?{!?X^\perp}$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash ?(1 \otimes 1) \parr ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), (\perp \parr \perp) \mathbin{\&} ?{!?X^\perp}$}
\end{prooftree}
$$

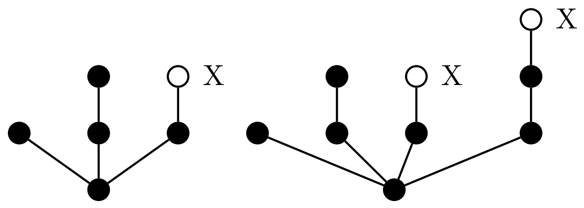
Figure 3: A proof in LL_{pol}

Translation of sequent calculus proofs. If we consider the proof of LL_{pol} in figure 3, the associated sliced proof-structure is $\mathcal{S}$.

Associated arenas. The arena associated with the conclusion:

$$\vdash ?(1 \otimes 1) \parr ?(1 \otimes ({!(\perp \mathbin{\&} (\top \parr N))} \oplus {!X})), (\perp \parr \perp) \mathbin{\&} ?{!?X^\perp}$$

of the proof (and of the proof-net $\mathcal{S}$) is:



with polarity O.

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