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Slicing polarized additive normalization

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Abstract

To attack the problem of "computing with the additives", we introduce a notion of sliced proof-net for the polarized fragment of linear logic. We prove that this notion yields computational objects, sequentializable in the absence of cuts. We then show how the injectivity property of denotational semantics guarantees the "canonicity" of sliced proof-nets, and prove injectivity for the fragment of polarized linear logic corresponding to the simply typed λ-calculus with pairing.

1.1 Introduction

The question of equality of proofs is an important one in the "proofs-as-programs" paradigm. Traditional syntaxes (sequent calculus, natural deduction, ...) distinguish proofs which are clearly the same as computational processes. On the other hand, denotational semantics identifies "too many" proofs (two different stages of the same computation are always identified). The seek of an object sticking as much as possible to the computational nature of proofs led to the introduction of a new syntax for logic: proof-nets, a graph-theoretic presentation which gives a more geometric account of proofs (see [Gir87]). This discovery was achieved by a sharp (syntactical and semantical) analysis of the cut-elimination procedure.

Any person with a little knowledge of the multiplicative framework of linear logic (LL), has no doubt that proof-nets are the canonical representation of proofs. But as soon as one moves from such a fragment, the notion of proof-net appears "less pure". A reasonable solution for the multiplicative and exponential fragment of LL (with quantifiers) does exist (combining [Dan90] and [Gir91b], like in [TdF00]). Turning to multiplicative and additive LL (MALL), the situation radically changes: since the introduction of proof-nets [Gir87], the additives were treated in an unsatisfactory way, by means of "boxes". Better solutions have been proposed in [Gir96] and [TdF03], until the paper [HvG03] introduced "the good notion" of proof-net for cut-free MALL. But still, trying to deal with the full propositional fragment means entering a true jungle. Of course, it is possible to survive (i.e. to compute) in this jungle, as shown in [Gir87, TdF00]. So what? The problem is that the objects (the proof-nets) used are definitely not *canonical*†.

Recently, a new fragment of LL appeared to have a great interest: in [Gir91a] and [DJS97] the *polarized fragment* of LL is shown to be enough to translate faithfully classical logic. A study of proof-nets for such a fragment was undertaken in [Lau99], and the notion of [Gir96] drastically simplified. In [LQTdF00] a proof of strong normalization and confluence of the cut-elimination procedure is given for polarized LL, using the syntax of [Gir87] (notice that for full LL confluence is wrong and strong normalization is still not completely proven). Despite these positive results, the notion of proof-net still appears as (more or less desperately, depending on the cases) non canonical.

The first contribution of the present paper is the proposal of a mathematical counterpart for the term "canonical". And here is where denotational semantics comes into the picture: in [TdF01], the question of *injectivity* of denotational semantics is addressed for proof-nets. Roughly speaking, denotational semantics is said to be injective when the equivalence relation it defines on proofs coincides with the one defined by the cut-elimination procedure. Our proposal is to let semantics decide on the canonicity of some notion of proof-net: this is canonical when there exists a (non contrived, obviously!) denotational semantics which is injective with respect to the would-be canonical notion of proof-net.

Notice that this is a rather severe notion of canonicity. Indeed, proof-nets for multiplicative LL are canonical (and this is probably true also for MALL using [HvG03]), but the previously mentioned extension to multiplicative and exponential LL is not guaranteed to be canonical: the time being we only know that coherent (set and multiset based) semantics *is not* injective for such proof-nets (see [TdF01]). Finally, the

† We will use the term canonical in an intuitive way, following the idea that a canonical representation of a proof is not sensitive to inessential commutations of rules.

known syntaxes for full LL (with additives) are obviously not canonical for the usual semantics of linear logic.

The notion of *slice* was first introduced in [Gir87]. The idea is very simple: instead of dealing with both the components of an additive box "at the same time", what about working with these two components separately? This attitude is tempting because it ignores the superimposition notion underlying the connective & (which is precisely the difficult point to understand). It is shown in [Gir96] that the correctness of the slices of a proof-structure does not imply the correctness of the proof-structure itself (see also [HvG03]). However, this turns out to be true in a polarized and cut-free framework (theorem 1.32).

In section 1.2, we give some intuitions on the original notion of slice for MALL coming from [Gir87].

We then define, in section 1.3, a notion of *sliced proof-structure* for polarized LL (definition 1.5), and we show how to translate sequent calculus proofs into sliced proof-structures. To obtain canonical objects, we deal with atomic axioms and proof-structures in the style of the "nouvelle syntaxe" of Danos and Regnier [Reg92]. For this purpose, we introduce $\flat$-formulas which do not occur in sequent calculus, but are very useful in our framework: a formula $\flat A$ is necessarily the premise of a ?-link. The notation (and the meaning) of $\flat A$ is clearly very much inspired from Girard's works on ludics [Gir01] and on light linear logic [Gir95].

We introduce in section 1.4 the relational semantics. We adapt the definition of experiment of [Gir87] to our framework, and we define the interpretation of a sliced proof-structure (definition 1.13). Particular experiments coming from [TdF01] are also introduced (injective 1-experiments), to be used later in section 1.8.

Section 1.5 is devoted to define and to study the notion of "correct" sliced proof-structure (or sliced *proof-net*). The polarization constraints allow to apply to our framework the *correctness criterion* of [Lau99]. We define a *sliced* cut-elimination procedure (definition 1.21), we prove that correctness is preserved by our sliced cut-elimination steps (theorem 1.24) and that our semantical interpretation is sound (theorem 1.26). Our sliced proof-nets are thus proven to be computational objects.

In section 1.6, we prove that in the absence of cuts, the correctness criterion (plus some obviously necessary conditions on sets of slices) is enough to "glue" in a unique way different slices: a sliced proof-net comes from a sequent calculus proof (theorem 1.32). This result follows [Lau99] (where the &-jumps of [Gir96] are removed) and [Lau03] (where the remaining jumps for weakenings are also removed).

Section 1.7 explains and justifies in details our method: the use of injective denotational semantics as a witness of canonicity of our sliced proof-nets.

The reader should notice that this is the very first time a notion of proof-net containing the additives and the exponentials can really pretend to be canonical.

Finally, section 1.8 shows that our method makes sense: there exist interesting fragments of polarized LL for which denotational semantics is injective (and thus the corresponding proof-nets are canonical), like the λ-calculus with pairing. The result that we prove is an extension of the result of [TdF01]. Thanks to a remark of L. Regnier on the λ-calculus (expressed by proposition 1.48), we could avoid to reproduce the entire proof. We thus get injectivity only for relational semantics, but in a quick and simple way.

Let us conclude by stressing the fact that the last section is simply an example to illustrate the method explained in section 1.7, and it is (very) likely that injectivity for coherent and relational semantics holds for the whole polarized fragment. This would give canonical proof-nets for polarized LL, that is for classical logic (see [LQTdF00]).

1.2 A little history of slices

Slices were first introduced in [Gir87], and the following examples come directly from the ideas of that work.

In this section, we only want to give some hints of what will be developed in the following ones. In particular, all the notions used here simply have an intuitive meaning, and will be formally defined later.

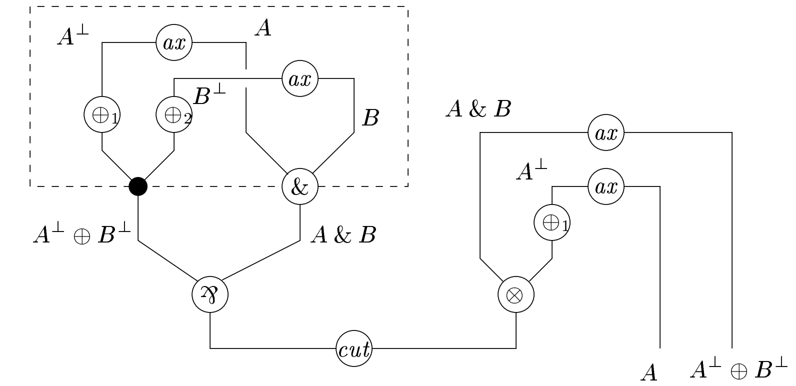
Intuitively, a slice of a proof is obtained by choosing, for every occurrence of the rule &, one of the two premises. With the sequent calculus proof obtained by adding a cut between

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,ax}{\vdash A^\perp \oplus B^\perp, A}\,\oplus_1 \quad \dfrac{\dfrac{}{\vdash B^\perp, B}\,ax}{\vdash A^\perp \oplus B^\perp, B}\,\oplus_2}{\vdash A^\perp \oplus B^\perp, A \mathbin{\&} B}\,\&}{\vdash (A^\perp \oplus B^\perp) \mathbin{⅋} (A \mathbin{\&} B)}\,⅋$$

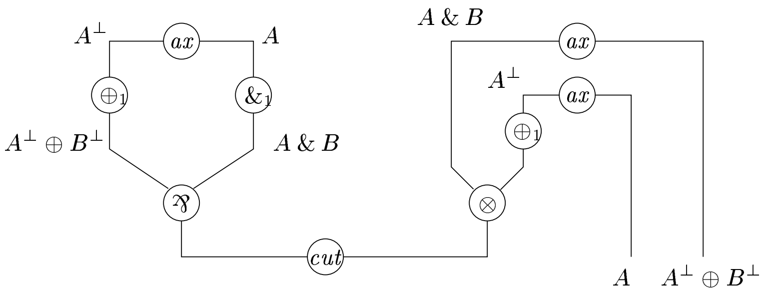
and

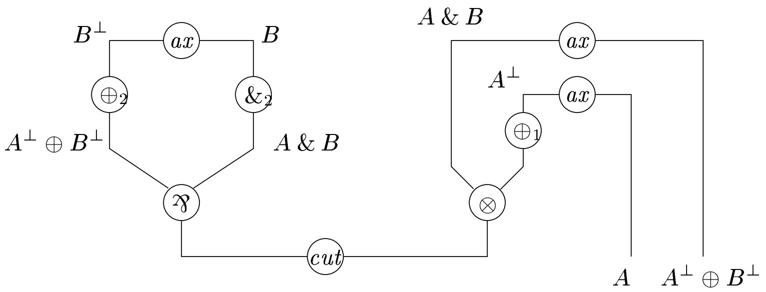
$$\dfrac{\dfrac{}{\vdash A \mathbin{\&} B, A^{\perp} \oplus B^{\perp}}\,ax \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,ax}{\vdash A^{\perp} \oplus B^{\perp}, A}\,\oplus_1}{\vdash (A \mathbin{\&} B) \otimes (A^{\perp} \oplus B^{\perp}), A, A^{\perp} \oplus B^{\perp}}\,\otimes$$

one would like to associate a graph, like:



where the dashed box is an attempt to express some kind of "superimposition" of two subgraphs. Choosing to work separately with each of these two subgraphs means "slicing" the proof-net into the two following slices (where the binary &-link is replaced by two unary &-links):

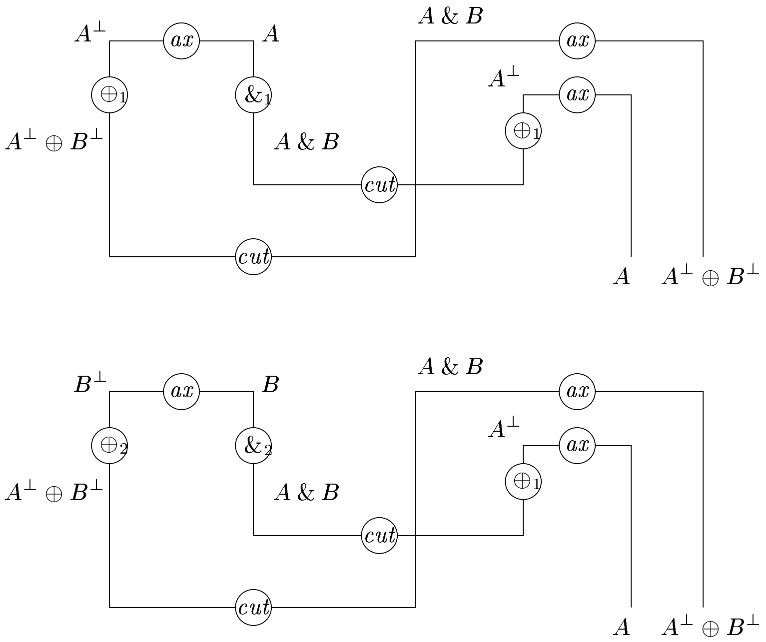




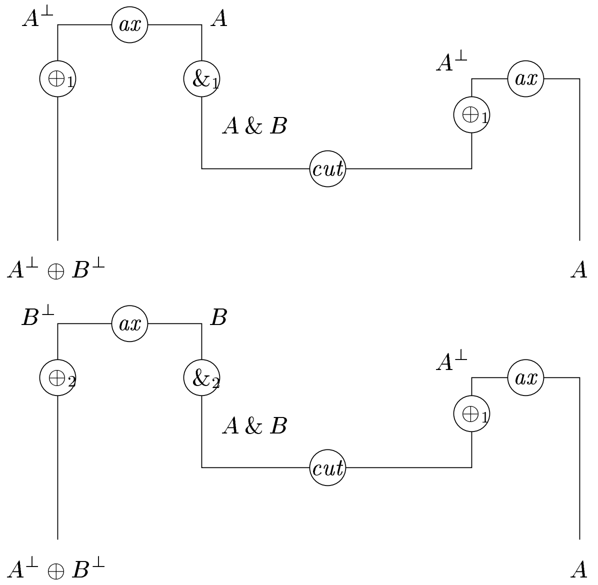
In [Gir96], Girard shows that the correctness of slices is not enough to ensure the correctness of the whole graph: it is easy to see that there exists a proof-structure with conclusion $A \otimes (B \& C), (A^\perp \parr B^\perp) \oplus (A^\perp \parr C^\perp)$, with two correct slices, which is itself not correct. We will come back to this point with our theorem 1.32.

Let's now give an intuition of a possible "sliced" cut-elimination procedure for the 2-sliced graph associated with the sequent calculus proof of $\vdash A, A^\perp \oplus B^\perp$ above.

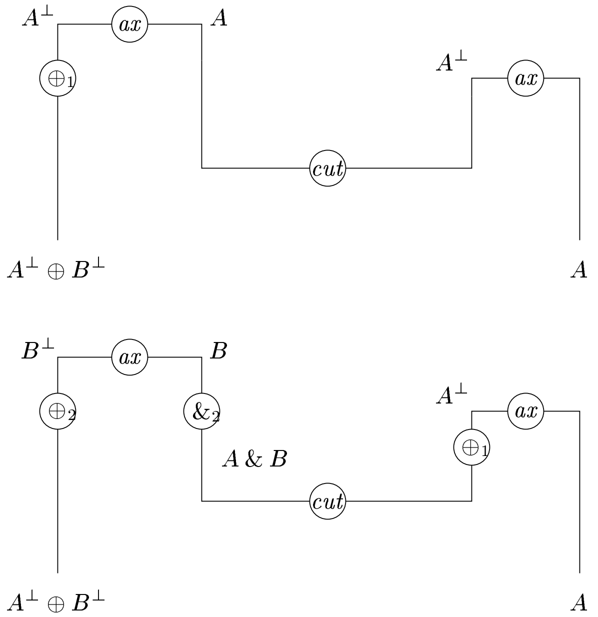
By eliminating the $\parr/\otimes$ cut in both the slices (notice that in a sliced perspective this corresponds to *two* steps), one gets the 2-sliced structure:



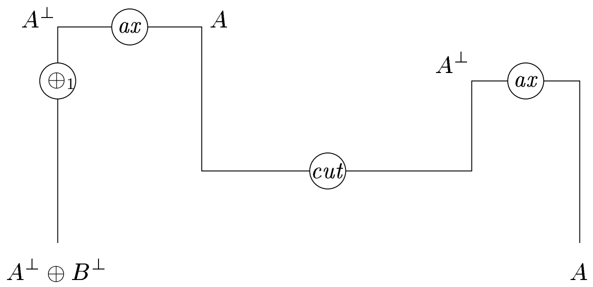
which after (two) axiom steps reduces to:



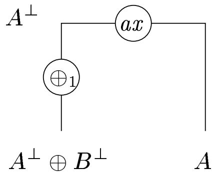
We meet here an important point: in one of the slices we have a $\&_1/\oplus_1$ cut which can be easily reduced, but in the second one we have a $\&_2/\oplus_1$ cut and no way of reducing it. By performing one step of cut-elimination (the only possible one), we obtain the 2-sliced structure:



and we now have to erase the slice containing the $\&_2/\oplus_1$ cut, thus obtaining the 1-sliced proof-structure:



which eventually reduces to:



1.3 Sliced proof-structures

In a polarized framework, we define sliced proof-structures and give the translation of sequent calculus proofs.

Definition 1.1 *A* polarized formula *is a linear propositional formula verifying the following constraints:*

$$
\begin{array}{lcccccccc}
N & ::= & X & \mid & N \parr N & \mid & N \,\&\, N & \mid & ?P \\
P & ::= & X^\perp & \mid & P \otimes P & \mid & P \oplus P & \mid & !N
\end{array}
$$

or a positive formula P prefixed by the symbol $\flat$ (considered as a negative formula).

LL_{pol} *[Lau99] is the fragment of* LL *using only polarized formulas.*

Lemma 1.2 *Every sequent $\vdash \Gamma$ provable in LL_{pol} contains at most one positive formula.*

Proof See [Lau99]. □

Definition 1.3 (Proof-structure) *A* proof-structure *is a finite oriented graph whose nodes are called links, and whose edges are typed by* *formulas of* $\mathsf{LL}_{\mathrm{pol}}$*. When drawing a proof-structure we represent edges oriented up-down so that we may speak of moving upwardly or downwardly in the graph, and of links or edges "above" or "under" a given link/edge. Links are defined together with an arity and a coarity, i.e. a given number of incident edges called the premises of the link and a given number of emergent edges called the conclusions of the link.*

- *an* axiom *link or ax-link has no premise and two conclusions typed by dual atomic formulas,*
- *a* cut *link has two premises typed by dual formulas (which are also called the active formulas of the cut link) and no conclusion,*
- *a* $\parr$*- (resp.* $\otimes$*-) link has two premises and one conclusion. If the left premise is typed by the formula* A *and the right premise is typed by the formula* B*, then the conclusion is typed by the formula* $A \parr B$ *(resp.* $A \otimes B$*),*
- *an* $!$*-link has no premise, exactly one conclusion of type* $!A$ *and some conclusions of* $\flat$*-types,*
- *a* $\flat$*-link has one premise of type* A *and one conclusion of type* $\flat A$*,*
- *a* $?$*-link has* $k \geq 0$ *premises of type* $\flat A$ *and one conclusion of type* $?A$*.*

Let G *be a set of links such that:*

(α) *every edge of* G *is the conclusion of a unique link;*
(β) *every edge of* G *is the premise of at most one link.*

We say that the edges which are not premise of a link are the conclusions of G*.*

We say that G *is a proof-structure if with every* $!$*-link with conclusions* $!A, \flat\Gamma$ *is associated a proof-structure with conclusions* $A, \flat\Gamma$ *(called its* box*).*

The links of the graph G *are called the links with depth* 0 *of the proof-structure* G*. If a link* n *has depth* k *in a box associated with an* $!$*-link of* G*, it has depth* $k+1$ *in* G*. The depth of an edge* a *is the depth of the link of which* a *is conclusion. The depth of* G *is the maximal depth of its links.*

Convention: In the sequel, proof-structures will always have a finite depth.

Remark 1.4 *Notice that, by definition, the boxes of a proof-structure satisfy a* nesting condition*: two boxes are either disjoint or contained one in the other.*

Notice also that the type of every conclusion of a box is a negative formula.

Definition 1.5 (Sliced proof-structure) *A* sliced proof-structure *is a finite set S of slices such that all the slices have the same conclusions, up to the ones of type $\flat$.*

If S contains n slices, and if $\Gamma, \flat\Delta_i$ are the conclusions of the slice s_i of S, then $\Gamma, \flat\Delta_1, \ldots, \flat\Delta_n$ are the conclusions of S.

A slice *s is a proof-structure possibly containing some unary $\&_1$-, $\&_2$- (resp. $\oplus_1$-, $\oplus_2$-) links, whose premise has type A, B and whose conclusion has type $A \& B$ (resp. $A \oplus B$). With every $!$-link n of s with main conclusion $!C$ is now associated a sliced proof-structure S_n (which is still called the box associated with n). This means, in particular, that C appears in every slice of S_n, while every $\flat$-conclusion of n appears in exactly one slice of S_n.*

Definition 1.6 (Single-threaded slice) *A* single-threaded slice *is a slice s such that the sliced proof-structures associated with the $!$-links of s contain only one slice, which is itself a single-threaded slice.*

The notions of *depth* in a single-threaded slice, in a slice, and in a sliced proof-structure are the straightforward generalizations of the same notions for proof-structures given in definition 1.3.

Remark 1.7 *With every sliced proof-structure S is naturally associated a set of single-threaded slices, to which we will refer as the set of the "single-threaded slices of S (or associated with S)" denoted by $sgth(S)$.*

Remark 1.8 *Every formula A of a sliced proof-structure is a conclusion of a unique link introducing A. (Notice that this is of course not the case in any version of proof-nets for the full propositional fragment of* LL*).*

We are now going to associate with every linear sequent calculus proof a sliced proof-structure.

Definition 1.9 (Translation of the sequent calculus) *Let R be the last rule of the (η-expanded) linear sequent calculus proof π. We define the sliced proof-structure S_π (with the same conclusions as π) by induction on π.*

- *If R is an axiom with conclusions $X, X^\perp$, then the unique slice of S_π is an axiom link with conclusions $X, X^\perp$.*

- *If R is a* $\parr$*- or a* $\oplus$*-rule, having as premise the subproof* π'*, then* S_π *is obtained by adding to every slice of* $S_{\pi'}$ *the link corresponding to* R*.*
- *If R is a* $\otimes$*- or a cut rule with premises the subproofs* π_1 *and* π_2*, then* S_π *is obtained by connecting every slice of* S_{π_1} *and every slice of* S_{π_2} *by means of the link corresponding to R. Notice that if* S_{π_1} *(resp.* S_{π_2}*) contains* k_1 *(resp.* k_2*) slices, then* S_π *contains* $k_1 \times k_2$ *slices.*
- *If R is a* $\&$*-rule with premises the subproofs* π_1 *and* π_2*, then* S_π *is obtained by adding a* $\&_1$*- (resp.* $\&_2$*-) link to every slice of* S_{π_1} *(resp.* S_{π_2}*) and by taking the union of these two sliced proof-structures.*
- *If R is a dereliction rule on A having as premise the subproof* π'*, then* S_π *is obtained by adding to each slice of* $S_{\pi'}$ *a* $\flat$*-link with premise A and conclusion* $\flat A$ *and a unary* $?$*-link with premise* $\flat A$ *and conclusion* $?A$*.*
- *If R is a weakening rule on* $?A$*, then* S_π *is obtained by adding a* $?$*-link with arity* 0 *and conclusion* $?A$*.*
- *If R is a contraction rule on* $?A$ *having as premise the subproof* π'*, then by induction hypothesis, every slice of* $S_{\pi'}$ *has two formulas* $?A$ *among its conclusions. By remark 1.8, these two formulas are both conclusions of a* $?$*-link. We replace the two* $?$*-links by a unique* $?$*-link with the required arity, and thus obtain the slices of* S_π*.*
- *If R is a promotion rule with conclusions* $!C, ?A_1, \ldots, ?A_n$ *having as premise the subproof* π'*, then let* s'_i *be one of the* $p \geq 1$ *slices of* $S_{\pi'}$*. For every slice* s'_i *of* $S_{\pi'}$ *with conclusions* $C, ?A_1, \ldots, ?A_n$*, we call* s_i *the graph obtained by erasing the* $?$*-links with conclusions* $?A_1, \ldots, ?A_n$*.* s_i *is a slice with conclusions:*

  $$C, \flat A^1_{1,i}, \ldots, \flat A^{q_{1,i}}_{1,i}, \ldots, \flat A^1_{n,i}, \ldots, \flat A^{q_{n,i}}_{n,i}$$

  with $q_{j,i} \geq 0$*. The unique slice of* S_π *is an* $!$*-link with conclusions* $!C, \flat A^1_{1,1}, \ldots, \flat A^{q_{n,1}}_{n,1}, \ldots, \flat A^1_{1,p}, \ldots, \flat A^{q_{n,p}}_{n,p}$*, to which we add for every* $1 \leq j \leq n$ *a* $?$*-link having as premises* $\flat A^k_{j,i}$ *(*$1 \leq i \leq p$ *and* $1 \leq k \leq q_{j,i}$*) and as conclusion* $?A_j$*. The sliced proof-structure associated with the unique* $!$*-link of* S_π *is the set of the* s_i *(*$1 \leq i \leq p$*).*

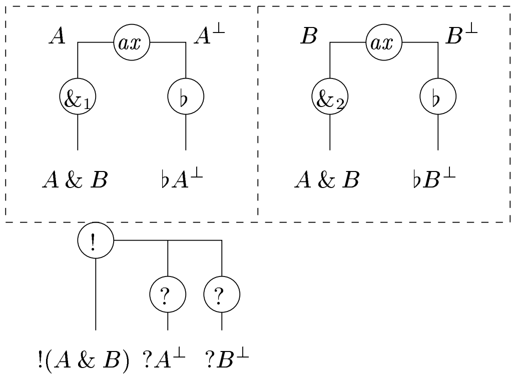
Remark 1.10 *Let's try to give a more informal (but, hopefully clearer) description of the last case of the previous definition. For every formula*

$?A_j$, we replace the $?$-link introducing it in each slice by a unique $?$-link in the (unique) slice of S_π.

Let us conclude the section by giving an example of the accuracy of our sliced structures. The following sequent calculus proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,ax}{\vdash A, ?A^\perp}\,?d}{\vdash A, ?A^\perp, ?B^\perp}\,?w \qquad \dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\,ax}{\vdash B, ?A^\perp, B^\perp}\,?w}{\vdash B, ?A^\perp, ?B^\perp}\,?d}{\vdash A \& B, ?A^\perp, ?B^\perp}\,\&}{\vdash !(A \& B), ?A^\perp, ?B^\perp}\,!$$

is translated as the sliced structure:



The previous structure is built inductively with respect to the depth: with the sequent calculus proof one associates the graph consisting in the !-link and in the two ?-links, and with the !-link are associated two slices (the ones inside the two dashed rectangles).

Notice that following the Danos-Regnier representation of proof-nets called “nouvelle syntaxe”, consisting in “pulling down” the structural rules, the two weakenings of the sequent calculus proof simply vanished.

1.4 Semantics

We consider the concrete semantics of experiments introduced in [Gir87]. We develop here only the case of relational semantics but the notion of experiment suits also very well coherent set-based and multiset-based semantics (see [TdF00]).

Our results (like the existence of an injective 1-experiment used in the proof of lemma 1.49) will be completely proven only in the relational case, but the extension to the coherent semantics is just a matter of checking some minor details, consisting in the extension to our framework of the results proven in [TdF01] without the additives.

Definition 1.11 (Relational interpretation of formulas) *The space interpreting a formula A will be denoted in the sequel by $\mathcal{A}$. It is a set, defined by induction on the complexity of A:*

- $\mathcal{X} = \mathcal{X}^\perp$ *is any set;*
- $\mathcal{A} \otimes \mathcal{B} = \mathcal{A} \parr \mathcal{B}$ *is the cartesian product of the sets $\mathcal{A}$ and $\mathcal{B}$;*
- $\mathcal{A} \,\&\, \mathcal{B} = \mathcal{A} \oplus \mathcal{B}$ *is the disjoint union of the sets $\mathcal{A}$ and $\mathcal{B}$;*
- $!\mathcal{A} = ?\mathcal{A} = \flat\mathcal{A}$ *is the set of finite multisets of elements of $\mathcal{A}$.*

Definition 1.12 (Experiment) *If S is a sliced proof-structure, an experiment of S is an experiment of one of the slices of S.*

An experiment e of a slice s of S is an application which associates with every edge a of type A with depth 0 of s an element $e(a)$ of $\mathcal{A}$, called the label *of a. We define such an application by induction on the depth p of s.*

If $p = 0$, then:

- *If $a = a_1$ is the conclusion of an axiom link with conclusions the edges a_1 and a_2 of type X and $X^\perp$ respectively, then $e(a_1) = e(a_2)$.*
- *If a is the conclusion of a $\parr$- (resp. $\otimes$-) link with premises a_1 and a_2, then $e(a) = (e(a_1), e(a_2))$.*
- *If a is the conclusion of a link $\oplus_i$ (resp. $\&_i$), $i \in \{1, 2\}$ with premise a_1, then $e(a) = (i, e(a_1))$.*
- *If a is the conclusion of a dereliction link with premise a_1, then $e(a) = \{e(a_1)\}$.*
- *If a is the conclusion of a $?$-link of arity $k \geq 0$, with premises $a_1, \ldots, a_k$, then $e(a) = e(a_1) \cup \cdots \cup e(a_k)$, and $e(a) \in ?\mathcal{C}$ (if $k = 0$ we have $e(a) = \emptyset$).*
- *If a is the premise of a cut link with premises a and b, then $e(a) = e(b)$.*

If the conclusions of S are the edges $a_1, \ldots, a_l$ of type, respectively, $A_1, \ldots, A_l$, and e is an experiment of S such that $\forall i \in \{1, \ldots, l\}$ $e(a_i) = x_i$, then we shall say that $(x_1, \ldots, x_l) \in \mathcal{A}_1 \parr \ldots \parr \mathcal{A}_l$ is the result *of the experiment e of S. We shall also denote it by $x_1, \ldots, x_l$.*

If $p > 0$, then e satisfies the same conditions as in case $p = 0$, and for every $!$-link n with depth 0 in s and with conclusions c of type $!C$

and $a_1, \dots, a_l$ *of type, respectively,* $\flat A_1, \dots, \flat A_l$*, there exist* $k \geq 0$ *experiments* $e_1, \dots, e_k$ *of the sliced proof-structure* S' *associated with* n *such that*

- $e(c) = \{x_1, \dots, x_k\}$*, where* x_j *is the label associated with the edge of type* C *by* e_j*,*
- *If* s' *is the (unique!) slice of* S' *containing the edge* a'_j *with the same type as* a_j*, then* $e(a_j)$ *is the union of the labels associated with* a'_j *by the* k *experiments of* s'*. Notice that it might be the case that none of the* k *experiments is defined on* a'_j*: in this case one has* $e(a_j) = \emptyset$*.*

Of course, we have that $e(c) \in !\mathcal{C}$*, and* $e(a_j) \in \flat\mathcal{A}_j$ *(this would be an extra requirement in the coherent case).*

Definition 1.13 (Interpretation) *The* interpretation *or* the semantics *of a sliced proof-structure* S *with conclusions* Γ *is the set:*

$[\![S]\!] := \{\gamma \in \parr\,\Gamma$: *there exists an experiment* e *of* S *with result* $\gamma\}$*, where* $\parr\,\Gamma$ *is the space interpreting the* $\parr$ *of the formulas of* Γ*.*

Remark 1.14 *The interpretation of a sliced proof-structure* S *depends on the interpretation chosen for the atoms of the formulas of* S*. Once this choice is made,* $[\![S]\!]$ *is (by definition) the union of the interpretations of the slices of* S*.*

The reader should notice that the union of the interpretations of the single-threaded *slices of* S is not *enough to recover* $[\![S]\!]$ *(except in some particular cases, for example when* S *is a cut-free proof-net, see section 1.8). This is a crucial point (behind which hide the complex relations between the additive and multiplicative worlds) showing the impossibility of working only with single-threaded slices.*

Indeed, were we working with a "single-threaded semantics", by cutting the single-threaded version of the example at the end of section 1.3 (on the formula $!(A\&B)$*) with the proof-net corresponding to the following proof (which is a single-threaded slice since there is no* $\&$*-rule):*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,ax}{\vdash A, A^\perp \oplus B^\perp}\,\oplus_1}{\vdash A, ?(A^\perp \oplus B^\perp)}\,?d}{\vdash !A, ?(A^\perp \oplus B^\perp)}\,! \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\,ax}{\vdash B, A^\perp \oplus B^\perp}\,\oplus_2}{\vdash B, ?(A^\perp \oplus B^\perp)}\,?d}{\vdash !B, ?(A^\perp \oplus B^\perp)}\,!}{\dfrac{\vdash !A \otimes !B, ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp)}{\vdash !A \otimes !B, ?(A^\perp \oplus B^\perp)}\,?c}\,\otimes$$

we would get a proof-net with an empty semantics. Moreover, applying the cut-elimination procedure described in the next section (to the set of single-threaded slices associated with that same net) would lead to an empty set of slices.

The following notion of 1-experiment is a particular case of the more general notion of n-obsessional experiment introduced in [TdF01].

Definition 1.15 (1-experiment) *An experiment e of a sliced proof-structure S is a* 1-experiment, *when with every $!$-link of S one has (using the notations of definition 1.12) $k = 1$, and e_1 is a 1-experiment.*

Remark 1.16 *Let S be a sliced proof-structure.*

(i) *Let e be a 1-experiment of S. If a is any edge of S of type A, then with a the experiment e associates at most one element of $\mathcal{A}$,* whatever the depth of a is. *In case e is not a 1-experiment, this is (in general) the case only for the edges with depth 0.*

(ii) *The 1-experiments of S are exactly the 1-experiments of the single-threaded slices of S.*

(iii) *We say that a 1-experiment e of a single-threaded slice s of S is* injective *when for every pair of (different) axiom links n_1 and n_2 of s, if x_1 (resp. x_2) is the (unique) label associated by e with the conclusions of n_1 (resp. n_2), then $x_1 \neq x_2$.*

(iv) *If S contains no cut links, then there always exists an injective experiment of any single-threaded slice of S (just associate distinct labels with the axiom links and "propagate" them downwardly). This is not that obvious in the coherent case (due to the presence of $?$-links): it is actually wrong in a non polarized framework, even for single-threaded slices coming from sequent calculus proofs (see [TdF01]).*

1.5 Proof-nets and cut-elimination

We now define a notion of correct sliced proof-structure: a *proof-net* is a sliced proof-structure satisfying some geometrical condition. For these sliced proof-nets, a "sliced" cut-elimination procedure is given: a cut-elimination step is a step in *one* of the slices.

We show that the cut-elimination steps preserve the correctness of the structures, and that the interpretation given by definition 1.13 is sound (i.e. invariant with respect to these steps).

1.5.1 Definitions

Definition 1.17 (Acyclic sliced proof-structure) *The* correction graph *(see [Lau99]) of a slice s is the directed graph obtained by erasing the edges conclusions of s, forgetting the sliced proof-structure associated with every $!$-link with depth 0 in s and by orienting negative (resp. positive) edges downwardly (resp. upwardly).*

A single-threaded slice satisfies (AC) when its correction graph, so as the correction graph of all its boxes, is acyclic.

A sliced proof-structure S is acyclic, *when every single-threaded slice associated with S satisfies (AC).*

Definition 1.18 (Proof-net) *Let S be an* acyclic *sliced proof-structure without any $\flat$-conclusion. S is a* proof-net *if every slice of S has exactly one $\flat$-link or one positive conclusion (at depth 0). Moreover, we require that the sliced proof-structures (the boxes) $S_1, \ldots, S_k$, recursively associated with the $!$-links of S also satisfy these properties.*

Remark 1.19 *More geometrically, notice that this only $\flat$-link (or link above the positive conclusion) is the only non-weakening initial node (without incident edge) of the correction graph.*

Remark 1.20 *It is easy (and standard) to show, by induction on the sequent calculus proof, that the sliced proof-structure associated by definition 1.9 with a sequent calculus proof is a proof-net.*

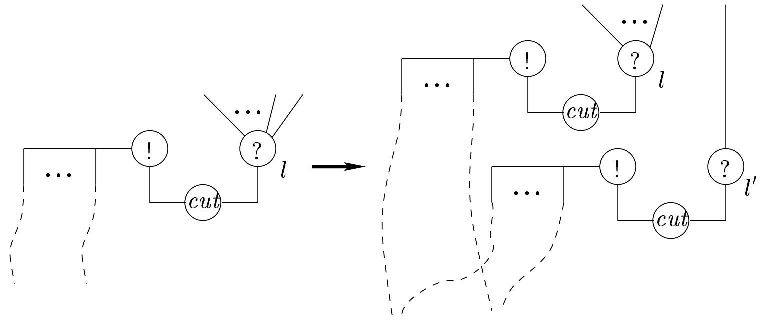
Notice that the condition given by definition 1.18 is nothing but the proof-net version of lemma 1.2.

We come now to the definition of the cut-elimination procedure. If the cut link c has depth n in the sliced proof-structure S, the cut-elimination step associated with c will be a step for the sliced proof-structure associated with the !-link (of depth $n-1$) the box of which contains c.

Definition 1.21 (Cut-elimination) *Let S be an acyclic sliced proof-structure without $\flat$-conclusions. We define a one step reduct S' of S. Let $s \in S$ and c be a cut link of s. We define $\{s'_i\}_{i\in I}$, obtained by applying some transformations to s. S' is the set of the slices obtained from S by substituting $\{s'_i\}_{i\in I}$ for s.*

- *If c is a cut link of type ax, then $\{s'\}$ is obtained, as usual, by erasing the axiom link and the cut link.*

- *If c is a cut link of type $⅋/\otimes$, let A and B (resp. $A^{\perp}$ and $B^{\perp}$) be the premises of the $⅋$-link (resp. $\otimes$-link). $\{s'\}$ is obtained by erasing the $⅋$-link, the $\otimes$-link and the cut link and by putting two new cut links between A and $A^{\perp}$, and B and $B^{\perp}$.*
- *If c is a cut link of type $\&_i/\oplus_i$, then $\{s'\}$ is obtained by erasing the two links and by moving up the cut link to their premises.*
- *If c is a cut link of type $\&_1/\oplus_2$ (or $\&_2/\oplus_1$), then $I = \emptyset$ (we simply erase s). Moreover, if s is the unique slice of the sliced proof-structure S_n associated with the $!$-link n, we also erase the slice containing n (and so on recursively...).*
- *If c is a cut link of type $!/?$ with a 0-ary $?$-link, then the $!$-link (together with its box) and its conclusion edges are erased. We then erase the 0-ary $?$-link (and the cut) thus obtaining $\{s'\}$ (notice that some $?$-links have lost some premises).*
- *If c is a cut link of type $!/?$ with a 1-ary $?$-link under a $\flat$-link, let T be the sliced proof-structure associated with the $!$-link. With each slice t_i of T, we associate the slice s'_i defined by erasing the $?$-link and the $\flat$-link, by replacing in s the $!$-link by t_i and by cutting the main conclusion of t_i with the premise of the $\flat$-link.*
- *If c is a cut link of type $!/?$ with a 1-ary $?$-link whose premise is a $\flat$-conclusion of an $!$-link l', let T be the sliced proof-structure associated with l' and l be the cut $!$-link. Let $?A/!A^{\perp}$ be the cut formula. $\{s'\}$ is obtained by erasing l and its conclusions and by replacing the conclusion $\flat A$ of l' by all the $\flat$-conclusions of l. And with this new $!$-link (which we still denote by l') is associated a sliced proof-structure T' obtained by replacing the (unique) slice t of T having $\flat A$ among its conclusions by the slice obtained by adding to the conclusion of type $\flat A$ of t a unary $?$-link and cutting its conclusion (of type $?A$) with the conclusion of type $!A^{\perp}$ of l. (The sliced proof-structure associated with l remains unchanged).*
- *If c is a cut link of type $!/?$ with a n-ary $?$-link l with $n > 1$, then $\{s'\}$ is obtained by creating a new unary $?$-link l' having as premise one of the premises of l (and erasing the corresponding edge above l), by duplicating the $!$-link and by cutting the copy with the conclusion of l', every $\flat$-conclusion of the copy of the $!$-link is premise of the same links as the edge it is a copy of (namely, they are intuitively premise of the same $?$-link). The sliced proof-structures associated with the two copies of the $!$-link are the same.*



Remark 1.22 *The attentive reader certainly noticed that there are exactly two cases in which the previous definition requires the acyclicity condition:*

(i) *when the two premises of a cut link are both conclusions of the same axiom link,*

(ii) *when the two premises of a cut link are both conclusions of the same* !*-link (in fact, in our framework, this means that the premise of type* ? *of the cut link is the conclusion of a* ?*-link whose premise is a conclusion of type* $\flat$ *of the* !*-link).*

In these two cases the cut-elimination procedure is not defined. By the following section, the acyclicity of a sliced proof-structure is a sufficient condition to ensure that cut-elimination never yields to these configurations.

1.5.2 Preservation of correctness

Proposition 1.23 (Preservation of acyclicity) *If S' is a sliced proof-structure obtained from the acyclic sliced proof-structure S (without $\flat$ conclusions) by performing some steps of cut-elimination, then S' is acyclic.*

Proof We study every cut-elimination step, using the notations of definition 1.21:

- For the $\&_i/\oplus_j$ $(i \neq j)$ and !/0-ary ? steps, we erase a part of the graphs, such an operation cannot create cycles.
- For the ax and $\&_i/\oplus_i$ steps, some paths are replaced by shorter ones changing nothing to cycles.

- For the $\parr/\otimes$ step, if p is a path containing a cycle in S', it must use one of the two new cut links starting from the premise A of the $\parr$-link and going to the premise $A^\perp$ of the $\otimes$-link, for example. If p exists, then replacing in S the part from A to $A^\perp$ by the path going from A through the $\parr$-link, the cut link and the $\otimes$-link to $A^\perp$ would give a cycle in S.
- For the !/1-ary ? step with a $\flat$-link just above the ?-link, if s'_i contains a cycle p, either it is inside t_i and thus comes from a cycle in S or it goes outside t_i, but due to the orientation, it is impossible for a path to go outside t_i and to come inside t_i since t_i has only emergent edges (since it has only negative conclusions from remark 1.4).
- For the !/1-ary ? step with an !-link just above the ?-link: at the depth p of the cut link, some paths are just replaced by shorter ones, and this cannot create any cycle. At depth $p+1$, adding a cut and an !-link to an acyclic graph cannot create any cycle.
- For the $!/n$-ary ? step $(n > 1)$, if p is a cycle in S', it has to cross one of the two residues of the cut link of S. But identifying the two ?-links, the two cut links and the two !-links in p would give a cycle in S thus p doesn't exist.

□

Theorem 1.24 (Preservation of correctness) *If S' is a sliced proof-structure obtained from the proof-net S by performing some steps of cut-elimination, then S' is a proof-net.*

Proof S' is acyclic by proposition 1.23. To conclude, we now prove that if S' is a one-step reduct of S, then (whatever reduction step has been performed) S' has exactly one positive conclusion or one $\flat$-link at depth 0 and in every slice of every box (assuming that the reduced cut has depth 0 in S):

- The multiplicative and additive steps are straightforward and the !/0-ary ? step, too.
- For the !/1-ary ? step with a $\flat$-link just above the ?-link, the $\flat$-link at depth 0 is erased and replaced by the one coming from every slice of the box of the !-link (which necessarily exists by remark 1.4 and definition 1.18).
- For the !/1-ary ? step with an !-link just above the ?-link, the

$\flat$-links and the positive conclusions at depth 0 are not modified and at depth 1, we just add an !-node to a slice.

- For the $!/n$-ary ? step ($n > 1$), some links are duplicated but the $\flat$-links (and the positive conclusions) are unchanged.

$\square$

1.5.3 Soundness of the interpretation

We are going to prove that the cut-elimination procedure previously defined preserves the semantical interpretation. We use exactly the same technique as in [Gir87], and give the details of the proof only in the most relevant cases. The proof is given for the relational semantics, and it can be straightforwardly extended to both the set and multiset based coherent semantics (see remark 1.27).

Remark 1.25 *By induction on the sequent calculus proof π, one can check that the semantics of π (as defined for example in [Gir87]) is the semantics of the sliced proof-structure S_π of definition 1.9.*

Theorem 1.26 (Semantical soundness) *If S' is a sliced proof-structure obtained from the acyclic proof-structure S without $\flat$-conclusions by performing some steps of cut-elimination. Then $[\![S]\!] = [\![S']\!]$.*

Proof Let Γ be the conclusions of S and S' and γ an element of $\parr\,\Gamma$. We show that there exist a slice s of S and an experiment e of s with result γ, iff there exist a slice s' of S' and an experiment e' of s' with result γ.

One has to check this is the case for every cut-elimination step defined in definition 1.21. We will use for these steps the notations of definition 1.21. Let c be a cut link of a slice s of S. Notice that our claim is obvious for the slices which are not concerned by the cut-elimination step that we consider, and we then restrict to the other ones: we prove that there exists an experiment e of s with result γ, iff there exist a slice s' of $\{s'_i\}_{i\in I}$ and an experiment e' of s' with result γ.

By induction on the depth of c in s, we can restrict to the case where c has depth 0. The steps associated with the ax and $\parr/\otimes$ cut links are the same as in [Gir87].

- If c is a cut link of type $\&_i/\oplus_i$, and e is an experiment of s, let (i, x) be the element of $\mathcal{A} \,\&\, \mathcal{B} = \mathcal{A}^\perp \oplus \mathcal{B}^\perp$ associated by e with the two edges premises of c. Then the experiment e' of s' we look for is the "restriction" of e to s': the label associated by e with the two premises of the unary $\&_i$ and $\oplus_i$ links of s is x, and x is also the label associated by e' with the two premises of the "residue" of c in s'. For the converse, one clearly proceeds in the same way.

- If c is a cut link of type $\&_1/\oplus_2$ (or $\&_2/\oplus_1$), then there exists no experiment of s (remember the condition of definition 1.12 on the label of the premises of a cut link), and no experiment of $\{s'_i\}_{i\in I}$ (remember $I = \emptyset$).
- If c is a cut link of type $!/?$ with a 0-ary ?-link, then we are simply applying the weakening step of [Gir87].
- If c is a cut link of type $!/?$ with a 1-ary ?-link whose premise is the conclusion of a $\flat$-link, let T be the sliced proof-structure associated with the !-link. With each slice t_i of T, this step associates a slice s'_i.
  Let e be an experiment of s, let $\{x\}$ be the element of $!\mathcal{A} = ?\mathcal{A}^\perp$ associated by e with the two edges premises of c. By definition of experiment, because the label of the conclusion of the !-link is a singleton, there is a unique slice t_i of the sliced proof-structure T (associated with the !-link), and a unique experiment e_i of t_i from which e is built. The label associated with the conclusion of type A of t_i will be $x \in \mathcal{A}$. Again by definition of experiment, the label associated by e with the premise of type $A^\perp$ of the $\flat$-link is $x \in \mathcal{A}^\perp$. We can then build (from e_i) an experiment e'_i of s'_i with the same result as e. For the converse, one proceeds in the same way: an experiment e' of some slice s'_i induces an experiment e_i of t_i, and an experiment e of s.
- If c is a cut link of type $!/?$ with a 1-ary ?-link whose premise is a conclusion (of type $\flat$) of the box associated with the !-link l' (different from l), then there is nothing new with respect to the commutative step of [Gir87].
- If c is a cut link of type $!/?$ with a n-ary ?-link l with $n > 1$, then let e be an experiment of s, let $\{x_1, \ldots, x_k\} = a_1 \cup \cdots \cup a_n$ be the element of $!\mathcal{A} = ?\mathcal{A}^\perp$ associated by e with the two edges premises of c. Suppose that a_1 is the label of the one among the premises of the ?-link of arity n, which becomes the conclusion of the new unary ?-link. We have $\{x_1, \ldots, x_k\} = a_1 \cup \{y_1, \ldots, y_h\}$. This splitting is actually a splitting of the k experiments of the sliced proof-structure associated with the !-link. This remark is enough

to conclude the existence of an experiment e' of s' with the same result as e. Conversely, let e' be an experiment of s'. Because the sliced proof-structure associated with the two !-links is the same we can build an experiment e of s with the same result as e'.

$\square$

Remark 1.27 *To prove the soundness of the (set and multiset based) coherent semantics, one first needs to generalize the following result of [Gir87] to* $\mathsf{LL}_{\mathrm{pol}}$*: "if S is an acyclic sliced proof-structure with conclusions Γ (where Γ contains no $\flat$ formula), then $[\![S]\!]$ is a* clique *of the coherent space $\parr\,\Gamma$."*

This result has to be used in the proof of the previous theorem in the cases of !/? cuts.

1.6 Sequentialization for (cut-free) slices

We show that the conditions on sliced proof-structures given in definitions 1.17 and 1.18 yield a *correctness criterion* for cut-free proof-structures (theorem 1.32): they allow to characterize exactly those proof-structures coming from sequent calculus proofs.

A novelty due to our sliced presentation is that we have to be able to glue together slices. Thanks to the polarization constraint this will be possible, provided one restricts to cut-free proof-structures. In the whole section, all our proof-structures will be cut-free.

Definition 1.28 (Equivalence of links) *Let s_1, s_2 be two slices of a sliced proof-structure S. Let n_1 and n_2 be two links of s_1 and s_2 at depth* 0 *having the same negative non-$\flat$ conclusion A. We define, by induction on the number of links under A in s_1, the meaning of n_1* and *n_2* are equivalent links *denoted by $n_1 \equiv n_2$.*

If A is a conclusion of s_1 then it is also a conclusion of s_2 and $n_1 \equiv n_2$ if they are the links introducing A in s_1 and s_2.

Let A be the premise of the unary link m_1 (resp. m_2) of s_1 (resp. s_2) and the conclusion of n_1 (resp. n_2): if $m_1 \equiv m_2$, then $n_1 \equiv n_2$.

Let A be the left or right premise of the binary link m_1 (resp. m_2) of s_1 (resp. s_2) and the conclusion of n_1 (resp. n_2): if $m_1 \equiv m_2$, then $n_1 \equiv n_2$.

It is clear that $\equiv$ is an equivalence relation on the negative links at depth 0 *of S.*

Remark 1.29 *If $n_1 \equiv n_2$ then n_1 and n_2 are links of the same kind except if $n_1 = \&_1$ and $n_2 = \&_2$.*

Definition 1.30 (Weights) *Let S be a sliced proof-structure and let $\&^1, \dots, \&^k$ be the equivalence classes for $\equiv$ of the $\&$-links at depth 0 of S. We associate with each $\&^i$ an* eigen weight *p_i that is a boolean variable (in the spirit of [Gir96]). The* weight *of a slice s of S is (with an empty product equal to 1 by convention):*

$$w(s) = \prod_{\&_1^i \in s} p_i \prod_{\&_2^i \in s} \bar{p}_i$$

and the weight of the set S is:

$$w(S) = \sum_{s \in S} w(s)$$

The sliced proof-structure S is full *if $w(S) = 1$ and* compatible *if we have $w(s)w(t) = 0$ for $s \neq t$.*

Remark 1.31 *We can now be more precise than in remark 1.20: the sliced proof-structure associated by definition 1.9 with a cut-free sequent calculus proof is a (cut-free) proof-net, which is full and compatible.*

Theorem 1.32 (Sequentialization) *If S is a cut-free sliced proof-structure, S is the translation of an $\mathsf{LL}_{\mathrm{pol}}$† sequent calculus proof if and only if S is a full and compatible proof-net.*

Proof We prove the second implication by induction on the size of S (the first one is remark 1.31). Since S has no $\flat$-conclusions, the conclusions of the slices of S are the same. The size of a slice s is the triple (depth(s),number of ?-links with arity at least 2 and depth 0,number of links with depth 0), lexicographically ordered, and the size of S is the sum (component by component) of the sizes of the slices of S.

Let s be a slice of S. We shall say that a link of s is terminal when its conclusion is a conclusion of s.

- If s has a terminal $⅋$-link, a corresponding link appears in each slice since they have the same conclusions. We can remove these links in each slice and we obtain a sliced proof-structure S' verifying the hypothesis of the theorem.

† The extension of this result to the multiplicative units is straightforward. The case of $\top$ presents no real difficulty but requires a heavier treatment (see [Lau02]).

- If s has a terminal &-link, a corresponding link appears in each slice. For some slices this link will be a $\&_1$-link (we call S_1 the set of slices obtained by erasing the $\&_1$-links in these slices) and for some others a $\&_2$-link (we call S_2 the corresponding set without the $\&_2$-links). We have to show that S_1 and S_2 are full and compatible. The weight of S_1 (resp. S_2) is obtained by taking $p = 1$ (resp. $p = 0$) in $p.w(S)$ (resp. $\bar{p}.w(S)$) thus this weight is 1. Let s and t be two slices of S_1 with weights $w_1(s)$ and $w_1(t)$, their weights in S are $p.w_1(s)$ and $p.w_1(t)$ thus $w_1(s)w_1(t) = 0$ (idem for S_2). We can now apply the induction hypothesis to S_1 and S_2.

Now, s has no terminal $\parr$-links and no terminal &-links thus it has no such links at depth 0 by *polarization*. This entails that s is the only slice of S by compatibility.

- If s has a terminal 0-ary ?-link, we can remove it: this corresponds to a weakening rule.
- If s has a terminal n-ary ?-link with $n \geq 2$, we break it into n unary links, we apply the induction hypothesis and perform $n-1$ contraction rules in the sequent calculus proof thus obtained.
- If s has a unary ?-link under a $\flat$-link, we remove both of them, and this corresponds to a dereliction rule. (Notice that we can apply the induction hypothesis, because when removing the two links we replace a $\flat$-link at depth 0 by a positive conclusion).
- If none of the previous conditions is satisfied then s has no $\parr$-, &-, ?-links at depth 0 (except unary ?-links under !-links). This means that if s has a terminal $\otimes$-link, it is the unique one and it is splitting: we can apply the induction hypothesis to the two sub-proof-structures.
- If s has a terminal $\oplus$-link, we just remove it and apply the induction hypothesis.

If s doesn't correspond to any of the cases above, either it is an axiom link (straightforward) or it is reduced to an !-link with a unary ?-link under each $\flat$-conclusion. Let S' be the box associated with the !-link. By adding to the slices of S' some 0-ary ?-links (like in example page 12) and a 1-ary ?-link under each $\flat$-conclusion, one gets a sliced proof-structure S''. Let π'' be the proof obtained by sequentializing S'', the sequentialization π of S is obtained by adding a promotion rule to π''. (As an exercise, the reader can apply this sequentialization method to the sliced proof-structure of page 12). $\square$

Remark 1.33 *Notice that (according to remark 1.8) every negative conclusion $M \parr N$ (resp. $M\&N$) of a proof-net S is the conclusion of a $\parr$ (resp. $\&$) link. The previous proof shows that there exists a sequentialization of S whose last rule introduces this formula. The reader might have recognized a proof-net version of the reversibility of the connectives $\parr$ and $\&$.*

Remark 1.34 *In fact, a(n apparently) stronger version of theorem 1.32 could be given: the reader certainly noticed that nowhere in the proof of the theorem we have used the acyclicity property of our proof-nets. This is simply due to the fact that every cut-free sliced proof-structure S is acyclic. Indeed, a path starting from a positive edge of S upwardly goes to an axiom link or an $!$-link and then goes down to a conclusion stopping there; while a path starting from a negative edge goes directly down to a conclusion and stops.*

1.7 Computing with slices

We now introduce a general method, allowing to use denotational semantics in order to guarantee the "canonicity" of our proof-nets. More precisely, we introduce the notion of *injective* semantics (which comes from [TdF00]), and show how the existence of such a semantics is a witness of the canonicity of our sliced proof-nets as computational objects.

Remark 1.35 *We will use in the sequel the strong normalization property for proof-nets with respect to the cut-elimination procedure. We do not give the proof of such a result, which is proven in [LQTdF00] (for $\mathsf{LL}_{\mathrm{pol}}$) in the framework of polarized proof-nets with additive boxes.*

Let F be a subsystem of our sliced proof-structures, and let $[\![.]\!]$ be an interpretation of the sliced proof-structures of F (satisfying theorem 1.26 and) injective: if S_1 and S_2 are two cut-free sliced proof-nets such that (for every interpretation of the atomic formulas) $[\![S_1]\!] = [\![S_2]\!]$, then $S_1 = S_2$.

Another way to speak of injectivity is the following: $[\![.]\!]$ is injective when the semantical equivalence class of every proof-net contains a unique cut-free proof-net. In this (strong) sense our objects are canonical. In particular, such a property entails confluence: if S_1^0 and S_2^0 are two normal forms of the proof-net S, then by theorem 1.26 and injectivity $S_1^0 = S_2^0$.

Another crucial point is that injectivity allows to compute with the sliced proof-structures of F coming from sequent calculus proofs. Indeed, let π be any linear propositional sequent calculus proof, let S_π be the sliced proof-structure associated with π by definition 1.9, and let S_0 be the normal form of S_π. Now compute a normal form π_0 of π semantically correct (i.e. satisfying $[\![\pi]\!] = [\![\pi_0]\!]$), which can be done by performing cut-elimination directly in sequent calculus in several different ways. By remark 1.25, $[\![\pi]\!] = [\![S_\pi]\!]$ and $[\![\pi_0]\!] = [\![S_{\pi_0}]\!]$, by theorem 1.26, $[\![S_\pi]\!] = [\![S_0]\!]$, and we know that $[\![\pi]\!] = [\![\pi_0]\!]$. By injectivity, we can then conclude that $S_{\pi_0} = S_0$. In fact, our approach to injectivity (in section 1.8, and more generally in [TdF01]) is "to rebuild" a cut-free proof from its semantics: on the one hand the injectivity property guarantees that any reasonable way of computing with sliced proof-structures coming from sequent calculus proof is sound ($S_{\pi_0} = S_0$), and on the other hand the technique used to prove injectivity suggests the possibility of semantically computing the normal form (S_0) of a proof (π_0). This last approach is very close to the so-called "normalization by evaluation" (see [BES98, DRR01]).

Summing up, one has:

$$\begin{tikzcd} \pi \arrow[r] \arrow[d] & \pi_0 \arrow[d] \\ S_\pi \arrow[r] & S_{\pi_0} \end{tikzcd}$$

This diagram expresses a simulation property of the cut-elimination (in sequent calculus) by proof-net reductions. The injectivity property of the semantics allows to obtain such a result by *semantical* means.

Notice that the mentioned argument holds for *any existing syntax for* LL_{pol} instead of sequent calculus (like proof-nets with additive boxes see [Gir87] and [LQTdF00], multiboxes see [TdF03], proof-nets with weights see [Gir96] and [Lau99]): let R be a proof in such a system, it will always be possible to translate R as a sliced proof-structure S_R with the same semantics as R (in the previously mentioned syntaxes, this is straightforward). Let S_0 be the normal form of S_R. Let R_0 be a normal form of R and let S_{R_0} be the sliced proof-structure associated with R_0. As before, we have $S_0 = S_{R_0}$.

We are claiming that our proof-nets are canonical computational objects: they are actually the first example of such objects in presence of the additive and exponential connectives. Indeed, (sliced) proof-nets are computational objects by theorem 1.24, and they are canonical by the injectivity property (as we already explained).

Notice that none of the previously mentioned polarized syntaxes can really claim to yield a canonical representation of proofs: denotational semantics is not injective for proof-nets with boxes nor multiboxes (even though this last syntax realizes a much greater quotient on proofs), and it is well-known that with a sequent calculus proof can be associated several proof-nets with weights (and the cut-elimination procedure is not always defined for such proof-nets).

We then have a new canonical syntax, independent from sequent calculus, allowing to make correct computations. Despite the fact that we don't have a procedure to sequentialize proof-nets with cuts, we know that if we start from a sequentializable proof-net S, we eventually reach a normal form S_0 which is itself sequentializable. This means on the one hand that nothing is lost, and on the other hand that the new objects which naturally appear (and which are not necessarily sequentializable) have a clear and well-structured computational behaviour. Actually, this is precisely the point where our approach differs from the one of [HvG03]: we mainly focus on the computational behaviour of our objects (cut-elimination), while [HvG03]'s main issue is correctness. Indeed, the "proof-nets" (i.e. the correct proof-structures) introduced by Hughes and Van Glabbeek are all sequentializable and this is not the case of ours. However, the translation of sequent calculus into sliced proof-structures is a function (this is not the case for [HvG03]'s nets), and our cut-elimination procedure is local (just perform it, separately, in each slice) while Hughes and Van Glabbeek have to reduce all the slices at the same time. The non-sequentializable sliced proof-structures naturally appearing during (sliced) cut-elimination have a perfectly well-understood computational behaviour, and we do not see any reason to reject them.

The equivalence relation on sequent calculus proofs defined by our (sliced) proof-nets can be very well compared to the one defined by ordinary proof-nets in the multiplicative fragment of linear logic.

But do there exist some (interesting) subsystems F of sliced proof-structures with an injective semantics?

Such systems and semantics certainly exist in the absence of the additives (see [TdF01]), it is very likely also the case for [HvG03]. The next section gives a positive answer to the previous question in presence of both additive and exponential connectives. We want to mention here that this is just a first (limited) result, and it is very likely that it can be extended to full $\mathsf{LL}_{\mathrm{pol}}$.

1.8 An application: λ-calculus with pairing

We prove that (relational) semantics is injective for the fragment $\lambda\mathsf{LL}_{\mathrm{pol}}$ of $\mathsf{LL}_{\mathrm{pol}}$, which corresponds to the simply typed λ-calculus with pairing.

Definition 1.36 (λ-calculus with pairing)

$$t \quad ::= \quad x \quad | \quad \lambda x.t \quad | \quad (t)t \quad | \quad \pi_1 t \quad | \quad \pi_2 t \quad | \quad <t,t>$$

Definition 1.37 (Girard's translation) *The types of the λ-calculus with pairing are translated as negative formulas as follows:*

$$\begin{array}{ccc} X & \rightsquigarrow & X \\ A \to B & \rightsquigarrow & ?A^{\perp} \parr B \\ A \wedge B & \rightsquigarrow & A \,\&\, B \end{array}$$

and terms are translated by the straightforward extension of Girard's translation [Gir87, Dan90] for the λ-calculus.

Let $\lambda\mathsf{LL}_{\mathrm{pol}}$ be the sub-system of $\mathsf{LL}_{\mathrm{pol}}$ containing only the following formulas:

$$\begin{array}{ccccccc} N & ::= & X & | & N \,\&\, N & | & ?P \parr N \\ P & ::= & X^{\perp} & | & P \oplus P & | & !N \otimes P \end{array}$$

(and their sub-formulas) together with the $\flat P$-formulas, and such that all the conclusions of proofs are negative formulas.

Terms are translated by proof-nets of $\lambda\mathsf{LL}_{\mathrm{pol}}$. The constraint that axiom links introduce only atomic formulas entails that the translation contains an implicit η-expansion of terms.

In the present section, in order to prove injectivity for $\lambda\mathsf{LL}_{\mathrm{pol}}$, we restrict to proof-structures, slices and sliced proof-structures of $\lambda\mathsf{LL}_{\mathrm{pol}}$ *without cut links* (corresponding to normal terms).

Definition 1.38 *Let s be a single-threaded slice. We denote by $L(s)$ (the "linearization" of s) the graph obtained by replacing every $!$-link n by the associated slice. More precisely, if n is an $!$-link having a conclusion of type $!A$ with an associated slice s_n, we replace n by a modified unary $!$-link with as premise the conclusion A of s_n; the $\flat$-conclusions of n are replaced by the corresponding $\flat$-conclusions of s_n.*

Remark 1.39 *If e is a 1-experiment of s, then with every edge a of type A of $L(s)$ is associated a unique label $e(a)$ of $\mathcal{A}$.*

For the 1-experiment e, we will denote by $e|_{L(s)}$ the labeling of the edges of $L(s)$ associated with e.

Lemma 1.40 *Let s and s' be two single-threaded slices. Let e (resp. e') be an injective 1-experiment of s (resp. s') with result γ (resp. γ').*

If $\gamma = \gamma'$, then $L(s) = L(s')$ and $e|_{L(s)} = e'|_{L(s')}$.

Proof Our claim is that the graph $L(s)$ so as the labels of its edges are completely determined by the types of the conclusions of s and by the result of an injective 1-experiment of s. Indeed, let's start from some edge a of $L(s)$, with its type A and its label $x \in \mathcal{A}$. There are exactly three cases in which either the type A of a is not enough to determine the link of $L(s)$ having a as conclusion or the link is known but the bottom-up propagation of the labels is not obviously deterministic:

(i) $A = C \,\&\, D$: then a might be conclusion of a $\&_1$- or of a $\&_2$-link. But the label of a tells us which of these two cases holds, and which is the label of the premise of the &-link.

(ii) $A = C \oplus D$: exactly like in the previous case.

(iii) $A = ?C$: then, *because e is a 1-experiment*, the cardinality of the label of a is the arity of the ?-link with conclusion a. This also implies that there is a unique way to determine the labels of the premises of the ?-link.

To conclude, notice that the fact that e is injective allows to uniquely determine the axiom links of $L(s)$. $\square$

1.8.1 Recovering boxes in $\lambda\mathsf{LL}_{\mathrm{pol}}$

We are now going to use in a strong way the particular shape of the (sliced) proof-nets of $\lambda\mathsf{LL}_{\mathrm{pol}}$. We show that for a single-threaded slice s of this fragment, the graph $L(s)$ contains as much information as s. In other terms, once $L(s)$ is known, the fact that s is a single-threaded slice of a sliced proof-structure which is the translation of a term, uniquely determines the way to "put" the boxes on the graph $L(s)$.

Lemma 1.41 *If s is a slice of a $\lambda\mathsf{LL}_{\mathrm{pol}}$ proof-net, there is exactly one $\flat$-link with depth 0 in every slice of every box of s.*

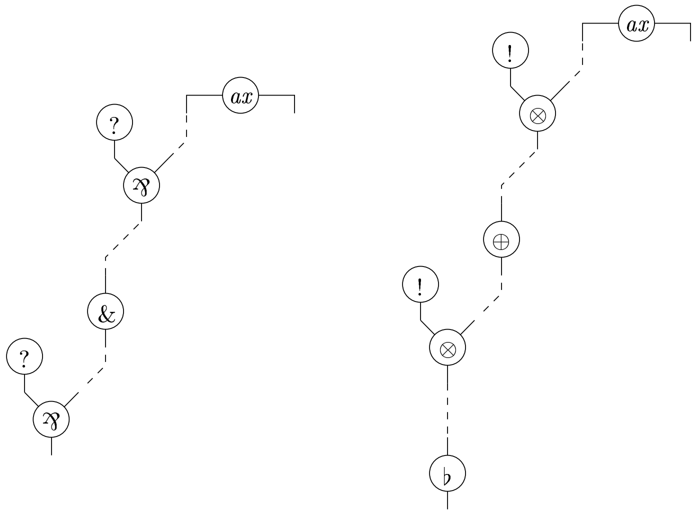


Fig. 1.1. Combs

Proof Just the correctness criterion (theorem 1.32). □

Lemma 1.42 *Let s be a single-threaded slice, and a an edge of type A of s.*

If A is a negative (resp. positive) formula, then the graph above a is a comb (see figure 1.1):

- *the teeth of the "negative comb" are edges of type $?$, while the backbone is made of unary $\&$-nodes and of $⅋$-nodes and moving upwards along it one necessarily ends in the unique (negative) atomic edge of the comb.*
- *dually, the teeth of the "positive comb" are edges of type $!$, while the backbone is made of $\oplus$-nodes and $\otimes$-nodes and moving upwards along it one necessarily ends in the unique (positive) atomic edge of the comb.*

We will speak of the comb associated with a. *Notice that a is considered as an edge of the comb.*

Proof Immediate consequence of the definition of $\lambda\mathsf{LL}_{\mathrm{pol}}$ and of the definition of single-threaded slice. □

Remark 1.43 *As a consequence of the previous lemma, with every negative edge α of a single-threaded slice s, is associated an oriented path Φ_α of s (see figure 1.2): it is the path with starting edge α (oriented upwardly), following the backbone of the negative comb up to the negative atomic edge X of the comb, crossing the axiom link and its positive conclusion $X^\perp$ (oriented now downwardly) and moving downwardly along the backbone of a positive comb (crossing $\oplus$- and $\otimes$-links) until a $\flat$-link is reached (there are no other possibilities).*

We will refer to Φ_α (in the sequel of the paragraph) as the oriented path associated with the negative edge α.

Until the end of this section 1.8.1, we will fix the following notations according to figure 1.2:

- α is the (negative) edge premise of an !-link l of the single-threaded slice s
- B_l is the box associated with l
- n is the last link of Φ_α which is a $\flat$-link (by remark 1.43)
- c is the (positive) premise of n
- $l_1, \ldots, l_k$ are the $k \geq 0$!-links of s whose conclusions are the teeth of the positive comb associated with c
- $B_1, \ldots, B_k$ are the boxes associated with $l_1, \ldots, l_k$

Remark 1.44 *Every edge of s "above α" is contained in B_l. Moreover, all the links of Φ_α (including n) are contained in B_l.*

Lemma 1.45 *Every edge with depth 0 of B_l is either an edge of Φ_α, or the conclusion of the $\flat$-link n, or a tooth of one of the two combs of Φ_α, or the $\flat$-conclusion of a !-link.*

Proof See figure 1.2.

Let G be the correction graph of B_l (see definition 1.17). The initial nodes of G are n and the 0-ary ?-links. Every link in G is accessible by an oriented path from an initial link, but any 0-ary ?-link is the premise of a $⅋$-link (in $\lambda\mathsf{LL}_{\text{pol}}$) that must be also accessible through its other premise. By induction on the number of links above this $⅋$-link we easily show that it is accessible from a non ?-link. So that every link (except 0-ary ?-links) at depth 0 in B_l is accessible from n. We said that every conclusion of a 0-ary ?-link is the premise of a $⅋$-link, and we just

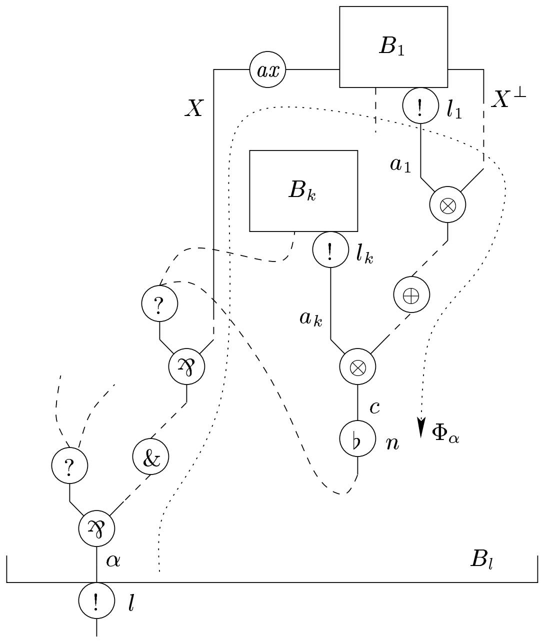


Fig. 1.2. Lemma 1.45 (dashed lines are given as examples)

proved that this ⅋-link is accessible from n: the conclusions of the 0-ary ?-links must then be teeth of the negative comb of Φ_α. □

We are now going to define a (partial) order relation on the !-links of $L(s)$, for every single-threaded slice s. We then show that this relation coincides with the nesting of boxes and it is enough to recover the boxes of s.

Definition 1.46 *Let s be a single-threaded slice, let l and m be two !-links of s and let α be the premise of l in $L(s)$. We define the relation $<_1$ on the !-links of s as follows: $m <_1 l$ iff the oriented path Φ_α crosses the $\otimes$-link of s having m as premise. We define the relation $\leq$ as the reflexive and transitive closure of $<_1$.*

Lemma 1.47 *Let s be a single-threaded slice. If l (resp. m) is an !-link of s and B_l (resp. B_m) is the box associated with l (resp. m), then $m \leq l$ iff B_l contains B_m.*

In particular, this implies that the relation $\leq$ is indeed a partial order relation.

Proof Suppose that $m \leq l$. From the nesting condition, it is clearly enough to show that if $m <_1 l$, then B_l contains m, and this is a consequence of remark 1.44.

Conversely, suppose that B_l contains B_m. It is again enough to consider the case in which B_l is the smallest box containing B_m. By lemma 1.45, the conclusion of m is one of the teeth of the positive comb of Φ_α. We just proved that $m <_1 l$. □

Proposition 1.48 *Let s and s' be two single-threaded slices of $\lambda\mathsf{LL}_{\text{pol}}$. If $L(s) = L(s')$, then $s = s'$.*

Proof The reason why this holds is that the paths of $L(s)$ are the same as the paths of s.

We still use figure 1.2 and show, by induction on the number of !-links of $L(s)$ smaller (with respect to $\leq$) than l (noting that it is a finite number by lemma 1.47, since our graphs are finite), that once $L(s)$ is known, we know how to recover B_l. By induction hypothesis, we know how to recover $B_1, \ldots, B_k$. By lemma 1.45, every edge with depth 0 of B_l is either an edge of one of the two combs of Φ_α, or the conclusion of n. By remark 1.44, all the just mentioned edges are edges of B_l. Then B_l can only be the graph containing $B_1, \ldots, B_k$, the two combs of Φ_α (including the ?-links) and the conclusion of n. □

1.8.2 Injectivity for $\lambda\mathsf{LL}_{\text{pol}}$

We prove the following lemma for relational semantics. It certainly holds in the coherent case too, but a detailed proof would require some more intermediate results.

In the sequel, we will write $[\![S]\!] = [\![S']\!]$, always meaning that the equality holds *for every interpretation of the atoms* of the formulas of S and S'.

Lemma 1.49 *Let S and S' be two sliced proof-structures with the same conclusions.*

If $[\![S]\!] = [\![S']\!]$, then $sgth(S) = sgth(S')$, where $sgth(S)$ (resp. $sgth(S')$) is the set of the single-threaded slices of S (resp. S').

Proof By contradiction, suppose that $sgth(S) \neq sgth(S')$. There exists a single-threaded slice s of S which is different from all the single-threaded slices of S'. Let e be an injective 1-experiment of s with result γ. Such an experiment obviously exists, at least in the case of relational semantics. From $[\![S]\!] = [\![S']\!]$, there exists an experiment e' of S' with result γ. It is easy to convince oneself that e' is a 1-experiment of S' (see [TdF01] for a proof without additives). From remark 1.16, e' is then a 1-experiment of a single-threaded slice s' of S'.

By lemma 1.40, we obtain that $L(s) = L(s')$, and then by proposition 1.48, $s = s'$ which is a contradiction. ☐

Definition 1.50 ($\flat$-free subgraph) *The* $\flat$-free subgraph *of a slice s is the graph obtained by keeping only the part of s at depth 0 and by replacing every $!$-link by an $!$-link without any $\flat$-conclusion. This erases some $\flat$-edges that are premises of $?$-links.*

Definition 1.51 (Non-contradiction of slices) *Let s and s' be two single-threaded slices with the same non-$\flat$ conclusions, the fact that s and s' are* non-contradictory *is defined by induction on the depth of s. s and s' are non-contradictory if either there exists a $\&_i$ (resp. $\&_j$) link n (resp. n') of s (resp. s') at depth 0 such that $n \equiv n'$ and $i \neq j$, or s and s' have the same $\flat$-free subgraph and the boxes of s and s' are non-contradictory.*

A sliced proof-structure S is non-contradictory *if for every pair of single-threaded slices s and s' of S, s and s' are non-contradictory.*

Theorem 1.52 (Injectivity) *Let S and S' be two non-contradictory proof-nets with the same conclusions.*

If $[\![S]\!] = [\![S']\!]$, then $S = S'$.

Proof By lemma 1.49, we have $sgth(S) = sgth(S')$. For a given set of non-contradictory single-threaded slices, there is only one way to reconstruct a sliced proof-structure: to glue (recursively with respect to the depth) the single-threaded slices with the same part at depth 0. ☐

Remark 1.53 *The reader should not think that the hypothesis of "non-contradiction" of proof-nets weakens our injectivity theorem: it is the opposite! Indeed, our requirement for a sliced proof-structure to deserve the name of proof-net is just that "it contains only correct slices" (see definition 1.18). This (minimal) requirement is already enough to make* *correct computations (theorem 1.24), which are also semantically sound (theorem 1.26). But it is obvious that a set of correct slices is not sequentializable (in general), and we could prove theorem 1.32 only by adding the "compatibility" and "fullness" conditions. A full and compatible proof-net is always non-contradictory, and the non-contradiction hypothesis (weaker than the compatibility and fullness one) is already enough to prove theorem 1.52.*

1.8.3 Computing with the λ-calculus slices

To apply the content of section 1.7 to $\lambda\mathsf{LL}_{\text{pol}}$, just notice that if S_{π_0} is the sliced proof-structure associated with the cut-free sequent calculus proof π_0, then by remark 1.31, S_{π_0} is full and compatible (thus non-contradictory). If π is a sequent calculus proof of $\lambda\mathsf{LL}_{\text{pol}}$ and S_0 is a normal form of S_π, then $[\![S_{\pi_0}]\!] = [\![S_0]\!]$ and by theorem 1.52, $S_0 = S_{\pi_0}$.

Acknowledgments: We thank Laurent Regnier who suggested us the property of the λ-calculus expressed by proposition 1.48, thus allowing us to (drastically) simplify the proof of theorem 1.52.

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