Polarities in Linear Logic

LL '02

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λ-calculus and Linear Logic

	LL	ILL	LLP
	$\Gamma \vdash \Delta$	$\Gamma \vdash A$	$\Gamma \vdash \Delta$
λ-calculus	$A \to B \rightsquigarrow\ !A \multimap B$ $\Gamma \vdash A \rightsquigarrow\ !\Gamma \vdash A$	$\vdash \overbrace{?\Gamma^{\perp}}^{i}, \overbrace{A}^{o}$	$\vdash \overbrace{?\Gamma^{\perp}, A}^{-}$
polarities		i/o	$+/-$
$\lambda\mu$-calculus LC	$A \to B \rightsquigarrow\ !?A \multimap ?B$ $A \to B \rightsquigarrow\ !(A \multimap ?B)$		$A \to B \rightsquigarrow\ !A \multimap B$ $A \to B \rightsquigarrow\ !(A \multimap ?B)$

Polarized connectives

Type translations:

Call By Name			Call By Value		
$A \wedge B$	$\Longmapsto$	$A \mathbin{\&} B$	$A \otimes B$	$\Longleftarrow\!\!\dashv$	$A \wedge B$
$A \to B$	$\Longmapsto$	$?A^{\perp} \parr B$	$!(A^{\perp} \parr ?B)$	$\Longleftarrow\!\!\dashv$	$A \to B$

Polarized formulas:

$$
\begin{array}{lllllllllllll}
P & ::= & X & \mid & P \otimes P & \mid & P \oplus P & \mid & 1 & \mid & 0 & \mid & !N \\
N & ::= & X^{\perp} & \mid & N \parr N & \mid & N \mathbin{\&} N & \mid & \perp & \mid & \top & \mid & ?P
\end{array}
$$

- negative $\leftrightarrow$ reversibility
- positive $\leftrightarrow$ focalization (Andreoli)

Polarized Linear Logic

$$\frac{}{\vdash N, N^{\perp}}\ ax \qquad \frac{\vdash \Gamma, N \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ cut$$

$$\frac{\vdash \Gamma, N, M}{\vdash \Gamma, N \parr M}\ \parr \qquad \frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\ \otimes$$

$$\frac{\vdash \Gamma, N \qquad \vdash \Gamma, M}{\vdash \Gamma, N \,\&\, M}\ \& \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\ \oplus_1 \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\ \oplus_2$$

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\ ! \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}\ ?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, N}\ ?w \qquad \frac{\vdash \Gamma, N, N}{\vdash \Gamma, N}\ ?c$$

$$\frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \frac{}{\vdash \Gamma, \top}\ \top$$

Properties

Focalization:

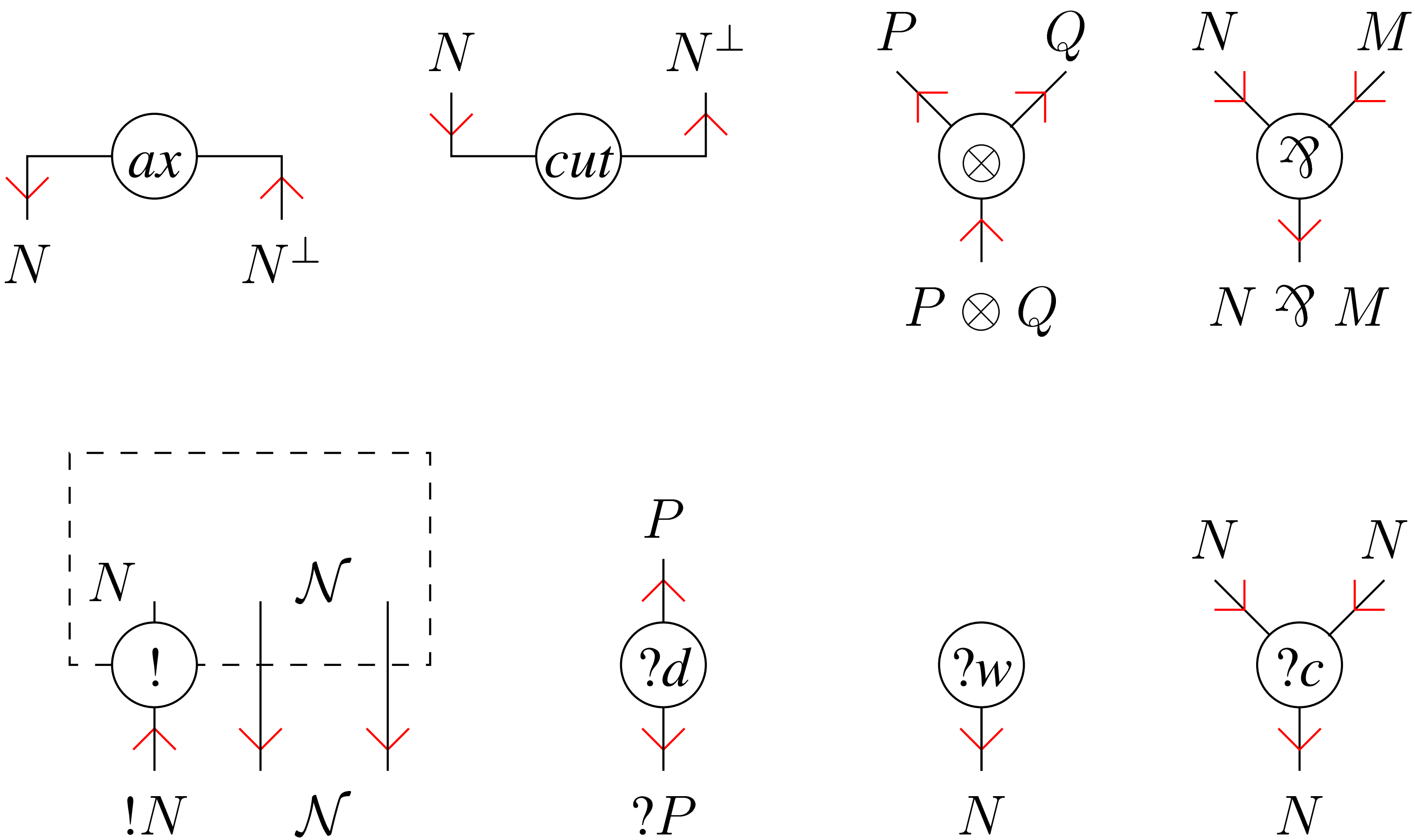
if $\vdash \Gamma$ in LLP, then Γ contains at most one positive formula

Internal translations:

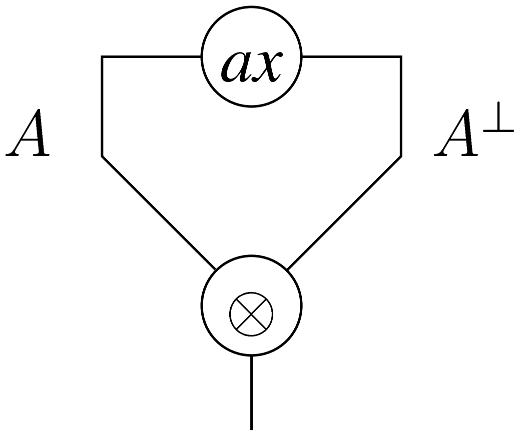
$$
\begin{array}{ccccccccc}
N & = & \parr \,\&\, \parr \,\& & ? & \otimes \oplus \otimes \oplus & ! & \parr \,\&\, \parr \,\& & ? & \ldots \\
 & \simeq & \&\, \parr & ? & \oplus\, \otimes & ! & \&\, \parr & ? & \ldots \\
 & \simeq & \& & ? & \oplus & ! & \& & ? & \ldots \\
 & & & & & & & & \\
P & \simeq & \oplus & ! & \& & ? & \oplus & ! & \ldots
\end{array}
$$

Proof-Nets

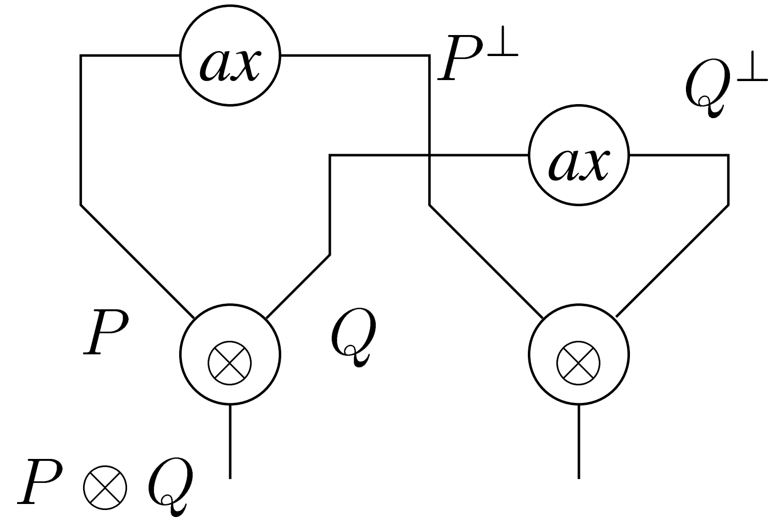
Polarized nodes

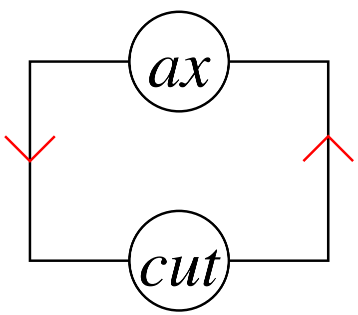


Invalid proof-nets



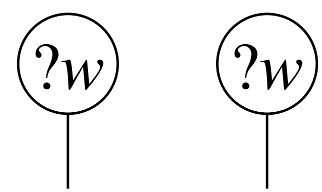
not typable in LLP





cyclic

not connected enough



Correctness

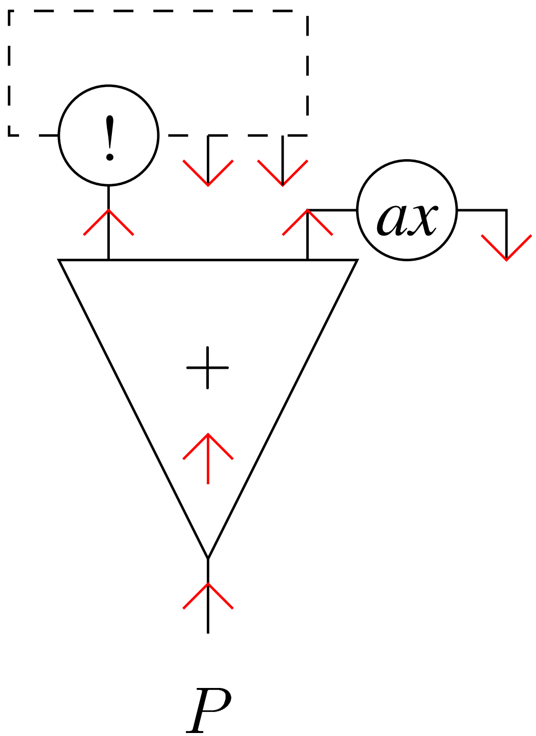
Correctness criterion:

- acyclic: as a directed graph
- connected: exactly one $?d$-node or one positive conclusion

Properties:

- linear complexity
- without cut $\Longrightarrow$ acyclic
- extensions to additives (boxes, weights, slices)

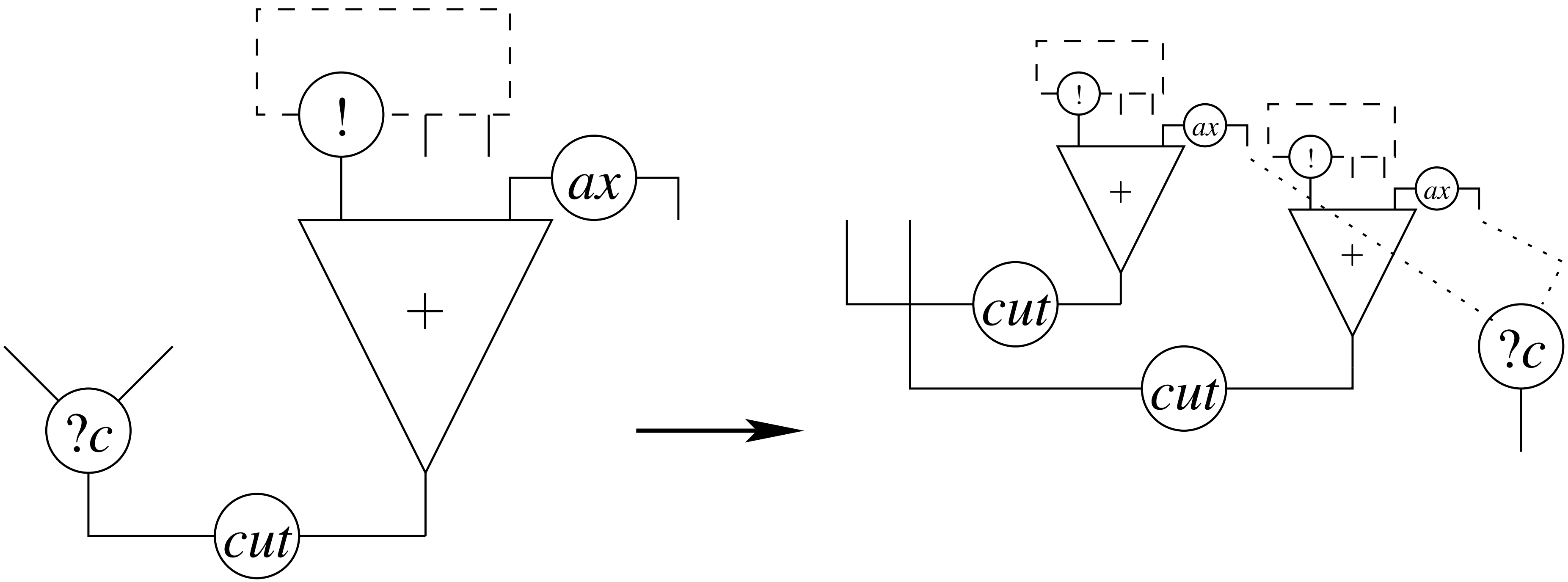
Cut elimination



Generalized box:

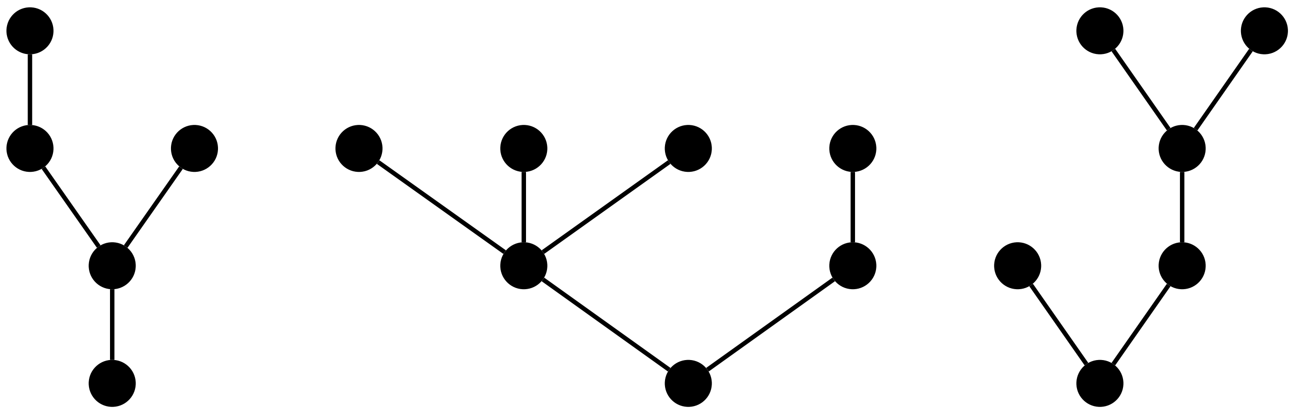
- sub proof-net
- one positive root
- only negative auxiliary doors

Cut elimination



Games

Arenas = forests

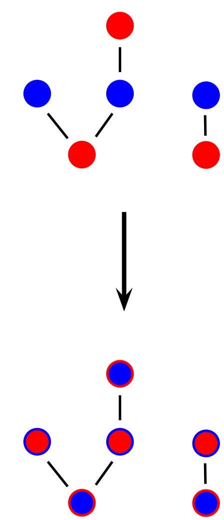


O / P

Arenas constructions

Definitions:

- $N^{\perp}$
- $\top$ (resp. 0)
- $\perp$ (resp. 1)
- $N \,\&\, M$ (resp. $P \oplus Q$)
- $N \parr M$ (resp. $P \otimes Q$)
- $!N$ (resp. $?P$)



Property:

same additive translation $\Longrightarrow$ same forest

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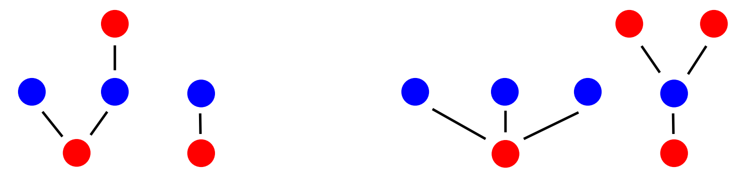
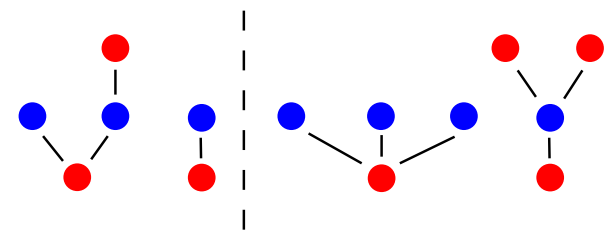


N M



$N \mathbin{\&} M$

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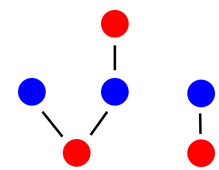
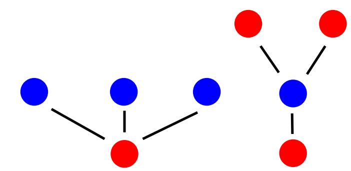
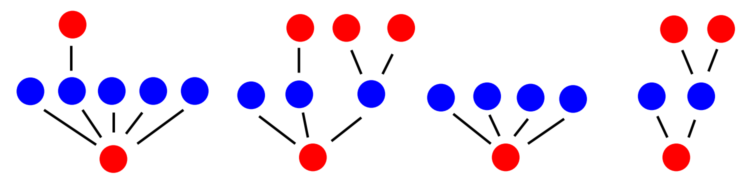


N



M



$N \parr M$

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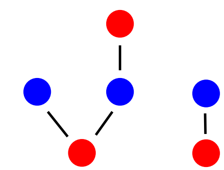
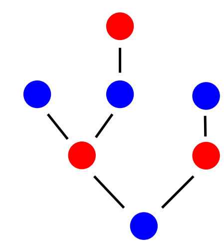


N



$!N$

Property:

same additive translation $\Longrightarrow$ same forest

Strategies

Definition:

non empty P-prefix closed set of plays $\mathcal{P}^O \times M^P$

Constraints:

- deterministic $\mathcal{P}^O \rightharpoonup M^P$
- total $\mathcal{P}^O \to M^P$
- visible
- innocent $\mathcal{V}^O \to M^P$
- finite

Game model for LLP

- **denotational model:**

$$\pi_1 =_{\beta\eta} \pi_2 \quad \Longrightarrow \quad \pi_1^\circ = \pi_2^\circ$$

- **surjective for formulas:**

$$\forall f,\ \exists A \quad \text{such that} \quad f = A^\circ$$

- **surjective for proofs (full completeness):**

$$\forall \sigma,\ \exists \pi \quad \text{such that} \quad \sigma = \pi^\circ$$

- **injective for formulas up to isomorphism:**

$$A^\circ = B^\circ \iff A^\circ \simeq B^\circ \iff A^{\mathrm{add}} = B^{\mathrm{add}} \iff A \simeq B$$

- **injective for sliced proof-nets (faithfulness):**

$$\mathcal{S}_1^\circ = \mathcal{S}_2^\circ \quad \Longrightarrow \quad \mathcal{S}_1 =_{\beta\eta} \mathcal{S}_2$$

Conclusions

- replacing intuitionistic polarities by classical ones
- embedding of classical logic (cbn and cbv)
- proof-nets
- game semantics
- categories
- coherent semantics
- ...

Current and future directions

- models of second order ($\mathsf{LLP} \neq \mathsf{LL}_{\mathrm{pol}}$)
- geometry of interaction
- application to lighter logics
- typing of the π-calculus (Berger-Honda-Yoshida)
- ...