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LAMBDA-CALCULUS
TYPES AND MODELS

Translated from french

by René Cori

To my daughter

Contents

INTRODUCTION

The lambda-calculus was invented in the early 1930's, by A. Church, and has been considerably developed since then. This book is an introduction to some aspects of the theory today : pure lambda-calculus, combinatory logic, semantics (models) of lambda-calculus, type systems. All these areas will be dealt with, only partially, of course, but in such a way, I think, as to illustrate their interdependence, and the essential unity of the subject.
No specific knowledge is required from the reader, but some familiarity with mathematical logic is expected ; in chapter 2, the concept of recursive function is used ; parts of chapters 6 and 7, as well as chapter 9, involve elementary topics in predicate calculus and model theory.

For about fifteen years, the typed lambda-calculus has provoked a great deal of interest, because of its close connections with programming languages, and of the link that it establishes between the concept of program and that of intuitionistic proof : this is known as the " Curry-Howard correspondence ". After the first type system, which was Curry's, many others appeared : for example, de Bruijn's Automath system, Girard's system $\mathscr{F}$, Martin-Löf's theory of intuitionistic types, Coquand-Huet's theory of constructions, Constable's Nuprl system...

This book will first introduce Coppo and Dezani's intersection type system. Here it will be called " system $\mathscr{D}\Omega$ ", and will be used to prove some fundamental theorems of pure lambda-calculus. It is also connected with denotational semantics : in Engeler and Scott's models, the interpretation of a term is essentially the set of its types. Next, Girard's system $\mathscr{F}$ of second order types will be considered, together with a simple extension, denoted by FA_2 (second order functional arithmetic). These types have a very transparent logical structure, and a great expressive power. They allow the Curry-Howard correspondence to be seen clearly, as well as the possibilities, and the difficulties, of using these systems as programming languages.

A programming language is a tool for writing a program in machine language (which is called the object code), in such a way as to keep control, as far as possible, on what will be done during its execution. To do so, the primitive method would be to write directly, in one column, machine language, and, alongside, comments indicating what the corresponding instructions are supposed to do. The result of this is called a " source program ". Here, the aim of the " compilation ", which transforms the source program into an object code, will be to get rid of the comments.

Such a language is said to be primitive, or " low level ", because the computer does not deal with the comments at all ; they are entirely intended for the programmer. In a higher level language, part of these comments would be checked by the computer, and the remainder left for the programmer ; the " mechanized " part of the comments is then called a " typing ". A language is considered high level if the type system is rich. In such a case, the aim of the compilation would be, first of all, to check the types, then, as before, to get rid of them, along with the rest of the comments.

The typed lambda-calculus can be used as a mathematical model for this situation ; the role of the machine language is played by the pure lambda-calculus. The type systems that are then considered are, in general, much more rich than those of the actual programming languages ; in fact, the types could almost be complete specifications of the programs, while the type checking (compilation) would be a " program proof ". These remarks are sufficient to explain the great interest there would be in constructing a programming language based on typed lambda-calculus ; but the problems, theoretical and practical, of such an enterprise are far from being fully resolved.

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Finally, it gives me much pleasure to dedicate this book to my daughter Sonia.

Paris, 1990

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Paris, 2011

Chapter 1

Substitution and beta-conversion

The terms of the λ-calculus (also called *λ-terms*) are finite sequences formed with the following symbols : variables $x, y, \ldots$ (the set of variables is assumed to be countable), left and right parenthesis, and the letter λ. They are obtained by applying, a finite number of times, the following rules :

- any variable x is a λ-term ;
- whenever t and u are λ-terms, then so is $(t)u$;
- whenever t is a λ-term and x is a variable, then $\lambda x\, t$ is a λ-term.

The set of all terms of the λ-calculus will be denoted by L.

The term $(t)u$ should be thought of as " t applied to u " ; it will also be denoted by tu if there is no ambiguity ; the term $(\ldots(((t)u_1)u_2)\ldots)u_k$ will also be written $(t)u_1u_2\ldots u_k$ or $tu_1u_2\ldots u_k$. Thus, for example, $(t)uv$, $(tu)v$ and tuv denote the same term.
By convention, when $k = 0$, $(t)u_1u_2\ldots u_k$ will denote the term t.

The *free occurrences* of a variable x in a term t are defined, by induction, as follows :

> if t is the variable x, then the occurrence of x in t is free ;
>
> if $t = (u)v$, then the free occurrences of x in t are those of x in u and v ;
>
> if $t = \lambda y\, u$, the free occurrences of x in t are those of x in u, except if $x = y$; in that case, no occurrence of x in t is free.

A *free variable* in t is a variable which has at least one free occurrence in t.
A term which has no free variable is called a *closed term*.
A *bound variable* in t is a variable which occurs in t just after the symbol λ.

1. Simple substitution

Let $t, t_1, \dots, t_k$ be terms and $x_1, \dots, x_k$ *distinct* variables ; we define the term $t{<}t_1/x_1, \dots, t_k/x_k{>}$ as the result of the replacement of every free occurrence of x_i in t by t_i $(1 \le i \le k)$. The definition is by induction on t, as follows :

if $t = x_i$ $(1 \le i \le k)$, then $t{<}t_1/x_1, \dots, t_k/x_k{>} = t_i$;
if t is a variable $\ne x_1, \dots, x_k$, then $t{<}t_1/x_1, \dots, t_k/x_k{>} = t$;
if $t = (u)v$, then
$$t{<}t_1/x_1, \dots, t_k/x_k{>} = (u{<}t_1/x_1, \dots, t_k/x_k{>})v{<}t_1/x_1, \dots, t_k/x_k{>} ;$$
if $t = \lambda x_i\, u$ $(1 \le i \le k)$, then
$$t{<}t_1/x_1, \dots, t_k/x_k{>} = \lambda x_i\, u{<}t_1/x_1, \dots, t_{i-1}/x_{i-1}, t_{i+1}/x_{i+1}, \dots, t_k/x_k{>} ;$$
if $t = \lambda x\, u$, with $x \ne x_1, \dots, x_k$, then
$$t{<}t_1/x_1, \dots, t_k/x_k{>} = \lambda x\, u{<}t_1/x_1, \dots, t_k/x_k{>}.$$

Such a substitution will be called a *simple* one, in order to distinguish it from the substitution defined further on, which needs a change of bound variables. Simple substitution corresponds, in computer science, to the notion of *macro-instruction.* It is also called *substitution with capture of variables.*

With the notation $t{<}t_1/x_1, \dots, t_k/x_k{>}$, it is understood that $x_1, \dots, x_k$ are distinct variables. Moreover, their order does not matter ; in other words :

$t{<}t_1/x_1, \dots, t_k/x_k{>} = t{<}t_{\sigma 1}/x_{\sigma 1}, \dots, t_{\sigma k}/x_{\sigma k}{>}$ *for any permutation* σ *of* $\{1, \dots, k\}$.

The proof is immediate by induction on the length of t ; also immediate is the following :

If $t_1, \dots, t_k$ *are variables, then the term* $t{<}t_1/x_1, \dots, t_k/x_k{>}$ *has the same length as* t.

Lemma 1.1. *If the variable* x_1 *is not free in the term* t *of* L*, then :*
$t{<}t_1/x_1, \dots, t_k/x_k{>} = t{<}t_2/x_2, \dots, t_k/x_k{>}$.

Proof by induction on t. The result is clear when t is either a variable or a term of the form $(u)v$. Now suppose $t = \lambda x\, u$; then :

if $x = x_1$, then :
$t{<}t_1/x_1, \dots, t_k/x_k{>} = \lambda x_1\, u{<}t_2/x_2, \dots, t_k/x_k{>} = t{<}t_2/x_2, \dots, t_k/x_k{>}$;

if $x = x_i$ with $i \ne 1$, say $x = x_k$, then :
$t{<}t_1/x_1, \dots, t_k/x_k{>} = \lambda x_k\, u{<}t_1/x_1, \dots, t_{k-1}/x_{k-1}{>}$
$= \lambda x_k\, u{<}t_2/x_2, \dots, t_{k-1}/x_{k-1}{>}$
(by induction hypothesis, since x_1 is not free in u)
$= t{<}t_2/x_2, \dots, t_k/x_k{>}$;

if $x \ne x_1, \dots, x_k$, then :
$t{<}t_1/x_1, \dots, t_k/x_k{>} = \lambda x\, u{<}t_1/x_1, \dots, t_k/x_k{>} = \lambda x\, u{<}t_2/x_2, \dots, t_k/x_k{>}$
(by induction hypothesis, since x_1 is not free in u) $= t{<}t_2/x_2, \dots, t_k/x_k{>}$.

Q.E.D.

Remark. Usually, in textbooks on λ-calculus (for example in [Bar84]), the simple substitution is considered for only one variable. In a substitution such as $t{<}u/x{>}$, the term t is then called a *context* or a *term with holes* ; the free occurrences of the variable x in t are called *holes* and denoted by [ ]. The term $t{<}u/x{>}$ is then denoted as $t[u]$ and is called the result of the " substitution of the term u in the holes of the context t ".

The major problem about simple substitution is that it is not *stable under composition* ; if you consider two substitutions :
$<t_1/x_1,\ldots,t_m/x_m>$ and $<u_1/y_1,\ldots,u_n/y_n>$
then the application $t \mapsto t{<}t_1/x_1,\ldots,t_m/x_m{>}{<}u_1/y_1,\ldots,u_n/y_n{>}$ is not, in general, given by a substitution. For instance, we have :
$y{<}y/x{>}{<}x/y{>} = x$ and $z{<}y/x{>}{<}x/y{>} = z$ for every variable $z \neq y$. Thus, if the operation $<y/x><x/y>$ was a substitution, it would be $<x/y>$. But this is false, because $\lambda y\, x{<}y/x{>}{<}x/y{>} = \lambda y\, y$ and $\lambda y\, x{<}x/y{>} = \lambda y\, x$.

In the following lemma, we give a partial answer to this problem. The definitive answer is given in the next section, with a new kind of substitution, which is stable by composition.

Lemma 1.2.
Let $\{x_1,\ldots,x_m\}$, $\{y_1,\ldots,y_n\}$ be two finite sets of variables, and suppose that their common elements are $x_1 = y_1,\ldots,x_k = y_k$. Let $t, t_1,\ldots,t_m,u_1,\ldots,u_n$ be terms of L, and assume that no free variable of $t_1,\ldots,t_m$ is bound in t. Then :
$$t{<}t_1/x_1,\ldots,t_m/x_m{>}{<}u_1/y_1,\ldots,u_n/y_n{>} = t{<}t'_1/x_1,\ldots,t'_m/x_m,u_{k+1}/y_{k+1},\ldots,u_n/y_n{>},$$
where $t'_i = t_i{<}u_1/y_1,\ldots,u_n/y_n{>}$.

Proof by induction on the length of t :
i) t is a variable : the possible cases are $t = x_i$ $(1 \le i \le m)$, $t = y_j$ $(k+1 \le j \le n)$, or t is another variable. In each of them, the result is immediate.
ii) $t = (u)v$; the result is obvious, by applying the induction hypothesis to u and v.
iii) $t = \lambda x\, u$; we first observe that the result follows immediately from the induction hypothesis for u, if $x \neq x_1,\ldots,x_m,y_1,\ldots,y_n$.

If $x = x_i$ $(1 \le i \le k)$, say x_1, then :
$t{<}t_1/x_1,\ldots,t_m/x_m{>} = \lambda x_1\, u{<}t_2/x_2,\ldots,t_m/x_m{>}$.
Since $x_1 = y_1$, we have :
$$t{<}t_1/x_1,\ldots,t_m/x_m{>}{<}u_1/y_1,\ldots,u_n/y_n{>} = \lambda x_1\, u{<}t_2/x_2,\ldots,t_m/x_m{>}{<}u_2/y_2,\ldots,u_n/y_n{>}.$$
By the induction hypothesis for u, we get :
$$u{<}t_2/x_2,\ldots,t_m/x_m{>}{<}u_2/y_2,\ldots,u_n/y_n{>} = u{<}t''_2/x_2,\ldots,t''_m/x_m,u_{k+1}/y_{k+1},\ldots,u_n/y_n{>}$$
with $t''_i = t_i{<}u_2/y_2,\ldots,u_n/y_n{>}$.

But, since $x_1 = y_1$ is bound in t, by hypothesis, it is not a free variable of t_i. From lemma 1.1, it follows that $t''_i = t_i{<}u_1/y_1,\dots,u_n/y_n{>} = t'_i$. Therefore :
$t{<}t_1/x_1,\dots,t_m/x_m{>}{<}u_1/y_1,\dots,u_n/y_n{>}$
$= \lambda x_1\, u{<}t'_2/x_2,\dots,t'_m/x_m,u_{k+1}/y_{k+1},\dots,u_n/y_n{>}$
$= t{<}t'_1/x_1,\dots,t'_m/x_m,u_{k+1}/y_{k+1},\dots,u_n/y_n{>}$.

If $x = x_i$ $(k+1 \le i \le m)$, say x_m, then :
$t{<}t_1/x_1,\dots,t_m/x_m{>} = \lambda x_m\, u{<}t_1/x_1,\dots,t_{m-1}/x_{m-1}{>}$,

and since $x_m \ne y_1,\dots,y_n$, we get :
$t{<}t_1/x_1,\dots,t_m/x_m{>}{<}u_1/y_1,\dots,u_n/y_n{>}$
$= \lambda x_m\, u{<}t_1/x_1,\dots,t_{m-1}/x_{m-1}{>}{<}u_1/y_1,\dots,u_n/y_n{>}$.
By the induction hypothesis for u, we get :
$u{<}t_1/x_1,\dots,t_{m-1}/x_{m-1}{>}{<}u_1/y_1,\dots,u_n/y_n{>}$
$= u{<}t'_1/x_1,\dots,t'_{m-1}/x_{m-1},u_{k+1}/y_{k+1},\dots,u_n/y_n{>}$,
Therefore $t{<}t_1/x_1,\dots,t_m/x_m{>}{<}u_1/y_1,\dots,u_n/y_n{>}$
$= \lambda x_m\, u{<}t'_1/x_1,\dots,t'_{m-1}/x_{m-1},u_{k+1}/y_{k+1},\dots,u_n/y_n{>}$
$= t{<}t'_1/x_1,\dots,t'_m/x_m,u_{k+1}/y_{k+1},\dots,u_n/y_n{>}$.

If $x = y_j$ $(k+1 \le j \le n)$, say y_n, then :
$t{<}t_1/x_1,\dots,t_m/x_m{>} = \lambda y_n\, u{<}t_1/x_1,\dots,t_m/x_m{>}$, since $y_n \ne x_1,\dots,x_m$.
Therefore $t{<}t_1/x_1,\dots,t_m/x_m{>}{<}u_1/y_1,\dots,u_n/y_n{>}$
$= \lambda y_n\, u{<}t_1/x_1,\dots,t_m/x_m{>}{<}u_1/y_1,\dots,u_{n-1}/y_{n-1}{>}$.
By the induction hypothesis for u, we get :
$u{<}t_1/x_1,\dots,t_m/x_m{>}{<}u_1/y_1,\dots,u_{n-1}/y_{n-1}{>}$
$= u{<}t''_1/x_1,\dots,t''_m/x_m,u_{k+1}/y_{k+1},\dots,u_{n-1}/y_{n-1}{>}$,
with $t''_i = t_i{<}u_1/y_1,\dots,u_{n-1}/y_{n-1}{>}$.
But, since y_n is bound in t, by hypothesis, it is not a free variable of t_i. From lemma 1.1, it follows that $t''_i = t_i{<}u_1/y_1,\dots,u_n/y_n{>} = t'_i$. Therefore :
$t{<}t_1/x_1,\dots,t_m/x_m{>}{<}u_1/y_1,\dots,u_n/y_n{>}$
$= \lambda y_n\, u{<}t'_1/x_1,\dots,t'_m/x_m,u_{k+1}/y_{k+1},\dots,u_{n-1}/y_{n-1}{>}$
$= t{<}t'_1/x_1,\dots,t'_m/x_m,u_{k+1}/y_{k+1},\dots,u_n/y_n{>}$.

Q.E.D.

Corollary 1.3. *Let $t, t_1,\dots,t_m$ be λ-terms, and $\{x_1,\dots,x_m\}$, $\{y_1,\dots,y_m\}$ two sets of variables such that none of the y_i's occur in t. Then :*
$t{<}y_1/x_1,\dots,y_m/x_m{>}{<}t_1/y_1,\dots,t_m/y_m{>} = t{<}t_1/x_1,\dots,t_m/x_m{>}$.

Suppose that $x_1,\dots,x_k \notin \{y_1,\dots,y_m\}$ and $x_{k+1},\dots,x_m \in \{y_1,\dots,y_m\}$.
Then $x_{k+1},\dots,x_m$ are not free in t and therefore, by lemma 1.1, we have :
$t{<}y_1/x_1,\dots,y_m/x_m{>} = t{<}y_1/x_1,\dots,y_k/x_k{>}$.
The two sets $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_m\}$ are disjoint, and the variables $y_1,\dots,y_m$ are not bound in t. Therefore, by lemma 1.2, we have :

$t<y_1/x_1,\dots,y_k/x_k><t_1/y_1,\dots,t_m/y_m> = t<t_1/x_1,\dots,t_k/x_k,t_1/y_1,\dots,t_m/y_m>$.
But $y_1,\dots,y_m$ are not free in t, and therefore, by lemma 1.1 :
$t<t_1/x_1,\dots,t_k/x_k,t_1/y_1,\dots,t_m/y_m> = t<t_1/x_1,\dots,t_k/x_k>$.
Now $x_{k+1},\dots,x_m$ are not free in t ; thus, again by lemma 1.1 :
$t<t_1/x_1,\dots,t_k/x_k> = t<t_1/x_1,\dots,t_m/x_m>$.

Q.E.D.

Let R be a binary relation on L ; we will say that R is λ*-compatible* if it is reflexive and satisfies :

$t\,R\,t' \Rightarrow \lambda x\,t\,R\,\lambda x\,t'$; $t\,R\,t'$, $u\,R\,u' \Rightarrow (t)u\,R\,(t')u'$.

Remark. A binary relation R is λ-compatible if and only if :
$x\,R\,x$ for each variable x ;
$t\,R\,t' \Rightarrow \lambda x\,t\,R\,\lambda x\,t'$; $t\,R\,t'$, $u\,R\,u' \Rightarrow (t)u\,R\,(t')u'$ for all terms t,u,t',u'.
Indeed, $t\,R\,t$ is easily proved, by induction on the length of t.

Lemma 1.4. *If R is λ-compatible and $t_1\,R\,t'_1,\dots,t_k\,R\,t'_k$, then :*
$t<t_1/x_1,\dots,t_k/x_k>\,R\,t<t'_1/x_1,\dots,t'_k/x_k>$.

Immediate proof by induction on the length of t.

Q.E.D.

Proposition 1.5. *Let R be a binary relation on L. Then, the least λ-compatible binary relation ρ containing R is defined by the following condition :*
(1) $t\rho\,t' \Leftrightarrow$ there exists terms $T,t_1,\dots,t_k,t'_1,\dots,t'_k$ and distinct variables $x_1,\dots,x_k$ such that $t_i\,R\,t'_i$ $(1 \le i \le k)$ and $t = T<t_1/x_1,\dots,t_k/x_k>$, $t' = T<t'_1/x_1,\dots,t'_k/x_k>$.

Let ρ' be the least λ-compatible binary relation containing R, and ρ the relation defined by condition (1) above. It follows from the previous lemma that $\rho' \supset \rho$. It is easy to see that $\rho \supset R$ (take $T = x_1$). It thus remains to prove that ρ is λ-compatible.
By taking $k = 0$ in condition (1), we see that ρ is reflexive.
Suppose $t = T<t_1/x_1,\dots,t_k/x_k>$, $t' = T<t'_1/x_1,\dots,t'_k/x_k>$. Let $y_1,\dots,y_k$ be distinct variables not occurring in T. Let $V = T<y_1/x_1,\dots,y_k/x_k>$. Then, it follows from corollary 1.3 that $t = V<t_1/y_1,\dots,t_k/y_k>$ and $t' = V<t'_1/y_1,\dots,t'_k/y_k>$. Thus the distinct variables $x_1,\dots,x_k$ in condition (1) can be arbitrarily chosen, except in some finite set.
Now suppose $t\rho t'$ and $u\rho u'$; then :

$t = T<t_1/x_1,\dots,t_k/x_k>$, $t' = T<t'_1/x_1,\dots,t'_k/x_k>$ with $t_iRt'_i$;
$u = U<u_1/y_1,\dots,u_l/y_l>$, $u' = U<u'_1/y_1,\dots,u'_l/y_l>$ with $u_jRu'_j$.

By the previous remark, we can assume that $x_1,\dots,x_k,y_1,\dots,y_l$ are distinct, different from x, and also that none of the x_i's occur in U, and none of the y_j's occur in T. Therefore :

$\lambda x\, t = (\lambda x\, T)<t_1/x_1, \ldots, t_k/x_k>,\ \ \lambda x\, t' = (\lambda x\, T)<t'_1/x_1, \ldots, t'_k/x_k>$
which proves that $\lambda x\, t\, \rho\, \lambda x\, t'$.
Also, by lemma 1.1 :
$t = T<t_1/x_1, \ldots, t_k/x_k, u_1/y_1, \ldots, u_l/y_l>$,
$t' = T<t'_1/x_1, \ldots, t'_k/x_k,\ u'_1/y_1, \ldots, u'_l/y_l>$
(since none of the y_j's occur in T) ;
and similarly :
$u = U<t_1/x_1, \ldots, t_k/x_k, u_1/y_1, \ldots, u_l/y_l>$,
$u' = U<t'_1/x_1, \ldots, t'_k/x_k, u'_1/y_1, \ldots, u'_l/y_l>$
(since none of the x_i's occur in U).
Let $V = (T)U$; then $(t)u = V<t_1/x_1, \ldots, t_k/x_k, u_1/y_1, \ldots, u_l/y_l>$,
$(t')u' = V<t'_1/x_1, \ldots, t'_k/x_k, u'_1/y_1, \ldots, u'_l/y_l>$ and thus $(t)u\, \rho\, (t')u'$.
Q.E.D.

2. Alpha-equivalence and substitution

We will now define an equivalence relation on the set L of all λ-terms. It is called *α-equivalence*, and denoted by $\equiv$.
Intuitively, $t \equiv t'$ means that t' is obtained from t by renaming the bound variables in t ; more precisely, $t \equiv t'$ if and only if t and t' have the same sequence of symbols (when all variables are considered equal), the same free occurrences of the same variables, and if each λ binds the same occurrences of variables in t and in t'.
We define $t \equiv t'$, on L, by induction on the length of t, by the following clauses :

if t is a variable, then $t \equiv t'$ if and only if $t = t'$;

if $t = (u)v$, then $t \equiv t'$ if and only if $t' = (u')v'$, with $u \equiv u'$ and $v \equiv v'$;

if $t = \lambda x\, u$, then $t \equiv t'$ if and only if $t' = \lambda x' u'$, with $u<y/x> \equiv u'<y/x'>$ for all variables y except a finite number.

(Note that $u<y/x>$ has the same length as u, thus is shorter than t, which guarantees the correctness of the inductive definition).

Proposition 1.6. *If $t \equiv t'$, then t and t' have the same length and the same free variables.*

The proof is done by induction on the length of t. The cases when t is a variable, or $t = uv$ are trivial.
Suppose now that $t = \lambda x\, u$ and therefore $t' = \lambda x'\, u'$. Thus, we have :
$u<y/x> \equiv u'<y/x'>$ for every variable y except a finite number.

We choose a variable $y \neq x, x'$ which, moreover, does not appear (free or bound) in u, u'. Let U (resp. U') be the set of free variables of u (resp. u').
The set V of free variables of $u{<}y/x{>}$ is U if $x \notin U$ and $(U \setminus \{x\}) \cup \{y\}$ if $x \in U$. Also, the set V' of free variables of $u'{<}y/x'{>}$ is U' if $x' \notin U'$ and $(U' \setminus \{x'\}) \cup \{y\}$ if $x' \in U'$. Now, we have $V = V'$, by the induction hypothesis.
If $x \notin U$, we have $y \notin V$, thus $y \notin V'$ and $x' \notin U'$. Thus $U = V = V' = U'$ and $\lambda x\, u$, $\lambda x'\, u'$ have the same set of free variables, which is U.
If $x \in U$, then $y \in V$, thus $y \in V'$ and therefore $x' \in U'$.
The set of free variables of $\lambda x\, u$ (resp. $\lambda x'\, u'$) is $U \setminus \{x\} = V \setminus \{y\}$ (resp. $U' \setminus \{x'\} = V' \setminus \{y\}$). Since $V = V'$, it is, once again, the same set.

Q.E.D.

The relation $\equiv$ is an equivalence relation on L.

Indeed, the proof of the three following properties is trivial, by induction on t : $t \equiv t$; $t \equiv t' \Rightarrow t' \equiv t$; $t \equiv t'$, $t' \equiv t'' \Rightarrow t \equiv t''$.

Proposition 1.7. *Let $t, t', t_1, t'_1 \dots, t_k, t'_k$ be λ-terms, and $x_1, \dots, x_k$ distinct variables. If $t \equiv t'$, $t_1 \equiv t'_1$, …, $t_k \equiv t'_k$ and if no free variable in $t_1, \dots, t_k$ is bound in t, t', then $t{<}t_1/x_1, \dots, t_k/x_k{>} \equiv t'{<}t'_1/x_1, \dots, t'_k/x_k{>}$.*

Note that, since $t \equiv t'$, t and t' have the same free variables. Thus it can be assumed that $x_1, \dots, x_k$ are free in t and t' ; indeed, if $x_1, \dots, x_l$ are those x_i variables which are free in t and t', then, by lemma 1.1 :
$t{<}t_1/x_1, \dots, t_k/x_k{>} = t{<}t_1/x_1, \dots, t_l/x_l{>}$ and
$t'{<}t'_1/x_1, \dots, t'_k/x_k{>} = t'{<}t'_1/x_1, \dots, t'_l/x_l{>}$.
Also, since $t_i \equiv t'_i$, t_i and t'_i have the same free variables. Therefore, no free variable in $t_1, t'_1, \dots, t_k, t'_k$ is bound in t, t'.

The proof of the proposition proceeds by induction on t. The result is immediate if t is a variable, or $t = (u)v$. Suppose $t = \lambda x\, u$. Then $t' = \lambda x' u'$ and $u{<}y/x{>} \equiv u'{<}y/x'{>}$ for all variables y except a finite number.
Since $x_1, \dots, x_k$ are free in t and t', x and x' are different from $x_1, \dots, x_k$. Thus $t{<}t_1/x_1, \dots, t_k/x_k{>} = \lambda x\, u{<}t_1/x_1, \dots, t_k/x_k{>}$ and
$t'{<}t'_1/x_1, \dots, t'_k/x_k{>} = \lambda x' u'{<}t'_1/x_1, \dots, t'_k/x_k{>}$.
Hence it is sufficient to show that :

$u{<}t_1/x_1, \dots, t_k/x_k{>}{<}y/x{>} \equiv u'{<}t'_1/x_1, \dots, t'_k/x_k{>}{<}y/x'{>}$

for all variables y except a finite number.
Therefore, we may assume that $y \neq x_1, \dots, x_k$. Since x, x' are respectively bound in t, t', they are not free in $t_1, \dots, t_k, t'_1, \dots, t'_k$; thus, it follows from lemma 1.2 that

$u{<}t_1/x_1, \dots, t_k/x_k{>}{<}y/x{>} = u{<}t_1/x_1, \dots, t_k/x_k, y/x{>}$ and
$u'{<}t'_1/x_1, \dots, t'_k/x_k{>}{<}y/x'{>} = u'{<}t'_1/x_1, \dots, t'_k/x_k, y/x'{>}$.

Since $y \neq x_1, \dots, x_k$, we get, applying again lemma 1.2 :

$u{<}y/x, t_1/x_1, \ldots, t_k/x_k{>} = u{<}y/x{>}{<}t_1/x_1, \ldots, t_k/x_k{>}$ and
$u'{<}y/x', t'_1/x_1, \ldots, t'_k/x_k{>} = u'{<}y/x'{>}{<}t'_1/x_1, \ldots, t'_k/x_k{>}$
and therefore :
$u{<}t_1/x_1, \ldots, t_k/x_k{>}{<}y/x{>} = u{<}y/x{>}{<}t_1/x_1, \ldots, t_k/x_k{>}$ and
$u'{<}t'_1/x_1, \ldots, t'_k/x_k{>}{<}y/x'{>} = u'{<}y/x'{>}{<}t'_1/x_1, \ldots, t'_k/x_k{>}$.
Now, since $u{<}y/x{>} \equiv u'{<}y/x'{>}$ for all variables y except a finite number, and $u{<}y/x{>}$ is shorter than t, the induction hypothesis gives :
$u{<}y/x{>}{<}t_1/x_1, \ldots, t_k/x_k{>} \equiv u'{<}y/x'{>}{<}t'_1/x_1, \ldots, t'_k/x_k{>}$, thus :
$u{<}t_1/x_1, \ldots, t_k/x_k{>}{<}y/x{>} \equiv u'{<}t'_1/x_1, \ldots, t'_k/x_k{>}{<}y/x'{>}$ for all variables y except a finite number.

Q.E.D.

Corollary 1.8. *The relation $\equiv$ is λ-compatible.*

Suppose $t \equiv t'$. We need to prove that $\lambda x\, t \equiv \lambda x\, t'$, that is to say :
$t{<}y/x{>} \equiv t'{<}y/x{>}$ for all variables y except a finite number. But this follows from proposition 1.7, provided that y is not a bound variable in t or in t'.

Q.E.D.

Corollary 1.9. *If $t, t_1, \ldots, t_k, t'_1, \ldots, t'_k$ are terms, and $x_1, \ldots, x_k$ are distinct variables, then :*

$$t_1 \equiv t'_1, \ldots, t_k \equiv t'_k \Rightarrow t{<}t_1/x_1, \ldots, t_k/x_k{>} \equiv t{<}t'_1/x_1, \ldots, t'_k/x_k{>}.$$

This follows from corollary 1.8 and lemma 1.4.

Q.E.D.

However, note that it is not true that $u \equiv u' \Rightarrow u{<}t/x{>} \equiv u'{<}t/x{>}$. For example, $\lambda y\, x \equiv \lambda z\, x$, while $\lambda y\, x{<}y/x{>} = \lambda y\, y \not\equiv \lambda z\, x{<}y/x{>} = \lambda z\, y$.

Lemma 1.10. *$\lambda x\, t \equiv \lambda y\, t{<}y/x{>}$ whenever y is a variable which does not occur in t.*

By corollary 1.3, $t{<}z/x{>} = t{<}y/x{>}{<}z/y{>}$ for any variable z, since y does not occur in t. Hence the result follows from the definition of $\equiv$.

Q.E.D.

Lemma 1.11. *Let t be a term, and $x_1, \ldots, x_k$ be variables. Then there exists a term t', $t' \equiv t$, such that none of $x_1, \ldots, x_k$ are bound in t'.*

The proof is by induction on t.
The result is immediate if t is a variable, or if $t = (u)v$.
If $t = \lambda x\, u$, then, by induction hypothesis, there exists a term u', $u' \equiv u$, in which none of $x_1, \ldots, x_k$ are bound. By the previous lemma, $t \equiv \lambda x\, u' \equiv \lambda y\, u'{<}y/x{>}$ with $y \neq x_1, \ldots, x_k$. Thus it is sufficient to take $t' = \lambda y\, u'{<}y/x{>}$.

Q.E.D.

From now on, α-equivalent terms will be identified ; hence we will deal with the quotient set $L/\equiv$; it is denoted by Λ.
For each variable x, its equivalence class will still be denoted by x (it is actually $\{x\}$). Furthermore, the operations $t, u \mapsto (t)u$ and $t, x \mapsto \lambda x\, t$ are compatible with $\equiv$ and are therefore defined in Λ.
Moreover, if $t \equiv t'$, then t and t' have the same free variables. Hence it is possible to define the free variables of a member of Λ.

Consider terms $t, t_1, \ldots, t_k \in \Lambda$ and distinct variables $x_1, \ldots, x_k$. Then the term $t[t_1/x_1, \ldots, t_k/x_k] \in \Lambda$ (being the result of the replacement of every free occurrence of x_i in t by t_i, for $i = 1, \ldots, k$) is defined as follows : let $\underline{t}, \underline{t}_1, \ldots, \underline{t}_k$ be terms of L, the equivalence classes of which are respectively $t, t_1, \ldots, t_k$. By lemma 1.11, we may assume that no bound variable of $\underline{t}$ is free in $t_1, \ldots, t_k$. Then $t[t_1/x_1, \ldots, t_k/x_k]$ is defined as the equivalence class of $\underline{t}<\underline{t}_1/x_1, \ldots, \underline{t}_k/x_k>$. Indeed, by proposition 1.7, this equivalence class does not depend on the choice of $\underline{t}, \underline{t}_1, \ldots, \underline{t}_k$.
So the substitution operation $t, t_1, \ldots, t_k \mapsto t[t_1/x_1, \ldots, t_k/x_k]$ is well defined in Λ. It corresponds to the replacement of the free occurrences of x_i in t by t_i $(1 \leq i \leq k)$, provided that a representative of t has been chosen such that no free variable in $t_1, \ldots, t_k$ is bound in it.
The substitution operation satisfies the following lemmas, already stated for the simple substitution :

Lemma 1.12. *If the variable x_1 is not free in the term t of Λ, then :*
$t[t_1/x_1, \ldots, t_k/x_k] = t[t_2/x_2, \ldots, t_k/x_k]$.

Immediate from lemma 1.1 and the definition of $t[t_1/x_1, \ldots, t_k/x_k]$.
Q.E.D.

The following lemma shows that the substitution behaves much better in Λ than in L (compare with lemma 1.2). In particular, it shows that *the composition of two substitutions gives a substitution.*

Lemma 1.13. *Let $\{x_1, \ldots, x_m\}$, $\{y_1, \ldots, y_n\}$ be two finite sets of variables, and suppose that their common elements are $x_1 = y_1, \ldots, x_k = y_k$.*
Let $t, t_1, \ldots, t_m, u_1, \ldots, u_n$ be terms of Λ. Then :
$t[t_1/x_1, \ldots, t_m/x_m][u_1/y_1, \ldots, u_n/y_n] = t[t'_1/x_1, \ldots, t'_m/x_m, u_{k+1}/y_{k+1}, \ldots, u_n/y_n]$
where $t'_i = t_i[u_1/y_1, \ldots, u_n/y_n]$.

Let $\underline{t}, \underline{t}_1, \ldots, \underline{t}_m, \underline{u}_1, \ldots, \underline{u}_n$ be some representatives of $t, t_1, \ldots, t_m, u_1, \ldots, u_n$. By lemma 1.11, we may assume that no bound variable of $\underline{t}$ is free in $t_1, \ldots, t_m$, $u_1, \ldots, u_n$, and that no bound variable of $\underline{t}_1, \ldots, \underline{t}_m$ is free in $u_1, \ldots, u_n$.

From lemma 1.2, we get :
$\underline{t}<\underline{t}_1/x_1,\dots,\underline{t}_m/x_m><\underline{u}_1/y_1,\dots,\underline{u}_n/y_n>$
$= \underline{t}<\underline{t}'_1/x_1,\dots,\underline{t}'_m/x_m,\underline{u}_{k+1}/y_{k+1},\dots,\underline{u}_n/y_n>$
where $\underline{t}'_i = \underline{t}_i<\underline{u}_1/y_1,\dots,\underline{u}_n/y_n>$.
The first member is a representative of $t[t_1/x_1,\dots,t_m/x_m][u_1/y_1,\dots,u_n/y_n]$, because $\underline{t}<\underline{t}_1/x_1,\dots,\underline{t}_m/x_m>$ is a representative of $t[t_1/x_1,\dots,t_m/x_m]$, and there is no bound variable of this term which is free in $\underline{u}_1,\dots,\underline{u}_n$. The second member is a representative of $t[t'_1/x_1,\dots,t'_m/x_m,u_{k+1}/y_{k+1},\dots,u_n/y_n]$, since no bound variable of $\underline{t}$ is free in $\underline{t}'_1,\dots,\underline{t}'_m,\underline{u}_{k+1},\dots,\underline{u}_n$.

Q.E.D.

Corollary 1.14. *Any free variable of $t[t_1/x_1,\dots,t_m/x_m]$ is free in t or in t_1 or … or in t_m.*

Let x be a variable which is not free in $t,t_1,\dots,t_m$. By lemma 1.13, we have $t[t_1/x_1,\dots,t_m/x_m][y/x] = t[t_1/x_1,\dots,t_m/x_m]$ for any variable y. This shows that x is not free in $t[t_1/x_1,\dots,t_m/x_m]$.

Q.E.D.

Lemma 1.15. *Let x,x' be variables and $u,u' \in \Lambda$ be such that $\lambda x\, u = \lambda x'\, u'$. Then $u[t/x] = u'[t/x']$ for every $t \in \Lambda$.*

Let $\underline{u},\underline{u}' \in L$ be representatives of u,u'. Then $\lambda x\,\underline{u} \equiv \lambda x'\,\underline{u}'$ and, by definition of the α-equivalence, we have $\underline{u}<y/x> \equiv \underline{u}'<y/x'>$ for every variable y but a finite number. If we suppose that y is not bound in $\underline{u},\underline{u}'$, we see that $u[y/x] = u'[y/x']$ for every variable y but a finite number ; therefore $u[y/x][t/y] = u'[y/x'][t/y]$. If we suppose that y is different from x,x', then, by lemma 1.13, we get :
$u[t/x,t/y] = u'[t/x',t/y]$. Assume now that y is not free in u,u' ; then, by lemma 1.12, we obtain $u[t/x] = u'[t/x']$.

Q.E.D.

Proposition 1.16. *Let $t \in \Lambda$ such that $t = \lambda x\, u$. Then, for every variable x' which is not free in t, there exists a unique $u' \in \Lambda$ such that $t = \lambda x'\, u'$; it is given by $u' = u[x'/x]$.*

Remark. Clearly, if x' is a free variable of t, we cannot have $t = \lambda x'\, u'$.

If $\lambda x\, u = \lambda x'\, u'$, then $u[x'/x] = u'[x'/x'] = u'$ by lemma 1.15.
We prove now that, if $u' = u[x'/x]$, then $\lambda x\, u = \lambda x'\, u'$. We may assume that x and x' are different, the result being trivial otherwise. Let $\underline{u}$ be a representative of u, in which the variable x' is not bound. Then $\underline{u}' = \underline{u}<x'/x>$ is a representative of u'. It is sufficient to show that $\lambda x\,\underline{u} \equiv \lambda x'\,\underline{u}'$, that is to say $\underline{u}<y/x> \equiv \underline{u}'<y/x'>$ for every variable y but a finite number.
Now $\underline{u}'<y/x'> = \underline{u}<x'/x><y/x'>$. By corollary 1.3, we get :

$\underline{u}<x'/x><y/x'> = \underline{u}<y/x>$ since the variable x' does not occur in $\underline{u}$: indeed, it is not bound in $\underline{u}$ by hypothesis, and it is not free in $\underline{u}$, because it is not free in $t = \lambda x\, u$.

Q.E.D.

We can now give the following inductive definition of the operation of substitution $[t_1/x_1, \ldots, t_k/x_k]$, which is useful for inductive reasoning :

$x_i[t_1/x_1, \ldots, t_k/x_k] = t_i$ for $1 \le i \le k$;

if x is a variable different from $x_1, \ldots, x_k$, then
$$x[t_1/x_1, \ldots, t_k/x_k] = x\ ;$$

if $t = uv$, then $t[t_1/x_1, \ldots, t_k/x_k]$
$$= (u[t_1/x_1, \ldots, t_k/x_k])\,v[t_1/x_1, \ldots, t_k/x_k]\ ;$$

if $t = \lambda x\, u$, we may assume that x is not free in $t_1, \ldots, t_k$ and different from $x_1, \ldots, x_k$ (proposition 1.16). Then
$$t[t_1/x_1, \ldots, t_k/x_k] = \lambda x(u[t_1/x_1, \ldots, t_k/x_k]).$$

We need only to prove the last case :
let $\underline{u}, \underline{t}_1, \ldots, \underline{t}_k$ be representatives of $u, t_1, \ldots, t_k$, such that no free variable of $t_1, \ldots, t_k$ is bound in $\underline{u}$.
Then $\underline{t} = \lambda x\, \underline{u}$ is a representative of t ; and $\tau = \underline{t}<\underline{t}_1/x_1, \ldots, \underline{t}_k/x_k>$ is a representative of $t[t_1/x_1, \ldots, t_k/x_k]$, since the bound variables of $\underline{t}$ are x and the bound variables of $\underline{u}$, and x is not free in $t_1, \ldots, t_k$. Now $\tau = \lambda x\, \underline{u}<\underline{t}_1/x_1, \ldots, \underline{t}_k/x_k>$ since $x \ne x_1, \ldots, x_k$. The result follows, because $\underline{u}<\underline{t}_1/x_1, \ldots, \underline{t}_k/x_k>$ is a representative of $u[t_1/x_1, \ldots, t_k/x_k]$.

We now define the notion of λ-compatibility on Λ : if R is a binary relation on Λ, we will say that R is *λ-compatible* if it satisfies :

$x\, R\, x$ for each variable x ;
$t\, R\, t' \Rightarrow \lambda x\, t\, R\, \lambda x\, t'$;
$t\, R\, t', u\, R\, u' \Rightarrow (t)u\, R\, (t')u'$.

A λ-compatible relation is necessarily reflexive. Indeed, we have :

Lemma 1.17. *If R is λ-compatible and $t_1 R\, t'_1, \ldots, t_k R\, t'_k$, then :*
$t[t_1/x_1, \ldots, t_k/x_k]\, R\, t[t'_1/x_1, \ldots, t'_k/x_k]$.

Immediate proof by induction on the length of t.

Q.E.D.

3. Beta-conversion

Let R be a binary relation, on an arbitrary set E ; the least transitive and reflexive binary relation which contains R is obviously the relation R' defined as follows : $t\,R'\,u \Leftrightarrow$ there exist a finite sequence $t = v_0, v_1, \dots, v_{n-1}, v_n = u$ of elements of E such that $v_i\,R\,v_{i+1}$ $(0 \le i < n)$.
R' is called the *transitive closure* of R.

We say that the binary relation R on E satisfies the *Church-Rosser (C.-R.) property* if and only if :
for every $t, u, u' \in E$ such that $t\,R\,u$ and $t\,R\,u'$, there exists some $v \in E$ such that $u\,R\,v$ and $u'\,R\,v$.

Lemma 1.18. *Let R be a binary relation which satisfies the Church-Rosser property. Then the transitive closure of R also satisfies it.*

Let R' be that transitive closure. We will first prove the following property :
$t\,R'\,u, t\,R\,u' \Rightarrow$ for some v, $u\,R\,v$ and $u'\,R'\,v$.
$t\,R'\,u$ means that there exists a sequence $t = v_0, v_1, \dots, v_{n-1}, v_n = u$ such that $v_i\,R\,v_{i+1}$ $(0 \le i < n)$.
The proof is by induction on n ; the case $n = 1$ is just the hypothesis of the lemma.
Now since $t\,R'\,v_{n-1}$ and $t\,R\,u'$, for some w, $v_{n-1}\,R\,w$ and $u'\,R'\,w$. But $v_{n-1}\,R\,u$, so $u\,R\,v$ and $w\,R\,v$ for some v (C.-R. property for R). Therefore $u'\,R'\,v$, which gives the result.
Now we can prove the lemma : the assumption is $t\,R'\,u$ and $t\,R'\,u'$, so there exists a sequence : $t = v_0, v_1, \dots, v_{n-1}, v_n = u'$ such that $v_i\,R\,v_{i+1}$ $(0 \le i < n)$.
The proof is by induction on n : the case $n = 1$ has just been settled.
Since $t\,R'\,u$ and $t\,R'\,v_{n-1}$, by induction hypothesis, we have $u\,R'\,w$ and $v_{n-1}\,R'\,w$ for some w. Now $v_{n-1}\,R\,u'$, so, by the previous property, $w\,R\,v$ and $u'\,R'\,v$ for some v. Thus $u\,R'\,v$.

Q.E.D.

In the following, we consider binary relations on the set Λ of λ-terms.

Proposition 1.19.
If $t, u, t', u' \in \Lambda$ *and* $(\lambda x\, u)t = (\lambda x'\, u')t'$, *then* $u[t/x] = u'[t'/x']$.

This is the same as lemma 1.15, since $(\lambda x\, u)t = (\lambda x'\, u')t'$ if and only if $t = t'$ and $\lambda x\, u = \lambda x'\, u'$.

Q.E.D.

A term of the form $(\lambda x\, u)t$ is called a *redex*, $u[t/x]$ is called its *contractum*. Proposition 1.19 shows that this notion is correctly defined on Λ.

A binary relation β_0 will now be defined on Λ ; $t\,\beta_0\,t'$ should be read as :
" t' is obtained by contracting a redex (or by a β-*reduction*) in t ".
The definition is by induction on t :

if t is a variable, then there is no t' such that $t\,\beta_0\,t'$;

if $t = (u)v$, then $t\,\beta_0\,t'$ if and only if

either $t' = (u)v'$ with $v\,\beta_0\,v'$,

or $t' = (u')v$ with $u\,\beta_0\,u'$,

or else $u = \lambda x\,w$ and $t' = w[v/x]$;

if $t = \lambda x\,u$, then $t\,\beta_0\,t'$ if and only if $t' = \lambda x\,u'$, with $u\,\beta_0\,u'$.

We must check that, in this last case, the definition of β_0 does not depend on the choice of the bound variable x. We show this by induction on the length of t, simultaneously with the following proposition 1.20.

We first remark, from the definition of β_0 and corollary 1.14, that whenever $t\,\beta_0\,t'$, any free variable in t' is also free in t.

Proposition 1.20. *If $t\,\beta_0\,t'$ then $t[t_1/x_1,\ldots,t_k/x_k]\,\beta_0\,t'[t_1/x_1,\ldots,t_k/x_k]$.*

For the sake of brevity, we use the notation $\hat{t}$ for $t[t_1/x_1,\ldots,t_k/x_k]$. It follows from the definition of β_0 that the different possibilities for t, t' are :
i) $t = (u)v$ and $t' = (u)v'$, with $v\,\beta_0\,v'$. Then, by induction hypothesis, we get $\hat{v}\,\beta_0\,\hat{v}'$; hence the result, by definition of β_0.
ii) $t = (u)v$ and $t' = (u')v$, with $u\,\beta_0\,u'$. Same proof.
iii) $t = (\lambda x\,u)v$ and $t' = u[v/x]$. By proposition 1.16, we may assume that x is not free in $t_1,\ldots,t_k$ and different from $x_1,\ldots,x_k$.
Then $\hat{t}' = u[v/x][t_1/x_1,\ldots,t_k/x_k] = u[\hat{v}/x, t_1/x_1,\ldots,t_k/x_k]$ (by lemma 1.13) $= u[t_1/x_1,\ldots,t_k/x_k][\hat{v}/x]$ (by lemma 1.13 and the choice of x) $= \hat{u}[\hat{v}/x]$.
Now $\hat{t} = (\lambda x\,\hat{u})\hat{v}$, and therefore $\hat{t}\,\beta_0\,\hat{t}'$.
iv) $t = \lambda x\,u$, $t' = \lambda x\,u'$, and $u\,\beta_0\,u'$. Let us check first that the definition of β_0 in this case does not depend on the choice of the bound variable x. Let y be a variable which is not free in t (and thus also not free in t'). By the induction hypothesis, we have $u[y/x]\,\beta_0\,u'[y/x]$, and therefore $\lambda y\,u[y/x]\,\beta_0\,\lambda y\,u'[y/x]$ which is the desired result.
Again, we may assume that x is not free in $t_1,\ldots,t_k$ and different from $x_1,\ldots,x_k$.
Then, by induction hypothesis, we get $\hat{u}\,\beta_0\,\hat{u}'$, and therefore $\lambda x\,\hat{u}\,\beta_0\,\lambda x\,\hat{u}'$.
Finally, by the choice of x, this is the same as :
$(\lambda x\,u)[t_1/x_1,\ldots,t_k/x_k]\,\beta_0\,(\lambda x\,u')[t_1/x_1,\ldots,t_k/x_k]$.

Q.E.D.

The β-*conversion* is the least binary relation β on Λ, which is reflexive, transitive, and contains β_0. Thus, we have :
$t\,\beta\,t' \Leftrightarrow$ there exists a sequence $t = t_0, t_1, \ldots, t_{n-1}, t_n = t'$ such that $t_i\,\beta_0\,t_{i+1}$ for $0 \le i \le n-1$ $(n \ge 0)$.

Therefore, whenever $t\,\beta\,t'$, any free variable in t' is also free in t.
The next two propositions give two simple characterizations of β.

Proposition 1.21. *The β-conversion is the least transitive λ-compatible binary relation β such that $(\lambda x\,u)t\,\beta\,u[t/x]$ for all terms t, u and variable x.*

Clearly, $t\,\beta_0\,t', u\,\beta_0\,u' \Rightarrow \lambda x\,t\,\beta_0\,\lambda x\,t'$ and $(u)t\,\beta\,(u')t'$. Hence β is λ-compatible. Conversely, if R is a λ-compatible binary relation and if $(\lambda x\,u)t\,R\,u[t/x]$ for all terms t, u, then it follows immediately from the definition of β_0 that $R \supset \beta_0$ (prove $t\,\beta_0\,t' \Rightarrow t\,R\,t'$ by induction on t). So, if R is transitive, then $R \supset \beta$.

Q.E.D.

Proposition 1.22. *β is the transitive closure of the binary relation ρ defined on Λ by : $u\,\rho\,u' \Leftrightarrow$ there exist a term v and redexes $t_1,\ldots,t_k$ with contractums $t'_1,\ldots,t'_k$ such that $u = v[t_1/x_1,\ldots,t_k/x_k], u' = v[t'_1/x_1,\ldots,t'_k/x_k]$.*

Since β is λ-compatible, it follows from lemma 1.17 that $\beta \supset \rho$, and therefore β contains the transitive closure of ρ. Conversely, the transitive closure of ρ clearly contains β_0, and therefore contains β.

Q.E.D.

Proposition 1.23. *If $t\,\beta\,t', t_1\,\beta\,t'_1,\ldots,t_k\,\beta\,t'_k$ then :*

$$t[t_1/x_1,\ldots,t_k/x_k]\,\beta\,t'[t'_1/x_1,\ldots,t'_k/x_k].$$

Since β is λ-compatible, we have, by lemma 1.17 :

$$t[t_1/x_1,\ldots,t_k/x_k]\,\beta\,t[t'_1/x_1,\ldots,t'_k/x_k].$$

Then, we get $t[t'_1/x_1,\ldots,t'_k/x_k]\,\beta\,t'[t'_1/x_1,\ldots,t'_k/x_k]$ by proposition 1.20.

Q.E.D.

A term t is said to be *normal*, or to be *in normal form*, if it contains no redex. So the normal terms are those which are obtained by applying, a finite number of times, the following rules :

any variable x is a normal term ;
whenever t is normal, so is $\lambda x\,t$;
if t, u are normal and if the first symbol in t is not λ, then $(t)u$ is normal.

This definition yields, immediately, the following properties :
A term is normal if and only if it is of the form $\lambda x_1\ldots\lambda x_k(x)t_1\ldots t_n$ (with $k, n \geq 0$), where x is a variable and $t_1,\ldots,t_n$ are normal terms.
A term t is normal if and only if there is no term t' such that $t\,\beta_0\,t'$.

Thus a normal term is " minimal " with respect to β, which means that, whenever t is normal, $t\,\beta\,t' \Rightarrow t = t'$. However the converse is not true :
take $t = (\lambda x(x)x)\lambda x(x)x$, then $t\,\beta\,t' \Rightarrow t = t'$ although t is not normal.

A term t is said to be *normalizable* if $t\,\beta\,t'$ for some normal term t'.
A term t is said to be *strongly normalizable* if there is no infinite sequence $t = t_0, t_1, \ldots, t_n, \ldots$ such that $t_i\,\beta_0\,t_{i+1}$ for all $i \geq 0$ (the term t is then obviously normalizable).
For instance, $\lambda x\,x$ is a normal term, $(\lambda x(x)x)\lambda x\,x$ is strongly normalizable, $(\lambda x\,y)\omega$ is normalizable but not strongly, and $\omega = (\lambda x(x)x)\lambda x(x)x$ is not normalizable at all.
For normalizable terms, the problem of the uniqueness of the normal form arises. It is solved by the following theorem :

Theorem 1.24 (Church-Rosser)**.**
The β-conversion satisfies the property of Church-Rosser.

This yields the uniqueness of the normal form : if $t\,\beta\,t_1$, $t\,\beta\,t_2$, with t_1, t_2 normal, then, according to the theorem, there exists a term t_3 such that $t_1\,\beta\,t_3$, $t_2\,\beta\,t_3$. Thus $t_1 = t_3 = t_2$.

In order to prove that β satisfies the Church-Rosser property, it is sufficient to exhibit a binary relation ρ on Λ which satisfies the Church-Rosser property and has the β-conversion as its transitive closure.
One could think of taking ρ to be the " reflexive closure " of β_0, which would be defined by $x\,\rho\,y \Leftrightarrow x = y$ or $x\,\beta_0\,y$. But this relation ρ does not satisfy the Church-Rosser property : for example, if $t = (\lambda x(x)x)r$, where r is a redex with contractum r', $u = (r)r$ and $v = (\lambda x(x)x)r'$, then $t\,\beta_0\,u$ and $t\,\beta_0\,v$, while there is no term w such that $u\,\beta_0\,w$ and $v\,\beta_0\,w$.

A suitable definition of ρ is as the least λ-compatible binary relation on Λ such that $t\,\rho\,t', u\,\rho\,u' \Rightarrow (\lambda x\,u)t\,\rho\,u'[t'/x]$.

To prove that $\beta \supset \rho$, it is enough to see that $t\,\beta\,t', u\,\beta\,u' \Rightarrow (\lambda x\,u)t\,\beta\,u'[t'/x]$; now : $(\lambda x\,u)t\,\beta\,(\lambda x\,u')t'$ (since β is λ-compatible) and $(\lambda x\,u')t'\beta\,u'[t'/x]$; then the expected result follows, by transitivity.
Therefore, β contains the transitive closure ρ' of ρ. But of course $\rho \supset \beta_0$, so $\rho' \supset \beta$.
Hence β is the transitive closure of ρ. It thus remains to prove that ρ satisfies the Church-Rosser property.

By definition, ρ is the set of all pairs of terms obtained by applying, a finite number of times, the following rules :
1. $x\,\rho\,x$ for each variable x ;
2. $t\,\rho\,t' \Rightarrow \lambda x\,t\,\rho\,\lambda x\,t'$;
3. $t\,\rho\,t'$ and $u\,\rho\,u' \Rightarrow (t)u\,\rho\,(t')u'$;
4. $t\,\rho\,t'$ and $u\,\rho\,u' \Rightarrow (\lambda x\,t)u\,\rho\,t'[u'/x]$.

Lemma 1.25.
i) If $x \,\rho\, t'$, where x is a variable, then $t' = x$.
ii) If $\lambda x\, u \,\rho\, t'$, then $t' = \lambda x\, u'$, and $u \,\rho\, u'$.
iii) If $(u)v \,\rho\, t'$, then either $t' = (u')v'$ with $u \,\rho\, u'$ and $v \,\rho\, v'$ or $u = \lambda x\, w$ and $t' = w'[v'/x]$ with $v \,\rho\, v'$ and $w \,\rho\, w'$.

i) $x \,\rho\, t'$ could only be obtained by applying rule 1, hence $t' = x$.
ii) Consider the last rule applied to obtain $\lambda x\, u \,\rho\, t'$; the form of the term on the left shows that it is necessarily rule 2 ; the result then follows.
iii) Same method : the last rule applied to obtain $(u)v \,\rho\, t'$ is 3 or 4 ; this yields the conclusion.

Q.E.D.

Lemma 1.26. *Whenever $t \,\rho\, t'$ and $u \,\rho\, u'$, then $t[u/x] \,\rho\, t'[u'/x]$.*

The proof proceeds by induction on the length of the derivation of $t \,\rho\, t'$ by means of rules 1, 2, 3, 4 ; consider the last rule used :

If it is rule 1, then $t = t'$ is a variable, and the result is trivial.

If it is rule 2, then $t = \lambda y\, v, t' = \lambda y\, v'$ and $v \,\rho\, v'$. By proposition 1.16, we may assume that y is different from x and is not free in u, u'. Since $u \,\rho\, u'$, the induction hypothesis implies $v[u/x] \,\rho\, v'[u'/x]$; hence $\lambda y\, v[u/x] \,\rho\, \lambda y\, v'[u'/x]$ (rule 2), that is to say $t[u/x] \,\rho\, t'[u'/x]$.

If it is rule 3, then $t = (v)w$ and $t' = (v')w'$, with $v \,\rho\, v'$ and $w \,\rho\, w'$. Thus, by induction hypothesis, $v[u/x] \,\rho\, v'[u'/x]$ and $w[u/x] \,\rho\, w'[u'/x]$. Therefore, by applying rule 3, we obtain $(v[u/x])w[u/x] \,\rho\, (v'[u'/x])w'[u'/x]$ that is $t[u/x] \,\rho\, t'[u'/x]$.

If it is rule 4, then $t = (\lambda y\, v)w$ and $t' = v'[w'/y]$, with $v \,\rho\, v'$ and $w \,\rho\, w'$. We assume that y is not free in u, u', and is different from x. By induction hypothesis, we have $v[u/x] \,\rho\, v'[u'/x]$ and $w[u/x] \,\rho\, w'[u'/x]$. By rule 4, we get :

$$(\star) \qquad (\lambda y\, v[u/x])w[u/x] \,\rho\, v'[u'/x][w'[u'/x]/y].$$

Now $\lambda y\, v[u/x] = (\lambda y\, v)[u/x]$, by hypothesis on y. It follows that :

$$t[u/x] = (\lambda y\, v[u/x])w[u/x].$$

On the other hand, we have $t'[u'/x] = v'[w'/y][u'/x] = v'[w'[u'/x]/y, u'/x]$ (by lemma 1.13) $= v'[u'/x][w'[u'/x]/y]$ (again by lemma 1.13, since the variable y is not free in u').
Then, $(\star)$ gives the wanted result : $t[u/x] \,\rho\, t'[u'/x]$.

Q.E.D.

Now the proof of the Church-Rosser property for ρ can be completed. So we assume that $t_0 \,\rho\, t_1, t_0 \,\rho\, t_2$, and we look for a term t_3 such that $t_1 \,\rho\, t_3, t_2 \,\rho\, t_3$. The proof is by induction on the length of t_0.

If t_0 is a variable, then by lemma 1.25(i), $t_0 = t_1 = t_2$; take $t_3 = t_0$.

If $t_0 = \lambda x\, u_0$, then, since $t_0\,\rho\,t_1, t_0\,\rho\,t_2$, by lemma 1.25(ii), we have : $t_1 = \lambda x\, u_1, t_2 = \lambda x\, u_2$ and $u_0\,\rho\,u_1, u_0\,\rho\,u_2$. By induction hypothesis, $u_1\,\rho\,u_3$ and $u_2\,\rho\,u_3$ hold for some term u_3. Hence it is sufficient to take $t_3 = \lambda x\, u_3$.

If $t_0 = (u_0)v_0$, then, since $t_0\,\rho\,t_1, t_0\,\rho\,t_2$, by lemma 1.25(iii), the different possible cases are :

a) $t_1 = (u_1)v_1, t_2 = (u_2)v_2$ with $u_0\,\rho\,u_1, v_0\,\rho\,v_1, u_0\,\rho\,u_2, v_0\,\rho\,v_2$. By induction hypothesis, $u_1\,\rho\,u_3, u_2\,\rho\,u_3, v_1\,\rho\,v_3, v_2\,\rho\,v_3$ hold for some u_3 and v_3. Hence it is sufficient to take $t_3 = (u_3)v_3$.

b) $t_1 = (u_1)v_1$, with $u_0\,\rho\,u_1, v_0\,\rho\,v_1$; $u_0 = \lambda x\, w_0$;
$t_2 = w_2[v_2/x]$, with $v_0\,\rho\,v_2, w_0\,\rho\,w_2$.

Since $u_0\,\rho\,u_1$, by lemma 1.25(ii), we have $u_1 = \lambda x\, w_1$, for some w_1 such that $w_0\,\rho\,w_1$. Thus $t_1 = (\lambda x\, w_1)v_1$.
Since $v_0\,\rho\,v_1, v_0\,\rho\,v_2$, and $w_0\,\rho\,w_1, w_0\,\rho\,w_2$, the induction hypothesis gives : $v_1\,\rho\,v_3, v_2\,\rho\,v_3$, and $w_1\,\rho\,w_3, w_2\,\rho\,w_3$ for some v_3 and w_3. Hence, by rule 4, we get $(\lambda x\, w_1)v_1\,\rho\,w_3[v_3/x]$, that is $t_1\,\rho\,w_3[v_3/x]$.
Now, by lemma 1.26, we get $w_2[v_2/x]\,\rho\,w_3[v_3/x]$.
Therefore we obtain the expected result by taking $t_3 = w_3[v_3/x]$.

c) $u_0 = \lambda x\, w_0, t_1 = w_1[v_1/x], t_2 = w_2[v_2/x]$ and we have : $v_0\,\rho\,v_1, v_0\,\rho\,v_2, w_0\,\rho\,w_1, w_0\,\rho\,w_2$.
By induction hypothesis, $v_1\,\rho\,v_3, v_2\,\rho\,v_3$, $w_1\,\rho\,w_3, w_2\,\rho\,w_3$ hold for some v_3 and w_3. Hence, by lemma 1.26, $w_1[v_1/x]\rho\,w_3[v_3/x], w_2[v_2/x]\rho\,w_3[v_3/x]$, that is to say $t_1\,\rho\,w_3[v_3/x], t_2\,\rho\,w_3[v_3/x]$. The result follows by taking $t_3 = w_3[v_3/x]$.

Q.E.D.

Remark. The intuitive meaning of the relation ρ is the following : $t\,\rho\,t'$ holds if and only if t' is obtained from t by contracting several redexes occurring in t. For example, $(\lambda x(x)x)\lambda x\, x\,\rho(\lambda x\, x)\lambda x\, x$; a new redex has been created, but it cannot be contracted ; $(\lambda x(x)x)\lambda x\, x\,\rho\,\lambda x\, x$ does not hold.
In other words, $t\,\rho\,t'$ means that t and t' are constructed simultaneously : for t the steps of the construction are those described in the definition of terms, while for t', the same rules are applied, except that the following alternative is allowed : whenever $t = (\lambda x\, u)v$, t' can be taken either as $(\lambda x\, u')v'$ or as $u'[v'/x]$. This is what lemma 1.25 expresses.

β-equivalence

The *β-equivalence* (denoted by $\simeq_\beta$) is defined as the least equivalence relation which contains β_0 (or β, which comes to the same thing). In other words :
$t \simeq_\beta t' \Leftrightarrow$ there exists a sequence $(t = t_1), t_2, \ldots, t_{n-1}, (t_n = t')$, such that $t_i\,\beta_0\,t_{i+1}$ or $t_{i+1}\,\beta_0\,t_i$ for $1 \le i < n$.
$t \simeq_\beta t'$ should be read as : t is β-equivalent to t'.

Proposition 1.27.
$t \simeq_\beta t'$ if and only if there exists a term u such that $t\,\beta\,u$ and $t'\,\beta\,u$.

The condition is obviously sufficient. For the purpose of proving that it is necessary, consider the relation $\simeq$ defined by : $t \simeq t' \Leftrightarrow t\,\beta\,u$ and $t'\,\beta\,u$ for some term u.
This relation contains β, and is reflexive and symmetric. It is also transitive, for if $t \simeq t'$, $t' \simeq t''$, then $t\,\beta\,u$, $t'\,\beta\,u$, and $t'\,\beta\,v$, $t''\,\beta\,v$ for suitable u and v. By theorem 1.24 (Church-Rosser's theorem), $u\,\beta\,w$ and $v\,\beta\,w$ hold for some term w ; thus $t\,\beta\,w$, $t''\,\beta\,w$.
Hence $\simeq$ is an equivalence relation which contains β, so it also contains $\simeq_\beta$.
Q.E.D.

Therefore, a non-normalizable term cannot be β-equivalent to a normal term.

4. Eta-conversion

Proposition 1.28. *If $\lambda x(t)x = \lambda x'(t')x'$ and x is not free in t, then $t = t'$.*

By proposition 1.16, we get $t'x' = (tx)[x'/x]$ which is tx' since x is not free in t. Therefore $t = t'$.
Q.E.D.

A term of the form $\lambda x(t)x$, where x is not free in t, is called an *η-redex*, its *contractum* being t.
A term of either of the forms $(\lambda x\,t)u$, $\lambda y(v)y$ (where y is not free in v) will be called a *$\beta\eta$-redex*.

We now define a binary relation η_0 on Λ ; $t\,\eta_0\,t'$ should be read as " t' is obtained by contracting an η-redex (or by an *η-reduction*) in the term t ". The definition is given by induction on t, as for β_0 :
if t is a variable, then there is no t' such that $t\,\eta_0\,t'$;
if $t = \lambda x\,u$, then $t\,\eta_0\,t'$ if and only if :
either $t' = \lambda x\,u'$, with $u\,\eta_0\,u'$, or $u = (t')x$, with x not free in t' ;
if $t = (u)v$, then $t\,\eta_0, t'$ if and only if :
either $t' = (u')v$ with $u\,\eta_0\,u'$ or $t' = (u)v'$ with $v\,\eta_0\,v'$.

The relation $t\,\beta\eta_0\,t'$ (which means : " t' is obtained from t by contracting a $\beta\eta$-redex ") is defined as : $t\,\beta_0\,t'$ or $t\,\eta_0\,t'$.

The *η-conversion* (resp. the *$\beta\eta$-conversion*) is defined as the least binary relation η (resp. $\beta\eta$) on Λ which is reflexive, transitive, and contains η_0 (resp. $\beta\eta_0$).

Proposition 1.29. *The $\beta\eta$-conversion is the least transitive λ-compatible binary relation $\beta\eta$ such that $(\lambda x\,t)u\,\beta\eta\,t[u/x]$ and $\lambda y(v)y\,\beta\eta\,v$ whenever y is not free in v.*

The proof is similar to that of proposition 1.21 (which is the analogue for β).

Q.E.D.

It can be proved, as for β, that $\beta\eta$ is the transitive closure of the binary relation ρ defined on Λ by : $u\,\rho\,u' \Leftrightarrow$ there exist a term v, and redexes $t_1,\ldots,t_k$ with contractums $t'_1,\ldots,t'_k$ such that $u = v[t_1/x_1,\ldots,t_k/x_k]$, $u' = v[t'_1/x_1,\ldots,t'_k/x_k]$.

Similarly : if $t\,\beta\eta\,t'$, then every free variable in t' is also free in t.

Proposition 1.30. *If $t\,\beta\eta_0\,t'$ then $t[t_1/x_1,\ldots,t_k/x_k]\,\beta\eta_0\,t'[t_1/x_1,\ldots,t_k/x_k]$.*

The proof is by induction on the length of t. For the sake of brevity, we use the notation $\hat{t}$ for $t[t_1/x_1,\ldots,t_k/x_k]$. It follows from the definition of $\beta\eta_0$ that the different possibilities for t, t' are :

i) $t = \lambda x\, u$, $t' = \lambda x\, u'$, and $u\beta\eta_0 u'$.

ii) $t = (u)v$ and $t' = (u')v$, with $u\beta\eta_0 u'$.

iii) $t = (u)v$ and $t' = (u)v'$, with $v\,\beta\eta_0\,v'$.

iv) $t = (\lambda x\, u)v$ and $t' = u[v/x]$.

v) $t = \lambda x(t')x$, with x not free in t'.

Cases i) to iv) are settled exactly as in proposition 1.20. In case v), assume that x is not free in $t_1,\ldots,t_k$ and different from $x_1,\ldots,x_k$. Then $\hat{t} = \lambda x(\hat{t}')x$, and therefore $\hat{t}\,\beta\eta_0\,\hat{t}'$.

Q.E.D.

Proposition 1.31. *If $t\,\beta\eta\,t'$, $t_1\,\beta\eta\,t'_1,\ldots,t_k\,\beta\eta\,t'_k$ then*

$$t[t_1/x_1,\ldots,t_k/x_k]\,\beta\eta\,t'[t'_1/x_1,\ldots,t'_k/x_k].$$

Since $\beta\eta$ is λ-compatible, we have $t[t_1/x_1,\ldots,t_k/x_k]\,\beta\eta\,t[t'_1/x_1,\ldots,t'_k/x_k]$, by lemma 1.17. Then, we get $t[t'_1/x_1,\ldots,t'_k/x_k]\,\beta\eta\,t'[t'_1/x_1,\ldots,t'_k/x_k]$ by proposition 1.30.

Q.E.D.

A term t is said to be $\beta\eta$-*normal* if it contains no $\beta\eta$-redex.

So the $\beta\eta$-normal terms are those obtained by applying, a finite number of times, the following rules :

any variable x is a $\beta\eta$-normal term ;

whenever t is $\beta\eta$-normal, then so is $\lambda x\, t$, except if $t = (t')x$, with x not free in t' ;

whenever t, u are $\beta\eta$-normal, then so is $(t)u$, except if the first symbol in t is λ.

Theorem 1.32. *The $\beta\eta$-conversion satisfies the Church-Rosser property.*

The proof is on the same lines as for the β-conversion. Here ρ is defined as the least λ-compatible binary relation on Λ such that :

$t\,\rho\,t'$, $u\,\rho\,u' \Rightarrow (\lambda x\,t)u\,\rho\,t'[u'/x]$;
$t\,\rho\,t' \Rightarrow \lambda x(t)x\,\rho\,t'$ whenever x is not free in t.
The first thing to be proved is : $\beta\eta \supset \rho$.
For that purpose, note that $t\,\beta\eta\,t'$, $u\,\beta\eta\,u' \Rightarrow (\lambda x\,t)u\,\beta\eta\,t'[u'/x]$;
indeed, since $\beta\eta$ is λ-compatible, we have $(\lambda x\,t)u\,\beta\eta\,(\lambda x\,t')u'$ and, on the other hand, $(\lambda x\,t')u'\,\beta\eta\,t'[u'/x]$; the result then follows, by transitivity.
Now we show that $t\,\beta\eta\,t' \Rightarrow \lambda x(t)x\,\beta\eta\,t'$ if x is not free in t ; this is immediate, by transitivity, since $\lambda x(t)x\,\beta\eta\,t$.
Therefore $\beta\eta$ is the transitive closure of ρ. It thus remains to prove that ρ satisfies the Church-Rosser property.
By definition, ρ is the set of all pairs of terms obtained by applying, a finite number of times, the following rules :
1. $x\,\rho\,x$ for each variable x ;
2. $t\,\rho\,t' \Rightarrow \lambda x\,t\,\rho\,\lambda x\,t'$;
3. $t\,\rho\,t'$ and $u\,\rho\,u' \Rightarrow (t)u\,\rho\,(t')u'$;
4. $t\,\rho\,t'$, $u\,\rho\,u' \Rightarrow (\lambda x\,t)u\,\rho\,t'[u'/x]$;
5. $t\,\rho\,t' \Rightarrow \lambda x(t)x\,\rho\,t'$ whenever x is not free in t.

The following lemmas are the analogues of lemmas 1.25 and 1.26.

Lemma 1.33. *i) If $x\,\rho\,t'$, where x is a variable, then $t' = x$.*

ii) If $\lambda x\,u\,\rho\,t'$, then either $t' = \lambda x\,u'$ and $u\,\rho\,u'$, or $u = (t)x$ and $t\,\rho\,t'$, with x not free in t.

iii) If $(u)v\,\rho\,t'$, then either $t' = (u')v'$ with $u\,\rho\,u'$ and $v\,\rho\,v'$, or $u = \lambda x\,w$ and $t' = w'[v'/x]$ with $v\,\rho\,v'$ and $w\,\rho\,w'$.

Same proof as for lemma 1.25.

Q.E.D.

Lemma 1.34. *Whenever $t\,\rho\,t'$ and $u\,\rho\,u'$, then $t[u/x]\,\rho\,t'[u'/x]$.*

The proof proceeds by induction on the length of the derivation of $t\,\rho\,t'$ by means of rules 1 through 5 ; consider the last rule used :

if it is one of rules 1, 2, 3, 4, then the proof is the same as in lemma 1.26 ;

if it is rule 5, then $t = \lambda y(v)y$ and $v\,\rho\,t'$, with y not free in v. We may assume that y is not free in u and different from x. By induction hypothesis, $v[u/x]\rho\,t'[u'/x]$, then, by applying rule 5, we obtain $\lambda y(v[u/x])y\,\rho\,t'[u'/x]$ (since y is not free in $v[u/x]$), that is $t[u/x]\rho\,t'[u'/x]$.

Q.E.D.

Now the proof of the Church-Rosser property for ρ can be completed. So we assume that $t_0\,\rho\,t_1$, $t_0\,\rho\,t_2$, and we look for a term t_3 such that $t_1\,\rho\,t_3$, $t_2\,\rho\,t_3$. The proof is by induction on the length of t_0.

If t_0 has length 1, then it is a variable ; hence, by lemma 1.33, $t_0 = t_1 = t_2$; take $t_3 = t_0$.

If $t_0 = \lambda x\, u_0$, then, since $t_0\, \rho\, t_1$, $t_0\, \rho\, t_2$, by lemma 1.33, the different possible cases are :

a) $t_1 = \lambda x\, u_1$, $t_2 = \lambda x\, u_2$, and $u_0\, \rho\, u_1$, $u_0\, \rho\, u_2$. By induction hypothesis, $u_1\, \rho\, u_3$ and $u_2\, \rho\, u_3$ hold for some term u_3. Then it is sufficient to take $t_3 = \lambda x\, u_3$.

b) $t_1 = \lambda x\, u_1$, and $u_0\, \rho\, u_1$; $u_0 = (t'_0)x$, with x not free in t'_0, and $t'_0 \rho\, t_2$. According to lemma 1.33, since $u_0\, \rho\, u_1$ and $u_0 = (t'_0)x$, there are two possibilities for u_1 :

i) $u_1 = (t'_1)x$, with $t'_0 \rho\, t'_1$. Now $t'_0 \rho\, t_2$, thus, by induction hypothesis, $t'_1 \rho\, t_3$ and $t_2\, \rho\, t_3$ hold for some term t_3. Note that, since $t'_0 \rho\, t'_1$, all free variables in t'_1 are also free in t'_0, so x is not free in t'_1. Hence, by rule 5, $\lambda x(t'_1)x\, \rho\, t_3$, that is $t_1\, \rho\, t_3$.

ii) $t'_0 = \lambda y\, u'_0$, $u_1 = u'_1[x/y]$ and $u'_0\, \rho\, u'_1$. By proposition 1.16, we may choose for y any variable which is not free in t'_0, x for example. Then $u_1 = u'_1$ and $u'_0\, \rho\, u_1$. Since ρ is λ-compatible, $\lambda x\, u'_0\, \rho\, \lambda x\, u_1$, that is $t'_0\, \rho\, t_1$. Since $t'_0\, \rho\, t_2$, there exists, by induction hypothesis, a term t_3 such that $t_1\, \rho\, t_3$, $t_2\, \rho\, t_3$.

c) $u_0 = (t'_0)x$, with x not free in t'_0, and $t'_0\, \rho\, t_1$, $t'_0\, \rho\, t_2$. The conclusion follows immediately from the induction hypothesis, since t'_0 is shorter than t_0.

If $t_0 = (v_0)u_0$, then, since $t_0\, \rho\, t_1$, $t_0\, \rho\, t_2$, by lemma 1.33, the different possible cases are :

a) $t_1 = (v_1)u_1$, $t_2 = (v_2)u_2$ with $u_0\, \rho\, u_1$, $v_0\, \rho\, v_1$, $u_0\, \rho\, u_2$, $v_0\, \rho\, v_2$. By induction hypothesis, $u_1\, \rho\, u_3$, $u_2\, \rho\, u_3$, $v_1\, \rho\, v_3$, $v_2\, \rho\, v_3$ hold for some u_3 and v_3. Then it is sufficient to take $t_3 = (v_3)u_3$.

b) $t_1 = (v_1)u_1$, with $u_0\, \rho\, u_1$, $v_0\, \rho\, v_1$; $v_0 \equiv \lambda x\, w_0$, $t_2 = w_2[u_2/x]$, with $u_0\, \rho\, u_2$, $w_0\, \rho\, w_2$. Since $v_0\, \rho\, v_1$, and $v_0 = \lambda x\, w_0$, by lemma 1.33, the different possible cases are :

i) $v_1 = \lambda x\, w_1$, with $w_0\, \rho\, w_1$. Then $t_1 = (\lambda x\, w_1)u_1$. Since $u_0\, \rho\, u_1$, $u_0\, \rho\, u_2$, and $w_0\, \rho\, w_1$, $w_0\, \rho\, w_2$, by induction hypothesis, $u_1\, \rho\, u_3$, $u_2\, \rho\, u_3$, and $w_1\, \rho\, w_3$, $w_2\, \rho\, w_3$ hold for some u_3, w_3. Thus, by rule 4, $(\lambda x\, w_1)u_1\, \rho\, w_3[u_3/x]$, that is $t_1\, \rho\, w_3[u_3/x]$. Hence, by lemma 1.34, $w_2[u_2/x]\, \rho\, w_3[u_3/x]$. The expected result is then obtained by taking $t_3 = w_3[u_3/x]$.

ii) $w_0 = (v'_0)x$, with x not free in v'_0, and $v'_0 \rho\, v_1$. Then $(v'_0)x\, \rho\, w_2$; since $u_0\, \rho\, u_2$, it follows from lemma 1.34 that $((v'_0)x)[u_0/x] \rho\, w_2[u_2/x]$. But x is not free in v'_0, so this is equivalent to $(v'_0)u_0\, \rho\, t_2$.
Now $v'_0 \rho\, v_1$ and $u_0\, \rho\, u_1$. Thus $(v'_0)u_0\, \rho\, (v_1)u_1$, in other words : $(v'_0)u_0\, \rho\, t_1$. Since $(v'_0)u_0$ is shorter than t_0 (because $v_0 = \lambda x(v'_0)x$), there exists, by induction hypothesis, a term t_3 such that $t_1\, \rho\, t_3$, $t_2\, \rho\, t_3$.

c) $v_0 = \lambda x\, w_0$, $t_1 = w_1[u_1/x]$, $t_2 = w_2[u_2/x]$, with $u_0\, \rho\, u_1$, $u_0\, \rho\, u_2$, $w_0\, \rho\, w_1$ and $w_0\, \rho\, w_2$. By induction hypothesis, $u_1\, \rho\, u_3$, $u_2\, \rho\, u_3$, $w_1\, \rho\, w_3$, $w_2\, \rho\, w_3$ hold

for some u_3 and w_3.
Thus, by lemma 1.34, we have $w_1[u_1/x]\,\rho\,w_3[u_3/x]$, $w_2[u_2/x]\,\rho\,w_3[u_3/x]$, that is to say $t_1\,\rho\,w_3[u_3/x]$, $t_2\,\rho\,w_3[u_3/x]$. The result follows by taking $t_3 = w_3[u_3/x]$.
Q.E.D.

The *$\beta\eta$-equivalence* (denoted by $\simeq_{\beta\eta}$) is defined as the least equivalence relation which contains $\beta\eta$. In other words :
$t \simeq_{\beta\eta} t' \Leftrightarrow$ there exists a sequence $t = t_1, t_2, \ldots, t_{n-1}, t_n = t'$, such that either $t_i\,\beta\eta\,t_{i+1}$ or $t_{i+1}\,\beta\eta\,t_i$, for $1 \leq i < n$.
As for the β-equivalence, it follows from Church-Rosser's theorem that :

Proposition 1.35. *$t \simeq_{\beta\eta} t' \Leftrightarrow t\,\beta\eta\,u$ and $t'\,\beta\eta\,u$ for some term u.*

The relation $\simeq_{\beta\eta}$ satisfies the " extensionality axiom ", that is to say :
If $(t)u \simeq_{\beta\eta} (t')u$ holds for all u, then $t \simeq_{\beta\eta} t'$.
Indeed, it is enough to take u as a variable x which does not occur in t, t'. Since $\simeq_{\beta\eta}$ is λ-compatible, we have $\lambda x(t)x \simeq_{\beta\eta} \lambda x(t')x$; therefore, by η-reduction, $t \simeq_{\beta\eta} t'$.

References for chapter 1

[Bar84], [Chu41], [Hin86].
(The references are in the bibliography at the end of the book).

Chapter 2

Representation of recursive functions

1. Head normal forms

In every λ-term, each subsequence of the form " $(\lambda$ " corresponds to a unique redex (this is obvious since redexes are terms of the form $(\lambda x\, t)u$). This allows us to define, in any non normal term t, the *leftmost redex* in t. Let t' be the term obtained from t by contracting that leftmost redex : we say that t' is obtained from t by a *leftmost β-reduction.*
Let t be an arbitrary λ-term. With t we associate a (finite or infinite) sequence of terms $t_0, t_1, \ldots, t_n, \ldots$ such that $t_0 = t$, and t_{n+1} is obtained from t_n by a leftmost β-reduction (if t_n is normal, then the sequence ends with t_n). We call it " the sequence obtained from t by leftmost β-reduction " ; it is uniquely determined by t.
The following theorem will be proved in chapter 4 (theorem 4.13) :

Theorem 2.1. *If t is a normalizable term, then the sequence obtained from t by leftmost β-reduction terminates with the normal form of t.*

We see that this theorem provides a " normalizing strategy ", which can be used for any normalizable term.

The next proposition is simply a remark about the form of the λ-terms :

Proposition 2.2. *Every term of the λ-calculus can be written, in a unique way, in the form $\lambda x_1 \ldots \lambda x_m (v)\, t_1 \ldots t_n$, where $x_1, \ldots, x_m$ are variables, v is either a variable or a redex ($v = (\lambda x\, t)u$) and $t_1, \ldots, t_n$ are terms ($m, n \geq 0$).*

Recall that $(v)\, t_1 \ldots t_n$ denotes the term $(\ldots((v)\, t_1) \ldots)\, t_n$.

We prove the proposition by induction on the length of the considered term τ : the result is clear if τ is a variable.
If $\tau = \lambda x\, \tau'$, then τ' is determined by τ, and can be written in a unique way in the indicated form, by induction hypothesis ; thus the same holds for τ.
If $\tau = (w)v$, then v and w are determined by τ. If w starts with λ, then τ is a redex, so it is of the second form, and not of the first one. If w does not start with λ, then, by induction hypothesis, $w = (w')t_1 \dots t_n$, where w' is a variable or a redex ; thus $\tau = (w')t_1 \dots t_n v$, which is in one and only one of the indicated forms.

Q.E.D.

Definitions. A term τ is *a head normal form* (or *in head normal form*) if it is of the first form indicated in proposition 2.2, namely if :

$$\tau = \lambda x_1 \dots \lambda x_m (x) t_1 \dots t_n,$$

where x is a variable.
In the second case, if $\tau = \lambda x_1 \dots \lambda x_m (\lambda x\ u) t t_1 \dots t_n$, then the redex $(\lambda x\, u)t$ is called the *head redex* of τ.
The head redex of a term τ, when it exists (namely when τ is not a head normal form), is clearly the leftmost redex in τ.

It follows from proposition 2.2 that a term t is normal if and only if it is a head normal form : $\tau = \lambda x_1 \dots \lambda x_m (x) t_1 \dots t_n$, where $t_1, \dots, t_n$ are normal terms. In other words, a term is normal if and only if it is " hereditarily in head normal form ".

The *head reduction* of a term τ is defined as the (finite or infinite) sequence of terms $\tau_0, \tau_1, \dots, \tau_n, \dots$ such that $\tau_0 = \tau$, and τ_{n+1} is obtained from τ_n by a β-reduction of the head redex of τ_n if such a redex exists ; if not, τ_n is in head normal form, and the sequence ends with τ_n.
The *weak head reduction* of a term τ is the initial part of its head reduction which stops as soon as we get a λ-term which begins with a λ. In other words, we reduce the head redex only if there is no λ in front of it.

Notation. We will write $t \succ u$ (resp. $t \succ_w u$) whenever u is obtained from t by a sequence of head β-reductions (resp. weak head β-reductions).
For example, we have $(\lambda x\, x)\lambda x(\lambda y\, y)z \succ_w \lambda x(\lambda y\, y)z \succ \lambda x\, z$.

A λ-term t is said to be *solvable* if, for any term u, there exist variables $x_1, \dots, x_k$ and terms $u_1, \dots, u_k, v_1, \dots, v_l$, $(k, l \geq 0)$ such that :
i) $(t[u_1/x_1, \dots, u_k/x_k])v_1 \dots v_l \simeq_\beta u$.

We have the following equivalent definitions :
(ii) t is solvable if and only if there exist variables $x_1, \dots, x_k$ and terms $u_1, \dots, u_k$, $v_1, \dots, v_l$ such that $(t[u_1/x_1, \dots, u_k/x_k])v_1 \dots v_l \simeq_\beta I$ (I is the term $\lambda x\, x$).
(iii) t is solvable if and only if, given any variable x which does not occur in t, there exist terms $u_1, \dots, u_k$, $v_1, \dots, v_l$ such that :

$$(t[u_1/x_1,\ldots,u_k/x_k])v_1\ldots v_l \simeq_\beta x.$$

Obviously, (i) ⇒ (ii) ⇒ (iii). Now if $(t[u_1/x_1,\ldots,u_k/x_k])v_1\ldots v_l \simeq_\beta x$, then :

$$(t[u_1/x_1,\ldots,u_k/x_k][u/x])v'_1\ldots v'_l \simeq_\beta u,$$

and therefore

$$(t[u'_1/x_1,\ldots,u'_k/x_k])v'_1\ldots v'_l \simeq_\beta u,$$

where $u'_i = u_i[u/x]$, $v'_j = v_j[u/x]$; so we also have (iii) ⇒ (i).

Remarks. The following properties are immediate :
1. Let t be a closed term. Then t is solvable if and only if there exist terms $v_1,\ldots,v_l$ such that $(t)v_1\ldots v_l \simeq_\beta I$.
2. A term t is solvable if and only if its closure $\bar{t}$ is solvable (the closure of t is, by definition, the term $\bar{t} = \lambda x_1\ldots\lambda x_n\, t$, where $x_1,\ldots,x_n$ are the free variables occurring in t).
3. If $(t)v$ is a solvable term, then t is solvable.
4. Of course, the head normal form of a term needs not be unique. Nevertheless :
If a term t has a head normal form $t_0 = \lambda x_1\ldots\lambda x_k(x)u_1\ldots u_n$, then any head normal form of t can be written $\lambda x_1\ldots\lambda x_k(x)u'_1\ldots u'_n$, with $u_i \simeq_\beta u'_i$.
Indeed, let $t_1 = \lambda y_1\ldots\lambda y_l(y)v_1\ldots v_p$ be another head normal form of t. By the Church-Rosser theorem 1.24, there exists a term t_2 which can be obtained by β-reduction from t_0 as well as from t_1. Now, in t_0 (resp. t_1) all possible β-reductions have to be made in $u_1,\ldots,u_n$ (resp. $v_1,\ldots,v_p$). Hence :

$$t_2 \equiv \lambda x_1\ldots\lambda x_k(x)u'_1\ldots u'_n \equiv \lambda y_1\ldots\lambda y_l(y)v'_1\ldots v'_p$$

with $u_i\,\beta\,u'_i$, $v_j\,\beta\,v'_j$. This yields the expected result.

The following theorem will be proved in chapter 4 (theorem 4.9) :

Theorem 2.3. *For every λ-term t, the following conditions are equivalent :*
i) t is solvable ;
ii) t is β-equivalent to a head normal form ;
iii) the head reduction of t terminates (with a head normal form).

2. Representable functions

We define the *Booleans :* $\mathbf{0} = \lambda x\lambda y\, y$ and $\mathbf{1} = \lambda x\lambda y\, x$. Then, for all terms t, u, $((\mathbf{0})t)u$ can be reduced (by head reduction) to u, while $((\mathbf{1})t)u$ can be reduced to t.

Given two terms t, u and an integer k, let $(t)^k u$ denote the term $(t)\ldots(t)u$ (with k occurrences of t) ; in particular, $(t)^0 u = u$.
Beware : the expression $(t)^k$ alone is not a λ-term.
We define the term $\underline{k} = \lambda f\lambda x(f)^k x$; $\underline{k}$ is called “ the numeral (or integer) k of the λ-calculus ” (also known as *Church numeral* k, or *Church integer* k).
Notice that the Boolean $\mathbf{0}$ is the same term as the numeral $\underline{0}$, while the Boolean $\mathbf{1}$ is different from the numeral $\underline{1}$.

Let φ be a partial function defined on $\mathbb{N}^n$, with values either in $\mathbb{N}$ or in $\{0,1\}$. Given a λ-term Φ, we say that Φ *represents* (resp. *strongly represents*) the function φ if, for all $k_1,\ldots,k_n \in \mathbb{N}$:

if $\varphi(k_1,\ldots,k_n)$ is undefined, then $(\Phi)\underline{k}_1 \ldots \underline{k}_n$ is not normalizable (resp. not solvable) ;

if $\varphi(k_1,\ldots,k_n) = k$, then $(\Phi)\underline{k}_1 \ldots \underline{k}_n$ is β-equivalent to $\underline{k}$ (or to $\mathbf{k}$, in case the range of φ is $\{0,1\}$).

Clearly, for total functions, these two notions of representation are equivalent.

Theorem 2.4. *Every partial recursive function from $\mathbb{N}^k$ to $\mathbb{N}$ is (strongly) representable by a term of the λ-calculus.*

Recall the definition of the class of partial recursive functions.
Given $f_1,\ldots,f_k$, partial functions from $\mathbb{N}^n$ to $\mathbb{N}$, and g, partial function from $\mathbb{N}^k$ to $\mathbb{N}$, the partial function h, from $\mathbb{N}^n$ to $\mathbb{N}$, obtained by composition, is defined as follows :

$$h(p_1,\ldots,p_n) = g(f_1(p_1,\ldots,p_n),\ldots,f_k(p_1,\ldots,p_n))$$

if $f_1(p_1,\ldots,p_n),\ldots,f_k(p_1,\ldots,p_n)$ are all defined, and $h(p_1,\ldots,p_n)$ is undefined otherwise.
Let h be a partial function from $\mathbb{N}$ to $\mathbb{N}$. If there exists an integer p such that $h(p) = 0$ and $h(q)$ is defined and different from 0 for all $q < p$, then we denote that integer p by $\mu n\{h(n) = 0\}$; otherwise $\mu n\{h(n) = 0\}$ is undefined.
We call *minimization* the operation which associates, with each partial function f from $\mathbb{N}^{k+1}$ to $\mathbb{N}$, the partial function g, from $\mathbb{N}^k$ to $\mathbb{N}$, such that :

$$g(n_1,\ldots,n_k) = \mu n\{f(n_1,\ldots,n_k,n) = 0\}.$$

The class of partial recursive functions is the least class of partial functions, with arguments and values in $\mathbb{N}$, closed under composition and minimization, and containing : the one argument constant function 0 and successor function ; the two arguments addition, multiplication, and characteristic function of the binary relation $x \leq y$; and the projections P_n^k, defined by $P_n^k(x_1,\ldots,x_n) = x_k$.

So it is sufficient to prove that the class of partial functions which are strongly representable by a term of the λ-calculus satisfies these properties.

The constant function 0 is represented by the term $\lambda d\,\underline{0}$.
The successor function on $\mathbb{N}$ is represented by the term :

$$suc = \lambda n \lambda f \lambda x((n)f)(f)x.$$

The addition and the multiplication (functions from $\mathbb{N}^2$ to $\mathbb{N}$) are respectively represented by the terms $\lambda m \lambda n \lambda f \lambda x((m)f)((n)f)x$ and $\lambda m \lambda n \lambda f(m)(n)f$.
The characteristic function of the binary relation $m \leq n$ on $\mathbb{N}$ is represented by the term $M = \lambda m \lambda n(((m)A)\lambda d\,\mathbf{1})((n)A)\lambda d\,\mathbf{0}$, where $A = \lambda f \lambda g(g)f$.
The function P_n^k is represented by the term $\lambda x_1 \ldots \lambda x_n\, x_k$.

From now on, we denote the term $(suc)^n 0$ by $\widehat{n}$; so we have :
$\widehat{n} \simeq_\beta \underline{n}$, and $(suc)\widehat{n} = \widehat{n+1}$.

Representation of composite functions

Given any two λ-terms t, u, and a variable x with no free occurrence in t, u, the term $\lambda x(t)(u)x$ is denoted by $t \circ u$.

Lemma 2.5. *$(\lambda g\, g \circ s)^k h \succ \lambda x(h)(s)^k x$ for all closed terms s, h and every integer $k \geq 1$.*

Recall that $t \succ u$ means that u is obtained from t by a sequence of head β-reductions.
We prove the lemma by induction on k. The case $k = 1$ is clear. Assume the result for k ; then
$(\lambda g\, g \circ s)^{k+1} h = (\lambda g\, g \circ s)^k (\lambda g\, g \circ s)h \succ \lambda x((\lambda g\, g \circ s)h)(s)^k x$
(by induction hypothesis, applied with $(\lambda g\, g \circ s)h$ instead of h)
$\succ \lambda x(h \circ s)(s)^k x \equiv \lambda x(\lambda y(h)(s)y)(s)^k x \succ \lambda x(h)(s)^{k+1} x$.
Q.E.D.

Lemma 2.6. *Let Φ, ν be two terms. Define $[\Phi, \nu] = (((\nu)\lambda g\, g \circ suc)\Phi)\underline{0}$. Then :*
if ν is not solvable, then neither is $[\Phi, \nu]$;
if $\nu \simeq_\beta \underline{n}$ (Church numeral), then $[\Phi, \nu] \simeq_\beta (\Phi)\underline{n}$; and if Φ is not solvable, then neither is $[\Phi, \nu]$.

The first statement follows from remark 3, page 31. If $\nu \simeq_\beta \underline{n}$, then :
$(\nu)\lambda g\, g \circ suc \simeq_\beta (\underline{n})\lambda g\, g \circ suc = (\lambda f \lambda h(f)^n h)\lambda g\, g \circ suc \simeq_\beta \lambda h(\lambda g\, g \circ suc)^n h$.
By lemma 2.5, this term gives, by head reduction, $\lambda h \lambda x(h)(suc)^n x$.
Hence $[\Phi, \nu] \simeq_\beta (\Phi)(suc)^n \underline{0} \simeq_\beta (\Phi)\underline{n}$. Therefore, if Φ is not solvable, then neither is $[\Phi, \nu]$ (remark 3, page 31).
Q.E.D.

The term $[\Phi, \nu_1, \ldots, \nu_k]$ is defined, for $k \geq 2$, by induction on k :
$[\Phi, \nu_1, \ldots, \nu_k] = [[\Phi, \nu_1, \ldots, \nu_{k-1}], \nu_k]$.

Lemma 2.7. *Let $\Phi, \nu_1, \ldots, \nu_k$ be terms such that each ν_i is either β-equivalent to a Church numeral, or not solvable. Then :*
if one of the ν_i's is not solvable, then neither is $[\Phi, \nu_1, \ldots, \nu_k]$;
if $\nu_i \simeq_\beta \underline{n}_i$ $(1 \leq i \leq k)$, then $[\Phi, \nu_1, \ldots, \nu_k] \simeq_\beta (\Phi)\underline{n}_1 \ldots \underline{n}_k$.

The proof is by induction on k : let $\Psi = [\Phi, \nu_1, \ldots, \nu_{k-1}]$; then :

$$[\Phi, \nu_1, \ldots, \nu_k] = [\Psi, \nu_k].$$

If ν_k is not solvable, then, by lemma 2.6, neither is $[\Psi, \nu_k]$. If ν_k is solvable (and β-equivalent to a Church numeral), and if one of the ν_i's $(1 \leq i \leq k-1)$ is not solvable, then Ψ is not solvable (induction hypothesis), and hence neither is $[\Psi, \nu_k]$ (lemma 2.6). Finally, if $\nu_i \simeq_\beta \underline{n}_i$ $(1 \le i \le k)$, then, by induction hypothesis, $\Psi \simeq_\beta (\Phi)\underline{n}_1 \ldots \underline{n}_{k-1}$; therefore, $[\Psi, \nu_k] \simeq_\beta (\Phi)\underline{n}_1 \ldots \underline{n}_k$ (lemma 2.6).

Q.E.D.

Proposition 2.8. *Let $f_1, \ldots, f_k$ be partial functions from $\mathbb{N}^n$ to $\mathbb{N}$, and g a partial function from $\mathbb{N}^k$ to $\mathbb{N}$. Assume that these functions are all strongly representable by λ-terms ; then so is the composite function $g(f_1, \ldots, f_k)$.*

Choose terms $\Phi_1, \ldots, \Phi_k$, Ψ which strongly represent respectively the functions $f_1, \ldots, f_k, g$. Then the term :

$\chi = \lambda x_1 \ldots \lambda x_n [\Psi, (\Phi_1)x_1 \ldots x_n, \ldots, (\Phi_k)x_1 \ldots x_n]$

strongly represents the composite function $g(f_1, \ldots, f_k)$.
Indeed, if $\underline{p}_1, \ldots, \underline{p}_n$ are Church numerals, then :

$(\chi)\underline{p}_1 \ldots \underline{p}_n \simeq_\beta [\Psi, (\Phi_1)\underline{p}_1 \ldots \underline{p}_n, \ldots, (\Phi_k)\underline{p}_1 \ldots \underline{p}_n]$.

Now each of the terms $(\Phi_i)\underline{p}_1 \ldots \underline{p}_n (1 \le i \le k)$ is, either unsolvable (and in that case $f_i(p_1, \ldots, p_n)$ is undefined), or β-equivalent to a Church numeral $\underline{q}_i$ (then $f_i(p_1, \ldots, p_n) = q_i$). If one of the terms $(\Phi_i)\underline{p}_1 \ldots \underline{p}_n$ is not solvable, then, by lemma 2.7, neither is $(\chi)\underline{p}_1 \ldots \underline{p}_n$. If $(\Phi_i)\underline{p}_1 \ldots \underline{p}_n \simeq_\beta \underline{q}_i$ for all i $(1 \le i \le k)$ where $\underline{q}_i$ is a Church numeral, then by lemma 2.7, we have :

$(\chi)\underline{p}_1 \ldots \underline{p}_n \simeq_\beta (\Psi)\underline{q}_1 \ldots \underline{q}_k$.

Q.E.D.

3. Fixed point combinators

A *fixed point combinator* is a closed term M such that $(M)F \simeq_\beta (F)(M)F$ for every term F. The main point is the existence of such terms. Here are two examples :

Proposition 2.9. *Let Y be the term $\lambda f(\lambda x(f)(x)x)\lambda x(f)(x)x$; then, for every term F, we have $(Y)F \simeq_\beta (F)(Y)F$.*

Indeed, $(Y)F \succ (G)G$, where $G = \lambda x(F)(x)x$; therefore :
$(Y)F \succ (\lambda x(F)(x)x)G \succ (F)(G)G \simeq_\beta (F)(Y)F$.

Q.E.D.

Y is known as *Curry's fixed point combinator.*
Note that we have neither $(Y)F \succ (F)(Y)F$, nor even $(Y)F \,\beta\, (F)(Y)F$.

Proposition 2.10. *Let Z be the term $(A)A$, where $A \equiv \lambda a \lambda f(f)(a)af$. Then, for any term F, we have $(Z)F \succ (F)(Z)F$.*

Indeed, $(Z)F \equiv (A)AF \succ (F)(A)AF \equiv (F)(Z)F$.

Q.E.D.

Z is called *Turing's fixed point combinator.*

Proposition 2.11.
Every fixed point combinator is solvable, but not normalizable.

Let M be a fixed point combinator and f a variable. Then :
$(M)\mathbf{0}f \simeq_\beta ((\mathbf{0})(M)\mathbf{0})f \simeq_\beta f$ and it follows that M is solvable.
If M is normalizable, then so is Mf. Let M' be the normal form of Mf. Since $Mf \simeq_\beta (f)(M)f$, it follows that $M' \simeq_\beta (f)M'$. But these terms are normal, so that $M' = (f)M'$ which is clearly impossible.

Q.E.D.

Representation of functions defined by minimization

The following lemma is an application of results in chapter 4.

Lemma 2.12.
Let b, t_0, t_1 be terms, and suppose $b \simeq_\beta \mathbf{1}$ (resp. $\mathbf{0}$). Then $(b)t_0t_1 \succ_w t_0$ (resp. t_1).

Recall that $\mathbf{1}, \mathbf{0}$ are respectively the booleans $\lambda x\lambda y\, x$ and $\lambda x\lambda y\, y$; and that $\succ_w$ denotes the weak head reduction (see page 30).

This lemma is the particular case of theorem 4.11, when $k = 2$ and $n = 0$.

Q.E.D.

Lemma 2.13. *There exists a closed term Δ such that, for all terms Φ, n :*
$(\Delta\Phi)n \succ ((\Phi n)(\Delta\Phi)(suc)n)n$.

Let $T = \lambda\delta\lambda\varphi\lambda\nu(\,(\varphi\nu)(\delta\varphi)(suc)\nu\,)\nu$. Then Δ is defined as a fixed point of T, by means, for example, of Curry's fixed point combinator : we take $\Delta = (D)D$, where $D = \lambda x(T)(x)x$. Then :
$(\Delta\Phi)n = (D)D\Phi n \succ ((T)(D)D)\Phi n = (T)\Delta\Phi n \succ ((\Phi n)(\Delta\Phi)(suc)n)n$.

We can also take $\Delta = D'D'$, where D' is the normal form of D, that is :
$D' = \lambda x\lambda\varphi\lambda\nu(\,(\varphi\nu)(xx\varphi)(suc)\nu\,)\nu$.

The Turing fixed point combinator gives another solution :
$\Delta = AAT$ with $A = \lambda a\lambda f(f)(a)af$.

Q.E.D.

Lemma 2.14. *Let Φ be a λ-term and $n \in \mathbb{N}$.*
If $\Phi\underline{n}$ is not solvable, then neither is $(\Delta\Phi)\underline{n}$.
If $\Phi\underline{n} \simeq_\beta \mathbf{0}$ (Boolean), then $(\Delta\Phi)\underline{n} \simeq_\beta \underline{n}$.
If $\Phi\underline{n} \simeq_\beta \mathbf{1}$ (Boolean), then $(\Delta\Phi)\widehat{n} \succ (\Delta\Phi)\widehat{p}$ with $p = n+1$.

(Recall that $\widehat{n} = (suc)^n\underline{0}$).

Indeed, it follows from lemma 2.13 that $(\Delta\Phi)\underline{n} \succ ((\Phi\underline{n})(\Delta\Phi)(suc)\underline{n})\underline{n}$. Hence, if $\Phi\underline{n}$ is not solvable, then neither is $(\Delta\Phi)\underline{n}$ (remark 3, page 31). Obviously, if $\Phi\underline{n} \simeq_\beta \mathbf{0}$ (Boolean), then $(\Delta\Phi)\underline{n} \simeq_\beta \underline{n}$.
On the other hand, according to the same lemma, we also have :
$(\Delta\Phi)\widehat{n} \succ ((\Phi\widehat{n})(\Delta\Phi)(suc)\widehat{n})\widehat{n}$; by lemma 2.12, if $\Phi\widehat{n} \simeq_\beta \mathbf{1}$ (Boolean), then :
$((\Phi\widehat{n})(\Delta\Phi)(suc)\widehat{n})\widehat{n} \succ (\Delta\Phi)(suc)\widehat{n}$.
Therefore $(\Delta\Phi)\widehat{n} \succ (\Delta\Phi)(suc)\widehat{n} = (\Delta\Phi)\widehat{p}$ with $p = n+1$.

Q.E.D.

Proposition 2.15. *Let $f(n_1,\dots,n_k,n)$ be a partial function from $\mathbb{N}^{k+1}$ to $\mathbb{N}$, and suppose that it is strongly representable by a term of the λ-calculus. Then the partial function defined by $g(n_1,\dots,n_k) = \mu n\{f(n_1,\dots,n_k,n) = 0\}$ is also strongly representable.*

Let ψ be the partial function from $\mathbb{N}^{k+1}$ to $\{0,1\}$, which has the same domain as f, and such that $\psi(n_1,\dots,n_k,n) = 0 \Leftrightarrow f(n_1,\dots,n_k,n) = 0$.
Then $g(n_1,\dots,n_k) = \mu n\{\psi(n_1,\dots,n_k,n) = 0\}$.
Let F denote a λ-term which strongly represents f ; consider the term :

$$\Psi = \lambda x_1 \dots \lambda x_k \lambda x((F x_1 \dots x_k x)\lambda d\,\mathbf{1})\mathbf{0}.$$

Then, it is easily seen that Ψ strongly represents ψ.
Now consider the term Δ constructed above (lemma 2.13).
We show that the term :

$$G = \lambda x_1 \dots \lambda x_k((\Delta)(\Psi)x_1 \dots x_k)\underline{0}$$

strongly represents the function g. Indeed, let $n_1,\dots,n_k \in \mathbb{N}$; we put :
$\Phi = (\Psi)\underline{n}_1 \dots \underline{n}_k$ and therefore, we get $G\underline{n}_1 \dots \underline{n}_k \succ (\Delta\Phi)\underline{0}$.

If $g(n_1,\dots,n_k)$ is defined and equal to p, then $\psi(n_1,\dots,n_k,n)$ is defined and equal to 1 for $n < p$ and to 0 for $n = p$. Thus $\Phi\underline{n} = (\Psi)\underline{n}_1 \dots \underline{n}_k\underline{n} \simeq_\beta \mathbf{1}$ for $n < p$, and $\Phi\underline{p} = (\Psi)\underline{n}_1 \dots \underline{n}_k\underline{p} \simeq_\beta \mathbf{0}$.
Now, we can apply lemma 2.14, and we get successively (since $\underline{0} = \widehat{0}$) :
$G\underline{n}_1 \dots \underline{n}_k \succ (\Delta\Phi)\underline{0} \succ (\Delta\Phi)\widehat{1} \succ \dots \succ (\Delta\Phi)\widehat{p} \simeq_\beta \underline{p}$.

If $g(n_1,\dots,n_k)$ is undefined, there are two possibilities :

i) $\psi(n_1,\dots,n_k,n)$ is defined and equal to 1 for $n < p$ and is undefined for $n = p$.
Then we can successively deduce from lemma 2.14 (since $\underline{0} = \widehat{0}$) :
$G\underline{n}_1 \dots \underline{n}_k \succ (\Delta\Phi)\underline{0} \succ (\Delta\Phi)\widehat{1} \succ \dots \succ (\Delta\Phi)\widehat{p}$; the last term obtained is not solvable, since neither is $\Phi\underline{p} = \Psi\underline{n}_1 \dots \underline{n}_k\underline{p}$ (lemma 2.14). Consequently, $G\underline{n}_1 \dots \underline{n}_k$ is not solvable (theorem 2.3,iii) ;

ii) $\psi(n_1,\dots,n_k,n)$ is defined and equal to 1 for all n.
Then (again by lemma 2.14) :
$G\underline{n}_1 \dots \underline{n}_k \succ (\Delta\Phi)\underline{0} \succ (\Delta\Phi)\widehat{1} \succ \dots \succ (\Delta\Phi)\widehat{n} \succ \cdots$

So the head reduction of $G\underline{n}_1 \ldots \underline{n}_k$ does not end. Therefore, by theorem 2.3, $G\underline{n}_1 \ldots \underline{n}_k$ is not solvable.

Q.E.D.

It is intuitively clear, according to Church's thesis, that any partial function from $\mathbb{N}^k$ to $\mathbb{N}$, which is representable by a λ-term, is partial recursive. We shall not give a formal proof of this fact. So we can state the

Theorem 2.16 (Church-Kleene theorem)**.** *The partial functions from $\mathbb{N}^k$ to $\mathbb{N}$ which are representable (resp. strongly representable) by a term of the λ-calculus are the partial recursive functions.*

The λ-terms which represent a given partial recursive function, that we obtain by this method, are not normal in general, and even not normalizable. Indeed, in the proof of lemma 2.13, we use a fixed point combinator, which is never a normalizable term (proposition 2.11). Let us show that we can get normal terms.

Lemma 2.17. *Let x be a variable and $t \in \Lambda$. Then, there exists a normal term t' such that $t[\underline{n}/x] \simeq_\beta t'[\underline{n}/x]$ for every $n \in \mathbb{N}$.*

We define t' by induction on the length of t :

if t is a variable, then $t' = t$;
if $t = \lambda y\, u$, then $t' = \lambda y\, u'$;
if $t = uv$, then $t' = (x)Iu'v'$ (with $I = \lambda y\, y$).

It is trivial to show, by induction on the length of t, that t' is normal and that $t[\underline{n}/x] \simeq_\beta t'[\underline{n}/x]$ for every $n \in \mathbb{N}$. We simply have to observe that $(\underline{n})I \simeq_\beta I$ if $n \in \mathbb{N}$.

Q.E.D.

Corollary 2.18. *For every partial recursive function φ, there exists a normal term which (strongly) represents φ.*

For simplicity, we suppose φ to be a unary function. Let Φ be a closed λ-term which strongly represents φ (theorem 2.16) and put $t = \Phi x$. Then $\Psi = \lambda x\, t'$ is normal, by lemma 2.17, and strongly represents φ : indeed, if $n \in \mathbb{N}$, we have $\Psi\underline{n} \simeq_\beta t'[\underline{n}/x] \simeq_\beta t[\underline{n}/x] = \Phi\underline{n}$.

Q.E.D.

4. The second fixed point theorem

Consider a recursive enumeration : $n \mapsto t_n$ of the terms of the λ-calculus. The inverse function will be denoted by $t \mapsto [[t]]$: more precisely, if t is a λ-term, then $[\![t]\!]$ is the Church numeral $\underline{n}$ such that $t_n = t$, which will be called *the numeral of* t.

The function $n \mapsto [\![(t_n)\underline{n}]\!]$ is thus recursive, from $\mathbb{N}$ to the set of Church numerals. By theorem 2.16, there exists a term δ such that $(\delta)\underline{n} \simeq_\beta [\![(t_n)\underline{n}]\!]$, for every integer n.

Now, given an arbitrary term F, let $B = \lambda x(F)(\delta)x$. Then, for any integer n, we have $(B)\underline{n} \simeq_\beta (F)[\![(t_n)\underline{n}]\!]$.

Take $\underline{n} = [\![B]\!]$, that is to say $t_n = B$; then $(t_n)\underline{n} = (B)[\![B]\!]$. If we denote the term $(B)[\![B]\!]$ by A, we obtain $A \simeq_\beta (F)[\![A]\!]$. So we have proved the :

Theorem 2.19.
For every λ-term F, there exists a λ-term A such that $A \simeq_\beta (F)[\![A]\!]$.

Remark. The intuitive meaning of theorem 2.19 is that we can write, as ordinary λ-terms, programs using a new instruction σ (for "self") which denotes the numeral of the program itself.

Indeed, if such a program is written as $\Phi[\sigma/x]$, where Φ is a λ-term, consider the λ-terms $F = \lambda x\Phi$, and A given by theorem 2.19. Then, we have $A \simeq_\beta (F)[\![A]\!]$ and therefore, $A \simeq_\beta \Phi[[\![A]\!]/x]$; thus, A is the λ-term we are looking for.

Theorem 2.20. *Let $\mathscr{X}, \mathscr{Y}$ be two non-empty disjoint sets of terms, which are saturated under the equivalence relation $\simeq_\beta$. Then $\mathscr{X}$ and $\mathscr{Y}$ are recursively inseparable.*

Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are recursively separable. This means that there exists a recursive set $\mathscr{A} \subset \Lambda$ such that $\mathscr{X} \subset \mathscr{A}$ and $\mathscr{Y} \subset \mathscr{A}^c$ (the complement of $\mathscr{A}$). By assumption, there exist terms ξ and η such that $\xi \in \mathscr{X}$ and $\eta \in \mathscr{Y}$. Since the characteristic function of $\mathscr{A}$ is recursive, there is a term Θ such that, for every integer n : $(\Theta)\underline{n} \simeq_\beta \mathbf{1} \Leftrightarrow t_n \in \mathscr{A}$ and $(\Theta)\underline{n} \simeq_\beta \mathbf{0} \Leftrightarrow t_n \notin \mathscr{A}$.

Now let $F = \lambda x(\Theta)x\eta\xi$. According to theorem 2.19, there exists a term A such that $(F)[\![A]\!] \simeq_\beta A$, which implies $(\Theta)[\![A]\!]\eta\xi \simeq_\beta A$.

If $A \in \mathscr{A}$, then, by the definition of Θ, $(\Theta)[\![A]\!] \simeq_\beta \mathbf{1}$, and it follows that : $(\Theta)[\![A]\!]\eta\xi \simeq_\beta \eta$. Therefore $A \simeq_\beta \eta$. Since $\eta \in \mathscr{Y} \subset \mathscr{A}^c$ and $\mathscr{Y}$ is saturated under the equivalence relation $\simeq_\beta$, we conclude that $A \in \mathscr{Y}$, thus $A \notin \mathscr{A}$, which is a contradiction.

Similarly, if $A \notin \mathscr{A}$, then $(\Theta)[\![A]\!] \simeq_\beta \mathbf{0}$, hence $(\Theta)[\![A]\!]\eta\xi \simeq_\beta \xi$, and $A \simeq_\beta \xi$. Since $\xi \in \mathscr{X} \subset \mathscr{A}$ and $\mathscr{X}$ is saturated under the equivalence relation $\simeq_\beta$, we conclude that $A \in \mathscr{X}$, thus $A \in \mathscr{A}$, which is again a contradiction.

Q.E.D.

Corollary 2.21. *The set of normalizable (resp. solvable) λ-terms is not recursive.*

Apply theorem 2.20 : take $\mathscr{X}$ as the set of normalizable (resp. solvable) terms, and $\mathscr{Y} = \mathscr{X}^c$.

Q.E.D.

The same method shows that, for instance, the set of λ-terms which are β-equivalent to a Church integer, or the set of λ-terms which are β-equivalent to a given one t_0, are not recursive.

The set of strongly normalizable λ-terms is also not recursive but, since it is not closed for β-equivalence, the above method does not work to prove this. The undecidability of strong normalization will be proved in chapter 10.

References for chapter 2

[Bar84], [Hin86].
(The references are in the bibliography at the end of the book).

Chapter 3

Intersection type systems

1. System $D\Omega$

A type system is a class of formulas in some language, the purpose of which is to express some properties of λ-terms. By introducing such formulas, as comments in the terms, we construct what we call *typed terms*, which correspond to programs in a high level programming language.
The main connective in these formulas is " $\to$ ", the type $A \to B$ being that of the " functions " from A to B, that is to say from the set of terms of type A to the set of terms of type B.
The first type system which we shall examine consists of propositional formulas. It uses the conjunction $\wedge$ in a very special way (this is why it is called *intersection type system*). It does not seem that this system can be used as a model for a programming language. However, it is very useful as a tool for studying pure λ-calculus.
We will call it *system* $\mathscr{D}\Omega$.
The types of this system are the formulas built with :

a constant Ω (type constant) ;
variables $X, Y, \ldots$ (type variables) ;
the connectives $\to$ and $\wedge$.

We will write $A_1, A_2, \ldots, A_k \to A$ instead of $A_1 \to (A_2 \to (\ldots(A_k \to A)\ldots))$.

The *positive* and *negative occurrences* of a variable X in a type A are defined by induction on the length of A :

> if A is a variable, or $A = \Omega$, then the possible occurrence of X in A is positive ;
>
> if $A = B \wedge C$, then any positive (resp. negative) occurrence of X in B or in C is positive (resp. negative) in A ;

if $A = B \to C$, then the positive (resp. negative) occurrences of X in A are the positive (resp. negative) occurrences of X in C, and the negative (resp. positive) occurrences of X in B.

We also define the *final occurrences* of the variable X in the type A :

if A is a variable, or $A = \Omega$, then the possible occurrence of X in A is final ;

if $A = B \wedge C$, then the final occurrences of X in A are its final occurrences in B and its final occurrences in C ;

if $A = B \to C$, then the final occurrences of X in A are its final occurrences in C.

Hence every final occurrence of a variable in a type is positive.

By a *variable declaration*, we mean an ordered pair (x, A), where x is a variable of the λ-calculus, and A is a type. It will be denoted by $x : A$ instead of (x, A).

A *context* Γ is a mapping from a finite set of variables to the set of all types. Thus it is a finite set $\{x_1 : A_1, \ldots, x_k : A_k\}$ of variable declarations, where $x_1, \ldots, x_k$ are distinct variables ; we will denote it by $x_1 : A_1, \ldots, x_k : A_k$ (without the braces). So, in such an expression, the order does not matter.
We will say that x_i is declared of type A_i in the context Γ.
The integer k may be 0 ; in that case, we have the *empty context.*
We will write $\Gamma,\ x : A$ in order to denote the context obtained by adding the declaration $x : A$ to the context Γ, provided that x is not already declared in Γ.

Given a λ-term t, a type A, and a context Γ, we define, by means of the following rules, the notion : *t is of type A in the context* Γ (we will also say : " t may be given type A in the context Γ ") ; this will be denoted by $\Gamma \vdash_{\mathscr{D}\Omega} t : A$ (or $\Gamma \vdash t : A$ if there is no ambiguity) :

1. If x is a variable, then $\Gamma,\ x : A \vdash_{\mathscr{D}\Omega} x : A$.
2. If $\Gamma,\ x : A \vdash_{\mathscr{D}\Omega} t : B$, then $\Gamma \vdash_{\mathscr{D}\Omega} \lambda x\, t : A \to B$.
3. If $\Gamma \vdash_{\mathscr{D}\Omega} t : A \to B$ and $\Gamma \vdash_{\mathscr{D}\Omega} u : A$, then $\Gamma \vdash_{\mathscr{D}\Omega} (t)u : B$.
4. If $\Gamma \vdash_{\mathscr{D}\Omega} t : A \wedge B$, then $\Gamma \vdash_{\mathscr{D}\Omega} t : A$ and $\Gamma \vdash_{\mathscr{D}\Omega} t : B$.
5. If $\Gamma \vdash_{\mathscr{D}\Omega} t : A$ and $\Gamma \vdash_{\mathscr{D}\Omega} t : B$, then $\Gamma \vdash_{\mathscr{D}\Omega} t : A \wedge B$.
6. $\Gamma \vdash_{\mathscr{D}\Omega} t : \Omega$ (for all t and Γ).

Any expression of the form $\Gamma \vdash_{\mathscr{D}_\Omega} t : A$ obtained by means of these rules will be called a *typing* of t in system $\mathscr{D}\Omega$. A *typable* term is a term which may be given some type in some context.

The notation $\vdash_{\mathscr{D}\Omega} t : A$ will mean that t is of type A in the empty context.

Note that, because of rule 6, there are terms which are typable in the context Γ, while not all of their free variables are declared in that context.

Proposition 3.1. *Suppose* $\Gamma \vdash_{\mathscr{D}\Omega} t : A$*, and let* $\Gamma' \subset \Gamma$ *which contains all those declarations in* Γ *which concern variables occurring free in* t*. Then* $\Gamma' \vdash_{\mathscr{D}\Omega} t : A$*.*

The proof is immediate, by induction on the number of rules used to obtain $\Gamma \vdash_{\mathscr{D}\Omega} t : A$.

Q.E.D.

Lemma 3.2. *If* $\Gamma,\, x : F \vdash_{\mathscr{D}\Omega} t : A$*, then for every variable* x' *which is not declared in* Γ *and not free in* t*, we have* $\Gamma,\, x' : F \vdash_{\mathscr{D}\Omega} t[x'/x] : A$*, and the length of the derivation is the same for both typings.*

We consider the derivation of $\Gamma,\, x : F \vdash_{\mathscr{D}\Omega} t : A$, and we perform on it an arbitrary permutation of variables. Obviously we obtain a correct derivation in $\mathscr{D}\Omega$. Now, we choose the permutation which swap x and x', and does not change any other variable. Since x' is not declared in Γ, we obtain a derivation of $\Gamma,\, x' : F \vdash_{\mathscr{D}\Omega} t[x'/x, x/x'] : A$. But x' is not free in t, and therefore $t[x'/x, x/x'] = t[x'/x]$.

Q.E.D.

Proposition 3.3. *If* $\Gamma \vdash_{\mathscr{D}\Omega} t : A$ *and* $\Gamma' \supset \Gamma$*, then* $\Gamma' \vdash_{\mathscr{D}\Omega} t : A$*.*

Proof by induction on the length of the derivation of $\Gamma \vdash_{\mathscr{D}\Omega} t : A$. Consider the last rule used in this derivation. If it is one of the rules 1, 3, 4, 5, 6, then the induction step is immediate.

If it is rule 2, then $t = \lambda x\, u$, $A = B \to C$, and we have $\Gamma,\, x : B \vdash_{\mathscr{D}\Omega} u : C$. Let x' be any variable not declared in Γ' and not free in u. By lemma 3.2, we get $\Gamma,\, x' : B \vdash_{\mathscr{D}\Omega} u[x'/x] : C$, and the derivation has the same length. By induction hypothesis, we get $\Gamma',\, x' : B \vdash_{\mathscr{D}\Omega} u[x'/x] : C$.

Therefore $\Gamma' \vdash_{\mathscr{D}\Omega} \lambda x' u[x'/x] : B \to C$ by rule 2. But, since x' is not free in u, we have $\lambda x' u[x'/x] = \lambda x\, u = t$, and therefore $\Gamma' \vdash_{\mathscr{D}\Omega} t : A$.

Q.E.D.

Normalization theorems

Since types can be thought of as properties of λ-terms, it seems natural to try and associate with each type a subset of Λ (the set of all λ-terms). We shall now describe a way of doing this.

Given any two subsets $\mathscr{X}$ and $\mathscr{Y}$ of Λ, we denote by $\mathscr{X} \to \mathscr{Y}$, the subset of Λ defined by the following condition :

$$u \in (\mathscr{X} \to \mathscr{Y}) \Leftrightarrow (u)t \in \mathscr{Y} \text{ for all } t \in \mathscr{X}.$$

Obviously :

If $\mathscr{X} \supset \mathscr{X}'$ *and* $\mathscr{Y} \subset \mathscr{Y}'$*, then* $(\mathscr{X} \to \mathscr{Y}) \subset (\mathscr{X}' \to \mathscr{Y}')$*.*

A subset $\mathcal{X}$ of Λ is said to be *saturated* if and only if, for all terms t, $t_1,\ldots,t_n$, u, we have $(u[t/x])t_1\ldots t_n \in \mathcal{X} \Rightarrow (\lambda x\, u)t t_1 \ldots t_n \in \mathcal{X}$.

The intersection of any set of saturated subsets of Λ is clearly saturated. Also clear is the fact that, for any subset $\mathcal{X}$ of Λ, the set of terms which reduce to an element of $\mathcal{X}$ by leftmost reduction is saturated. Similarly, the set of terms which reduce to an element of $\mathcal{X}$ by head reduction is saturated.

Proposition 3.4. *Let $\mathcal{Y}$ be a saturated subset of Λ ; then $\mathcal{X} \to \mathcal{Y}$ is saturated for all $\mathcal{X} \subset \Lambda$.*

Assume $(u[t/x])t_1\ldots t_n \in \mathcal{X} \to \mathcal{Y}$; then for all v in $\mathcal{X}$, $(u[t/x])t_1\ldots t_n v \in \mathcal{Y}$, and, since $\mathcal{Y}$ is saturated, $(\lambda x\, u)t t_1\ldots t_n v \in \mathcal{Y}$.
Therefore, $(\lambda x\, u)t t_1 \ldots t_n \in \mathcal{X} \to \mathcal{Y}$.

Q.E.D.

An *interpretation* $\mathcal{I}$ is, by definition, a function which associates, with each type variable X, a saturated subset of Λ, denoted by $|X|_{\mathcal{I}}$ (or $|X|$ if there is no ambiguity). Given such a function, we can extend it and associate with each type A a saturated subset of Λ, denoted by $|A|_{\mathcal{I}}$ (or simply $|A|$), defined as follows, by induction on the length of A :

if A is a type variable, then $|A|$ is given with the interpretation $\mathcal{I}$;
$|\Omega| = \Lambda$;
if $A = B \to C$, then $|A| = |B| \to |C|$;
if $A = B \wedge C$, then $|A| = |B| \cap |C|$.

Lemma 3.5 (Adequacy lemma)**.**
Let $\mathcal{I}$ be an interpretation, and u a λ-term, such that :

$$x_1 : A_1,\ldots,x_k : A_k \vdash_{\mathcal{D}\Omega} u : A.$$

If $t_1 \in |A_1|_{\mathcal{I}},\ldots, t_k \in |A_k|_{\mathcal{I}}$, then $u[t_1/x_1,\ldots,t_k/x_k] \in |A|_{\mathcal{I}}$.

The proof proceeds by induction on the number of rules used to obtain the typing of u. Consider the last one :

If it is rule 1, then u is one of the variables x_i, and $A = A_i$; in that case $u[t_1/x_1,\ldots,t_k/x_k] = t_i$, and the conclusion is immediate.

If it is rule 2, then $A = B \to C$ and $u = \lambda x\, v$. We can assume that x does not occur free in $t_1,\ldots,t_k$ and is different from $x_1,\ldots,x_k$; moreover :
$x : B, x_1 : A_1,\ldots,x_k : A_k \vdash_{\mathcal{D}\Omega} v : C$.
By induction hypothesis, $v[t/x, t_1/x_1,\ldots,t_k/x_k] \in |C|$ holds for every $t \in |B|$. But it then follows from our assumptions about x that :

$$v[t/x, t_1/x_1,\ldots,t_k/x_k] = v[t_1/x_1,\ldots,t_k/x_k][t/x].$$

Then we have $(\lambda x\, v[t_1/x_1,\ldots,t_k/x_k])t \in |C|$, since C is saturated. Now this holds for all $t \in |B|$, so $\lambda x\, v[t_1/x_1,\ldots,t_k/x_k] \in (|B| \to |C|) = |A|$.

If it is rule 3, then $u = (w)v$, where w is of type $B \to A$ and v is of type B in the context $x_1 : A_1, \ldots, x_k : A_k$. By induction hypothesis, we have :
$w[t_1/x_1, \ldots, t_k/x_k] \in |B \to A|$, and $v[t_1/x_1, \ldots, t_k/x_k] \in |B|$, thus :
$(w[t_1/x_1, \ldots, t_k/x_k])v[t_1/x_1, \ldots, t_k/x_k] \in |A|$.

If it is rule 4, then we know that a previous typing of u gave it the type $A \wedge B$ (or $B \wedge A$), in the same context. By induction hypothesis :
$u[t_1/x_1, \ldots, t_k/x_k] \in |A \wedge B| = |A| \cap |B|$, and therefore :
$u[t_1/x_1, \ldots, t_k/x_k] \in |A|$.

If it is rule 5, then $A = B \wedge C$, and, by previous typings (in the same context), u is of type B as well as of type C. By induction hypothesis, we have $u[t_1/x_1, \ldots, t_k/x_k] \in |B|, |C|$, and therefore $u[t_1/x_1, \ldots, t_k/x_k] \in |B \wedge C|$.

If it is rule 6, then the result is obvious.

Q.E.D.

A type A is said to be *trivial* if no variable has a final occurrence in A. (For example $A \to \Omega \wedge (B \to \Omega)$ is a trivial type, for all A and B).
The trivial types are those obtained by applying the following rules :

Ω is trivial ;
if A is trivial, then $B \to A$ is trivial for every B ;
if A, B are trivial, then so is $A \wedge B$.

As an immediate consequence, we have :

If A is a trivial type, then its value $|A|_{\mathscr{I}}$ under any interpretation $\mathscr{I}$ is the whole set Λ.

Lemma 3.6. *Let $\mathcal{N}_0$, $\mathcal{N}$ be subsets of Λ, with the following properties :*
$\mathcal{N}$ is saturated, $\mathcal{N}_0 \subset \mathcal{N}$, $\mathcal{N}_0 \subset (\Lambda \to \mathcal{N}_0)$, $\mathcal{N} \supset (\mathcal{N}_0 \to \mathcal{N})$.
Let $\mathscr{I}$ be the interpretation such that $|X|_{\mathscr{I}} = \mathcal{N}$ for every type variable X. Then $|A|_{\mathscr{I}} \supset \mathcal{N}_0$ for every type A, and $|A|_{\mathscr{I}} \subset \mathcal{N}$ for every non-trivial type A.

We first prove, by induction on A, that $|A|_{\mathscr{I}} \supset \mathcal{N}_0$; this is obvious whenever A is a type variable, or $A = \Omega$, or $A = B \wedge C$.
If $A = B \to C$, then $|A| = |B| \to |C|$, and $|B| \subset \Lambda$, $|C| \supset \mathcal{N}_0$ (induction hypothesis) ; hence $|A| \supset \Lambda \to \mathcal{N}_0$, and since it has been assumed that $\Lambda \to \mathcal{N}_0 \supset \mathcal{N}_0$, we have $|A| \supset \mathcal{N}_0$.

Now we prove, by induction on A, that $|A| \subset \mathcal{N}$ for every non-trivial type A. The result is immediate whenever A is a type variable, or $A = \Omega$, or $A = B \wedge C$.
If $A = B \to C$, then C is not trivial ; we have $|A| = |B| \to |C|$, $|B| \supset \mathcal{N}_0$ (this has just been proved), and $|C| \subset \mathcal{N}$ (induction hypothesis). Hence $|A| \subset (\mathcal{N}_0 \to \mathcal{N})$, and since we assumed that $(\mathcal{N}_0 \to \mathcal{N}) \subset \mathcal{N}$, we can conclude that $|A| \subset \mathcal{N}$.

Q.E.D.

Theorem 3.7 (Head normal form theorem)**.** *Let t be a term which is typable with a non-trivial type A, in system $\mathscr{D}\Omega$. Then the head reduction of t is finite.*

The converse of this theorem is true and will be proved later (theorem 4.9).

Let $\mathscr{N}_0 = \{(x)v_1 \ldots v_p\ ;\ x$ is a variable, $v_1, \ldots, v_p \in \Lambda\}$ and $\mathscr{N} = \{t \in \Lambda$; the head reduction of t is finite$\}$.

Lemma 3.8. *$\mathscr{N}_0$ and $\mathscr{N}$ satisfy the hypotheses of lemma 3.6.*

Clearly, $\mathscr{N}_0 \subset \mathscr{N}$ and $\mathscr{N}_0 \subset \Lambda \to \mathscr{N}_0$. Also, $\mathscr{N}$ is saturated :
indeed, if $(u[t/x])t_1 \ldots t_n$ has a finite head reduction, then the head reduction of $(\lambda x\, u)t\, t_1 \ldots t_n$ is also finite.
We now prove that $\mathscr{N} \supset \mathscr{N}_0 \to \mathscr{N}$: let $u \in \mathscr{N}_0 \to \mathscr{N}$; then, for any variable x, $(u)x$ has a finite head reduction (since $x \in \mathscr{N}_0$). Suppose that the head reduction of u is infinite, namely : $u, u_1, \ldots, u_n, \ldots$ Then there is an n such that u_n starts with λ ; otherwise the head reduction of $(u)x$ would be :
$(u)x, (u_1)x, \ldots, (u_n)x, \ldots$ which is infinite.
Let k be the least integer such that u_k starts with λ ; for instance $u_k = \lambda y\, v_k$, and then $u_n = \lambda y\ v_n$ for every $n \geq k$.
Thus the head reduction of v_k is : $v_k, v_{k+1}, \ldots$ Therefore, the head reduction of $(u)x$ is : $(u)x, (u_1)x, \ldots, (u_k)x,\ v_k[x/y], v_{k+1}[x/y], \ldots$ Again, it is infinite and we have a contradiction.

Q.E.D.

Now we can prove theorem 3.7 : let t be a term which is typable with a non-trivial type A in the context $x_1 : A_1, \ldots, x_k : A_k$. Consider the interpretation $\mathscr{I}$ such that $|X|_{\mathscr{I}} = \mathscr{N}$ for every type variable X. It follows from the adequacy lemma that, whenever $a_i \in |A_i|_{\mathscr{I}}$, $t[a_1/x_1, \ldots, a_k/x_k] \in |A|_{\mathscr{I}}$. By lemma 3.6, $|A_i|_{\mathscr{I}} \supset \mathscr{N}_0$, so all variables are in $|A_i|_{\mathscr{I}}$, and therefore $t \in |A|_{\mathscr{I}}$.
Also by lemma 3.6, $|A|_{\mathscr{I}} \subset \mathscr{N}$, thus $t \in \mathscr{N}$ and the head reduction of t is finite.

Q.E.D.

An ordered pair $(\mathscr{N}_0, \mathscr{N})$ of subsets of Λ is said to be *adapted* if it satisfies the following properties :

i) $\mathscr{N}$ is saturated ;
ii) $\mathscr{N}_0 \subset \mathscr{N}$; $\mathscr{N}_0 \subset (\mathscr{N} \to \mathscr{N}_0)$; $(\mathscr{N}_0 \to \mathscr{N}) \subset \mathscr{N}$.

An equivalent way of stating condition (ii) is :

ii') $\mathscr{N}_0 \subset (\mathscr{N} \to \mathscr{N}_0) \subset (\mathscr{N}_0 \to \mathscr{N}) \subset \mathscr{N}$.

Indeed, the inclusion $(\mathscr{N} \to \mathscr{N}_0) \subset (\mathscr{N}_0 \to \mathscr{N})$ is an immediate consequence of $\mathscr{N}_0 \subset \mathscr{N}$.

Lemma 3.9. *Let $(\mathscr{N}_0, \mathscr{N})$ be an adapted pair, and $\mathscr{I}$ an interpretation such that, for every type variable X, $|X|_{\mathscr{I}}$ is a saturated subset of $\mathscr{N}$ containing $\mathscr{N}_0$. Then,* *for every type A with no negative (resp. positive) occurrence of the symbol Ω, we have the inclusion $|A|_{\mathscr{I}} \supset \mathscr{N}_0$ (resp. $|A|_{\mathscr{I}} \subset \mathscr{N}$).*

The proof is by induction on A. The conclusion is immediate whenever A is a type variable or $A = \Omega$.
If $A = B \wedge C$, and if there is no negative (resp. positive) occurrence of Ω in A, then the situation is the same in B, and in C. Therefore, by induction hypothesis, we have $|B|_{\mathscr{I}}$, $|C|_{\mathscr{I}} \supset \mathscr{N}_0$ (resp. $\subset \mathscr{N}$). Thus $|B \wedge C|_{\mathscr{I}} = |B|_{\mathscr{I}} \cap |C|_{\mathscr{I}} \supset \mathscr{N}_0$ (resp. $\subset \mathscr{N}$).
If $A = B \to C$, and if Ω has no negative occurrence in A, then Ω has no positive (resp. negative) occurrence in B (resp. C). By induction hypothesis, $|B|_{\mathscr{I}} \subset \mathscr{N}$ and $|C|_{\mathscr{I}} \supset \mathscr{N}_0$. Hence $|B|_{\mathscr{I}} \to |C|_{\mathscr{I}} \supset \mathscr{N} \to \mathscr{N}_0$. Since $(\mathscr{N}_0, \mathscr{N})$ is an adapted pair, we have $\mathscr{N} \to \mathscr{N}_0 \supset \mathscr{N}_0$, and therefore $|A|_{\mathscr{I}} \supset \mathscr{N}_0$.
If $A = B \to C$ and Ω has no positive occurrence in A, then Ω has no negative (resp. positive) occurrence in B (resp. C). By induction hypothesis, $|B|_{\mathscr{I}} \supset \mathscr{N}_0$ and $|C|_{\mathscr{I}} \subset \mathscr{N}$. Therefore, $|B|_{\mathscr{I}} \to |C|_{\mathscr{I}} \subset \mathscr{N}_0 \to \mathscr{N}$. Now $(\mathscr{N}_0, \mathscr{N})$ is an adapted pair, so $\mathscr{N}_0 \to \mathscr{N} \subset \mathscr{N}$, and, finally, $|A|_{\mathscr{I}} \subset \mathscr{N}$.
Q.E.D.

Now we shall prove that the pair $(\mathscr{N}_0, \mathscr{N})$ defined below is adapted :
$\mathscr{N}$ is the set of all terms which are normalizable by leftmost β-reduction :
Namely, we have $t \in \mathscr{N}$ if and only if the sequence obtained from t by leftmost β-reduction ends with a normal term.
$\mathscr{N}_0$ is the set of all terms of the form $(x)t_1 \dots t_n$, where $t_1, \dots, t_n \in \mathscr{N}$ and x is a variable. In particular, all variables are in $\mathscr{N}_0$ (take $n = 0$).
We now check conditions (i) and (ii) in the definition of adapted pairs (page 46) :
i) $\mathscr{N}$ is saturated : clearly, if $(u[t/x])t_1 \dots t_n$ is normalizable by leftmost β-reduction, then so is $(\lambda x\, u)t t_1 \dots t_n$.
ii) $\mathscr{N}_0 \subset \mathscr{N}$: if $t \in \mathscr{N}_0$, then $t = (x)t_1 \dots t_n$ for some variable x and $t_1, \dots, t_n$ are all normalizable by leftmost β-reduction. Thus t clearly has the same property.
The inclusion $\mathscr{N}_0 \subset (\mathscr{N} \to \mathscr{N}_0)$ is obvious.
Now we come to $(\mathscr{N}_0 \to \mathscr{N}) \subset \mathscr{N}$: let $t \in \mathscr{N}_0 \to \mathscr{N}$ and x be some variable not occurring in t ; since $x \in \mathscr{N}_0$, $(t)x \in \mathscr{N}$, thus $(t)x$ is normalizable by leftmost β-reduction. We need to prove that the same property holds for t ; this is done by induction on the length of the normalization of $(t)x$ by leftmost β-reduction.
If t does not start with λ, then the first step of this normalization is a leftmost β-reduction in t, which produces a term t' ; thus the term $(t')x$ has a normalization by leftmost β-reduction which is shorter than that of $(t)x$. Hence, by induction hypothesis, t' is normalizable by leftmost β-reduction, and therefore so is t.
If $t = \lambda y\, u$, then the first leftmost β-reduction in $(t)x$ produces the term $u[x/y]$, which is therefore normalizable by leftmost β-reduction. Hence u satisfies the

same property, and so does $t = \lambda y\, u$: let $u = u_0, u_1, \ldots, u_n$ be the normalization of u by leftmost β-reduction, then that of $\lambda y\, u$ is : $\lambda y\, u, \lambda y\, u_1, \ldots, \lambda y\, u_n$.

Theorem 3.10 (Normalization theorem)**.** *Let t be a typable term in system $\mathscr{D}\Omega$, of type A in the context $x_1 : A_1, \ldots, x_k : A_k$. Suppose that the symbol Ω has no positive occurrence in A, and no negative occurrence in $A_1, \ldots, A_k$. Then t is normalizable by leftmost β-reduction.*

Define an interpretation $\mathscr{I}$ by taking $|X|_{\mathscr{I}} = \mathscr{N}$ for every type variable X. It follows from lemma 3.9 that $|A_i|_{\mathscr{I}} \supset \mathscr{N}_0$; now $x_i \in \mathscr{N}_0$ (by definition of $\mathscr{N}_0$), thus $x_i \in |A_i|_{\mathscr{I}}$; by the adequacy lemma, we have :
$t = t[x_1/x_1, \ldots, x_n/x_n] \in |A|_{\mathscr{I}}$.
Now by lemma 3.9, $|A|_{\mathscr{I}} \subset \mathscr{N}$ and therefore $t \in \mathscr{N}$.

Q.E.D.

The converse of this theorem will be proved later (theorem 4.13).

Corollary 3.11. *Suppose that $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{D}\Omega} t : A$, and Ω does not occur in A, $A_1, \ldots, A_k$. Then t is normalizable by leftmost β-reduction.*

An *infinite quasi leftmost reduction* of a term $t \in \Lambda$ is an infinite sequence of terms $t = t_0, t_1, \ldots, t_n, \ldots$ such that :

for every $n \geq 0$, $t_n\, \beta_0\, t_{n+1}$ (t_{n+1} is obtained by reducing a redex in t_n) ;

for every $n \geq 0$, there exists a $p \geq n$ such that t_{p+1} is obtained by reducing the leftmost redex in t_p.

We can state a strengthened normalization theorem :

Theorem 3.12 (Quasi leftmost normalization theorem)**.**
Suppose $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{D}\Omega} t : A$, and Ω does not occur in $A, A_1, \ldots, A_k$. Then there is no infinite quasi leftmost reduction of t.

In order to prove it, we again define an adapted pair $(\mathscr{N}_0, \mathscr{N})$:
$\mathscr{N}$ is the set of all terms which do not admit an infinite quasi leftmost reduction ; $\mathscr{N}_0$ is the set of all terms of the form $(x)t_1 \ldots t_n$, where x is some variable, and $t_1, \ldots, t_n \in \mathscr{N}$. In particular, all variables are in $\mathscr{N}_0$ (take $n = 0$). We check conditions (i) and (ii) of the definition of adapted pairs (page 46) :

i) $\mathscr{N}$ is saturated : given $(\lambda x\, u)t t_1 \ldots t_n = \tau_0$, we assume the existence of an infinite quasi leftmost β-reduction $\tau_0, \tau_1, \ldots, \tau_n, \ldots$, and we prove :
$(u[t/x])t_1 \ldots t_n \notin \mathscr{N}$ by induction on the least integer k such that τ_{k+1} is obtained from τ_k by reducing the leftmost redex.
If $k = 0$, then $\tau_1 = (u[t/x])t_1 \ldots t_n$, and, therefore, this term admits an infinite quasi leftmost β-reduction. If $k > 0$, then τ_1 is obtained by a reduction either in u, or in $t, t_1, \ldots, t_n$, so it can be written $\tau_1 = (\lambda x\, u')t' t'_1 \ldots t'_n$ (with either

$u = u'$ or $u\,\beta_0\,u'$, and the same for $t, t_1, \dots, t_n$). Now the induction hypothesis applies to τ_1 (since the integer corresponding to its quasi leftmost β-reduction is $k-1$), so $(u'[t'/x])t'_1 \dots t'_n \notin \mathcal{N}$. But we have $(u[t/x])t_1 \dots t_n\,\beta\,(u'[t'/x])t'_1 \dots t'_n$, and therefore there exists an infinite quasi leftmost β-reduction for the term $(u[t/x])t_1 \dots t_n$.
ii) $\mathcal{N}_0 \subset \mathcal{N}$: let $\tau \in \mathcal{N}_0$, say $\tau = (x)t_1 \dots t_n$, where $t_1, \dots, t_n \in \mathcal{N}$ and x is some variable. Suppose that τ admits an infinite quasi leftmost β-reduction, say $\tau = \tau_0, \tau_1, \dots, \tau_k, \dots$; then $\tau_k = (x)t_1^k \dots t_n^k$, with either $t_i^k = t_i^{k+1}$ or $t_i^k\,\beta_0\,t_i^{k+1}$. Clearly, there exists $i \le n$ such that t_i^k contains the leftmost redex of τ_k for every large enough k. Hence t_i admits an infinite quasi leftmost β-reduction, contradicting our assumption.
The inclusion $\mathcal{N}_0 \subset (\mathcal{N} \to \mathcal{N}_0)$ is obvious.
It remains to prove that $(\mathcal{N}_0 \to \mathcal{N}) \subset \mathcal{N}$: let $\tau \in \mathcal{N}_0 \to \mathcal{N}$ and x be a variable which does not occur in τ ; since $x \in \mathcal{N}_0$, $(\tau)x \in \mathcal{N}$. If τ admits an infinite quasi leftmost β-reduction, say $\tau = \tau_0, \tau_1, \dots, \tau_k, \dots$, then so does $(\tau)x$ (contradicting the definition of $\mathcal{N}$) : indeed, if none of the τ_n's start with λ, then $(\tau_0)x, (\tau_1)x, \dots, (\tau_k)x, \dots$ is an infinite quasi leftmost β-reduction of $(\tau)x$. If $\tau_k = \lambda y\,\tau'_k$, then τ'_k admits an infinite quasi leftmost reduction, and so does $\tau'_k[x/y]$. Hence $(\tau_0)x, (\tau_1)x, \dots, (\tau_k)x, \tau'_k[x/y]$ is an initial segment of an infinite quasi leftmost reduction of the term $(\tau)x$.

Now the end of the proof of the quasi leftmost normalization theorem 3.12 is the same as that of the normalization theorem 3.10.

Q.E.D.

The following theorem is another application of the same method.

Theorem 3.13. *Suppose* $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathcal{D}\Omega} t : A$, *and* Ω *does not occur in* $A, A_1, \dots, A_k$. *Then there exists a* $\beta\eta$*-normal term* u *such that, if* $t\,\beta\eta\,t'$ *for some* t'*, then* $t'\,\beta\eta\,u$.

Remark. In particular, t is $\beta\eta$-normalizable (take $t' = t$) and its $\beta\eta$-normal form is unique. The interesting fact is that the proof does not use the Church-Rosser theorems of chapter 1 (theorems 1.24 and 1.32).

We define a new adapted pair $(\mathcal{N}_0, \mathcal{N})$.
$\mathcal{N}$ is the set of all terms with the desired property ; in other words :
$t \in \mathcal{N} \Leftrightarrow$ there exists a $\beta\eta$-normal term u such that, if $t\,\beta\eta\,t'$ for some t', then $t'\,\beta\eta\,u$.
$\mathcal{N}_0 = \{(x)t_1 \dots t_n;\ x$ is any variable, $t_1 \dots t_n \in \mathcal{N}\}$.
We now check conditions (i) and (ii) of the definition of adapted pairs (page 46) :

i) $\mathcal{N}$ is saturated : suppose that $(u[t/x])t_1 \dots t_n \in \mathcal{N}$, and let τ be its (unique) $\beta\eta$-normal form. Let $v \in \Lambda$ be such that :
$(\star)$ $\quad (\lambda x\,u)t\,t_1 \dots t_n\,\beta\eta\,v$.

We show that $v\,\beta\eta\,\tau$. Consider, at the beginning of the $\beta\eta$-reduction $(\star)$, the longest possible sequence of $\beta\eta$-reductions which take place inside u or t or t_1 or ... or t_n ; this gives $(\lambda x\, u')\,t'\,t'_1 \dots t'_n$, with $u\,\beta\eta\,u'$, $t\,\beta\eta\,t'$ and $t_i\,\beta\eta\,t'_i$.
Then, there are three possibilities :
• The $\beta\eta$-reduction $(\star)$ stops there.
Thus, $v = (\lambda x\, u')\,t'\,t'_1 \dots t'_n$ so that $v\,\beta\eta\,(u'[t'/x])\,t'_1 \dots t'_n$.
But we have $(u[t/x])\,t_1 \dots t_n\,\beta\eta\,(u'[t'/x])\,t'_1 \dots t'_n$, because the relation $\beta\eta$ is λ-compatible. Since $(u[t/x])\,t_1 \dots t_n \in \mathcal{N}$, it follows from the definition of $\mathcal{N}$ that $(u'[t'/x])\,t'_1 \dots t'_n\,\beta\eta\,\tau$; therefore $v\,\beta\eta\,\tau$.
• The following step consists in reducing the β-redex $(\lambda x\, u')\,t'$ and gives : $(u'[t'/x])\,t'_1 \dots t'_n$. Therefore, we have $(u'[t'/x])\,t'_1 \dots t'_n\,\beta\eta\,v$ and it follows that $(u[t/x])\,t_1 \dots t_n\,\beta\eta\,v$. Since $(u[t/x])\,t_1 \dots t_n \in \mathcal{N}$, it follows from the definition of $\mathcal{N}$ that $v\,\beta\eta\,\tau$.
• $\lambda x\, u'$ is an η-redex, i.e. $u' = (u'')x$ and x is not free in u'' ; moreover, the following step consists in reducing this η-redex. This gives $(u'')\,t'\,t'_1 \dots t'_n$, i.e. $(u'[t'/x])\,t'_1 \dots t'_n$. Thus, the result follows as in the previous case.

ii) $\mathcal{N}_0 \subset \mathcal{N}$: let $t = (x)\,t_1 \dots t_n \in \mathcal{N}_0$, where x is some variable, and $t_1, \dots, t_n \in \mathcal{N}$. Suppose that $t\,\beta\eta\,t'$. We have $t' = (x)\,t'_1 \dots t'_n$ with $t_i\,\beta\eta\,t'_i$. Therefore $t'_i\,\beta\eta\,u_i$, where u_i is the (unique) $\beta\eta$-normal form of t_i. It follows that $t'\,\beta\eta\,(x)\,u_1 \dots u_n$.
The inclusion $\mathcal{N}_0 \subset (\mathcal{N} \to \mathcal{N}_0)$ is obvious, by definition of $\mathcal{N}_0$.
It remains to prove that $(\mathcal{N}_0 \to \mathcal{N}) \subset \mathcal{N}$: let $t \in (\mathcal{N}_0 \to \mathcal{N})$ and x be a variable which does not occur in t ; since $x \in \mathcal{N}_0$, we have $(t)x \in \mathcal{N}$.
Let u be the (unique) $\beta\eta$-normal form of $(t)x$ and define $w \in \Lambda$ as follows : $w = \lambda x\, u$ if $\lambda x\, u$ is not a η-redex, and $w = v$ if $u = (v)x$ with x not free in v ; then w is $\beta\eta$-normal.
Consider a $\beta\eta$-reduction $t\,\beta\eta\,t'$; we show that $t'\,\beta\eta\,w$.
We have $(t)x\,\beta\eta\,(t')x\,\beta\eta\,u$. If the $\beta\eta$-reduction from $(t')x$ to u takes place inside t', we have $u = (v)x$ and $t'\,\beta\eta\,v$; thus, x is not free in v (because it is not free in t') and $t'\,\beta\eta\,w = v$. Otherwise, we have $t'\,\beta\eta\,\lambda x\, t''$ and $t''\,\beta\eta\,u$, so that $t'\,\beta\eta\,\lambda x\, u$; and in case $u = (v)x$ with x not free in v, we get $t'\,\beta\eta\,\lambda x(v)x\,\beta\eta\,v$. Thus, we have again $t'\,\beta\eta\,w$ in any case, and this shows that $t \in \mathcal{N}$.

Now, the end of the proof of theorem 3.13 is the same as that of the normalization theorem 3.10.

2. System D

In order to study the strongly normalizable terms, we shall deal with the same type system, but without using the constant Ω. Here it will be called *system* $\mathcal{D}$. The definitions below are quite the same as in the previous section, except for those about saturated sets and interpretations.

So the types of system $\mathscr{D}$ are formulas built with :

variables $X, Y, \dots$ (type variables) ;

the connectives $\to$ and $\wedge$.

As before, a *context* Γ is a set of the form $x_1 : A_1, x_2 : A_2, \dots, x_k : A_k$ in which $x_1, x_2, \dots, x_k$ are distinct variables of the λ-calculus and $A_1, A_2, \dots, A_k$ are types of system $\mathscr{D}$.

Given a λ-term t, a type A, and a context Γ, we define, by means of the following rules, the notion : *t is of type A in the context Γ* (or *t may be given type A in the context Γ*) ; this will be denoted by $\Gamma \vdash_{\mathscr{D}} t : A$ (or $\Gamma \vdash t : A$ if there is no ambiguity) :

1. If x is a variable, then $\Gamma, x : A \vdash_{\mathscr{D}} x : A$.
2. If $\Gamma, x : A \vdash_{\mathscr{D}} t : B$, then $\Gamma \vdash_{\mathscr{D}} \lambda x\, t : A \to B$.
3. If $\Gamma \vdash_{\mathscr{D}} t : A \to B$ and $\Gamma \vdash_{\mathscr{D}} u : A$, then $\Gamma \vdash_{\mathscr{D}} (t)u : B$.
4. If $\Gamma \vdash_{\mathscr{D}} t : A \wedge B$, then $\Gamma \vdash_{\mathscr{D}} t : A$ and $\Gamma \vdash_{\mathscr{D}} t : B$.
5. If $\Gamma \vdash_{\mathscr{D}} t : A$ and $\Gamma \vdash_{\mathscr{D}} t : B$, then $\Gamma \vdash_{\mathscr{D}} t : A \wedge B$.

Any expression of the form $\Gamma \vdash_{\mathscr{D}} t : A$ obtained by means of these rules will be called a *typing* of t in system $\mathscr{D}$. A term is *typable* if it may be given some type in some context.

Clearly, if a term t is typed in the context $x_1 : A_1, \dots, x_k : A_k$, then the free variables of t are among $x_1, \dots, x_k$ (this was not true in system $\mathscr{D}\Omega$).

As in $\mathscr{D}\Omega$, we have :

Proposition 3.14. *If $\Gamma \vdash_{\mathscr{D}} t : A$ and $\Gamma' \supset \Gamma$, then $\Gamma' \vdash_{\mathscr{D}} t : A$.*
If $\Gamma \vdash_{\mathscr{D}} t : A$, and if $\Gamma' \subset \Gamma$ is the set of those declarations in Γ which concern variables occurring free in t, then $\Gamma' \vdash_{\mathscr{D}} t : A$.

The strong normalization theorem

Consider a fixed subset $\mathscr{N}$ of Λ (in fact, we shall mostly deal with the case where $\mathscr{N}$ is the set of strongly normalizable terms).
A subset $\mathscr{X}$ of Λ is said to be $\mathscr{N}$*-saturated* if, for all terms $t_1, \dots, t_n, u$:
$(u[t/x])t_1 \dots t_n \in \mathscr{X} \Rightarrow (\lambda x\, u)t t_1 \dots t_n \in \mathscr{X}$ *for every* $t \in \mathscr{N}$.

Proposition 3.15. *If $\mathscr{Y}$ is an $\mathscr{N}$-saturated subset of Λ, then $\mathscr{X} \to \mathscr{Y}$ is $\mathscr{N}$-saturated for all $\mathscr{X}$.*

Indeed, suppose $t \in \mathscr{N}$ and $(u[t/x])t_1 \dots t_n \in \mathscr{X} \to \mathscr{Y}$.
For any t_0 in $\mathscr{X}$, $(u[t/x])t_1 \dots t_n t_0 \in \mathscr{Y}$, and therefore $(\lambda x\, u)t t_1 \dots t_n t_0 \in \mathscr{Y}$, since $\mathscr{Y}$ is $\mathscr{N}$−saturated. Hence $(\lambda x\, u)t t_1 \dots t_n \in \mathscr{X} \to \mathscr{Y}$.

Q.E.D.

An $\mathscr{N}$*-interpretation* $\mathscr{I}$ is, by definition, a function which associates with each type variable X an $\mathscr{N}$-saturated subset of Λ, denoted by $|X|_{\mathscr{I}}$ (or simply $|X|$ if there is no ambiguity). Given such a function, we can extend it and associate with each type A an $\mathscr{N}$-saturated subset of Λ, denoted by $|A|_{\mathscr{I}}$ (or simply $|A|$), defined as follows, by induction on the length of A :

if A is a type variable, then $|A|_{\mathscr{I}}$ is given with the interpretation $\mathscr{I}$;
if $A = B \to C$, then $|A|_{\mathscr{I}} = |B|_{\mathscr{I}} \to |C|_{\mathscr{I}}$;
if $A = B \wedge C$, then $|A|_{\mathscr{I}} = |B|_{\mathscr{I}} \cap |C|_{\mathscr{I}}$.

Lemma 3.16 (Adequacy lemma)**.**
Let $\mathscr{I}$ be an $\mathscr{N}$-interpretation such that $|F|_{\mathscr{I}} \subset \mathscr{N}$ for every type F of system $\mathscr{D}$, and u a λ-term, such that :

$$x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{D}} u : A.$$

If $t_1 \in |A_1|_{\mathscr{I}}, \ldots, t_k \in |A_k|_{\mathscr{I}}$ then $u[t_1/x_1, \ldots, t_k/x_k] \in |A|_{\mathscr{I}}$.

The proof proceeds by induction on the number of rules used to obtain the typing of u. Consider the last one :

If it is rule 1, 3, 4 or 5, then we can repeat the proof of the adequacy lemma (lemma 3.5), for the corresponding rules.

If it is rule 2, then $A = B \to C$ and $u = \lambda x\, v$; we can assume that x does not occur free in $t_1, \ldots, t_k$ and is different from $x_1, \ldots, x_k$. Moreover :
$x : B, x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{D}} v : C$.
By induction hypothesis, $v[t/x, t_1/x_1, \ldots, t_k/x_k] \in |C|$ holds for any $t \in |B|$.
It then follows from our assumptions about x that :
$v[t/x, t_1/x_1, \ldots, t_k/x_k] = v[t_1/x_1, \ldots, t_k/x_k][t/x]$.
Since C is $\mathscr{N}$-saturated and $t \in |B| \subset \mathscr{N}$, we have : $(\lambda x\, v[t_1/x_1, \ldots, t_k/x_k])t \in |C|$.
Now since t is an arbitrary element of $|B|$, we obtain :
$\lambda x\, v[t_1/x_1, \ldots, t_k/x_k] \in (|B| \to |C|) = |A|$.

Q.E.D.

We now give a method which will provide a set $\mathscr{N}$ such that $|F|_{\mathscr{I}} \subset \mathscr{N}$ for every $\mathscr{N}$-interpretation $\mathscr{I}$ and every type F of system $\mathscr{D}$.

In this context, an ordered pair $(\mathscr{N}_0, \mathscr{N})$ of subsets of Λ is said to be *adapted* if and only if :

i) $\mathscr{N}$ is $\mathscr{N}$-saturated ;
ii) $\mathscr{N}_0 \subset \mathscr{N}$; $\mathscr{N}_0 \subset (\mathscr{N} \to \mathscr{N}_0)$; $(\mathscr{N}_0 \to \mathscr{N}) \subset \mathscr{N}$.

The difference with the definition page 46 lies in condition (i).

As above, condition (ii) can also be stated this way :

ii') $\mathscr{N}_0 \subset (\mathscr{N} \to \mathscr{N}_0) \subset (\mathscr{N}_0 \to \mathscr{N}) \subset \mathscr{N}$.

Lemma 3.17. *Let $(\mathscr{N}_0, \mathscr{N})$ be an adapted pair, and $\mathscr{I}$ an $\mathscr{N}$-interpretation such that, for every type variable X, $|X|_{\mathscr{I}}$ is an $\mathscr{N}$-saturated subset of $\mathscr{N}$ containing*

$\mathcal{N}_0$. Then, for every type A, $|A|_{\mathcal{I}}$ is an $\mathcal{N}$-saturated subset of $\mathcal{N}$ which contains $\mathcal{N}_0$.

Proof by induction on A. The result is clear whenever A is a type variable or $A = B \wedge C$.
If $A = B \to C$, then $|A| = |B| \to |C|$, thus $|A|$ is $\mathcal{N}$-saturated since $|C|$ is (proposition 3.15). Moreover, by induction hypothesis, $|B| \supset \mathcal{N}_0$, and $|C| \subset \mathcal{N}$. Hence $|B| \to |C| \subset \mathcal{N}_0 \to \mathcal{N}$. Now $\mathcal{N}_0 \to \mathcal{N} \subset \mathcal{N}$ according to the definition of adapted pairs ; therefore $|B \to C| \subset \mathcal{N}$.
Similarly, we have $|B| \subset \mathcal{N}$, and $|C| \supset \mathcal{N}_0$. Hence $|B \to C| \supset \mathcal{N} \to \mathcal{N}_0$; since $\mathcal{N} \to \mathcal{N}_0 \supset \mathcal{N}_0$ (definition of adapted pairs), we obtain $|B \to C| \supset \mathcal{N}_0$.
Q.E.D.

Now we define two sets $\mathcal{N}$ and $\mathcal{N}_0$ and show that $(\mathcal{N}_0, \mathcal{N})$ is an adapted pair :

$\mathcal{N}$ is the set of strongly normalizable terms ; in other words, $t \in \mathcal{N} \Leftrightarrow$ there is no infinite sequence $t = t_0, t_1, \ldots, t_n, \ldots$ such that $t_i \,\beta_0\, t_{i+1}$ for all i ; therefore each maximal sequence of this form (called normalization of t) ends with the normal form of t.

$\mathcal{N}_0$ is the set of all terms of the form $(x)t_1 \ldots t_n$, where x is some variable, and $t_1, \ldots, t_n \in \mathcal{N}$.

Proposition 3.18. *A strongly normalizable term admits only finitely many normalizations.*

(This is an application of the well known *König's lemma*). Let t be a term which admits infinitely many normalizations. Then at least one of the terms obtained by contracting a redex in t admits infinitely many normalizations. Let t_1 be such a term ; we have $t \,\beta_0\, t_1$. Now the same argument applies to t_1 ; so we can carry on and construct an infinite sequence $t = t_0, t_1, \ldots, t_n, \ldots$ such that $t_n \,\beta_0\, t_{n+1}$ for all n ; therefore t is not strongly normalizable.
Q.E.D.

Proposition 3.19. *$\mathcal{N}$ is $\mathcal{N}$-saturated.*

Let $t \in \mathcal{N}$, $(u[t/x])t_1 \ldots t_n \in \mathcal{N}$. We need to prove that $(\lambda x\, u)t t_1 \ldots t_n \in \mathcal{N}$.
Let p (resp. q) be the sum of all the lengths of the normalizations of t (resp. $(u[t/x])t_1 \ldots t_n$).
The proof is by induction on p, and, for each fixed p, by induction on q.
Consider the terms obtained by contracting a redex in $(\lambda x\, u)t t_1 \ldots t_n$. It is sufficient to prove that all of them are in $\mathcal{N}$. The redex on which the contraction is done may be :
1. The redex $(\lambda x\, u)t$; then the reduced term is $(u[t/x])t_1 \ldots t_n$, which is in $\mathcal{N}$;

2. A redex in u, the reduced term being u', with $u\,\beta_0\,u'$; we want to prove that $(\lambda x\, u')\, t\, t_1 \dots t_n \in \mathcal{N}$. But we have $u[t/x]\,\beta_0\,u'[t/x]$ (proposition 1.20), and therefore $u[t/x]\,t_1 \dots t_n\,\beta_0\,u'[t/x]\,t_1 \dots t_n$; thus, the sum of the lengths of the normalizations of $(u'[t/x])\,t_1 \dots t_n$ is $< q$, and the induction hypothesis yields the expected result ;
3. A redex in t_i ; same proof ;
4. A redex in t, the reduced term being t' ; then the sum of the lengths of the normalizations of t' is $p' < p$. On the other hand, we have $u[t/x]\,\beta\,u[t'/x]$ (proposition 1.23), so there is a normalization of $(u[t/x])\,t_1 \dots t_n$ which involves the term $(u[t'/x])\,t_1 \dots t_n$; therefore, $(u[t'/x])\,t_1 \dots t_n \in \mathcal{N}$. With the induction hypothesis, we conclude that $(\lambda x\, u)\, t'\, t_1 \dots t_n \in \mathcal{N}$.

Q.E.D.

Now we prove that $(\mathcal{N}_0, \mathcal{N})$ is an adapted pair : condition (i) was checked in proposition 3.19 ; we have obviously $\mathcal{N}_0 \subset \mathcal{N}$ and $\mathcal{N}_0 \subset \mathcal{N} \to \mathcal{N}_0$; in order to prove that $\mathcal{N}_0 \to \mathcal{N} \subset \mathcal{N}$, suppose that u is not strongly normalizable, and let x be some variable ($x \in \mathcal{N}_0$) ; there exists an infinite sequence $u = u_0, u_1, \dots, u_n, \dots$ such that $u_i\,\beta_0\,u_{i+1}$ for all i ; then the sequence $(u)x = (u_0)x, (u_1)x, \dots, (u_n)x, \dots$ attests that $(u)x$ is not strongly normalizable.

Theorem 3.20 (Strong normalization theorem)**.** *Every term which is typable in system $\mathcal{D}$ is strongly normalizable.*

Indeed, let t be a term of type A, in the context $x_1 : A_1, \dots, x_k : A_k$. Define an $\mathcal{N}$-interpretation $\mathcal{I}$ by taking $|X|_{\mathcal{I}} = \mathcal{N}$ for every type variable X. We have $x_i \in \mathcal{N}_0$ by definition of $\mathcal{N}_0$, so $x_i \in |A_i|$; by the adequacy lemma, $t = t[x_1/x_1, \dots, x_n/x_n] \in |A|$. Now by lemma 3.17, $|A| \subset \mathcal{N}$; thus $t \in \mathcal{N}$.

Q.E.D.

Remark. Proposition 3.19 provides the following algorithm for checking whether or not a term is strongly normalizable :
if t is a head normal form, say $t = \lambda x_1 \dots \lambda x_n (x)\, t_1 \dots t_k$: then do the checking for $t_1, \dots, t_k$; otherwise, we have $t = \lambda x_1 \dots \lambda x_n (\lambda x\; u)\, v\, t_1 \dots t_k$: then do the checking for v and for $(u[v/x])\,t_1 \dots t_k$. The algorithm terminates if and only if t is strongly normalizable.

3. Typings for normal terms

We intend to show that head normal forms and normal forms are typable, in a notable way : a head normal form is typable in system $\mathcal{D}\Omega$, with a non-trivial type ; a normal form is typable in system $\mathcal{D}$ (and therefore also in system $\mathcal{D}\Omega$, with a type in which the symbol Ω does not occur).

Proposition 3.21. *Let t be a term in head normal form. Then t is typable in system $\mathscr{D}\Omega$, with a type of the form $U_1, \ldots, U_n \to X$ (where X is a type variable, and $n \geq 0$).*

Indeed, $t = \lambda x_1 \ldots \lambda x_n (y) u_1 \ldots u_k$. Now, $(y) u_1 \ldots u_k$ is of type X in the context $y : U$ (where $U = \Omega, \Omega, \ldots, \Omega \to X$).
Thus t is of type $U_1, \ldots, U_n \to X$ in the context $y : U$ ($U_1, \ldots, U_n$ may be arbitrarily chosen, except when $y = x_i$; in that case, take $U_i = U$).

Q.E.D.

Lemma 3.22. *If $x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k \vdash t : A$, then :*
$x_1 : A_1 \wedge A'_1, x_2 : A_2, \ldots, x_k : A_k \vdash t : A$.

Proof by induction on the number of rules used to obtain :
$x_1 : A_1,\ x_2 : A_2, \ldots,\ x_k : A_k \vdash t : A$ (either rules 1 to 6, page 42 or rules 1 to 5, page 51). Consider the last one. The only non-trivial case is that of rule 1, when $t = x_1$. Then we have $A = A_1$. Now, by rule 1, $x_1 : A_1 \wedge A'_1, \ldots \vdash x_1 : A_1 \wedge A'_1$; therefore $x_1 : A_1 \wedge A'_1, \ldots \vdash x_1 : A_1$ (rule 4).

Q.E.D.

Proposition 3.23. *Given any two contexts Γ, Γ', there exists a context Γ'' such that, if $\Gamma \vdash t : A$ and $\Gamma' \vdash u : B$, then $\Gamma'' \vdash t : A, u : B$.*

Even if it means extending both contexts, we may assume that :
Γ is $x_1 : A_1, \ldots, x_k : A_k$ and Γ' is $x_1 : B_1, \ldots, x_k : B_k$.
Then it suffices to take for Γ'' the context $x_1 : A_1 \wedge B_1, \ldots, x_k : A_k \wedge B_k$ and apply the previous lemma.

Q.E.D.

The next proposition shows that every normal term is typable in system $\mathscr{D}$.

Proposition 3.24. *For every normal term t, there exist a type A and a context Γ such that $\Gamma \vdash_{\mathscr{D}} t : A$. Moreover, if t does not start with λ, then, for every type A, there exists a context Γ such that $\Gamma \vdash_{\mathscr{D}} t : A$.*

Recall that the normal terms are defined by the following conditions :

any variable x is a normal term ;

if t is a normal term, and if x is a variable, then $\lambda x\, t$ is a normal term ;

if t, u are normal terms, and if t does not start with λ, then $(t)u$ is a normal term.

The proof of the proposition is by induction on the length of t. If t is a variable, then t is of type A in the context $t : A$.
If $t = \lambda x\, u$, then u is of type A in a context Γ ; we may assume that the declaration $x : B$ occurs in Γ, for some type B (otherwise we add it).
Hence $\Gamma \vdash_{\mathscr{D}} t : B \to A$.
Now suppose that $t = (u)v$, and u does not start with λ. Let A be any type of system $\mathscr{D}$. By induction hypothesis, v is of some type B, in some context Γ. Moreover, there exists a context Γ' such that $\Gamma' \vdash_{\mathscr{D}} u : B \to A$. By the previous proposition, there exists a context Γ'' such that $\Gamma'' \vdash_{\mathscr{D}} v : B$, $u : B \to A$.
Thus $\Gamma'' \vdash_{\mathscr{D}} (u)v : A$.

Q.E.D.

Principal typings of a normal term in system $\mathscr{D}$

We have just shown that every normal term t is typable in system $\mathscr{D}$. We shall improve this result and see that, actually, there is a type which characterizes t up to η-equivalence.

Recall that, if $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathscr{D}} t : A$, then the free variables of t are among $x_1, \dots, x_k$, and the symbol Ω does not occur in the types $A_1, \dots, A_k, A$.

Let t be a *normal* term and $\{x_1, \dots, x_k\}$ a finite set of variables, containing all the free variables of t. We shall define a special kind of typings of t in system $\mathscr{D}$, of the form $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathscr{D}} t : A$, which will be called *principal typings* of t. The definition is by induction on t :

If t is a variable x_i, we take distinct type variables $X_1, \dots, X_k$. The principal typings are $x_1 : X_1, \dots, x_k : X_k \vdash_{\mathscr{D}} x_i : X_i$.

If $t = \lambda x\, u$, let $x : A$, $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathscr{D}} u : B$ be a principal typing of u. Then $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathscr{D}} t : A \to B$ is a principal typing of t.

If t does not start with λ, we have $t = (x)t_1 \dots t_n$, where x is a variable, and $t_1, \dots, t_n$ are normal terms. Let $x : A_i$, $x_1 : A_i^1, \dots, x_k : A_i^k \vdash_{\mathscr{D}} t_i : B_i$ be a principal typing of t_i $(1 \le i \le n)$. Even if it means changing the type variables, we may assume that, whenever $i \neq j$, the typings of t_i and t_j have no type variable in common. Then we take a new type variable X, and we obtain a principal typing of t, which is $\Gamma \vdash_{\mathscr{D}} t : X$, where Γ is the context :
$x : \bigwedge_{i=1}^n A_i \wedge (B_1, \dots, B_n \to X)$, $x_1 : \bigwedge_{i=1}^n A_i^1, \dots, x_k : \bigwedge_{i=1}^n A_i^k$.
This is indeed a typing of t : it follows from lemma 3.22 that

$$\Gamma \vdash_{\mathscr{D}} t_i : B_i \text{ and } \Gamma \vdash_{\mathscr{D}} x : (B_1, \dots, B_n \to X) ;$$

then it remains to apply rule 3, page 51.

Lemma 3.25. *Let $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathscr{D}} t : A$ be a principal typing of a normal term t, and $y_1, \dots, y_l$ be new variables. Then there exist types $B_1, \dots, B_l$ such that $x_1 : A_1, \dots, x_k : A_k, y_1 : B_1, \dots, y_l : B_l \vdash_{\mathscr{D}} t : A$ is a principal typing of t.*

Immediate proof by induction on the length of t.

Q.E.D.

Definition. Given any λ-term t, every term u such that $t\,\eta\,u$ will be called an *η-reduced image* of t.

Theorem 3.26.
Let $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{D}} t : A$ be a principal typing of a normal term t, and let u be a typed term in system $\mathscr{D}\Omega$, of type A in the context $x_1 : A_1, \ldots, x_k : A_k$. Then there exists an η-reduced image of t which can be obtained from u by leftmost β-reduction.

Examples : $t = \lambda x(x)x$; the principal type is $X \wedge (X \to Y) \to Y$; any term of that type can therefore be reduced to t by leftmost β-reduction ;
$t = \lambda f \lambda x(f)x$; the principal type is $(X \to Y) \to (X \to Y)$; any term of that type can be reduced either to t, or to $\lambda f\, f$ (which is an η-reduced image of t), by leftmost β-reduction ;
$t = \lambda f \lambda x(f)(f)x$; the principal type is $(X \to Y) \wedge (Y \to Z) \to (X \to Z)$.

Lemma 3.27. *Suppose t is normal and $t\,\eta\,t'$; then t' is normal. Moreover, if λ is not the first symbol in t, then neither is it in t'.*

We can assume that $t\,\eta_0\,t'$ (t' is obtained by one single η-reduction in t).
The proof is by induction on t. If t is a variable, then $t = t'$ and the result is obvious. If t starts with λ, then there are two possibilities :

$t = \lambda x\, u$, $t' = \lambda x\, u'$, and $u\,\eta_0\,u'$; then u' is normal, thus so is t'.

$t = \lambda x(t')x$, and x does not occur free in t' ; then t' needs to be normal, since t is.

If t does not start with λ, then $t = (u)v$, and the first symbol in u is not λ. In that case, either $t' = (u)v'$ or $(u')v$, with $u\,\eta_0\,u'$ or $v\,\eta_0\,v'$. By induction hypothesis, u' and v' are normal and u' does not start with λ. Thus t' is normal (and does not start with λ).

Q.E.D.

Lemma 3.28. *Consider two terms t, v, and a variable x with no free occurrence in v. Suppose $(v)x >> t$. Then there exists an η-reduced image u of $\lambda x\, t$ such that $v >> u$.*

Recall that $t_0 >> t_1$ means that t_1 is obtained from t_0 by leftmost β-reduction.
The proof proceeds by induction on the number of steps of leftmost β-reduction which transform $(v)x$ in t.
1. $(v)x = t$; then $\lambda x\, t\,\eta\,v$ (definition of η) ; take $u = v$.
2. $(v)x \neq t$ and v does not start with λ. Then the first leftmost β-reduction in $(v)x$ is done in the subterm v ; it gives a term $(v')x$, where v' is obtained from v

by a leftmost β-reduction. By induction hypothesis, there exists a term u such that $\lambda x\, t \eta\, u$ and $v' >> u$. Thus $v >> u$.

3. $(v)x \neq t$ and v starts with λ. Since x is not free in v, we may write $v = \lambda x\, w$; therefore, a leftmost β-reduction in $(v)x$ produces the term w. Thus it follows from our assumption that $w >> t$. Hence $v = \lambda x\, w >> \lambda x\, t$.

Q.E.D.

Theorem 3.29. *Let t be a normal term, and $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{D}} t : A$ a principal typing of t. Then there exists an interpretation $\mathscr{I}$ such that :*

i) $x_1 \in |A_1|_{\mathscr{I}}, \ldots, x_k \in |A_k|_{\mathscr{I}}$;

ii) for every term $v \in |A|_{\mathscr{I}}$ having all its free variables among $x_1, \ldots, x_k$, there exists an η-reduced image u of t such that $v >> u$.

We first show how theorem 3.26 easily follows from theorem 3.29 : indeed, let v be any typed term in system $\mathscr{D}\Omega$, of type A in the context $x_1 : A_1, \ldots, x_k : A_k$; by lemma 3.25, we may assume that the free variables of v are all among $x_1, \ldots, x_k$. By the adequacy lemma (lemma 3.5), we have $v[a_1/x_1, \ldots, a_k/x_k] \in |A|_{\mathscr{I}}$ whenever $a_i \in |A_i|_{\mathscr{I}}$; now $x_i \in |A_i|_{\mathscr{I}}$, and therefore $v \in |A|_{\mathscr{I}}$. Then theorem 3.29 ensures the existence of an η-reduced image of t which can be obtained from v by leftmost β-reduction.

Now we prove theorem 3.29 by induction on the length of t :

If t is a variable, say x_1, then the given typing is $x_1 : X_1, \ldots, x_k : X_k \vdash_{\mathscr{D}} x_1 : X_1$, where the X_i's are type variables. The interpretation $\mathscr{I}$ can be defined by $v \in |X_i|_{\mathscr{I}} \Leftrightarrow v >> x_i$.

If $t = \lambda x\, u$, then we have a principal typing of u of the form :
$x : A,\ x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{D}} u : B$; by induction hypothesis, there exists an interpretation $\mathscr{I}$ such that $x \in |A|_{\mathscr{I}},\ x_1 \in |A_1|_{\mathscr{I}}, \ldots, x_k \in |A_k|_{\mathscr{I}}$. Now the given principal typing of $t = \lambda x\, u$ is $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{D}} t : A \to B$. Let $v \in |A \to B|_{\mathscr{I}}$ be a term with no free variables but $x_1, \ldots, x_k$ (so x does not occur free in v). Since $x \in |A|_{\mathscr{I}}$, $(v)x \in |B|_{\mathscr{I}}$. Therefore, by induction hypothesis, $(v)x >> w$, where w is an η-reduced image of u. By lemma 3.28, there exists a term t' such that $v >> t'$ and $\lambda x\, w \eta\, t'$; thus $v >> t'$ and $\lambda x\, u \eta\, t'$.

If t does not start with λ, then $t = (x)t_1 \ldots t_n$, where x is some variable and $t_1, \ldots, t_n$ are normal terms. We also have principal typings for the t_i's : $x : A_i$, $x_1 : A_i^1, \ldots, x_k : A_i^k \vdash_{\mathscr{D}} t_i : B_i$, and interpretations $\mathscr{I}_i$. Observe that the typings of the t_i's have no type variable in common, so it is possible to define one single interpretation $\mathscr{I}$ such that for every i, $\mathscr{I}_i$ and $\mathscr{I}$ have the same restriction to the type variables occurring in the typing of t_i. Now the given principal typing of t is $\Gamma \vdash_{\mathscr{D}} t : X$, where Γ is the context :
$x : \bigwedge_{i=1}^n A_i \wedge (B_1, \ldots, B_n \to X),\ x_1 : \bigwedge_{i=1}^n A_i^1, \ldots, x_k : \bigwedge_{i=1}^n A_i^k$.

By induction hypothesis, $x \in |A_i|_{\mathscr{I}}$, thus $x \in |\bigwedge_{i=1}^n A_i|_{\mathscr{I}}$;
similarly, we have $x_j \in |\bigwedge_{i=1}^n A_i^j|_{\mathscr{I}}$.
We define the value of X in the interpretation $\mathscr{I}$ by taking :
$|X|_{\mathscr{I}} = \{v \in \Lambda;$ there exist $t'_1 \in |B_1|_{\mathscr{I}}, \dots, t'_n \in |B_n|_{\mathscr{I}}$ such that $v \succ\succ (x)t'_1 \dots t'_n\}$
(this is indeed a saturated subset of Λ).
It follows from this definition that $x \in |B_1, \dots, B_n \to X|_{\mathscr{I}}$. Thus :

$x \in |\bigwedge_{i=1}^n A_i \wedge (B_1, \dots, B_n \to X)|_{\mathscr{I}}$.

Let $v \in |X|_{\mathscr{I}}$, with no free variables but $x_1, \dots, x_k$. Then v reduces to $(x)t'_1 \dots t'_n$ by leftmost β-reduction ; we have $t'_i \in |B_i|_{\mathscr{I}}$ and therefore, by induction hypothesis, $t'_i \succ\succ t''_i$, where t''_i is an η-reduced image of t_i. Hence $v \succ\succ (x)t''_1 \dots t''_n$, which is clearly an η-reduced image of $t = (x)t_1 \dots t_n$.
So we have shown that the interpretation $\mathscr{I}$ satisfies all the required properties with respect to the given principal typing of t.

Q.E.D.

Corollary 3.30. *Let t, t' be two normal terms ;*
i) Suppose that $\Gamma \vdash_{\mathscr{D}\Omega} t : A \Rightarrow \Gamma \vdash_{\mathscr{D}\Omega} t' : A$, for any type A and any context Γ ; then $t\,\eta\,t'$.
ii) Suppose that $\Gamma \vdash_{\mathscr{D}\Omega} t : A \Leftrightarrow \Gamma \vdash_{\mathscr{D}\Omega} t' : A$, for any type A and any context Γ ; then $t = t'$.

i) Take Γ and A such that $\Gamma \vdash_{\mathscr{D}\Omega} t : A$ is a principal typing of t. By assumption, we have $\Gamma \vdash_{\mathscr{D}\Omega} t' : A$; by theorem 3.26, there exists a term u such that $t\,\eta\,u$ and $t' \succ\succ u$. Now since t' is normal, this implies $t' = u$.
ii) It follows from (i) that $t\,\eta\,t'$ and $t'\eta\,t$; therefore $t = t'$ (indeed, if $t\,\eta\,t'$ and $t \neq t'$, then t' is strictly shorter than t).

Q.E.D.

References for chapter 3

[Hin78], [Hin86], [Cop78], [Pot80], [Ron84].
(The references are in the bibliography at the end of the book).

Chapter 4

Normalization and standardization

1. Typings for normalizable terms

Notation. In this chapter, the notation $\vdash$ refers to system $\mathscr{D}$ or system $\mathscr{D}\Omega$ (the result hold in both cases). Of course, the notation $\vdash_{\mathscr{D}\Omega}$ refers to system $\mathscr{D}\Omega$ only, and the notation $\vdash_{\mathscr{D}}$ refers to system $\mathscr{D}$ only.

Proposition 4.1.
Let Γ be a context and $x_1, \ldots, x_k$ variables which are not declared in Γ. Suppose that $\Gamma, x_1 : A_1, \ldots, x_k : A_k \vdash u : B$, and $\Gamma \vdash t_i : A_i$ for all i such that x_i occurs free in u ($1 \leq i \leq k$). Then $\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : B$.

Proof by induction on the number of rules used for the typing
$\Gamma, x_1 : A_1, \ldots, x_k : A_k \vdash u : B$. Consider the last one :

If it is rule 1, then u is a variable ;

if $u = x_i$, then $B = A_i$, and $u[t_1/x_1, \ldots, t_k/x_k] = t_i$, which is of type B in the context Γ.

if u is a variable and $u \neq x_1, \ldots, x_k$, then $u[t_1/x_1, \ldots, t_k/x_k] = u$, and Γ contains the declaration $u : B$; thus $\Gamma \vdash u : B$.

If it is rule 2, then $u = \lambda y\, v$, $B = C \to D$, and :

$$\Gamma,\ x_1 : A_1, \ldots, x_k : A_k,\ y : C \vdash v : D.$$

By induction hypothesis, we have $\Gamma,\ y : C \vdash v[t_1/x_1, \ldots, t_k/x_k] : D$. Therefore, by rule 2, we obtain $\Gamma \vdash \lambda y\, v[t_1/x_1, \ldots, t_k/x_k] : C \to D$, that is to say:
$\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : C \to D$.

If it is rule 3, then $u = v\,w$ and :

$$\Gamma,\ x_1 : A_1, \ldots, x_k : A_k \vdash v : C \to B,\ w : C.$$

By induction hypothesis :

$$\Gamma \vdash v[t_1/x_1, \ldots, t_k/x_k] : C \to B \text{ and } \Gamma \vdash w[t_1/x_1, \ldots, t_k/x_k] : C.$$

Hence $\Gamma \vdash (v[t_1/x_1, \ldots, t_k/x_k])\,w[t_1/x_1, \ldots, t_k/x_k] : B$.

In other words $\Gamma \vdash u[t_1/x_1,\dots,t_k/x_k] : B$.

The other cases are obvious.

Q.E.D.

We will say that a type A is *prime* if $A \neq \Omega$ and A is not a conjunction. So a prime type is either a type variable or a type of the form $A \to B$.

Any type A is a conjunction of prime types and of Ω (when A is prime, this conjunction reduces to one single element). These prime types will be called the *prime factors* of A. The formal definition, by induction on the length of A, of the prime factors of A, is as follows :

- if $A = \Omega$, it has no prime factor ;
- if A is a variable, or $A = B \to C$, it has exactly one prime factor, which is A itself ;
- if $A = B \wedge C$, the prime factors of A are the prime factors of B and the prime factors of C.

Lemma 4.2. *Suppose* $\Gamma \vdash t : A$, *where* A *is a prime type.*
i) If t *is some variable* x, *then* x *is declared of type* A' *in* Γ, A *being a prime factor of* A'.
ii) If $t = \lambda x\, u$, *then* $A = B \to C$, *and* $\Gamma, x : B \vdash u : C$.
iii) If $t = uv$, *then* $\Gamma \vdash v : B$, $\Gamma \vdash u : B \to A'$, *and* A *is a prime factor of* A'.

In case (ii), recall that the notation “ $\Gamma, x : B$ ” implies that x is not declared in Γ (otherwise, one should rename the bound variables of $\lambda x\, u$).

The given typing of t (with a prime type A in the context Γ) is obtained by the rules listed on p. 42 or p. 51. Consider the first step when one of these rules produces a typing $\Gamma \vdash t : A'$, where A is a prime factor of A'.

The rule applied at that step is neither rule 4 nor rule 5 :

Indeed, rule 4 requires a previous typing of the form $\Gamma \vdash t : A' \wedge B$, and A would already be a prime factor of $A' \wedge B$. As for rule 5, it requires previous typings of the form $\Gamma \vdash t : A'_1$, and $\Gamma \vdash t : A'_2$, with $A' = A'_1 \wedge A'_2$; then A would already be either a prime factor of A'_1 or of A'_2.

In case (i), the rule applied may only be 1, 4 or 5, since the term obtained is a variable. But 4 and 5 have just been eliminated ; so it is rule 1, and therefore x is declared of type A' in Γ.

In case (ii), the rule applied may only be 2, 4, or 5, since the term obtained is $\lambda x\, u$. So it is rule 2, which implies that A' is of the form $B \to C$; now this is a prime type, thus $A' = A = B \to C$. Moreover, in this case, rule 2 requires as a previous typing : $\Gamma, x : B \vdash u : C$.

In case (iii), the rule applied may only be 3, 4 or 5, since the term obtained is uv. So it is rule 3, and therefore we have : $\Gamma \vdash v : B$ and $\Gamma \vdash u : B \to A'$.

Q.E.D.

Proposition 4.3. *If* $\Gamma \vdash t : A$ *and* $t\,\beta\,t'$*, then* $\Gamma \vdash t' : A$.

We may assume $t\,\beta_0\,t'$ (that is to say that t' is obtained by contracting one redex in t). The proposition is proved by induction on the number of rules used to obtain $\Gamma \vdash t : A$. Consider the last one :
It cannot be rule 1, since $t\,\beta_0\,t'$ is impossible when t is a variable.
If it is rule 2, then $t = \lambda x\, u$, $A = B \to C$, and $\Gamma, x : B \vdash u : C$. In this case, $t' = \lambda x\, u'$ and $u\,\beta_0 u'$. By induction hypothesis, we have $\Gamma, x : B \vdash u' : C$;
thus $\Gamma \vdash \lambda x\, u' : B \to C$, that is to say $\Gamma \vdash t' : A$.
If it is rule 3, then $t = uv$, $\Gamma \vdash u : B \to A$, and $\Gamma \vdash v : B$. Here there are three possible situations for t' :
i) $t' = u'v$, with $u\,\beta_0\,u'$; by induction hypothesis, we have $\Gamma \vdash u' : B \to A$, and therefore $\Gamma \vdash t' : A$.
ii) $t' = uv'$, with $v\,\beta_0\,v'$; by induction hypothesis, $\Gamma \vdash v' : B$; thus $\Gamma \vdash t' : A$.
iii) $u = \lambda x\, w$ and $t' = w[v/x]$; so we have $\Gamma \vdash \lambda x\, w : B \to A$. Therefore, by lemma 4.2(ii), $\Gamma, x : B \vdash w : A$; now, since $\Gamma \vdash v : B$, proposition 4.1 proves that $\Gamma \vdash w[v/x] : A$, that is to say $\Gamma \vdash t' : A$.
If the last rule used is 4, 5 or 6, then the result is obvious.

Q.E.D.

Proposition 4.4. *Let* Γ *be a context and* $x_1, \ldots, x_k$ *variables which are not declared in* Γ. *If* $\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : B$, *and if* $t_1, \ldots, t_k$ *are typable in the context* Γ, *then there exist types* $A_1, \ldots, A_k$ *such that* $\Gamma \vdash t_i : A_i$ $(1 \le i \le k)$ *and* $\Gamma, x_1 : A_1, \ldots, x_k : A_k \vdash u : B$.

Remarks.
1. If the type system is $\mathscr{D}\Omega$, then the condition " t_i is typable in the context Γ " is satisfied anyway ($\Gamma \vdash t_i : \Omega$).
2. The necessity of introducing the conjunction symbol $\wedge$, with its specific syntax, appears in this proposition ; the result is characteristic of this kind of type systems.

First, observe that the proposition is obvious when $u = x_i$. Indeed, in that case, we have $\Gamma \vdash t_i : B$, and, of course, $\Gamma, x_i : B \vdash x_i : B$. Thus we can take $A_i = B$, and, for $j \neq i$, take A_j as any type satisfying $\Gamma \vdash t_j : A_j$.
Now suppose $u \neq x_1, \ldots, x_k$. The proof is by induction on the number of rules used to obtain $\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : B$. Consider the last one.

If it is rule 1, then $u[t_1/x_1, \ldots, t_k/x_k]$ is a variable y, and Γ contains the declaration $y : B$. Thus u is also a variable. Now since $u \neq x_1, \ldots, x_k$, we have $u[t_1/x_1, \ldots, t_k/x_k] = u$, and $u = y$. Therefore $\Gamma \vdash u : B$; besides, it has been assumed that $\Gamma \vdash t_i : A_i$ for appropriate types A_i.

If it is rule 2, then we have $B = C \to D$, $u[t_1/x_1, \ldots, t_k/x_k] = \lambda y\, u'$ and $\Gamma, y : C \vdash u' : D$. Since $u \neq x_1, \ldots, x_k$, we have $u = \lambda y\, v$. As usual, we may suppose that y does not occur free in $\Gamma, u, t_1, \ldots, t_k$, and $y \neq x_1, \ldots, x_k$. We have

$u' = v[t_1/x_1,\ldots,t_k/x_k]$ and therefore $\Gamma,\ y:C \vdash v[t_1/x_1,\ldots,t_k/x_k]:D$. By induction hypothesis, there exist types A_i such that $\Gamma,\ y:C \vdash t_i:A_i$, and $\Gamma,\ y:C,\ x_1:A_1,\ldots,x_k:A_k \vdash v:D$. Consequently :
$\Gamma,\ x_1:A_1,\ldots,x_k:A_k \vdash u:C \to D$.
Moreover, since y does not occur in t_i, we have $\Gamma \vdash t_i:A_i$ (propositions 3.1, 3.3 and 3.14).

If it is rule 3, then $u[t_1/x_1,\ldots,t_k/x_k] = v'w'$, and $\Gamma \vdash v':C \to B$, $\Gamma \vdash w':C$. Since $u \neq x_1,\ldots,x_k$, we have $u = vw$, and therefore :
$v' = v[t_1/x_1,\ldots,t_k/x_k]$, $w' = w[t_1/x_1,\ldots,t_k/x_k]$. Consequently :
$\Gamma \vdash v[t_1/x_1,\ldots,t_k/x_k]:C \to B$, and $\Gamma \vdash w[t_1/x_1,\ldots,t_k/x_k]:C$.
By induction hypothesis, there exist types A'_i, A''_i such that :
$\Gamma \vdash t_i:A'_i$; $\Gamma \vdash t_i:A''_i$;
$\Gamma,\ x_1:A'_1,\ldots,x_k:A'_k \vdash v:C \to B$; $\Gamma,\ x_1:A''_1,\ldots,x_k:A''_k \vdash w:C$.
Let $A_i = A'_i \wedge A''_i$; then we have :
$\Gamma,\ x_1:A_1,\ldots,x_k:A_k \vdash v:C \to B,\ w:C$. Thus :
$\Gamma,\ x_1:A_1,\ldots,x_k:A_k \vdash u:B$. Moreover, $\Gamma \vdash t_i:A_i$.

If it is rule 4 or rule 6, then the result is trivial.

If it is rule 5, then :
$B = B' \wedge B''$, and $\Gamma \vdash u[t_1/x_1,\ldots,t_k/x_k]:B'$, $\Gamma \vdash u[t_1/x_1,\ldots,t_k/x_k]:B''$.
By induction hypothesis, there exist types A'_i, A''_i such that :
$\Gamma \vdash t_i:A'_i$; $\Gamma \vdash t_i:A''_i$;
$\Gamma,\ x_1:A'_1,\ldots,x_k:A'_k \vdash u:B'$; $\Gamma,\ x_1:A''_1,\ldots,x_k:A''_k \vdash u:B''$.
Let $A_i = A'_i \wedge A''_i$; then we have $x_1:A_1,\ldots,x_k:A_k \vdash u:B' \wedge B''$, that is to say $u:B$. Moreover, $\Gamma \vdash t_i:A_i$.

Q.E.D.

Corollary 4.5.
If $\Gamma \vdash u[t/x]:B$ and if t is typable in the context Γ, then $\Gamma \vdash (\lambda x\, u)t:B$.

Remark. In system $\mathscr{D}\Omega$, the condition about t is satisfied anyway, since $\Gamma \vdash t:\Omega$.

Proof. By proposition 4.4, we have $\Gamma \vdash t:A$ and $\Gamma, x:A \vdash u:B$ for some type A. Hence $\Gamma \vdash \lambda x\, u:A \to B$ (rule 2), and therefore, by rule 3, $\Gamma \vdash (\lambda x\, u)t:B$.

Q.E.D.

Theorem 4.6. *Let t and t' be two λ-terms such that t' is obtained from t by β-reduction (in other words $t\,\beta\,t'$). If $\Gamma \vdash_{\mathscr{D}\Omega} t':A$, then $\Gamma \vdash_{\mathscr{D}\Omega} t:A$.*

We may suppose $t\,\beta_0\,t'$ (i.e. t' is obtained by contracting a redex in t).
The proof is by induction on the length of t and, for each fixed t, by induction on the length of A.
If $A = \Omega$, the result is trivial.

If $A = A_1 \wedge A_2$, then $\Gamma \vdash t' : A_1$ and $\Gamma \vdash t' : A_2$. By induction hypothesis, we have $\Gamma \vdash t : A_1$, and $\Gamma \vdash t : A_2$, therefore $\Gamma \vdash t : A$.

So we may now suppose that A is a prime type. There are three possible cases for t :

i) t is a variable ; this is impossible since $t\,\beta_0\,t'$.

ii) $t = \lambda x\, u$; then $t' = \lambda x\, u'$ and $u\,\beta_0\,u'$. Since $\lambda x\, u'$ is of prime type A in the context Γ, by lemma 4.2(ii), we have $A = B \to C$, and $\Gamma, x : B \vdash u' : C$. Now u is shorter than t, so by induction hypothesis, $\Gamma, x : B \vdash u : C$. Thus $t = \lambda x\, u$ is of type $A = B \to C$ in the context Γ.

iii) $t = uv$; then we have three possible situations for t' :

a) $t' = uv'$, with $v\,\beta_0\,v'$; by assumption uv' is of prime type A in the context Γ. By lemma 4.2(iii), we have $\Gamma \vdash v' : B$ and $\Gamma \vdash u : B \to A'$, A being a prime factor of A'. Now v is shorter than t so, by induction hypothesis, $\Gamma \vdash v : B$. Thus $t = uv$ is of type A', and hence also of type A, in the context Γ.

b) $t' = u'v$, with $u\,\beta_0\,u'$; similarly, we have :
$\Gamma \vdash v : B$ and $\Gamma \vdash u' : B \to A'$, A being a prime factor of A'. By induction hypothesis, $\Gamma \vdash u : B \to A'$. Thus $t = uv$ is of type A', and hence also of type A in the context Γ.

c) $u = \lambda x\, w$, (so $t = (\lambda x\, w)v$) and $t' = w[v/x]$.
The assumption is $\Gamma \vdash w[v/x] : A$. By corollary 4.5, *and since we are in system* $\mathscr{D}\Omega$, we also have $\Gamma \vdash (\lambda x\, w)v : A$.

Q.E.D.

As an immediate consequence of theorem 4.6 and proposition 4.3, we obtain :

Theorem 4.7.
If t is β-equivalent to t', and if $\Gamma \vdash_{\mathscr{D}\Omega} t : A$, then $\Gamma \vdash_{\mathscr{D}\Omega} t' : A$.

We are then able to give an alternative proof of the uniqueness of the normal form :

Corollary 4.8. *Suppose t and t' are normal and $t \simeq_\beta t'$. Then $t = t'$.*

Apply theorem 4.7 and corollary 3.30.

Q.E.D.

Theorem 4.9. *For every λ-term t, the following conditions are equivalent :*
i) t is solvable ;
ii) t is β-equivalent to a head normal form ;
iii) the head reduction of t is finite ;
iv) t is typable with a non-trivial type in system $\mathscr{D}\Omega$.

Recall that the trivial types are those obtained by the following rules :
Ω is trivial ;
if A is trivial, then so is $B \to A$ for every B ;
if A, B are trivial, then so is $A \wedge B$.

Lemma 4.10. *If $\lambda x\, t$ (resp. tu) is typable with a non-trivial type in system $\mathscr{D}\Omega$, then the same property holds for t.*

We may assume that this type is non-trivial and *prime*, since any non-trivial type has a prime factor which is also non-trivial.
Suppose that $\Gamma \vdash \lambda x\, t : A$, where A is a prime non-trivial type. By lemma 4.2(ii), we get $A = B \to C$ and $\Gamma, x : B \vdash t : C$. Moreover, C is non-trivial since A is.
Suppose that $\Gamma \vdash tu : A$, where A is a prime non-trivial type.
By lemma 4.2(iii), we get $\Gamma \vdash t : B \to A'$ and A is a prime factor of A'. It follows that A' is non-trivial.

Q.E.D.

We are now able to prove theorem 4.9.

(i) $\Rightarrow$ (iv) : Let $u = \lambda x_1 \ldots \lambda x_k\, t$ be the closure of t. Then u is solvable (remark 2, p. 31, chapter 2), and therefore $u v_1 \ldots v_n \simeq_\beta x$, where x is some variable with no occurrence in u. Since x can obviously be typed with a non-trivial type, the same holds for $u v_1 \ldots v_n$ (theorem 4.7), and hence also for u, according to lemma 4.10. Applying this lemma again, we can see that t itself is typable with a non-trivial type.

(iv) $\Rightarrow$ (iii) : This is the head normal form theorem 3.7.

(iii) $\Rightarrow$ (ii) : Obvious.

(ii) $\Rightarrow$ (i) : We may suppose that t is a closed term (otherwise, take its closure).
We have $t \simeq_\beta \lambda x_1 \ldots \lambda x_k (x_i) u_1 \ldots u_l$ (closed term in head normal form).
Let $v_i = \lambda y_1 \ldots \lambda y_l\, x$ (where x is a new variable), and v_j be arbitrary terms for $j \neq i$, $1 \leq j \leq k$. Then $(t) v_1 \ldots v_k \simeq_\beta x$, which proves that t is solvable.

Q.E.D.

As an application of theorem 4.9, we now prove the following property of solvable terms, which we have used in chapter 2 (namely, lemma 2.12) :

Theorem 4.11. *If $t \simeq_\beta \lambda x_1 \ldots \lambda x_k (x_i) t_1 \ldots t_n$ (with $1 \leq i \leq k$) then, there exist $t'_j \simeq_\beta t_j$ ($1 \leq j \leq n$) such that, for any $u_1, \ldots, u_k \in \Lambda$, we have :*
$(t) u_1 \ldots u_k \succ_w (u_i) t''_1 \ldots t''_n$ with $t''_j = t'_j [u_1/x_1, \ldots, u_k/x_k]$.

Recall that $\succ_w$ denotes *weak head reduction* (see page 30).

Lemma 4.12. *If $t \succ (x) t_1 \ldots t_n$, then $t[u/x, u_1/x_1, \ldots, u_k/x_k] \succ_w (u) t'_1 \ldots t'_n$ where $t'_j = t_j [u/x, u_1/x_1, \ldots, u_k/x_k]$ for $1 \leq j \leq k$.*

Proof by induction on the length of the head reduction from t to $(x)t_1 \dots t_n$. Note that this reduction is, indeed, a *weak* head reduction, because the final term does not begin with a λ.

The result is trivial if this length is 0, i.e. if $t = (x)t_1 \dots t_n$. Otherwise, by proposition 2.2, we have $t = (\lambda z\, w)v v_1 \dots v_p$ (since t does not begin with a λ). Let $t^* = (w[v/z])v_1 \dots v_p$; we can apply the induction hypothesis to t^*, so that $t^*[u/x, u_1/x_1, \dots, u_k/x_k] \succ_w (u)t'_1 \dots t'_n$.

Define $v' = v[u/x, u_1/x_1, \dots, u_k/x_k]$, and the same for $v_1, \dots, v_p, w$.

Thus, we have :

$t^*[u/x, u_1/x_1, \dots, u_k/x_k] = (w[v/z][u/x, u_1/x_1, \dots, u_k/x_k])v'_1 \dots v'_p$

$= (w[u/x, u_1/x_1, \dots, u_k/x_k, v'/z])v'_1 \dots v'_p$ (by lemma 1.13)

$= (w'[v'/z])v'_1 \dots v'_p$ (again by lemma 1.13, since z is not free in $u, u_1, \dots, u_k$).

Therefore, we have $(w'[v'/z])v'_1 \dots v'_p \succ_w (u)t'_1 \dots t'_n$.

It follows trivially that $(\lambda z\, w')v' v'_1 \dots v'_p \succ_w (u)t'_1 \dots t'_n$. This gives the result, because $t[u/x, u_1/x_1, \dots, u_k/x_k] = (\lambda z\, w')v' v'_1 \dots v'_p$.

Q.E.D.

We can now prove theorem 4.11. The hypothesis gives :

$(t)x_1 \dots x_k \simeq_\beta (x_i)t_1 \dots t_n$ and the variables $x_1, \dots, x_k$ are not free in t.

By theorem 4.9, the head reduction of $(t)x_1 \dots x_k$ is finite and gives a λ-term which is β-equivalent to $(x_i)t_1 \dots t_n$. In other words :

$(t)x_1 \dots x_k \succ (x_i)t'_1 \dots t'_n$, with $t'_j \simeq_\beta t_j$ $(1 \le j \le n)$.

We now use lemma 4.12, with the substitution $[u_1/x_1, \dots, u_k/x_k]$, and we obtain $(t)u_1 \dots u_k \succ_w (u_i)t''_1 \dots t''_n$ with $t''_j = t'_j[u_1/x_1, \dots, u_k/x_k]$.

Q.E.D.

Theorem 4.13. *For every λ-term t, the following conditions are equivalent :*
i) t is normalizable ;
ii) t is normalizable by leftmost β-reduction ;
iii) there exist a type A and a context Γ, both containing no occurrence of the symbol Ω, such that $\Gamma \vdash_{\mathscr{D}\Omega} t : A$;
iv) there exist a type A with no positive occurrence of Ω, and a context Γ with no negative occurrence of Ω, such that $\Gamma \vdash_{\mathscr{D}\Omega} t : A$.

Clearly, (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv). We already know that (iv) $\Rightarrow$ (ii) : this is the normalization theorem 3.10.

It remains to prove that (i) $\Rightarrow$ (iii) :

If t is normalizable, then $t \simeq_\beta t'$ for some normal term t' ; by proposition 3.24, there exist a type A and a context Γ, both containing no occurrence of the symbol Ω, such that $\Gamma \vdash_{\mathscr{D}} t' : A$. It then follows from theorem 4.7 that we also have $\Gamma \vdash_{\mathscr{D}\Omega} t : A$.

Q.E.D.

Theorem 4.14. *A λ-term t is normalizable if and only if it admits no infinite quasi leftmost reduction.*

The condition is obviously sufficient. Conversely, if t is normalizable, then, by theorem 4.13, there exist a type A and a context Γ, both containing no occurrence of the symbol Ω, such that $\Gamma \vdash_{\mathscr{D}\Omega} t : A$. Thus, it follows from the quasi leftmost normalization theorem 3.12 that t admits no infinite quasi leftmost reduction.

Q.E.D.

With the help of the results above, we can now give yet another proof of the uniqueness of the normal form (the third, see corollary 4.8) which makes no use of the Church-Rosser theorem 1.24.

Theorem 4.15. *If t is normalizable, then it has only one normal form. In other words, if $t\,\beta\,u$, $t\,\beta\,u'$ and u, u' are normal, then $u = u'$.*

By theorem 4.13(i)(ii), t is normalizable by leftmost β-reduction. We prove the theorem by induction on the *total length* of this reduction (i.e. the total number of symbols which appear in it).
By proposition 2.2, we have $t = \lambda x_1 \dots \lambda x_k(\xi)\, t_1 \dots t_n$ where ξ is a variable or a redex.
If ξ is a variable, the leftmost β-reduction of t is exactly the succession of the leftmost β-reductions of $t_1, \dots, t_n$. Therefore, we can apply the induction hypothesis to $t_1, \dots, t_n$ and we see that t has only one normal form, which is :
$\lambda x_1 \dots \lambda x_k(\xi)\, t_1^* \dots t_n^*$ where t_i^* is the (unique) normal form of t_i.
If $\xi = (\lambda x\, u)\, v$ is a redex, the first step of leftmost β-reduction in t gives :
$t^{**} = \lambda x_1 \dots \lambda x_k\, u[v/x]\, t_1 \dots t_n$.
By the induction hypothesis, t^{**} has a unique normal form t^*.
Consider now any β-reduction of t, which gives a normal form. We show that it gives t^*. Since $t = \lambda x_1 \dots \lambda x_k(\lambda x\, u)\, v\, t_1 \dots t_n$, this reduction begins with some β-reductions in $u, v, t_1, \dots, t_n$, which give :
$\lambda x_1 \dots \lambda x_k(\lambda x\, u')\, v'\, t_1' \dots t_n'$, with $u\,\beta\,u'$, $v\,\beta\,v'$, $t_1\,\beta\,t_1', \dots, t_n\,\beta\,t_n'$.
Then, the head redex is reduced, which gives $\lambda x_1 \dots \lambda x_k\, u'[v'/x]\, t_1' \dots t_n'$.
But β-reduction is a λ-compatible relation, and therefore, we have :
$t^{**}\,\beta\,\lambda x_1 \dots \lambda x_k\, u'[v'/x]\, t_1' \dots t_n'$.
This shows that this β-reduction will finally give a normal form of t^{**}, i.e. t^*.

Q.E.D.

Strong normalization

The next proposition is a generalization of corollary 4.5. It holds for both systems $\mathscr{D}$ and $\mathscr{D}\Omega$ (in the case of system $\mathscr{D}\Omega$, the condition “ t is typable in the context Γ ” is satisfied anyway, since $\Gamma \vdash_{\mathscr{D}\Omega} t : \Omega$).

Proposition 4.16. *For all terms $u, t, t_1, \dots, t_n$, and any variable x, if $\Gamma \vdash (u[t/x])t_1 \dots t_n : B$, and if t is typable in the context Γ, then $\Gamma \vdash (\lambda x\ u)t t_1 \dots t_n : B$.*

The proof is by induction on n and, for each fixed n, by induction on the length of B. The case $n = 0$ is precisely corollary 4.5.
If $B = B_1 \wedge B_2$, then$(u[t/x])t_1 \dots t_n$ may be given both type B_1 and type B_2 in the context Γ ; by induction hypothesis, the same holds for $(\lambda x\ u)t t_1 \dots t_n$, which is thus typable in the context Γ, with type $B_1 \wedge B_2$.
Now we may suppose that B is a prime type and that $n \geq 1$.
We have $\Gamma \vdash u[t/x]t_1 \dots t_n : B$; it follows from lemma 4.2(iii) that t_n is of type C, and $(u[t/x])t_1 \dots t_{n-1}$ of type $C \to B'$, in the context Γ, B being a prime factor of B'.
By induction hypothesis, we have $\Gamma \vdash (\lambda x\ u)t t_1 \dots t_{n-1} : C \to B'$. Therefore $(\lambda x\, u)t t_1 \dots t_n$ is of type B', and hence also of type B, in the context Γ.
Q.E.D.

Theorem 4.17. *Every strongly normalizable term is typable in system $\mathscr{D}$.*

Consider a strongly normalizable term τ, and let $N(\tau)$ be the sum of the lengths of all possible normalizations of τ (proposition 3.18 ensures the correctness of this definition). The proof is by induction on $N(\tau)$. By proposition 2.2, we have : $\tau = \lambda x_1 \dots \lambda x_m(v)t_1 \dots t_n$, where v is either a variable or a redex.
If v is a variable, then $t_1, \dots, t_n$ are strongly normalizable and we have : $N(\tau) > N(t_1), \dots, N(t_n)$. Thus $t_1, \dots, t_n$ are typable, with types $A_1, \dots, A_n$, respectively, in system $\mathscr{D}$; we may suppose that all these typings are in the same context Γ (proposition 3.23) and that Γ contains a declaration for each of the variables $x_1, \dots, x_m, v$, say $x_1 : U_1, \dots, x_m : U_m, v : V$ (with $V = U_i$ whenever $v = x_i$).
Let X be a new type variable, $V' = V \wedge (A_1, \dots, A_n \to X)$, and Γ' the context obtained by replacing in Γ the declaration of v with : $v : V'$.
Then we have $\Gamma' \vdash_{\mathscr{D}} t_i : A_i$ $(1 \leq i \leq n)$, and thus $\Gamma' \vdash_{\mathscr{D}} (v)t_1 \dots t_n : X$;
hence τ may be given :
either type $U_1, \dots, U_m \to X$ (if $v \neq x_1, \dots, x_m$)
or type $U_1, \dots, U_{i-1}, V', U_{i+1}, \dots, U_m \to X$ (if $v = x_i$).
If $v = (\lambda x\, u)t$ (v is a redex), then $\tau = \lambda x_1 \dots \lambda x_m(\lambda x\, u)t t_1 \dots t_n$;
let $\tau' = u[t/x]t_1 \dots t_n$. Clearly, $N(\tau) > N(\tau')$ (every normalization of τ' is strictly included in a normalization of τ) ; it is also clear that $N(\tau) > N(t)$ (since t is a subterm of τ). Thus, by induction hypothesis, τ' and t are typable in system $\mathscr{D}$; moreover, proposition 3.23 allows us to assume that they are typable in the same context. It then follows from proposition 4.16 that $(\lambda x\ u)t t_1 \dots t_n$ is typable, with some type B, in some context Γ : even if it means extending it,

we may assume that Γ contains a declaration for each of the variables $x_1,\dots,x_m$, say $x_1 : U_1,\dots,x_m : U_m$. Finally, τ is seen to be typable, with type $U_1,\dots,U_m \to B$.
Q.E.D.

Corollary 4.18. *A term is strongly normalizable if and only if it is typable in system $\mathscr{D}$.*

Indeed, by the strong normalization theorem 3.20, every term which is typable in system $\mathscr{D}$ is strongly normalizable.

Remarks.

1. Theorem 4.6 does not hold any more if we replace system $\mathscr{D}\Omega$ with system $\mathscr{D}$. For instance, the term $t = \lambda y(\lambda x\, y)(y)y$ is β-equivalent to $\lambda y\, y$, which is of type $Y \to Y$, where Y is any type variable. Now t may not be given type $Y \to Y$:
Indeed, if $\vdash_{\mathscr{D}} t : Y \to Y$, then, by lemma 4.2(ii), we have :
$y : Y \vdash_{\mathscr{D}} (\lambda x\, y)(y)y : Y$; therefore, by lemma 4.2(iii), $y : Y \vdash_{\mathscr{D}} (y)y : A$ for some type A ; hence $y : Y \vdash_{\mathscr{D}} y : B \to C$ (by lemma 4.2(iii)) ; but this is in contradiction with lemma 4.2(i).
Nevertheless, t is typable ; for example, it may be given type
$Y \wedge (Y \to X) \to Y \wedge (Y \to X)$.
There is an analogue of theorem 4.6 for system $\mathscr{D}$, which uses βI-reduction instead of β-reduction (see below theorem 4.21).

2. A normalizable term, of which every proper subterm is strongly normalizable, need not be strongly normalizable. For instance, the term :
$t = (\lambda x(\lambda y\, z)(x)\delta)\delta$, where $\delta = \lambda x\, xx$, is normalizable (it is β-equivalent to z), but not strongly normalizable (t reduces to $(\lambda y\, z)(\delta)\delta$, and $(\delta)\delta$ is not normalizable).

βI-reduction

A λ-term of the form $(\lambda x\, t)u$ will be called a *I-redex* if x is a *free* variable of t. Reducing a I-redex will be called a *step of βI-reduction.* A finite sequence of such steps will be called a *βI-reduction.* The notation $t\, \beta I\, t'$ means that t' is obtained by βI-reduction from t.
We will now prove the following result *(Barendregt's conservation theorem)* :

Theorem 4.19. *If t' is strongly normalizable and if $t\, \beta I\, t'$, then t is strongly normalizable.*

Lemma 4.20. *If $\Gamma \vdash_{\mathscr{D}} u[v/x] : A$ and if x is free in u, then v is typable, in system $\mathscr{D}$, in the context Γ.*

We first observe that the result is trivial if u is a variable : indeed, this variable must be x. Therefore, from now on, we assume that u is not a variable.

We prove the lemma by induction on the length of the proof of the typing : $\Gamma \vdash_{\mathscr{D}} u[v/x]{:}A$ in system $\mathscr{D}$. Consider the last rule used in this proof (page 51).
If it is rule 1, $u[v/x]$ is a variable, thus u must also be a variable.
If it is rule 2, then $u[v/x] = \lambda y\, w$ and we have $A = B \to C$ and $\Gamma, y{:}B \vdash w{:}C$. Now, u is not an application ($u[v/x]$ would also be an application) and we assumed it is not a variable. Therefore, we have $u = \lambda y\, u'$ and $w = u'[v/x]$.
Thus $\Gamma, y{:}B \vdash u'[v/x]{:}C$ is the previous step of the proof. Now, the variable x is free in u', since it is free in u. By the induction hypothesis, we see that v is typable, in system $\mathscr{D}$, in the context $\Gamma, y{:}B$. But y is not free in v and it follows from proposition 3.14 that v is typable in the context Γ.
If it is rule 3, then $u[v/x] = w_0 w_1$ and we have :
$\Gamma \vdash w_0{:}B \to A$, $\Gamma \vdash w_1{:}B$. Now, u is not an abstraction ($u[v/x]$ would also be an abstraction) and we assumed it is not a variable. Therefore, we have $u = u_0 u_1$ and $w_0 = u_0[v/x], w_1 = u_1[v/x]$. Thus, some previous steps of the proof are $\Gamma \vdash u_0[v/x]{:}B \to A$, $\Gamma \vdash u_1[v/x]{:}B$. But x is free in $u = u_0 u_1$, and therefore, it is free in u_0 or in u_1. We may thus apply the induction hypothesis, and we see that v is typable, in system $\mathscr{D}$, in the context Γ.
The case of the rules 4 and 5 is trivial.

Q.E.D.

Theorem 4.21.
Let t and t' be two λ-terms such that $t\, \beta I\, t'$. If $\Gamma \vdash_{\mathscr{D}} t' : A$, then $\Gamma \vdash_{\mathscr{D}} t : A$.

Remark. Thus, the typings in system $\mathscr{D}$ are preserved by inverse βI-reduction. This theorem is close to theorem 4.6, which says that, in system $\mathscr{D}\Omega$, the typings are preserved by inverse β-reduction.

We may assume that t' is obtained from t by one step of βI-reduction.
The proof is by induction on the length of t and, for each fixed t, by induction on the length of A. It is exactly the same as for theorem 4.6, except for :

- the very first step : of course, the case $A = \Omega$ is not considered.
- the very last step (iii)(c), which is managed as follows :

c) $u = \lambda x\, w$, (so $t = (\lambda x\, w)v$) and $t' = w[v/x]$. Since we have a step of βI-reduction, the variable x is free in w.
Now, the assumption is : $\Gamma \vdash_{\mathscr{D}} w[v/x] : A$. By lemma 4.20, v is typable in the context Γ, in system $\mathscr{D}$. By corollary 4.5, we also have $\Gamma \vdash_{\mathscr{D}} (\lambda x\, w)v : A$.

Q.E.D.

We can now prove theorem 4.19 : if t' is strongly normalizable, it is typable in system $\mathscr{D}$ (corollary 4.18). By theorem 4.21, t is also typable in system $\mathscr{D}$; thus, by corollary 4.18, t is strongly normalizable.

Q.E.D.

Two redexes $(\lambda x\, t)u$ and $(\lambda x'\, t')u'$ will be called *equivalent* if :
$u = u'$ and $t[u/x] = t'[u'/x']$ (they have identical arguments and reducts).
A redex which is equivalent to a I-redex will be called a I'-redex.
For example, $(\lambda x\, uv)u$ is always a I'-redex, even when x is not free in u, v. Indeed, in this case, it is equivalent to the I-redex $(\lambda x\, xv)u$.
We shall write $t\,\beta I'\,t'$ if t' is obtained from t by a sequence of reductions of I'-redexes.

We can strengthen theorems 4.21 and 4.19 in the following way, with exactly the same proof :

Theorem 4.22. *Let t and t' be two λ-terms such that $t\,\beta I'\,t'$. If $\Gamma \vdash_{\mathscr{D}} t' : A$, then $\Gamma \vdash_{\mathscr{D}} t : A$.*

Theorem 4.23. *If t' is strongly normalizable and if $t\,\beta I'\,t'$, then t is strongly normalizable.*

The λI-calculus

The terms of the *λI-calculus* form a subset Λ_I of Λ, which is defined as follows :

- If x is a variable, then $x \in \Lambda_I$.
- If $t, u \in \Lambda_I$, then $tu \in \Lambda_I$.
- If $t \in \Lambda_I$ and x is a variable *which appears free in* t, then $\lambda x\, t \in \Lambda_I$.

The typical example of a closed λ-term which is not in Λ_I is $\lambda x \lambda y\, x$.

If $t \in \Lambda_I$, then every subterm of t is in Λ_I (trivial proof, by induction on the length of t).

Proposition 4.24. *If $t, t_1, \ldots, t_n \in \Lambda_I$, then $t[t_1/x_1, \ldots, t_n/x_n] \in \Lambda_I$.*

Proof by induction on the length of t : the result is immediate if t is a variable, or if $t = uv$, with $u, v \in \Lambda_I$. If $t = \lambda x\, u$, then :
$t[t_1/x_1, \ldots, t_n/x_n] = \lambda x\, u[t_1/x_1, \ldots, t_n/x_n]$ (we suppose $x \neq x_1, \ldots, x_n$).
By hypothesis, there is a free occurrence of x in u and therefore, there is also one in $u[t_1/x_1, \ldots, t_n/x_n]$. By induction hypothesis, we have $u[t_1/x_1, \ldots, t_n/x_n] \in \Lambda_I$.
It follows that $\lambda x\, u[t_1/x_1, \ldots, t_n/x_n] \in \Lambda_I$.

Q.E.D.

Proposition 4.25. *Λ_I is closed by β-reduction. More precisely, if $t \in \Lambda_I$ and $t\,\beta\,t'$, then $t' \in \Lambda_I$ and t' has the same free variables as t.*

Suppose $t \in \Lambda_I$ and $t\,\beta_0\,t'$; we show the result by induction on the length of t ; observe that t cannot be a variable.
If $t = \lambda x\,u$, then $t' = \lambda x\,u'$ with $u\,\beta_0\,u'$. Since $u \in \Lambda_I$ and x is a free variable of u, by induction hypothesis, u' has the same properties. It follows that $t' \in \Lambda_I$ and t' has the same free variables as t.
If $t = u\,v$, we have three possibilities for t' :

$t' = u'\,v$ with $u\,\beta_0\,u'$; by induction hypothesis, we have $u' \in \Lambda_I$ and u' has the same free variables as u. Hence, $t' \in \Lambda_I$ and t' has the same free variables as t.

$t' = u\,v'$ with $v\,\beta_0\,v'$; same proof.

$u = \lambda x\,w$ (so that $t = (\lambda x\,w)v$), and $t' = w[v/x]$; we have $v, w \in \Lambda_I$ and therefore, by proposition 4.24, we have $t' \in \Lambda_I$. Now, let F_v (resp. F_w) the set of free variables of v (resp. w) ; thus, we have $x \in F_w$. The set of free variables of t is $F_v \cup (F_w \setminus \{x\})$. The set of free variables of t' is the same, because v is really a subterm of $t' = w[v/x]$.

Q.E.D.

Theorem 4.26. *If $t \in \Lambda_I$ is normalizable, then t is strongly normalizable.*

We prove first the following lemma on strong normalization :

Lemma 4.27. *Let $t_1, \ldots, t_n, u, v \in \Lambda$ be such that $u[v/x]\,t_1 \ldots t_n$ and v are strongly normalizable. Then $(\lambda x\,u)\,v\,t_1 \ldots t_n$ is strongly normalizable.*

By corollary 4.18, we know that $u[v/x]\,t_1 \ldots t_n$ and v are typable in system $\mathscr{D}$. By proposition 3.23, they are typable in the *same context*. Then, we apply proposition 4.16, which shows that $(\lambda x\,u)\,v\,t_1 \ldots t_n$ is typable in system $\mathscr{D}$. Applying again corollary 4.18, we see that $(\lambda x\,u)\,v\,t_1 \ldots t_n$ is strongly normalizable.

We can give a more direct proof, which does not use types. Suppose that there exists an infinite sequence of β-reductions for the λ-term $(\lambda x\,u)\,v\,t_1 \ldots t_n$. There are two possible cases :

- Each β-reduction takes place in one of the terms $u, v, t_1, \ldots, t_n$.

Thus, there is an infinite sequence of β-reductions in *one* of these terms. But it cannot be v, which is strongly normalizable ; and it can be neither u, nor $t_1, \ldots,$ nor t_n, because $u[v/x]\,t_1 \ldots t_n$ is strongly normalizable.

- The sequence begins with a finite number of β-reductions in the terms $u, v, t_1, \ldots, t_n$ and then, the head redex is reduced. This gives $(\lambda x\,u')\,v'\,t'_1 \ldots t'_n$ with $u\,\beta\,u', v\,\beta\,v', t_1\,\beta\,t'_1, \ldots, t_n\,\beta\,t'_n$ and then $u'[v'/x]\,t'_1 \ldots t'_n$. Therefore, this term is not strongly normalizable. But β-reduction is a λ-compatible relation, and it follows that $u[v/x]\,t_1 \ldots t_n\,\beta\,u'[v'/x]\,t'_1 \ldots t'_n$. Therefore, $u[v/x]\,t_1 \ldots t_n$ is also not strongly normalizable, which is a contradiction.

Q.E.D.

Now, we prove theorem 4.26 : by theorem 4.13, we know that t is normalizable by leftmost reduction. We prove the result by induction on the *total length* of this leftmost reduction (i.e. the sum of the lengths of the λ-terms which appear in it).

By proposition 2.2, there are two possibilities for t :

- $t = \lambda x_1 \dots \lambda x_m (y) t_1 \dots t_n$ where y is a variable.

Then, we have $t_1, \dots, t_n \in \Lambda_I$ and their leftmost reductions are stricly shorter than the one of t. By induction hypothesis, they are all strongly normalizable, and so is t.

- $t = \lambda x_1 \dots \lambda x_m (\lambda x\, u) v t_1 \dots t_n$;

we have to show that $(\lambda x\, u) v t_1 \dots t_n$ is strongly normalizable. By lemma 4.27, it suffices to show that $u[v/x] t_1 \dots t_n$ and v are strongly normalizable. Now, $u[v/x] t_1 \dots t_n$ is obtained by β-reduction from $(\lambda x\, u) v t_1 \dots t_n \in \Lambda_I$.
Thus, $u[v/x] t_1 \dots t_n \in \Lambda_I$ (proposition 4.25).
It is clear that its leftmost reduction is strictly shorter than the one of :
$t = \lambda x_1 \dots \lambda x_m (\lambda x\, u) v t_1 \dots t_n$.
Thus, by induction hypothesis, we see that $u[v/x] t_1 \dots t_n$ is strongly normalizable. But $\lambda x\, u \in \Lambda_I$, because it is a subterm of t ; thus, x is a free variable of u. It follows that v is a subterm of $u[v/x] t_1 \dots t_n$, and therefore v is also strongly normalizable.

Q.E.D.

There is a short proof of theorem 4.26, by means of the above results on βI-reduction : suppose that $t \in \Lambda_I$ is normalizable and let t' be its normal form. Thus, t' is typable in sytem $\mathscr{D}$ (proposition 3.24). But we have $t\, \beta I\, t'$, since the reduction of t takes place in Λ_I. Therefore, by theorem 4.21, t is typable in sytem $\mathscr{D}$ and thus, t is strongly normalizable (theorem 3.20).

Q.E.D.

$\beta\eta$-reduction

Let $X_1, \dots, X_k$ be distinct type variables, A a type, Γ a context, and $U_1, \dots U_k$ arbitrary types. The type (resp. the context) obtained by replacing, in A (resp. in Γ), each occurrence of X_i by U_i ($1 \le i \le k$) will be denoted by : $A[U_1/X_1, \dots, U_k/X_k]$ (resp. $\Gamma[U_1/X_1, \dots, U_k/X_k]$).

The next two propositions hold for both systems $\mathscr{D}$ and $\mathscr{D}\Omega$.

Proposition 4.28.
If $\Gamma \vdash t : A$, *then* $\Gamma[U_1/X_1, \dots, U_k/X_k] \vdash t : A[U_1/X_1, \dots, U_k/X_k]$.

Immediate, by induction on the number of rules used to obtain $\Gamma \vdash t : A$.

Q.E.D.

Proposition 4.29. *Suppose* $t\eta_0 t'$ *and* $\Gamma \vdash t' : A$*, and let* $X_1, \ldots, X_k$ *be the type variables which occur either in* Γ *or in* A*. Then :*
$\Gamma[U_1/X_1, \ldots, U_k/X_k] \vdash t : A[U_1/X_1, \ldots, U_k/X_k]$ *for all types* $U_1, \ldots, U_k$ *of the form* $V \to W$.

Recall that $t\eta_0 t'$ means that t' is obtained from t by *one* η-reduction.

The proof of the proposition is by induction on the length of t and, for a given t, by induction on the length of A.
If $A = \Omega$, the result is trivial.
If $A = A_1 \wedge A_2$, then $\Gamma \vdash t' : A_1$, $\Gamma \vdash t' : A_2$. By induction hypothesis, we have $\Gamma[U_1/X_1, \ldots, U_k/X_k] \vdash t : A_i[U_1/X_1, \ldots, U_k/X_k] (i = 1, 2)$; therefore, by rule 5, $\Gamma[U_1/X_1, \ldots, U_k/X_k] \vdash t : A[U_1/X_1, \ldots, U_k/X_k]$.

So we now may suppose that A is a prime type. The three possible situations for t are :

i) t is a variable : this is impossible since $t\eta_0 t'$.

ii) $t = \lambda x\, u$; then we have two possible cases for t' :

a) $t' = \lambda x\, u'$, with $u\eta_0 u'$. Since $\Gamma \vdash t' : A$ (prime type), it follows from lemma 4.2(ii) that $A = B \to C$, and $\Gamma, x : B \vdash u' : C$. By induction hypothesis :
$\Gamma[U_1/X_1, \ldots, U_k/X_k], x : B[U_1/X_1, \ldots, U_k/X_k] \vdash u : C[U_1/X_1, \ldots, U_k/X_k]$ for all types U_i of the form $V \to W$. Thus t is of type :
$B[U_1/X_1, \ldots, U_k/X_k] \to C[U_1/X_1, \ldots, U_k/X_k] = A[U_1/X_1, \ldots, U_k/X_k]$
in the context $\Gamma[U_1/X_1, \ldots, U_k/X_k]$.

b) $t = \lambda x\, t'x$, and x does not occur free in t'. By assumption, we have :
$\Gamma \vdash t' : A$, A being a prime type. According to the definition of prime types, we have two cases :

If $A = B \to C$, then, $x : B \vdash t'x : C$; hence $\Gamma \vdash \lambda x\, t'x : B \to C$, in other words $\Gamma \vdash t : A$; by proposition 4.28, we have :
$\Gamma[U_1/X_1, \ldots, U_k/X_k] \vdash t : A[U_1/X_1, \ldots, U_k/X_k]$.

If A is a type variable X_i, then $\Gamma \vdash t' : X_i$; therefore, by proposition 4.28, we have $\Gamma[U_1/X_1, \ldots, U_k/X_k] \vdash t' : U_i$. Now, by assumption, $U_i = V \to W$. It then follows that $\Gamma[U_1/X_1, \ldots, U_k/X_k], x : V \vdash t'x : W$ and, consequently, $\Gamma[U_1/X_1, \ldots, U_k/X_k] \vdash \lambda x\, t'x : U_i$, that is to say
$\Gamma[U_1/X_1, \ldots, U_k/X_k] \vdash t : U_i$.

iii) $t = uv$; again, we have two possible cases for t' :

a) $t' = uv'$, with $v\eta_0 v'$; since uv' is of prime type A in the context Γ, it follows from lemma 4.2(iii) that v' is of type B and u of type $B \to A'$ in the context Γ, A being a prime factor of A'. By induction hypothesis :
$\Gamma[U_1/X_1, \ldots, U_k/X_k] \vdash v : B[U_1/X_1, \ldots, U_k/X_k]$
for all types U_i of the form $V \to W$.
By proposition 4.28, we have :

$\Gamma[U_1/X_1,\ldots,U_k/X_k] \vdash u : B[U_1/X_1,\ldots,U_k/X_k] \to A'[U_1/X_1,\ldots,U_k/X_k]$. Thus $t = uv$ is of type $A'[U_1/X_1,\ldots,U_k/X_k]$ in the context $\Gamma[U_1/X_1,\ldots,U_k/X_k]$, and hence is also of type $A[U_1/X_1,\ldots,U_k/X_k]$.

b) $t = u'v$, with $u\,\eta_0\,u'$; the proof is the same as in case (a).

Q.E.D.

Theorem 4.30. *A λ-term is $\beta\eta$-normalizable if and only if it is normalizable.*

Necessity : let t be a $\beta\eta$-normalizable term ; we prove that t is normalizable, by induction on the length of its $\beta\eta$-normalization. Consider the first $\beta\eta$-reduction done in t : it produces a term t', which is normalizable (induction hypothesis). If it is a β-reduction, then $t\,\beta_0\,t'$, thus t is also normalizable. If it is an η-reduction, then $t\,\eta_0\,t'$; since t' is normalizable (induction hypothesis), we have $\Gamma \vdash_{\mathscr{D}\Omega} t' : A$, where both A and Γ contain no occurrence of the symbol Ω (theorem 4.13). By proposition 4.29, there exist a type A' and a context Γ', with no occurrence of Ω, such that $\Gamma' \vdash_{\mathscr{D}\Omega} t : A'$; it then follows from theorem 4.13 that t is normalizable.

Sufficiency : if t is normalizable, then $t\,\beta\,t'$ for some normal term t' ; consider a maximal sequence of η-reductions starting with t' (such a sequence needs to be finite, since the length of terms strictly decreases under η-reduction) : it produces a term which is still normal (lemma 3.27) and contains no η-redex, in other words a $\beta\eta$-normal term.

Q.E.D.

We can now give an alternative proof of the uniqueness of the $\beta\eta$-normal form :

Theorem 4.31. *If $t \in \Lambda$ is $\beta\eta$-normalizable, then it has only one $\beta\eta$-normal form. More precisely, there exists a $\beta\eta$-normal term u such that, if $t\,\beta\eta\,t'$ for some t', then $t'\,\beta\eta\,u$.*

Remark. This is exactly the Church-Rosser property for t.

By theorem 4.30, t is normalizable ; by theorem 4.13(i)(iii), there exist a type A and a context Γ, both containing no occurrence of the symbol Ω, such that $\Gamma \vdash_{\mathscr{D}\Omega} t : A$. Then the result follows immediately from theorem 3.13.

Q.E.D.

Theorem 4.32. *A λ-term t is solvable if, and only if there exists a head normal form u such that $t\,\beta\eta\,u$.*

If t is solvable, then $t\,\beta\,u$ for some head normal form u and, therefore, $t\,\beta\eta\,u$. Conversely, suppose that $t\,\beta\eta\,u$, u being a head normal form. Then, there exists a sequence $t_0, t_1, \ldots, t_n$ such that $t_0 = t$, t_n is solvable and, for each $i = 0, \ldots, n$ we have $t_i\,\beta\,t_{i+1}$ or $t_i\,\eta_0\,t_{i+1}$.

We show that t is solvable by induction on n. This is trivial if $n = 0$. If $n \geq 1$, then t_1 is solvable, by induction hypothesis and there are two cases :

i) $t_0\,\beta\,t_1$; then $t = t_0$ is solvable.

ii) $t_0\,\eta_0\,t_1$; since t_1 is solvable, by theorem 4.9(i)(iv), it is typable with a non-trivial type in system $\mathscr{D}\Omega$. By proposition 4.29, $t = t_0$ has the same property ; it is therefore solvable, again by theorem 4.9(i)(iv).

Q.E.D.

2. The finite developments theorem

Remark. Until the end of this chapter, we shall only use the Church-Rosser theorem 1.24 and the strong normalization theorem 3.20.

Let $t \in \Lambda$; recall that a redex in t is, by definition, an occurrence, in t, of a subterm of the form $(\lambda x\,u)v$. In other words, a redex is defined by a subterm of the form $(\lambda x\,u)v$, together with its position in t. So we clearly have the following inductive definition for the redexes of a term t :

if t is a variable, then there is no redex in t ;

if $t = \lambda x\,u$, the redexes in t are those in u ;

if $t = uv$, the redexes in t are those in u, those in v, and, if u starts with λ, t itself.

We add to the λ-calculus a new variable, denoted by c, and we define $\Lambda(c)$ as the least set of terms satisfying the following rules :

1. If x is a variable $\neq c$, then $x \in \Lambda(c)$;
2. If x is a variable $\neq c$, and if $t \in \Lambda(c)$, then $\lambda x\,t \in \Lambda(c)$;
3. If $t, u \in \Lambda(c)$, then $(c)tu \in \Lambda(c)$;
4. If $t, u \in \Lambda(c)$, and if t starts with λ, then $tu \in \Lambda(c)$.

Lemma 4.33. *If $t, u \in \Lambda(c)$, and if x is a variable $\neq c$, then $u[t/x] \in \Lambda(c)$.*

The proof is by induction on u. The result is obvious whenever u is a variable $\neq c$, or $u = \lambda y\,v$, or $u = (c)vw$. If $u = (\lambda y\ v)w$, then $u[t/x] = (\lambda y\,v[t/x])w[t/x]$. By induction hypothesis, $v[t/x], w[t/x] \in \Lambda(c)$, and therefore $u[t/x] \in \Lambda(c)$.

Q.E.D.

Lemma 4.34. *If $t \in \Lambda(c)$ and $t\,\beta_0\,t'$, then $t' \in \Lambda(c)$.*

By induction on t. If $t = \lambda x\,u$, then $t' = \lambda x\,u'$, with $u\,\beta_0\,u'$; then the conclusion follows from the induction hypothesis.

If $t = (c)uv$, then $t' = (c)u'v$ or $(c)uv'$, with $u\,\beta_0\,u'$ or $v\,\beta_0\,v'$. By induction hypothesis, $u', v' \in \Lambda(c)$, and therefore $t' \in \Lambda(c)$.

If $t = (\lambda x\,u)v$, there are three possibilities for t' :

$t' = (\lambda x\, u')v$, or $(\lambda x\, u)v'$, with $u\,\beta_0\,u'$ or $v\,\beta_0\,v'$. By induction hypothesis, $u', v' \in \Lambda(c)$, and then $t' \in \Lambda(c)$.

$t' = u[v/x]$; then $t' \in \Lambda(c)$ by lemma 4.33.

Q.E.D.

We see that $\Lambda(c)$ is invariant under β-reduction (if $t \in \Lambda(c)$ and $t\,\beta\,t'$, then $t' \in \Lambda(c)$).

Lemma 4.35. *Let $t \in \Lambda(c)$, and Γ be any context in which all the variables of t, except c, are declared. Then there exist two types C, T of system $\mathscr{D}$ such that Γ, $c : C \vdash_{\mathscr{D}} t : T$.*

Proof by induction on t : this is obvious when t is a variable $\neq c$.
If $t = \lambda x\, u$, we can assume that the variable x is not declared in Γ (otherwise, we change the name of this variable in t). By induction hypothesis, we have $\Gamma, x : A, c : C \vdash u : U$, and therefore $\Gamma, c : C \vdash \lambda x\, u : A \to U$.
If $t = (c)uv$, with $u, v \in \Lambda(c)$, then, by induction hypothesis :
$\Gamma, c : C \vdash u : U$, and $\Gamma, c : C' \vdash v : V$. Hence :
$\Gamma, c : C \wedge C' \wedge (U, V \to W) \vdash (c)uv : W$.
If $t = (\lambda x\, u)v$, with $u, v \in \Lambda(c)$, we may assume that the variable x is not declared in Γ (otherwise, we change the name of this variable in $\lambda x\, u$). By induction hypothesis :
$\Gamma, x : A, c : C \vdash u : U$, and $\Gamma, c : C' \vdash v : V$; but here A is an arbitrary type, so we can take $A = V$. Then $\Gamma, c : C \vdash \lambda x\, u : V \to U$, and therefore :
$\Gamma, c : C \wedge C' \vdash (\lambda x\, u)v : U$.

Q.E.D.

Corollary 4.36. *Every term in $\Lambda(c)$ is strongly normalizable.*

This is immediate, according to the strong normalization theorem 3.20.

Q.E.D.

We define a mapping from $\Lambda(c)$ onto Λ, denoted by $T \mapsto |T|$, by induction on T :

if T is a variable $\neq c$, then $|T| = T$;
if $T = \lambda x\, U$, with $U \in \Lambda(c)$, then $|T| = \lambda x|U|$;
if $T = (c)UV$, with $U, V \in \Lambda(c)$, then $|T| = (|U|)|V|$;
if $T = (\lambda x\, U)V$, with $U, V \in \Lambda(c)$, then $|T| = (\lambda x|U|)|V|$;

Roughly speaking, one obtains $|T|$ by " forgetting " c in T.

Let $T \in \Lambda(c)$ and $t = |T|$; there is an obvious way of associating, with each redex R in T, a redex $r = |R|$ in t, called the *image* of R. Distinct redexes in T have distinct images in t ; this property, like the next ones, is immediate, by induction on T :

If $T, U \in \Lambda(c)$, and $|T| = t$, $|U| = u$, then $|T[U/x]| = t[u/x]$.
Let $T \in \Lambda(c)$, R be a redex in T, T' the term obtained by contracting R in T, $t = |T|$, $r = |R|$, and $t' = |T'|$; then t' is the term obtained by contracting the redex r in t.

Lemma 4.37. *Let $t \in \Lambda$ and $\mathscr{R}$ be a set of redexes of t. Then there exists a unique term $T \in \Lambda(c)$ such that $t = |T|$ and $\mathscr{R}$ is the set of all images of the redexes of T.*

This term T will be called the *representative* of $(t, \mathscr{R})$. So we have a one-to-one correspondence between $\Lambda(c)$ and the set of ordered pairs $(t, \mathscr{R})$ such that $t \in \Lambda$ and $\mathscr{R}$ is a set of redexes of t.

We define T by induction on t. If t is a variable, then $\mathscr{R} = \emptyset$; the only way of obtaining a term $T \in \Lambda(c)$ such that $|T|$ is a variable is to use rule 1 in the inductive definition of $\Lambda(c)$ given above. Thus $T = t$.
If $t = \lambda x\, u$, then $\mathscr{R}$ is a set of redexes of u. Only rule 2 can produce a term T such that $|T|$ starts with λ. So $T = \lambda x\, U$, and U needs to be the representative of $(u, \mathscr{R})$.
If $t = t_1 t_2$, let $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) be the subset of $\mathscr{R}$ consisting of those redexes which occur in t_1 (resp. t_2). T is obtained by rule 3 or rule 4, thus either $T = (c)T_1 T_2$, or $T = T_1 T_2$, T_i being the representative of $(t_i, \mathscr{R}_i)$.
If t itself is not a member of $\mathscr{R}$, then T cannot be obtained by rule 4 ; otherwise T would be a redex, and its image t would be in $\mathscr{R}$. Thus $T = (c)T_1 T_2$.
If t is a member of $\mathscr{R}$, then T needs to be a redex, so T cannot be obtained by rule 3, and therefore $T = T_1 T_2$.

Q.E.D.

Intuitively, the representative of $(t, \mathscr{R})$ is obtained by using the variable c to " destroy " those redexes of t which are not in $\mathscr{R}$, and to " neutralize " the applications in such a way that they cannot be transformed in redexes via β-reduction.

Let $t \in \Lambda$, $\mathscr{R}$ be a set of redexes of t, r_0 a redex of t, and t' the term obtained by contracting r_0 in t. We define a set $\mathscr{R}'$ of redexes of t' called *residues of $\mathscr{R}$ relative to* r_0 : let $\mathscr{S} = \mathscr{R} \cup \{r_0\}$, T be the representative of $(t, \mathscr{S})$, R_0 the redex of T of which r_0 is the image, and T' the term obtained by contracting R_0 in T ; so we have $t' = |T'|$. Then $\mathscr{R}'$ is, by definition, the set of images in t' of the redexes of T'.

Remark. The set of residues of $\mathscr{R}$ relative to r_0 does not only depend on t and t', but also on the redex r_0. For example, take $t = (\lambda x\, x)(\lambda x\, x)x$, $t' = (\lambda x\, x)x$, $r_0 = t$ and $r_1 = t'$: clearly, t' is obtained by contracting either the redex r_0 or the redex r_1 in t ; but $\{r_0\}$ has a residue relative to r_1, while it has no residue relative to r_0.

Let $t \in \Lambda$; a *reduction B* starting with t consists, by definition, of a finite sequence of terms $(t_0 = t), t_1, \ldots, t_n$, together with a sequence of redexes :

$r_0, r_1, \ldots, r_{n-1}$, such that each r_i is a redex of t_i, and t_{i+1} is obtained by contracting the redex r_i in the term $t_i (0 \leq i < n)$. The term t_n is called the *result* of the reduction B. We shall also say that the reduction B leads from t to t_n.
Now let $\mathscr{R}$ be a set of redexes of t. We define *the set of residues of $\mathscr{R}$ in t_n, relative to the reduction B*, by induction on n : we just gave the definition for the case $n = 1$; suppose $n > 1$, and let $\mathscr{R}_{n-1}$ be the set of residues of $\mathscr{R}$ in t_{n-1} relative to B ; then the residues of $\mathscr{R}$ in t_n relative to B are the residues of $\mathscr{R}_{n-1}$ in t_n relative to r_{n-1}.

Let $t \in \Lambda$ and $\mathscr{R}$ be a set of redexes of t. A *development* of $(t, \mathscr{R})$ is, by definition, a reduction D starting with t such that its redexes $r_0, r_1, \ldots, r_{n-1}$ satisfy the following conditions : $r_0 \in \mathscr{R}$, and r_i is a residue of $\mathscr{R}$ relative to the reduction $r_0, r_1, \ldots, r_{i-1} (0 < i < n)$. The development is said to be *complete* provided that $\mathscr{R}$ has no residue in t_n relative to the reduction D.

The main purpose of the next theorem is to prove that the lengths of the developments of a set of redexes are bounded.

Theorem 4.38 (Finite developments theorem). *Let $t \in \Lambda$, and $\mathscr{R}$ be a set of redexes of t. Then :*
i) There exists an integer N such that the length of every development of $(t, \mathscr{R})$ is $\leq N$.
ii) Every development of $(t, \mathscr{R})$ can be extended to a complete development.
iii) All complete developments of $(t, \mathscr{R})$ have the same result.

Let D be a development of $(t, \mathscr{R})$, $(t_0 = t), t_1, \ldots, t_n$ its sequence of terms, $r_0, r_1, \ldots, r_{n-1}$ its sequence of redexes, $\mathscr{R}_i$ the set of residues of $\mathscr{R}$ in t_i relative to the β-reduction $r_0, \ldots, r_{i-1} (1 \leq i \leq n)$, and $\mathscr{R}_0 = \mathscr{R}$.
We have $r_0 \in \mathscr{R}_0$, each t_i $(1 \leq i \leq n)$ is obtained by contracting the redex r_{i-1} in t_{i-1}, and $r_i \in \mathscr{R}_i$. Therefore $\mathscr{R}_i$ is the set of residues of $\mathscr{R}_{i-1}$ relative to r_{i-1}.
Let $T \in \Lambda(c)$ be the representative of $(t, \mathscr{R})$ and $T_i \in \Lambda(c)$ $(0 \leq i \leq n)$ the representative of $(t_i, \mathscr{R}_i)$ $(T_0 = T)$. Since $r_i \in \mathscr{R}_i$, r_i is the image of a redex R_i in T_i. Let $U_{i+1} \in \Lambda(c)$ $(0 \leq i \leq n-1)$ be the term obtained by contracting the redex R_i in T_i. Then $|U_{i+1}| = t_{i+1}$ (the term obtained by contracting the redex r_i in t_i). The set of all images of the redexes of U_{i+1} is therefore the set of residues of $\mathscr{R}_i$ in t_{i+1} relative to r_i (by definition of this set of residues), that is to say $\mathscr{R}_{i+1}$. Consequently, U_{i+1} is the representative of $(t_{i+1}, \mathscr{R}_{i+1})$, and therefore $U_{i+1} = T_{i+1}$. So we have proved that the sequence of terms $(T_0 = T), T_1, \ldots, T_n$ and the sequence of redexes $R_0, R_1, \ldots, R_{n-1}$ form a reduction $B(D)$ of T.
Clearly, the mapping $D \to B(D)$ is a one-to-one correspondence between the developments of $(t, \mathscr{R})$ and the reductions of its representative T. In particular, the length of any development of $(t, \mathscr{R})$ is that of some reduction of T. Thus it is $\leq N$, where N is the maximum of the lengths of the reductions of T $(T \in \Lambda(c)$

is strongly normalizable). Moreover, every reduction of T can be extended to a reduction which reaches the normal form of T. Because of the correspondence defined above, this implies that every development of $(t,\mathscr{R})$ can be extended to a development in which the last term contains no residue of $\mathscr{R}$, in other words to a complete development.
Finally,if $(t_0 = t), t_1, \dots t_n$ is a complete development of $(t,\mathscr{R})$, and if the corresponding reduction of T is $(T_0 = T), T_1, \dots, T_n$, then T_n is the normal form of T ; therefore, $t_n = |T_n|$ does not depend on the development.

Q.E.D.

3. The standardization theorem

Let t be a λ-term. Any redex of t which is not the head redex will be called an *internal redex* of t. An *internal reduction* (resp. *head reduction*) is, by definition, a sequence $t_1, \dots, t_n$ of λ-terms such that t_{i+1} is obtained by contracting an internal redex (resp. the head redex) of t_i.
A *standard reduction* consists of a head reduction followed by an internal one.

Theorem 4.39 (Standardization theorem)**.** *If $t\,\beta\,t'$, then there is a standard reduction leading from t to t'.*

Let t be a λ-term, $\mathscr{R}$ a set of redexes of t, and $N_{\mathscr{R}}$ the sum of the lengths of all complete developments of $(t,\mathscr{R})$. Consider the result u of any complete development of $(t,\mathscr{R})$; we shall write $t \xrightarrow{\mathscr{R}} u$. The finite developments theorem ensures that $N_{\mathscr{R}}$ and u are uniquely determined (if $\mathscr{R} = \emptyset$, then $N_{\mathscr{R}} = 0$ and $u \equiv t$).
We shall say that the set $\mathscr{R}$ is *internal* if all the members of $\mathscr{R}$ are internal redexes of t.

Lemma 4.40. *Let r be an internal redex of t, and t' the term obtained by contracting r. If t' has a head redex, then this is the only residue, relative to r, of the head redex of t.*

The term t cannot be a head normal form, otherwise t' would also be one. So we have $t \equiv \lambda x_1 \dots \lambda x_m (\lambda y\, u) v t_1 \dots t_n$. The result of the contraction of the redex r is the term : $t' \equiv \lambda x_1 \dots \lambda x_m (\lambda y\, u') v' t'_1 \dots t'_n$, and the head redex of t' can be seen to be the only residue (relative to r) of the head redex of t.

Q.E.D.

Corollary 4.41. *Let $\mathscr{R}$ be an internal set of redexes of t. Then every development of $(t,\mathscr{R})$ is an internal reduction of t ; if t' is the result of a development of $(t,\mathscr{R})$, then the head redex of t' (if there is one) is the only residue of the head redex of t.*

By lemma 4.40, every residue of an internal redex of t relative to an internal redex of t is an internal redex ; this proves the first part of the corollary. For the second one, it is enough to apply repeatedly the same lemma.

Q.E.D.

We shall call *head reduced image* of a term t any term obtained from t by head reduction.

Theorem 4.42. *Consider a sequence $t_0, t_1, \ldots, t_n$ of λ-terms, and, for each i, a set $\mathscr{R}_i$ of redexes of t_i, such that : $t_0 \xrightarrow{\mathscr{R}_0} t_1 \xrightarrow{\mathscr{R}_1} t_2 \cdots t_{n-1} \xrightarrow{\mathscr{R}_{n-1}} t_n$. Then there exist a sequence $u_0, u_1, \ldots, u_n$ of terms, and, for each i, a set $\mathscr{S}_i$ of internal redexes of u_i, such that : $u_0 \xrightarrow{\mathscr{S}_0} u_1 \xrightarrow{\mathscr{S}_1} u_2 \cdots u_{n-1} \xrightarrow{\mathscr{S}_{n-1}} u_n$, u_0 is a head reduced image of t_0, and $u_n \equiv t_n$.*

The proof is by induction on the n-tuple $(N_{\mathscr{R}_{n-1}}, \ldots, N_{\mathscr{R}_0})$, with the lexicographical order on the n-tuples of integers. The result is obvious if all the $\mathscr{R}_i$'s are internal. Otherwise, consider the least integer k such that t_k has a head redex, which is in $\mathscr{R}_k$.

If $k = 0$, then t_0 has a head redex ρ, which is in $\mathscr{R}_0$. Let t'_0 be the term obtained by contracting the redex ρ, and $\mathscr{R}'_0$ the set of residues of $\mathscr{R}_0$ relative to ρ. We have $t_0 \xrightarrow{\mathscr{R}_0} t_1$, and therefore $t'_0 \xrightarrow{\mathscr{R}'_0} t_1$. Moreover, it is clear that $N_{\mathscr{R}'_0} < N_{\mathscr{R}_0}$. Thus we obtain the expected conclusion by applying the induction hypothesis to the sequence : $t'_0 \xrightarrow{\mathscr{R}'_0} t_1 \xrightarrow{\mathscr{R}_1} t_2 \cdots t_{n-1} \xrightarrow{\mathscr{R}_{n-1}} t_n$.

Now suppose $k > 0$, and let ρ_k be the head redex of t_k, t'_k the term obtained by contracting that redex, and $\mathscr{R}'_k$ the set of residues of $\mathscr{R}_k$ relative to ρ_k. Since $\rho_k \in \mathscr{R}_k$, and $t_k \xrightarrow{\mathscr{R}_k} t_{k+1}$, we clearly have $N_{\mathscr{R}'_k} < N_{\mathscr{R}_k}$ and $t'_k \xrightarrow{\mathscr{R}'_k} t'_{k+1}$.

On the other hand, $\mathscr{R}_{k-1}$ is an internal set of redexes of t_{k-1}, so by the previous corollary there is an internal reduction which leads from t_{k-1} to t_k. Thus t_{k-1} has a head redex, which we denote by ρ_{k-1}. Now let $\mathscr{R}'_{k-1} = \mathscr{R}_{k-1} \cup \{\rho_{k-1}\}$; the result of a complete development of t_{k-1} relative to $\mathscr{R}'_{k-1}$ can be obtained by taking the result t_k of a complete development of t_{k-1} relative to $\mathscr{R}_{k-1}$, then the result of a complete development of t_k relative to the set of residues of ρ_{k-1} relative to $\mathscr{R}_{k-1}$. But there is only one such residue, namely the head redex of t_k. So the result is t'_k, and therefore we have :

$t_0 \xrightarrow{\mathscr{R}_0} t_1 \cdots t_{k-1} \xrightarrow{\mathscr{R}'_{k-1}} t'_k \xrightarrow{\mathscr{R}'_k} t_{k+1} \cdots t_{n-1} \xrightarrow{\mathscr{R}_{n-1}} t_n$.

This yields the conclusion, since the induction hypothesis applies ; indeed, we have :

$$(N_{\mathscr{R}_{n-1}}, \ldots, N_{\mathscr{R}_{k+1}}, N_{\mathscr{R}'_k}, N_{\mathscr{R}'_{k-1}}, \ldots, N_{\mathscr{R}_0}) < (N_{\mathscr{R}_{n-1}}, \ldots, N_{\mathscr{R}_{k+1}}, N_{\mathscr{R}_k}, N_{\mathscr{R}_{k-1}}, \ldots, N_{\mathscr{R}_0}),$$

since $N_{\mathscr{R}'_k} < N_{\mathscr{R}_k}$.

Q.E.D.

Now we are able to complete the proof of the standardization theorem : consider a reduction $(t_0 = t), t_1, \ldots, t_{n-1}, (t_n = t')$ which leads from t to t'. One obtains t_{i+1} from t_i by contracting a redex r_i of t_i, that is by a complete development of the set $\mathscr{R}_i = \{r_i\}$. Thus, by theorem 4.42, there exists a sequence $u_0 \xrightarrow{\mathscr{S}_0} u_1 \xrightarrow{\mathscr{S}_1} u_2 \cdots u_{n-1} \xrightarrow{\mathscr{S}_{n-1}} u_n$ such that u_0 is a head reduced image of t_0, $u_n \equiv t_n$ and $\mathscr{S}_i$ is an internal set of redexes of u_i. Hence there is an internal reduction which leads from u_0 to t_n and therefore, there is a standard reduction which leads from t_0 to t_n.

Q.E.D.

As a consequence, we obtain an alternative proof of part of theorem 4.9 :

Corollary 4.43. *A λ-term is β-equivalent to a head normal form if and only if its head reduction is finite.*

If t is β-equivalent to a head normal form, then, by the Church-Rosser theorem, we have $t\,\beta\,u$, where u is a head normal form. By the standardization theorem, there exists a head reduced image of t, say t', such that some internal reduction leads from t' to u. If t' has a head redex, then also u has a head redex (an internal reduction does not destroy the head redex) : this is a contradiction. Thus the head reduction of t ends with t'.
The converse is obvious.

Q.E.D.

Corollary 4.44. *If $t \simeq_\beta \lambda x\, u$, then there exists a head reduced image of t of the form $\lambda x\, v$.*

Indeed, by the Church-Rosser theorem, we have $t\,\beta\,\lambda x\, u'$. By the standardization theorem, there exists a head reduced image t' of t, such that some internal reduction leads from t' to $\lambda x\, u'$. Now an internal reduction cannot introduce an occurrence of λ in a head position. Therefore t' starts with λ.

Q.E.D.

A term t is said to be *of order 0* if no term starting with λ is β-equivalent to t. Therefore, corollary 4.44 can be restated this way : a term t is of order 0 if and only if no head reduced image of t starts with λ.

Remark.

The standardization theorem is very easy to prove *with the hypothesis that the head reduction of t is finite* or, more generally, that there exists an upper bound for the lengths of those head reductions of t which lead to a term which can be reduced to t'.
Indeed, in such a case, it is enough to consider, among all the reductions which lead from t to t', any of those starting with a head reduction of maximal length, let us say

$(t_0 = t), t_1, \dots, t_k$. The proof of the theorem will be completed if we show that all the reductions which lead from t_k to t' are internal.
This is obvious if t_k is a head normal form.
Now suppose that $t_k = \lambda x_1 \dots \lambda x_m (\lambda x\, u) v v_1 \dots v_n$ and consider a reduction, leading from t_k to t', which is not internal ; it cannot start with a head reduction (otherwise we would have a reduction, leading from t to t', starting with a head reduction of length $> k$). Consequently, it starts with an internal reduction, which leads from $t_k = \lambda x_1 \dots \lambda x_m (\lambda x\, u) v v_1 \dots v_n$ to $\lambda x_1 \dots \lambda x_m (\lambda x\, u') v' v'_1 \dots v'_n$ (with $u\, \beta\, u'$, $v\, \beta\, v'$, $v_i\, \beta\, v'_i$). This internal reduction is followed by at least one step of head reduction, which leads to $\lambda x_1 \dots \lambda x_m\, u'[v'/x] v'_1 \dots v'_n$. Now this term can be obtained from t_k by the following path : first one step of head reduction, which gives $\lambda x_1 \dots \lambda x_m\, u[v/x] v_1 \dots v_n$; then a β-reduction applied to $u, v, v_1, \dots, v_n$, which leads to $\lambda x_1 \dots \lambda x_m\, u'[v'/x] v'_1 \dots v'_n$. Since $\lambda x_1 \dots \lambda x_m\, u'[v'/x] v'_1 \dots v'_n\, \beta\, t'$, what we have obtained is a reduction which leads from t_k to t' and starts with a head reduction : this is impossible.

Q.E.D.

The standardization theorem is usually stated in a (slightly) stronger form.
First, we define the *rank* of a redex ρ in a λ-term t, by induction on the length of t.
If $t = \lambda x\, u$, then ρ is a redex of u ; the rank of ρ in t is the same as in u.
If $t = (u)v$ then either $\rho = t$, or ρ is a redex of u, or ρ is a redex of v ;
if $\rho = t$, then the rank of ρ in t is 0 ;
if ρ is in u, then its rank in t is the same as in u ;
if ρ is in v, then its rank in t is its rank in v plus the number of redexes in u.
Remark. The rank describes the order of redexes in t, from left to right (the position of a redex is given by the position of its leading λ).

Consider a reduction $t_0, \dots, t_k$ and let n_i be the rank, in t_i, of the redex ρ_i which is reduced at this step. The reduction will be called *strongly standard* if we have $n_0 \le n_1 \le \dots \le n_{k-1}$.

Remark. A strongly standard reduction is clearly a standard one. Indeed, if there is a head redex, then its rank is 0.

Theorem 4.45 (Standardization theorem, 2nd form)**.** *If $t\, \beta\, t'$, then there is a strongly standard reduction leading from t to t'.*

The proof is by induction on the length of t'. By theorem 4.39, we consider a standard reduction from t to t'. This standard reduction begins with a head reduction from t to u, which is followed by an internal reduction from u to t'.
By proposition 2.2, we have $u = \lambda x_1 \dots \lambda x_k (\rho) u_1 \dots u_n$ where ρ is a redex or a variable ; therefore, we have :
$t' = \lambda x_1 \dots \lambda x_k (\rho') u'_1 \dots u'_n$, with $\rho\, \beta\, \rho'$, $u_1\, \beta\, u'_1, \dots, u_n\, \beta\, u'_n$.
Then, there are two possibilities :

i) If $\rho = (\lambda x\, v)\, w$ is a redex, then $\rho' = (\lambda x\, v')\, w'$ (because the reduction from u to t' is internal) and we have $v\, \beta\, v'$, $w\, \beta\, w'$.
By induction hypothesis, there are strongly standard reductions leading from v to v', w to w', u_1 to u'_1, ..., u_n to u'_n. By putting these reductions in sequence, we get a strongly standard reduction from u to t' ; and therefore, also a strongly standard reduction from t to t'.

ii) If ρ is a variable, then $\rho = \rho'$ and we have $u_1\, \beta\, u'_1, \ldots, u_n\, \beta\, u'_n$. The end of the proof is the same as in case (i).

Q.E.D.

References for chapter 4

[Bar83], [Bar84], [Cop78], [Hin86], [Mit79], [Pot80].
(The references are in the bibliography at the end of the book).
The proof given above of the finite developments theorem was communicated to me by M. Parigot.

Chapter 5

The Böhm theorem

Let $\alpha_n = \lambda z_1 \ldots \lambda z_n \lambda z(z) z_1 \ldots z_n$ for every $n \geq 0$ (in particular, $\alpha_0 = \lambda z z$) ; α_n is the " applicator " of order n (it applies an n-ary function to its arguments).
Propositions 5.1 and 5.8 show that, in some weak sense, applicators behaves like variables with respect to normal terms.

Proposition 5.1.
Let t be a normal λ-term and $x_1, \ldots, x_k$ variables ; then $t[\alpha_{n_1}/x_1, \ldots, \alpha_{n_k}/x_k]$ is normalizable provided that $n_1, \ldots, n_k \in \mathbb{N}$ are large enough.

The proof is by induction on the length of t. If t is a variable, then the result is clear, since α_n is normal.
If $t = \lambda y\, u$, then $t[\alpha_{n_1}/x_1, \ldots, \alpha_{n_k}/x_k] = \lambda y\, u[\alpha_{n_1}/x_1, \ldots, \alpha_{n_k}/x_k]$; by induction hypothesis, $u[\alpha_{n_1}/x_1, \ldots, \alpha_{n_k}/x_k]$ is normalizable provided that $n_1, \ldots, n_k$ are large enough, thus so is $t[\alpha_{n_1}/x_1, \ldots, \alpha_{n_k}/x_k]$.
Now we can assume that t does not start with λ. Since t is normal, by proposition 2.2, we have $t = (y) t_1 \ldots t_p$, where y is a variable. Now t_i is shorter than t, so $t_i[\alpha_{n_1}/x_1, \ldots, \alpha_{n_k}/x_k]$ is normalizable provided that $n_1, \ldots, n_k$ are large enough. Let u_i be its normal form.
If $y \notin \{x_1, \ldots x_k\}$, then $t[\alpha_{n_1}/x_1, \ldots, \alpha_{n_k}/x_k] \simeq_\beta (y) u_1 \ldots u_p$, which is a normal form.
If $y \in \{x_1, \ldots x_k\}$, say $y = x_1$, then :

$$t[\alpha_{n_1}/x_1, \ldots, \alpha_{n_k}/x_k] \simeq_\beta (\alpha_{n_1}) u_1 \ldots u_p$$
$$\simeq_\beta (\lambda x_1 \ldots \lambda x_{n_1} \lambda x(x) x_1 \ldots x_{n_1}) u_1 \ldots u_p \ ;$$

if $n_1 \geq p$, this term becomes, after β-conversion :

$$\lambda x_{p+1} \ldots \lambda x_{n_1} \lambda x(x) u_1 \ldots u_p x_{p+1} \ldots x_{n_1}$$

which is in normal form.

Q.E.D.

Remark. In proposition 5.1, the condition " provided that $n_1, \ldots, n_k$ are large enough " is indispensable : if $\delta = \lambda y(y) y$ and $t = (x)\delta\delta$, then $t[\alpha_0/x]$ is not normalizable.

The main result in this chapter is the following theorem, due to C. Böhm :

Theorem 5.2. *Let t, t' be two closed normal λ-terms, which are not $\beta\eta$-equivalent ; then there exist closed λ-terms $t_1, \dots, t_k$ such that :*

$$(t)t_1 \dots t_k \simeq_\beta \mathbf{0}, \text{ and } (t')t_1 \dots t_k \simeq_\beta \mathbf{1}.$$

Recall that, by definition, $\mathbf{0} = \lambda x \lambda y\, y$ and $\mathbf{1} = \lambda x \lambda y\, x$.

Corollary 5.3. *Let t, t' be two closed normal λ-terms, which are not $\beta\eta$-equivalent, and v, v' two arbitrary λ-terms. Then there exist λ-terms $t_1, \dots, t_k$ such that $(t)t_1 \dots t_k \simeq_\beta v$ and $(t')t_1 \dots t_k \simeq_\beta v'$.*

Indeed, by theorem 5.2, we have $(t)t_1 \dots t_k \simeq_\beta \mathbf{0}$ and $(t')t_1 \dots t_k \simeq_\beta \mathbf{1}$;
thus $(t)t_1 \dots t_k v' v \simeq_\beta v$ and $(t')t_1 \dots t_k v' v \simeq_\beta v'$.
Q.E.D.

The following corollary shows that the $\beta\eta$-equivalence is maximal, among the λ-compatible equivalence relations on Λ which contain the β-equivalence.

Corollary 5.4. *Let $\simeq$ be an equivalence relation on Λ, containing $\simeq_\beta$, such that : $t \simeq t' \Rightarrow (t)u \simeq (t')u$ and $\lambda x\, t \simeq \lambda x\, t'$, for every term t, t', u and every variable x. If there exist two normalizable non $\beta\eta$-equivalent terms t_0, t'_0 such that $t_0 \simeq t'_0$, then $v \simeq v'$ for all terms v, v'.*

Indeed, let $x_1, \dots, x_k$ be the free variables of t_0, t'_0, let $t = \lambda x_1 \dots \lambda x_k t_0$ and $t' = \lambda x_1 \dots \lambda x_k t'_0$. Then $t \simeq t'$ and t is not $\beta\eta$-equivalent to t'. Thus, by corollary 5.3, we have $(t)t_1 \dots t_k \simeq_\beta v$ and $(t')t_1 \dots t_k \simeq_\beta v'$; therefore $v \simeq v'$.
Q.E.D.

We will call *Böhm transformation* any function from Λ into Λ, obtained by composing " elementary " functions of the form : $t \mapsto (t)u_0$ or $t \mapsto t[u_0/x]$ (where u_0 and x are given term and variable).

The function $t \mapsto (t)u_0$, from Λ to Λ, will be denoted by B_{u_0}.
The function $t \mapsto t[u_0/x]$ will be denoted by $B_{u_0,x}$.

Note that every Böhm transformation F is compatible with both β- and $\beta\eta$-equivalence : $t \simeq_\beta t' \Rightarrow F(t) \simeq_\beta F(t')$ and $t \simeq_{\beta\eta} t' \Rightarrow F(t) \simeq_{\beta\eta} F(t')$.

Lemma 5.5. *For every Böhm transformation F, there exist terms $t_1, \dots, t_k$ such that $F(t) = (t)t_1 \dots t_k$ for every closed term t.*

The proof is immediate, by induction on the number of elementary functions of which F is the composite. Indeed, if $F(t)$ is in the indicated form, then so are $(F(t))u_0$ and $(F(t))[u_0/x]$: the former is $(t)t_1 \dots t_k u_0$, and the latter $(t)t'_1 \dots t'_k$ where $t'_i = t_i[u_0/x]$, since t is closed.
Q.E.D.

Theorem 5.6. *Let $x_1,\dots,x_k$ be distinct variables and t, t' be two normal non-$\beta\eta$-equivalent terms. Then, for all distinct integers $n_1,\dots,n_k$, provided that they are large enough, there exists a Böhm transformation F such that :*
$F(t[\alpha_{n_1}/x_1,\dots,\alpha_{n_k}/x_k]) \simeq_\beta \mathbf{0}$ *and* $F(t'[\alpha_{n_1}/x_1,\dots,\alpha_{n_k}/x_k]) \simeq_\beta \mathbf{1}$.

Theorem 5.2 is an immediate consequence of theorem 5.6 : indeed, if t is a closed term, and F a Böhm transformation, then, by lemma 5.5, we have $F(t) = (t)t_1\dots t_n$, where $t_1,\dots,t_n$ depend only on F. By applying theorem 5.6, we therefore obtain $(t)t_1\dots t_n \simeq_\beta \mathbf{0}$, and $(t')t_1\dots t_n \simeq_\beta \mathbf{1}$. We may suppose that $t_1,\dots,t_n$ are closed terms (in case they have free variables $x_1,\dots,x_p$, simply replace t_i by $t_i[a_1/x_1,\dots,a_p/x_p]$, where $a_1,\dots,a_p$ are fixed closed terms, for instance $\mathbf{0}$).
We also deduce :

Corollary 5.7. *Let $\simeq$ be an equivalence relation on Λ, containing $\simeq_\beta$, such that $t \simeq t' \Rightarrow (t)u \simeq (t')u$ and $t[u/x] \simeq t'[u/x]$ for every term t, t', u and every variable x. If there exist two normalizable non-$\beta\eta$-equivalent terms t_0, t'_0 such that $t_0 \simeq t'_0$, then $t \simeq t'$ for all terms t, t'.*

By theorem 5.6 (where we take $k = 0$), there exists a Böhm transformation F such that $F(t_0) \simeq_\beta \mathbf{0}$, and $F(t'_0) \simeq_\beta \mathbf{1}$. Thus it follows from the assumptions about relation $\simeq$ that $t_0 \simeq t'_0 \Rightarrow F(t_0) \simeq F(t'_0)$. Therefore $\mathbf{0} \simeq \mathbf{1}$, and hence $(\mathbf{0})t't \simeq (\mathbf{1})t't$, that is $t \simeq t'$.
Q.E.D.

Proposition 5.8. *Let $x_1,\dots,x_k$ be distinct variables and t, t' be two normal non-$\beta\eta$-equivalent terms. Then, for all distinct integers $n_1,\dots,n_k$, provided that they are large enough, the terms :*
$t[\alpha_{n_1}/x_1,\dots,\alpha_{n_k}/x_k]$ *and* $t'[\alpha_{n_1}/x_1,\dots,\alpha_{n_k}/x_k]$ *are not $\beta\eta$-equivalent.*

Immediate from theorem 5.6.
Q.E.D.

Corollary 5.9. *Let t, t' be two normalizable terms :*
i) if $t[\alpha_n/x] \simeq_{\beta\eta} t'[\alpha_n/x]$ for infinitely many integers n, then $t \simeq_{\beta\eta} t'$;
ii) if $(t)\alpha_n \simeq_{\beta\eta} (t')\alpha_n$ for infinitely many integers n, then $t \simeq_{\beta\eta} t'$.

Proof of (i) : it is the particular case $k = 1$ of proposition 5.8.
Proof of (ii) : let x be a variable with no occurrence in t, t' ; by applying (i) to the terms $(t)x$ and $(t')x$, we obtain : $(t)x \simeq_{\beta\eta} (t')x$, thus $\lambda x(t)x \simeq_{\beta\eta} \lambda x(t')x$, and therefore $t \simeq_{\beta\eta} t'$.
Q.E.D.

The following result will be used to prove theorem 5.6 :

Lemma 5.10. *Let t, u be two λ-terms. If one of the following conditions hold, then there exists a Böhm transformation F such that :*

$$F(t) \simeq_\beta \mathbf{0} \text{ and } F(u) \simeq_\beta \mathbf{1}.$$

i) $t = (x)t_1 \dots t_p$, $u = (y)u_1 \dots u_q$, where $x \neq y$ or $p \neq q$;
ii) $t = \lambda x_1 \dots \lambda x_m \lambda x(x)t_1 \dots t_p$, $u = \lambda x_1 \dots \lambda x_n \lambda x(x)u_1 \dots u_q$ where $m \neq n$ or $p \neq q$.

Proof of (i).
Case 1 : $x \neq y$; let $\sigma_0 = \lambda z_1 \dots \lambda z_p \mathbf{0}$, $\sigma_1 = \lambda z_1 \dots \lambda z_q \mathbf{1}$. By β-reduction, we obtain immediately $B_{\sigma_0,x}B_{\sigma_1,y}(t) \simeq_\beta \mathbf{0}$ and $B_{\sigma_0,x}B_{\sigma_1,y}(u) \simeq_\beta \mathbf{1}$. Thus $B_{\sigma_0,x}B_{\sigma_1,y}$ is the desired Böhm transformation.
Case 2 : $x = y$ and $p \neq q$, say $p < q$; then we have :
$B_{\alpha_q,x}(t) = (\alpha_q)t'_1 \dots t'_p$ and $B_{\alpha_q,x}(u) = (\alpha_q)u'_1 \dots u'_q$ (where $\tau' = \tau[\alpha_q/x]$ for every term τ). By β-reduction, we obtain :
$B_{\alpha_q,x}(t) \simeq_\beta \lambda z_{p+1} \dots \lambda z_q \lambda z(z)t'_1 \dots t'_p z_{p+1} \dots z_q$ and
$B_{\alpha_q,x}(u) \simeq_\beta \lambda z(z)u'_1 \dots u'_q$.
Then the result follows from case 1 of part (ii), treated below.

Proof of (ii).
Case 1 : $m \neq n$, say $m < n$; take distinct variables $z_1, \dots, z_n, z$ not occurring in t, u. Let $B = B_z B_{z_n} \dots B_{z_1}$. Then, by β-reduction, we have :
$B(t) \simeq_\beta (z_{m+1})t'_1 \dots t'_p z_{m+2} \dots z_n z$, and $B(u) \simeq_\beta (z)u''_1 \dots u''_q$ (where τ' is the term $\tau[z_1/x_1, \dots, z_m/x_m, z_{m+1}/x]$, and τ'' is the term $\tau[z_1/x_1, \dots, z_n/x_n, z/x]$).
Since $z_{m+1} \neq z$, the result follows from case 1 of part (i) above.
Case 2 : $m = n$ and $p \neq q$; let $B = B_x B_{x_m} \dots B_{x_1}$. We have :
$B(t) = (x)t_1 \dots t_p$ and $B(u) = (x)u_1 \dots u_q$.
Since $p \neq q$, the result follows from case 2 of (i).
Q.E.D.

The length $lg(t)$ of a term t is inductively defined as follows (actually, it is the length of the expression obtained from t by erasing all the parentheses) :
if t is a variable, then $lg(t) = 1$;
$lg((t)u) = lg(t) + lg(u)$; $lg(\lambda x\, t) = lg(t) + 2$.

We now prove theorem 5.6 by induction on $lg(t) + lg(t')$.

Take a variable $y \neq x_1, \dots, x_k$, with no occurrence in t, t', and let w, w' be the terms obtained from $(t)y$, $(t')y$ by normalization. If $w \simeq_{\beta\eta} w'$, then $\lambda y\, w \simeq_{\beta\eta} \lambda y\, w'$, thus $\lambda y(t)y \simeq_{\beta\eta} \lambda y(t')y$ and hence $t \simeq_{\beta\eta} t'$, which contradicts the hypothesis. Thus w and w' are not $\beta\eta$-equivalent.

If both t, t' start with λ, say $t = \lambda x\, u$, $t' = \lambda x' u'$, then :
$w = u[y/x]$, $w' = u'[y/x']$ and $lg(w) + lg(w') = lg(t) + lg(t') - 4$.
If t starts with λ, say $t = \lambda x\, u$, while t' does not, then either $t' = (v')u'$ or t' is a variable. Thus, $w = u[y/x]$, $w' = (t')y$ and $lg(w) + lg(w') = lg(t) + lg(t') - 1$.

Therefore, in both cases, we can apply the induction hypothesis to w, w'.

Thus, given large enough distinct integers $n_1, \dots, n_k$, there exists a Böhm transformation F such that :

$F(w[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k]) \simeq_\beta \mathbf{0}$ and $F(w'[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k]) \simeq_\beta \mathbf{1}$.

Now we have :

$w[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k] \simeq_\beta (t[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k])y$ and
$w'[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k] \simeq_\beta (t'[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k])y$.

It follows that Böhm transformation FB_y have the required properties :
$FB_y(t[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k]) \simeq_\beta \mathbf{0}$ and $FB_y(t'[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k])) \simeq_\beta \mathbf{1}$.

Now we may suppose that none of t, t' start with λ (note that this happens at the first step of the induction, since we then have $lg(t) = lg(t') = 1$, so t and t' are variables).

Since t, t' are normal, we have $t = (x)t_1 \dots t_p$ and $t' = (y)t'_1 \dots t'_q$, where x, y are variables, and $t_1, \dots, t_p, t'_1, \dots, t'_q$ are normal terms.

We now fix *distinct* integers $n_1, \dots, n_k$ and distinct variables $x_1, \dots, x_k$. We will use the notation $\tau[]$ as an abbreviation for $\tau[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k]$, for every λ-term τ.

Now, there are the following three possibilities :

1. Suppose that $x, y \notin \{x_1, \dots, x_k\}$. Then we have :

$$t[] = (x)t_1[] \dots t_p[] \text{ and } t'[] = (y)t'_1[] \dots t'_q[].$$

If $x \neq y$ or $p \neq q$, then, by lemma 5.10(i), there exists a Böhm transformation F such that $F(t[]) \simeq_\beta \mathbf{0}$ and $F(t'[]) \simeq_\beta \mathbf{1}$: this is the expected result.

In case $x = y$ and $p = q$, take any integer $n > n_1, \dots, n_k, p$. Then :

$B_{\alpha_n,x}(t[]) = (\alpha_n)t_1[[]] \dots t_p[[]]$ and $B_{\alpha_n,x}(t'[]) = (\alpha_n)t'_1[[]] \dots t'_p[[]]$

(the notation $\tau[[]]$ stands for $\tau[\alpha_{n_1}/x_1, \dots, \alpha_{n_k}/x_k, \alpha_n/x]$, for every term τ).

Since $\alpha_n = \lambda z_1 \dots \lambda z_n \lambda z(z)z_1 \dots z_n$, we therefore obtain, by β-reduction :
$B_{\alpha_n,x}(t[]) \simeq_\beta \lambda z_{p+1} \dots \lambda z_n \lambda z(z)t_1[[]] \dots t_p[[]]z_{p+1} \dots z_n$ and
$B_{\alpha_n,x}(t'[]) \simeq_\beta \lambda z_{p+1} \dots \lambda z_n \lambda z(z)t'_1[[]] \dots t'_q[[]]z_{p+1} \dots z_n$.

Note that the terms $t_i[[]]$ and $t'_i[[]]$ contain none of the variables $z, z_1, \dots, z_n$.

We have :
$B_z B_{z_n} \dots B_{z_{p+1}} B_{\alpha_n,x}(t[]) \simeq_\beta (z)t_1[[]] \dots t_p[[]]z_{p+1} \dots z_n$ and
$B_z B_{z_n} \dots B_{z_{p+1}} B_{\alpha_n,x}(t'[]) \simeq_\beta (z)t'_1[[]] \dots t'_p[[]]z_{p+1} \dots z_n$.

Now, by hypothesis, $t = (x)t_1 \dots t_p$ and $t' = (x)t'_1 \dots t'_p$, and t and t' are not $\beta\eta$-equivalent. Thus, for some $i (1 \leq i \leq p)$, t_i and t'_i are not $\beta\eta$-equivalent.

Let $\pi_i = \lambda x_1 \dots \lambda x_n\, x_i$ and $B = B_z B_{z_n} \dots B_{z_{p+1}} B_{\alpha_n,x}$. Since the variable z occurs neither in $t_i[[]]$ nor in $t'_i[[]]$, we have :

$B_{\pi_i,z} B(t[]) \simeq_\beta t_i[[]]$; $B_{\pi_i,z} B(t'[]) \simeq_\beta t'_i[[]]$.

Now $lg(t_i) + lg(t'_i) < lg(t) + lg(t')$. Thus, by induction hypothesis, provided that $n_1, \dots, n_k, n$ are large enough distinct integers, there exists a Böhm transformation, say F, such that $F(t_i[[]]) \simeq_\beta \mathbf{0}$ and $F(t'_i[[]]) \simeq_\beta \mathbf{1}$. Therefore, $FB_{\pi_i,z}B(t[]) \simeq_\beta \mathbf{0}$ and $FB_{\pi_i,z}B(t'[]) \simeq_\beta 1$, which is the expected result.

2. Now suppose that $x \in \{x_1,\dots,x_k\}$, for instance $x = x_1$, while $y \notin \{x_1,\dots,x_k\}$. Then we have $t[] = (\alpha_{n_1})t_1[]\dots t_p[]$ and $t'[] = (y)t'_1[]\dots t'_q[]$. For every $n_1 \geq p$, we have, by β-reduction :

$$t[] \simeq_\beta \lambda z_{p+1}\dots\lambda z_{n_1}\lambda z(z)t_1[]\dots t_p[]z_{p+1}\dots z_{n_1}.$$

Therefore, if we let $B = B_z B_{z_{n_1}}\dots B_{z_{p+1}}$, we have :

$$B(t[]) \simeq_\beta (z)t_1[]\dots t_p[]z_{p+1}\dots z_{n_1} \text{ and}$$
$$B(t'[]) = (y)t'_1[]\dots t'_q[]z_{p+1}\dots z_{n_1}z.$$

Since y and z are distinct variables, lemma 5.10(i) provides a Böhm transformation F such that $FB(t[]) \simeq_\beta \mathbf{0}$ and $FB(t'[]) \simeq_\beta \mathbf{1}$, which is the expected result.

3. Finally, suppose that $x, y \in \{x_1,\dots,x_k\}$.
If $x \neq y$, say, for instance, $x = x_1$, $y = x_2$, then :

$$t[] = (\alpha_{n_1})t_1[]\dots t_p[] \text{ and } t'[] = (\alpha_{n_2})t'_1[]\dots t'_q[].$$

For all $n_1 \geq p$ and $n_2 \geq q$, we have, by β-reduction :

$$t[] \simeq_\beta \lambda z_{p+1}\dots\lambda z_{n_1}\lambda z(z)t_1[]\dots t_p[]z_{p+1}\dots z_{n_1} \text{ and}$$
$$t'[] \simeq_\beta \lambda z_{q+1}\dots\lambda z_{n_2}\lambda z(z)t'_1[]\dots t'_q[]z_{q+1}\dots z_{n_2}.$$

Since $n_1 \neq n_2$ (by hypothesis), the result follows from lemma 5.10(ii).

If $x = y$, say, for instance, $x = y = x_1$, then :

$$t[] = (\alpha_{n_1})t_1[]\dots t_p[] \text{ and } t'[] = (\alpha_{n_1})t'_1[]\dots t'_q[].$$

For every $n_1 \geq p, q$, we have, by β-reduction :

$$t[] \simeq_\beta \lambda z_{p+1}\dots\lambda z_{n_1}\lambda z(z)t_1[]\dots t_p[]z_{p+1}\dots z_{n_1} \text{ and}$$
$$t'[] \simeq_\beta \lambda z_{q+1}\dots\lambda z_{n_1}\lambda z(z)t'_1[]\dots t'_q[]z_{q+1}\dots z_{n_1}.$$

If $p \neq q$, then the results follows from lemma 5.10(ii) $(n_1 - p \neq n_1 - q)$.

If $p = q$, then :

$$t[] \simeq_\beta \lambda z_{p+1}\dots\lambda z_{n_1}\lambda z(z)t_1[]\dots t_p[]z_{p+1}\dots z_{n_1} \text{ and}$$
$$t'[] \simeq_\beta \lambda z_{p+1}\dots\lambda z_{n_1}\lambda z(z)t'_1[]\dots t'_p[]z_{p+1}\dots z_{n_1}.$$

Now, by hypothesis, $t = (x)t_1\dots t_p$ and $t' = (x)t'_1\dots t'_p$, and t and t' are not $\beta\eta$-equivalent. Thus, for some $i\,(1 \leq i \leq p)$, t_i and t'_i are not $\beta\eta$-equivalent.
Let $\pi_i = \lambda x_1\dots\lambda x_{n_1}x_i$ and $B = B_z B_{z_{n_1}}\dots B_{z_{p+1}}$. Since the variables z, z_j occur neither in $t_i[]$ nor in $t'_i[]$, we have :

$$B_{\pi_i,z}B(t[]) \simeq_\beta t_i[]\,;\, B_{\pi_i,z}B(t'[]) \simeq_\beta t'_i[].$$

Now $lg(t_i)+lg(t'_i) < lg(t)+lg(t')$. Thus, by induction hypothesis, provided that $n_1,\dots,n_k$ are large enough distinct integers, there exists a Böhm transformation, say F, such that $F(t_i[]) \simeq_\beta \mathbf{0}$ and $F(t'_i[]) \simeq_\beta \mathbf{1}$. Therefore, $FB_{\pi_i,z}B(t[]) \simeq_\beta \mathbf{0}$ and $FB_{\pi_i,z}B(t'[]) \simeq_\beta \mathbf{1}$.
This completes the proof.

Q.E.D.

References for chapter 5

[Bar84], [Boh68].
(The references are in the bibliography at the end of the book).

Chapter 6

Combinatory logic

1. Combinatory algebras

In this chapter, we shall deal with theories in the first order predicate calculus with equality and we assume that the reader has some familiarity with elementary model theory. We consider a language $\mathscr{L}_0$ consisting of one binary function symbol Ap (for " application "). Given terms f, t, u, v, ..., the term $Ap(f,t)$ will be written $(f)t$ or ft ; the terms $((f)t)u$, $(((f)t)u)v$, ... will be respectively written $(f)tu$, $(f)tuv$, ... or even ftu, $ftuv$, ...
A model for this language (that is a non-empty set A, equipped with a binary function) is called an *applicative structure*.
Let $\mathscr{L}$ be the language obtained by adding to $\mathscr{L}_0$ two constant symbols K, S.
We shall use the following notations :
$t \equiv u$ will mean that t and u are identical terms of $\mathscr{L}$;
$\mathscr{M} \models F$ will mean that the closed formula F is satisfied in the model $\mathscr{M}$ (of $\mathscr{L}$) ;
$\mathscr{A} \vdash F$ will mean that F is a consequence of the set $\mathscr{A}$ of formulas, in other words, that every model of $\mathscr{A}$ satisfies F.
Given terms t, u of $\mathscr{L}$, and a variable x, $t[u/x]$ denotes the term obtained from t by replacing every occurrence of x with u.

Consider the following axioms :

(C_0) $\qquad (K)xy = x$; $(S)xyz = ((x)z)(y)z$.

Actually, we consider the closure of these formulas, namely, the axioms :

$$\forall x \forall y\{(K)xy = x\} \; ; \; \forall x \forall y \forall z\{(S)xyz = ((x)z)(y)z\}.$$

The term $(S)KK$ is denoted by I. Thus $C_0 \vdash (I)x = x$.
A model of this system of axioms is called a *combinatory algebra*. The combinatory algebra consisting of one single element is said to be *trivial*.

For every term t of $\mathscr{L}$, and every variable x, we now define a term of $\mathscr{L}$, denoted by $\lambda x\, t$, by induction on the length of t :

- if x does not occur in t, then $\lambda x\, t \equiv (K)t$;
- $\lambda x\, x \equiv (S)KK \equiv I$;
- if $t \equiv (u)v$ and x occurs in t, then $\lambda x\, t \equiv ((S)\lambda x\, u)\lambda x\, v$.

Proposition 6.1. *For every term t of $\mathscr{L}$, the term $\lambda x\, t$ does not contain the variable x, and we have $C_0 \vdash \forall x\{(\lambda x\, t)x = t\}$.*

It follows that $C_0 \vdash (\lambda x\, t)u = t[u/x]$, for all terms t, u of $\mathscr{L}$.

It is obvious that x does not occur in $\lambda x\, t$. The second part of the statement is proved by induction on the length of t :
If x does not occur in t, then $(\lambda x\, t)x \equiv (K)tx$, and $C_0 \vdash (K)tx = t$.
If $t \equiv x$, then $(\lambda x\, t)x \equiv (I)x$, and $C_0 \vdash (I)x = x$.
If $t \equiv (u)v$ and x occurs in t, then $(\lambda x\, t)x \equiv (((S)\lambda x\, u)\lambda x\, v)x$. By the second axiom of C_0, we have $C_0 \vdash (\lambda x\, t)x = ((\lambda x\, u)x)(\lambda x\, v)x$. Now, by induction hypothesis : $C_0 \vdash (\lambda x\, u)x = u$ and $(\lambda x\, v)x = v$. Therefore, $C_0 \vdash (\lambda x\, t)x = (u)v = t$.

Q.E.D.

It follows immediately that :

$C_0 \vdash (\lambda x_1 \ldots \lambda x_k\, t)x_1 \ldots x_k = t$ *for all variables* $x_1, \ldots, x_k$.

Proposition 6.2. *All non-trivial combinatory algebras are infinite.*

Let A be a finite combinatory algebra, and n its cardinality. For $0 \le i \le n$, let $a_i \in A$ be the interpretation in A of the term $\lambda x_0 \lambda x_1 \ldots \lambda x_n\, x_i$. Then there exist two distinct integers $i, j \le n$ such that $a_i = a_j$. Suppose, for example, that $i = 0$ and $j = 1$. We therefore have :
$a_0 b_0 b_1 \ldots b_n = a_1 b_0 b_1 \ldots b_n$, for all $b_0, b_1, \ldots, b_n \in A$.
Thus $b_0 = b_1$ for all $b_0, b_1 \in A$, which means that A is trivial.

Q.E.D.

An applicative structure A is said to be *combinatorially complete* if, for every term t of $\mathscr{L}_0$, with variables among $x_1, \ldots, x_k$, and parameters in A, there exists an element $f \in A$ such that $A \models (f)x_1 \ldots x_k = t$, that is to say :
$(f)a_1 \ldots a_k = t[a_1/x_1, \ldots, a_k/x_k]$ for all $a_1, \ldots, a_k \in A$).
This property is therefore expressed by the following axiom scheme :

(CC) $\quad\quad \exists f \forall x_1 \ldots \forall x_n \{(f)x_1 \ldots x_n = t\}$

where t is an arbitrary term of $\mathscr{L}_0$, and $n \ge 0$.

Proposition 6.3. *An applicative structure A is combinatorially complete if and only if A can be given a structure of combinatory algebra.*

In other words, A is combinatorially complete if and only if the constant symbols K and S may be interpreted in A in such a way as to satisfy C_0.

Indeed, if A is a combinatory algebra, and t is any term with variables among $x_1,\ldots,x_n$, then it suffices to take $f = \lambda x_1 \ldots \lambda x_n t$.
Conversely, if A is combinatorially complete, then there exist $k, s \in A$ satisfying C_0 : it is enough to apply CC, first with $n = 2$ and $t = x_1$, then with $n = 3$ and $t = ((x_1)x_3)(x_2)x_3$.

Q.E.D.

The axiom scheme CC is thus equivalent to the conjunction of two particular cases :

(CC') $\quad \exists k\forall x\forall y\{(k)xy = x\}$; $\exists s\forall x\forall y\forall z\{(s)xyz = ((x)z)(y)z\}$

Let E denote the term $\lambda x\lambda y(x)y$. By proposition 6.1, we therefore have :

$$C_0 \vdash (E)xy = (x)y.$$

By definition of λ, we have $\lambda y(x)y \equiv ((S)(K)x)I$, and hence :

$$E \equiv \lambda x((S)(K)x)I.$$

Thus, by proposition 6.1 : $C_0 \vdash (E)x = ((S)(K)x)I$.
Let t be a term containing no occurrence of the variable x. Then, by definition of λ : $\lambda x(t)x \equiv ((S)\lambda x\, t)I \equiv ((S)(K)t)I$. We have thus proved :

Proposition 6.4. *Let t be a term and x a variable not occurring in t ; then :* $C_0 \vdash \lambda x(t)x = (E)t = ((S)(K)t)I$.

We now consider the axioms :

(C_1) $\quad K = \lambda x\lambda y(K)xy$; $S = \lambda x\lambda y\lambda z(S)xyz$.

According to proposition 6.1, the following formulas are consequences of the axioms $C_0 + C_1$:

$$(K)x = \lambda y(K)xy\ ;\ (S)xy = \lambda z(S)xyz\ ;$$

thus, by proposition 6.4, so are the formulas :

(C_1^0) $\quad (E)(K)x = (K)x$; $(E)(S)xy = (S)xy$.

Proposition 6.5. *The following formulas are consequences of $C_0 + C_1^0$:*
i) $\lambda x\, t = (E)\lambda x\, t = \lambda x(\lambda x\, t)x$ *for every term t of $\mathscr{L}$;*
ii) $(E)E = E$ *;* $(E)(E)x = (E)x$.

i) The second identity follows from proposition 6.4, since x does not occur in $\lambda x\, t$. On the other hand, by definition of $\lambda x\, t$, we have either $\lambda x\, t \equiv (K)t$, or $\lambda x\, t \equiv (S)KK$, or $\lambda x\, t \equiv (S)uv$ for suitable terms u, v. It follows immediately that $C_1^0 \vdash (E)\lambda x\, t = \lambda x\, t$.
ii) We have $E = \lambda x\lambda y(x)y$, and hence $C_0 + C_1^0 \vdash (E)E = E$, by (i). Now, by proposition 6.4, $C_0 \vdash (E)x = \lambda y(x)y$, and therefore, by (i) again :
$C_0 + C_1^0 \vdash (E)(E)x = (E)x$.

Q.E.D.

2. Extensionality axioms

The following axiom scheme :

(WEXT) $\forall x(t = u) \to \lambda x\, t = \lambda x\, u$

(where t, u are arbitrary terms of $\mathscr{L}$, allowed to contain variables) is called the *weak extensionality scheme.*

As a consequence of this axiom, we obtain (by induction on n) :

$$\forall x_1 \ldots \forall x_n \{t = u\} \to \lambda x_1 \ldots \lambda x_n t = \lambda x_1 \ldots \lambda x_n u.$$

The *weak extensionality axiom* is the following formula :

(Wext) $\forall y \forall z \{\forall x[(y)x = (z)x] \to (E)y = (E)z\}$.

Proposition 6.6. *WEXT and Wext are equivalent modulo* $C_0 + C_1^0$.

Indeed, let A be a model of $C_0 + C_1^0$ + WEXT, and $b, c \in A$ such that :
$(b)x = (c)x$ for every $x \in A$. Applying WEXT with $t \equiv (b)x$ and $u \equiv (c)x$, we obtain $\lambda x(b)x = \lambda x(c)x$. Now both $\lambda x(b)x = (E)b$ and $\lambda x(c)x = (E)c$ hold in A, since $A \models C_0$ (proposition 6.4). Thus $(E)b = (E)c$.
Conversely, let A be a model of $C_0 + C_1^0$ + Wext, and t, u two terms with parameters in A, where x is the only variable ; assume that $A \models \forall x(t = u)$. Since $A \models C_0$, we have $A \models (\lambda x\, t)x = t$, $(\lambda x\, u)x = u$ (proposition 6.1).
Thus $A \models \forall x\{(\lambda x\, t)x = (\lambda x\, u)x\}$.
By Wext, we obtain $A \models (E)\lambda x\, t = (E)\lambda x\, u$, and hence $A \models \lambda x\, t = \lambda x\, u$ (by proposition 6.5).

Q.E.D.

We shall denote by CL *(combinatory logic)* the system of axioms :
$C_0 + C_1$ + Wext (or, equivalently, $C_0 + C_1$ + WEXT).

Now we consider the axioms :

(C_1') $(E)K = K$; $(E)(K)x = (K)x$;
$(E)S = S$; $(E)(S)x = (S)x$; $(E)(S)xy = (S)xy$.

Proposition 6.7. CL *is equivalent to* $C_0 + C_1'$ + *Wext.*

The following formulas (in fact, their closures) are obviously consequences of $C_0 + C_1$:

$K = \lambda x \lambda y(K)xy$; $(K)x = \lambda y(K)xy$;
$S = \lambda x \lambda y \lambda z(S)xyz$; $(S)x = \lambda y \lambda z(S)xyz$; $(S)xy = \lambda z(S)xyz$.

In view of proposition 6.5, we deduce immediately that C_1' is a consequence of $C_0 + C_1$, and therefore of CL.
Conversely, we have $C_1' \vdash (S)xy = (E)(S)xy$, and hence :
$C_0 + C_1' \vdash (S)xy = \lambda z(S)xyz$.
Now we also have : $C_0 \vdash (\lambda y \lambda z(S)xyz)y = \lambda z(S)xyz$.

Thus $C_0 + C_1' \vdash (S)xy = (\lambda y\lambda z(S)xyz)y$. By Wext, we obtain first :
$(E)(S)x = (E)\lambda y\lambda z(S)xyz$, then $(S)x = \lambda y\lambda z(S)xyz$ (by C_1' and proposition 6.5) ;
thus $(S)x = (\lambda x\lambda y\lambda z(S)xyz)x$. By applying Wext again, we conclude that :
$(E)S = (E)\lambda x\lambda y\lambda z(S)xyz$, and hence $S = \lambda x\lambda y\lambda z(S)xyz$ (by C_1' and proposition 6.5 again). The same kind of proof gives the equation $K = \lambda x\lambda y(K)xy$.

Q.E.D.

The *extensionality axiom* is the formula :

(Ext) $\forall y\forall z\{\forall x[(y)x = (z)x] \to y = z\}$.

As a consequence of this axiom, we obtain (by induction on n) :

(Ext_n) $\forall y\forall z\{\forall x_1 \ldots \forall x_n[(y)x_1 \ldots x_n = (z)x_1 \ldots x_n] \to y = z\}$.

We now prove that, modulo C_0, the extensionality axiom is equivalent to :

$$\text{Wext} + (E = I).$$

Indeed, it is clear that Wext $+(E = I) + C_0 \vdash$ Ext (since $C_0 + (E = I) \vdash (E)x = x$).
Conversely, we have $C_0 \vdash (E)xy = (I)xy = (x)y$. With Ext_2, we obtain C_0+ Ext $\vdash E = I$.

We shall denote by *ECL (extensional combinatory logic)* the system of axioms C_0+ Ext.
Note that C_0+ Ext $\vdash C_1$, and thus $ECL \vdash CL$; indeed, by proposition 6.1, for every term T, we have :
$C_0 \vdash (T)x_1 \ldots x_n = (\lambda x_1 \ldots \lambda x_n(T)x_1 \ldots x_n)x_1 \ldots x_n$;
then, by Ext_n, we can deduce $T = \lambda x_1 \ldots \lambda x_n(T)x_1 \ldots x_n$.

Scott-Meyer's axioms

Let A be an applicative structure, with a distinguished element e, satisfying the following axioms, known as *Scott-Meyer's axioms* :
i) Combinatorial completeness
$\exists k\forall x\forall y[(k)xy = x]$; $\exists s\forall x\forall y\forall z[(s)xyz = ((x)z)(y)z]$;
ii) $\forall x\forall y[(e)xy = (x)y]$;
iii) Weak extensionality
$\forall y\forall z\{\forall x[(y)x = (z)x] \to (e)y = (e)z\}$

Theorem 6.8. *Let A be an applicative structure satisfying the Scott-Meyer's axioms. Then there is a unique way of assigning values in A to the symbols K, S of $\mathscr{L}$ so that A becomes a model of CL satisfying $\forall x[(E)x = (e)x]$. Moreover, in that model, we have $E = (e)e$.*

Notice that E is a term of $\mathscr{L}$, not a symbol.

Unicity : suppose that values have been assigned to K, S so that CL is satisfied. We have $(E)x = (e)x$, thus $(E)E = (e)E$ (take $x = E$), and hence $E = (e)E$ (we have seen that $CL \vdash E = (E)E$). Now the above weak extensionality axiom gives : $\forall x[(E)x = (e)x] \to (e)E = (e)e$. Therefore, $E = (e)e$.
Let K_1, S_1 and K_2, S_2 be two possible interpretations of K, S in A such that the required conditions hold, and let E_1, E_2 be the corresponding interpretations of E (actually, we have seen that $E_1 = E_2 = (e)e$) ; thus $(E_1)x = (E_2)x = (e)x$ and $(S_1)xyz = (S_2)xyz = ((x)z)(y)z$; by weak extensionality, it follows that : $(e)(S_1)xy = (e)(S_2)xy$, and we therefore obtain : $(E_1)(S_1)xy = (E_2)(S_2)xy$. Since CL holds, the axioms of C'_1 are satisfied and we have : $(E_i)(S_i)xy = (S_i)xy (i = 1, 2)$; therefore $(S_1)xy = (S_2)xy$.
By weak extensionality again, it follows that $(e)(S_1)x = (e)(S_2)x$, that is : $(E_1)(S_1)x = (E_2)(S_2)x$, and hence $(S_1)x = (S_2)x$ (by C'_1). Using the weak extensionality once more, we obtain $(e)S_1 = (e)S_2$, that is to say $(E_1)S_1 = (E_2)S_2$, and hence $S_1 = S_2$ (by C'_1). The proof of $K_1 = K_2$ is similar.

Existence : take $k, s \in A$ such that $(k)xy = y$ and $(s)xyz = ((x)z)(y)z$ for all $x, y, z \in A$; this is possible according to the first two axioms of Scott-Meyer. For every term t with parameters in A (and containing variables), we now define, inductively, a term $\lambda' x\, t$:
$\lambda' x\, t = (e)(k)t$ if x does not occur in t ;
$\lambda' x\, x = (e)i$ with $i = (s)kk$ (thus $(i)x = x$ for every $x \in A$) ;
$\lambda' x\, t = (e)((s)\lambda' x\, u)\lambda' x\, v$ if $t = (u)v$ and x occurs in t.

Notice that $(e)xy = xy$, and hence, by weak extensionality (Scott-Meyer's axioms), $(e)(e)x = (e)x$. It follows immediately that $(e)\lambda' x\, t = \lambda' x\, t$ for every term t.
Moreover, we have $(\lambda' x\, t)x = t$ (by induction on t, as in proposition 6.1).

Let $K = \lambda' x \lambda' y\, x$; $S = \lambda' x \lambda' y \lambda' z((x)z)(y)z$.
We do have $(K)xy = x$, $(S)xyz = ((x)z)(y)z$; moreover, since $(S)xy = \lambda' z \ldots$, we also have $(S)xy = (e)(S)xy$; similarly, $(e)(S)x = (S)x$ and $(e)S = S$. On the other hand, since $(S)xyz = (s)xyz$, we obtain $(e)(s)xy = (e)(S)xy = (S)xy$ by weak extensionality ; similarly, $(e)(k)x = (K)x$. Therefore, we may restate the definition of $\lambda' x\, t$ this way :
$\lambda' x\, t = (K)t$ if x does not occur in t ;
$\lambda' x\, x = I$ with $I = (S)KK$ (indeed, we have $(I)x = (i)x$, thus $(e)I = (e)i$; but $(e)I = I$ by definition of I) ;
$\lambda' x\, t = ((S)\lambda' x\, u)\lambda' x\, v$ if $t = (u)v$ and x occurs in t.
We see that this definition is the same as that of the term $\lambda x\, t$; thus $\lambda' x\, t = \lambda x\, t$.
Now let $E = \lambda x \lambda y(x)y$; thus $(E)x = \lambda y \ldots$, and therefore $(e)(E)x = (E)x$; now $(E)xy = (x)y$, and hence, by weak extensionality, $(e)(E)x = (e)x$, that is to say $(E)x = (e)x$.

This proves that the axiom Wext holds, as well as C_0. Besides, we have : $(E)\lambda x\,t = \lambda x\,t$ for every term t (since $(e)\lambda' x\,t = \lambda' x\,t$).
Since $K = \lambda x\lambda y\,x$ and $S = \lambda x\lambda y\lambda z((x)z)(y)z$, we may deduce, using C_0, that $(K)x = \lambda y\,x$; $(S)x = \lambda y\lambda z((x)z)(y)z$; $(S)xy = \lambda z((x)z)(y)z$. Thus $(E)K = K$, $(E)(K)x = (K)x$, $(E)S = S$, $(E)(S)x = (S)x$ and $(E)(S)xy = (S)xy$. Thus the axioms C_1' hold, and finally our model satisfies $C_0 + C_1' +$ Wext, that is to say CL.

Q.E.D.

3. Curry's equations

Let A be a model of $C_0 + C_1^0$. We wish to construct an embedding of A in a model of Wext.
Let k, s, e denote the interpretations in A of the symbols K, S and the closed term E of $\mathscr{L}$. Define $B = (e)A = \{(e)a\ ;\ a \in A\} = \{a \in A,\ (e)a = a\}$ (indeed, $(e)(e)a = (e)a$). We shall define an applicative structure over B : its binary operation will be denoted by $[a]b$, and defined by $[a]b = (s)ab$ (note that we do have $(s)ab \in B$ since $(e)(s)ab = (s)ab$, by C_1^0).
We define a one-one function $j : A \to B$ by taking $j(a) = (k)a$ (let us note that $(k)a \in B$ since $(e)(k)a = (k)a$, by C_1^0) : indeed, if $(k)a = (k)b$, then $(k)ax = (k)bx$ for arbitrary $x \in A$, which implies $a = b$.
Let $A' \subset B$ be the range of this function. We want j to be an isomorphism of applicative structures from A into B. This happens if and only if : $[(k)a](k)b = (k)(a)b$ for all $a, b \in A$. In other words, j is an isomorphism if and only if A satisfies the following axiom :

(C_2) $$((S)(K)x)(K)y = (K)(x)y\ ;$$

this will be assumed from now on.
Notice that :
B is a proper extension of A' (that is $B \supset A'$ and $B \neq A'$) if and only if A is non-trivial (that is A has at least two elements). In that case, $i \in B \setminus A'$ (where $i = (s)kk$ is the interpretation of I).
Indeed, if $i \in A'$, then $i = (k)a$, thus $(i)b = (k)ab$, that is to say $b = a$, for every $b \in A$, and A is trivial. Conversely, if A contains only one element, then, obviously, $A = B = A'$.

The interpretations of K, S in B are the same as in A', namely : $(k)k$ and $(k)s$. B satisfies C_0 if and only if :

i) $[[(k)k](e)a](e)b = (e)a$ and

ii) $[[[(k)s](e)a](e)b](e)c = [[(e)a](e)c][(e)b](e)c$

for all $a, b, c \in A$.
(i) can be written $((s)((s)(k)k)(e)a)(e)b = (e)a$. Now consider the axiom :

(C_3) $$((S)((S)(K)K)x)y = (E)x.$$

It implies (i) since, by proposition 6.5, we have $C_0 + C_1^0 \vdash (E)(E)x = (E)x$.
C_3 is equivalent, modulo C_0, to :

(C_3') $$((S)((S)(K)K)x)y = \lambda z(x)z.$$

(ii) can be written :

$$((s)((s)((s)(k)s)(e)a)(e)b)(e)c = ((s)((s)(e)a)(e)c)((s)(e)b)(e)c.$$

Now consider the axiom :

(C_4) $$((S)((S)((S)(K)S)x)y)z = ((S)((S)x)z)((S)y)z.$$

At this point, we have proved the first part of :

Lemma 6.9. *Let A be a combinatory algebra satisfying $C_0 + C_1^0 + C_2 + C_3 + C_4$. Then B is an extension of A' (a combinatory algebra, isomorphic with A) which satisfies C_0. Moreover, if $a \in A$, then $[ka]i = (e)a$.*

Indeed, we have $[ka]i = ((s)(k)a)i = (e)a$ (by proposition 6.4).

Q.E.D.

Let t, u be two terms of $\mathscr{L}$, and $\{x_1, \ldots, x_n\}$ the set of variables occurring in t or u. The formula $t = u$ (in fact, its closure $\forall x_1 \ldots \forall x_n\{t = u\}$) is called an *equation*; this equation is said to be *closed* if both t and u contain no variables ($n = 0$) ; the equation $\lambda x_1 \ldots \lambda x_n\, t = \lambda x_1 \ldots \lambda x_n\, u$ will be called the λ-*closure* of the equation $t = u$.

For each axiom C_i ($2 \le i \le 4$), let CL_i denote its λ-closure, that is to say :

(CL_2) $$\lambda x \lambda y((S)(K)x)(K)y = \lambda x \lambda y(K)(x)y$$

(CL_3) $$\lambda x \lambda y((S)((S)(K)K)x)y = \lambda x \lambda y \lambda z(x)z$$

(CL_4) $$\lambda x \lambda y \lambda z((S)((S)((S)(K)S)x)y)z = \lambda x \lambda y \lambda z((S)((S)x)z)((S)y)z.$$

Proposition 6.10. *Let A be a combinatory algebra, and Q a set of closed equations such that $C_0 + Q \vdash C_1^0$. If $A \models C_0 + Q + CL_2 + CL_3 + CL_4$, then there exist an extension A_1 of A satisfying the same axioms, and an element $\xi_1 \in A_1$ such that, for all $a, b \in A$: $(a)\xi_1 = (b)\xi_1 \Rightarrow (e)a = (e)b$.*

Indeed, $C_0 + CL_i \vdash C_i$ (proposition 6.1), thus $A \models C_0 + C_1^0 + C_2 + C_3 + C_4$. By lemma 6.9, there exists an extension B of A' satisfying C_0. Since $A \models CL_i$ and $A \models Q$, and CL_i and Q are closed equations, we have $B \models CL_i$ et $B \models Q$. Now j is an isomorphism from A onto A', and hence there exist an extension A_1 of A and an isomorphism J from A_1 onto B extending j. Let $\xi_1 = J^{-1}(i)$; for every $a, b \in A$ such that $(a)\xi_1 = (b)\xi_1$, we have $[Ja]J\xi_1 = [Jb]J\xi_1$, that is $[ka]i = [kb]i$, and therefore $(e)a = (e)b$, by lemma 6.9.

Q.E.D.

Theorem 6.11. *Let A be a combinatory algebra, and Q a set of closed equations such that $C_0 + Q \vdash C_1^0$. Then there exists an extension A^* of A satisfying $C_0 + Q+$ Wext if and only if $A \models C_0 + Q + CL_2 + CL_3 + CL_4$.*

First, notice that the systems of axioms $C_0 + Q+$ Wext and $C_0 + Q+$ WEXT are equivalent (since $C_0 + Q \vdash C_1^0$, and $C_0 + C_1^0 \vdash$ Wext $\Leftrightarrow$ WEXT). We shall denote by $\mathscr{Q}$ this system of axioms.

The condition is necessary : it suffices to prove that $\mathscr{Q} \vdash CL_i$ $(2 \le i \le 4)$. By definition of the axiom scheme WEXT, we have WEXT $\vdash C_i \Rightarrow CL_i$, thus it is enough to prove : $\mathscr{Q} \vdash C_i$. We have :

$C_0 \vdash (((S)(K)x)(K)y)z = ((Kx)z)(Ky)z = (x)y$;

thus $C_0 \vdash (((S)(K)x)(K)y)z = ((K)(x)y)z$. By weak extensionality, it follows that $\mathscr{Q} \vdash (E)((S)(K)x)(K)y = (E)(K)(x)y$, and then, by C_1^0, that :
$\mathscr{Q} \vdash ((S)(K)x)(K)y = (K)(x)y$; therefore $\mathscr{Q} \vdash C_2$.
The equation (C_3) is written $((S)((S)(K)K)x)y = (E)x$. Now we have

$$C_0 \vdash (((S)((S)(K)K)x)y)z = ((((S)(K)K)x)z)(y)z = (((K)Kz)(x)z)(y)z = ((K)(x)z)(y)z = (x)z.$$

Hence C_0+ Wext $\vdash (E)((S)((S)(K)K)x)y = (E)x$.
Thus $C_0 + C_1^0+$ Wext $\vdash ((S)((S)(K)K)x)y = (E)x$, that is to say $\mathscr{Q} \vdash C_3$.
The axiom (C_4) is written $((S)((S)((S)(K)S)x)y)z = ((S)((S)x)z)((S)y)z$.
Now we have

$$C_0 \vdash (((S)((S)((S)(K)S)x)y)z)a = \{[((S)((S)(K)S)x)y]a\}(z)a$$
$$= \{[((((S)(K)S)x)a](y)a\}(z)a = \{[((((K)S)a)(x)a](y)a\}(z)a$$
$$= \{[(S)(x)a](y)a\}(z)a = (((x)a)(z)a)((y)a)(z)a.$$

On the other hand :

$$C_0 \vdash (((S)((S)x)z)((S)y)z)a = ((S)xza)(S)yza = (((x)a)(z)a)((y)a)(z)a.$$

Therefore, $C_0 \vdash (((S)((S)((S)(K)S)x)y)z)a = (((S)((S)x)z)((S)y)z)a$.
Thus C_0+ Wext $\vdash (E)((S)((S)((S)(K)S)x)y)z = (E)((S)((S)x)z)((S)y)z$.
It follows that

$C_0 + C_1^0+$ Wext $\vdash ((S)((S)((S)(K)S)x)y)z = ((S)((S)x)z)((S)y)z$;
that is to say $\mathscr{Q} \vdash C_4$.

The condition is sufficient : Let A be a model of $C_0 + Q + CL_2 + CL_3 + CL_4$.
By proposition 6.10, we may define an increasing sequence :
$A = A_0 \subset A_1 \subset \ldots \subset A_n \subset \ldots$ of combinatory algebras which satisfy the same axioms, and such that, for each n, there exists $\xi_{n+1} \in A_{n+1}$ such that :
if $a, b \in A_n$ and $(a)\xi_{n+1} = (b)\xi_{n+1}$, then $(e)a = (e)b$.
Let $A^* = \cup_n A_n$. Then $A^* \models C_0 + Q + CL_i$ $(2 \le i \le 4)$ as well as the weak extensionality axiom : if $a, b \in A^*$ and $(a)x = (b)x$ for every $x \in A^*$, then we have $a, b \in A_n$ for some n ; hence $(a)\xi_{n+1} = (b)\xi_{n+1}$ and therefore $(e)a = (e)b$.

Q.E.D.

Intuitively, the extension of A constructed here is obtained by adding infinitely many " variables " which are the ξ_n 's.

Now we consider the system of axioms :

$(CL_=)$ $$C_0 + C_1 + CL_2 + CL_3 + CL_4.$$

Theorem 6.12. *Let A be a combinatory algebra. Then there exists an extension of A satisfying CL if and only if A satisfies $CL_=$.*

It suffices to apply theorem 6.11, where Q is taken as the system of axioms C_1.

Q.E.D.

Corollary 6.13. *The universal consequences of CL are those of $CL_=$.*

Indeed, let A be a model of $CL_=$, and F a universal formula which is a consequence of CL (see chapter 9). We need to prove that $A \models F$. By theorem 6.12, A can be embedded in some model B of CL. Thus $B \models F$ and, since F is universal and A is a submodel of B, we deduce that $A \models F$.

Conversely, it follows from theorem 6.12 that every model of CL is a model of $CL_=$.

Q.E.D.

We now consider the axiom :

(CL_5) $$E = I$$

that is to say (by definition of E) :

(CL_5) $$\lambda x((S)(K)x)I = I.$$

Clearly, $C_0 + CL_5 \vdash C_1^0$. Moreover, C_3 is obviously equivalent, modulo $C_0 + CL_5$, to :

(C_3'') $$((S)((S)(K)K)x)y = x.$$

Let CL_3'' denote the λ-closure of C_3'', that is to say :

(CL_3'') $$\lambda x\lambda y((S)((S)(K)K)x)y = \lambda x\lambda y\, x.$$

We also define the following system of axioms $ECL_=$:

$(ECL_=)$ $$C_0 + CL_2 + CL_3'' + CL_4 + CL_5.$$

Theorem 6.14. *Let A be a combinatory algebra. Then there exists an extension of A satisfying ECL if and only if A satisfies $ECL_=$.*

This follows immediately from theorem 6.11, where Q is taken as the axiom $E = I$.

Q.E.D.

Corollary 6.15. *The universal consequences of ECL are those of $ECL_=$.*

Let A be a model of $\mathscr{L}$. The *diagram* of A, denoted by D_A, is defined as the set of all formulas of the form $t = u$ or $t \neq u$ which hold in A, t and u being arbitrary closed terms with parameters in A. The models of D_A are those models of $\mathscr{L}$ which are extensions of A.

Theorem 6.16. *Let A be a model of $CL_=$, and t,u two terms with parameters in A (and variables). Then :*
i) if $D_A + C_0 \vdash t = u$, then $D_A + C_0 \vdash \lambda x\, t = \lambda x\, u$;
ii) if $D_A + C_0 \vdash (t)x = (u)x$, where x is a variable which does not occur in t,u, then $D_A + C_0 \vdash (E)t = (E)u$.

$D_A + C_0 \vdash F$ means : every extension of A satisfying C_0 satisfies F.

Proof of (i) : let B be an extension of A satisfying C_0. Then B satisfies $CL_=$ and, by theorem 6.12, there exists an extension B' of B which satisfies CL. By hypothesis, we have $D_A + C_0 \vdash t = u$, and hence $B' \models t = u$; by weak extensionality, it follows that :
$B' \models \lambda x\, t = \lambda x\, u$; therefore, $B \models \lambda x\, t = \lambda x\, u$.
Same proof for (ii).

Q.E.D.

A similar proof yields the following theorem :

Theorem 6.17. *Let A be a model of $ECL_=$, and t,u two terms with parameters in A where x does not occur. If $D_A + C_0 \vdash (t)x = (u)x$, then $D_A + C_0 \vdash t = u$.*

4. Translation of λ-calculus

We define a model $\mathscr{M}_0$ of $\mathscr{L}$, called " model over λ-terms ", as follows :
the domain M_0 is the quotient set $\Lambda/\simeq_\beta$;
the constant symbols K,S are respectively interpreted by the (equivalence classes of) λ-terms $\lambda x \lambda y\, x$ and $\lambda x \lambda y \lambda z((x)z)(y)z$;
the function symbol Ap is interpreted by the function $u,t \mapsto (u)t$ from $M_0 \times M_0$ to M_0.

Lemma 6.18. *$\mathscr{M}_0$ is a model of CL. For every term $t \in \Lambda$, we have $(E)t \simeq_\beta \lambda x(t)x$, where x is any variable which does not occur free in t.*

Here we will only use the definition of β-equivalence, not its properties shown in chapter 1.
We first prove that $\mathscr{M}_0 \models C_0$:
that is to say that $(K)uv \simeq_\beta u$ and $(S)uvw \simeq_\beta ((u)w)(v)w$ for all $u,v,w \in \Lambda$, which is clear in view of the interpretations of K and S.

Now we come to the second part of the lemma : since $\mathscr{M}_0 \models C_0$, we have, by proposition 6.4 : $(E)t = ((S)(K)t)I$, with $I = (S)KK$.
Looking again to the interpretations of K and S in $\mathscr{M}_0$, we obtain easily :
$I \simeq_\beta \lambda x\, x$, and then $((S)(K)t)I \simeq_\beta \lambda x(t)x$, which gives the desired result.
Then we prove that $\mathscr{M}_0 \models$ Wext : suppose that $\mathscr{M}_0 \models \forall x[(t)x = (u)x]$, with $t, u \in \Lambda/\simeq_\beta$. Take a variable x which does not occur in t, u ; then $(t)x \simeq_\beta (u)x$, and therefore $\lambda x(t)x \simeq_\beta \lambda x(u)x$. Now we have seen that $\lambda x(t)x \simeq_\beta (E)t$ and $\lambda x(u)x \simeq_\beta (E)u$. Thus $(E)t \simeq_\beta (E)u$ and hence $\mathscr{M}_0 \models (E)t = (E)u$.
Finally, we show that $\mathscr{M}_0$ is a model of C_1', in other words, of the formulas :

$$(E)K = K\ ;\ (E)(K)x = (K)x\ ;$$
$$(E)S = S\ ;\ (E)(S)x = (S)x\ ;\ (E)(S)xy = (S)xy.$$

So we need to prove that

$$(E)K \simeq_\beta K\ ;\ (E)(K)x \simeq_\beta (K)x\ ;$$
$$(E)S \simeq_\beta S\ ;\ (E)(S)x \simeq_\beta (S)x\ ;\ (E)(S)xy \simeq_\beta (S)xy.$$

We have seen that $(E)t \simeq_\beta \lambda z(t)z$, where z does not occur in t. Thus it remains to prove that : $\lambda x(K)x \simeq_\beta K$; $\lambda y(K)xy \simeq_\beta (K)x$; $\lambda x(S)x \simeq_\beta S$; $\lambda y(S)xy \simeq_\beta (S)x$; $\lambda z(S)xyz \simeq_\beta (S)xy$. Now all these equivalences are trivial, in view of the interpretations of K and S in $\mathscr{M}_0$.

Q.E.D.

We define similarly a model $\mathscr{M}_1$ of $\mathscr{L}$, over the domain $M_1 = \Lambda/\simeq_{\beta\eta}$ (the quotient set of Λ by the $\beta\eta$-equivalence relation) ;
again, the constant symbols K and S are interpreted by the (equivalence classes of) terms $\lambda x \lambda y\, x$ and $\lambda x\lambda y\lambda z((x)z)(y)z$, and the function symbol Ap is interpreted by the function $u, t \mapsto (u)t$ from $M_1 \times M_1$ to M_1.

Lemma 6.19. *$\mathscr{M}_1$ is a model of ECL.*

We only prove that $\mathscr{M}_1 \models$ Ext (the other axioms are checked as above). Let $t, u \in \Lambda/\simeq_{\beta\eta}$ be such that $\mathscr{M}_1 \models \forall x[(t)x = (u)x]$. Take a variable x not occurring in t, u : we have $(t)x \simeq_{\beta\eta} (u)x$, thus $\lambda x(t)x \simeq_{\beta\eta} \lambda x(u)x$, and hence $t \simeq_{\beta\eta} u$. Therefore $\mathscr{M}_1 \models t = u$.

Q.E.D.

Recall that a combinatory algebra A (that is a model of C_0) is *trivial* if it contains only one element. Actually, A is trivial if and only if it is a model of the axiom $\mathbf{0} = \mathbf{1}$, where $\mathbf{0} \equiv \lambda x\lambda y\, y \equiv (K)I$ and $\mathbf{1} \equiv \lambda x\lambda y\, x = (E)K$: indeed, if $A \models \mathbf{0} = \mathbf{1}$, then, for all $a, b \in A$, we have $A \models (\mathbf{0})ab = (\mathbf{1})ab$, thus $A \models b = a$, and hence A has only one element.
The axiom $\mathbf{0} = \mathbf{1}$ is equivalent to $K = S$: indeed, from $K = S$, we deduce :
$(K)abc = (S)abc$, thus $(a)c = ((a)c)(b)c$ for all $a, b, c \in A$.
Taking $a = (K)I$, $b = (K)d$, we obtain $I = d$ for every $d \in A$, and therefore A is trivial.

Theorem 6.20. *CL and ECL have a non-trivial model, and are not equivalent theories.*

We have seen that $\mathscr{M}_1 \models ECL$, thus both CL and ECL have a non-trivial model (to make sure that $\mathscr{M}_1$ is not trivial, notice, for instance, that two distinct variables of the λ-calculus may not be $\beta\eta$-equivalent, according to the Church-Rosser property for $\beta\eta$).
On the other hand, $\mathscr{M}_0$ is a model of CL, but not of ECL : indeed, let $\xi, \zeta \in \Lambda$ be such that ξ is a variable of the λ-calculus and $\zeta = \lambda x(\xi)x$, where $x \neq \xi$. Then $(\xi)t \simeq_\beta (\zeta)t$ and hence $\mathscr{M}_0 \models (\xi)t = (\zeta)t$ for every $t \in \Lambda$. Now ξ and ζ are not β-equivalent and therefore $\mathscr{M}_0 \models \xi \neq \zeta$.

Q.E.D.

For every λ-term t, we define, inductively, a term $t_{\mathscr{L}}$ of the language of combinatory logic :

if t is a variable, then $t_{\mathscr{L}} = t$ (by convention, we identify variables of the λ-calculus and variables of the language $\mathscr{L}$) ;

if $t = (u)v$, then $t_{\mathscr{L}} = (u_{\mathscr{L}})v_{\mathscr{L}}$;

if $t = \lambda x\, u$, then $t_{\mathscr{L}} = \lambda x\, u_{\mathscr{L}}$.

Notice that the symbol λ is used here in two different ways : on the one hand in the λ-terms, and on the other hand in the terms of $\mathscr{L}$.

Conversely, with each term t of the language $\mathscr{L}$, we associate a λ-term t_Λ, defined by induction on t :

$K_\Lambda = \lambda x \lambda y\, x$; $S_\Lambda = \lambda x \lambda y \lambda z((x)z)(y)z$;

if $t = (u)v$, then $t_\Lambda = (u_\Lambda)v_\Lambda$.

Clearly, for every term t of $\mathscr{L}$ (with or without variables), t_Λ is the value of t in both models $\mathscr{M}_0$ and $\mathscr{M}_1$, when each variable of $\mathscr{L}$ is interpreted by itself (considered as an element of Λ). Therefore :

Lemma 6.21. *Let t, u be two terms of $\mathscr{L}$. If $CL \vdash t = u$, then $t_\Lambda \simeq_\beta u_\Lambda$; if $ECL \vdash t = u$, then $t_\Lambda \simeq_{\beta\eta} u_\Lambda$.*

Lemma 6.22. *For every λ-term t, $t_{\mathscr{L}\Lambda} \simeq_\beta t$.*

The proof is by induction on t. This is obvious in case t is a variable or $t = (u)v$. Suppose that $t = \lambda x\, u$; then $t_{\mathscr{L}} = \lambda x\, u_{\mathscr{L}}$. Therefore, $CL \vdash (t_{\mathscr{L}})x = u_{\mathscr{L}}$ (proposition 6.1). Thus, by lemma 6.21, we have $(t_{\mathscr{L}\Lambda})x \simeq_\beta u_{\mathscr{L}\Lambda}$ and, by induction hypothesis, $u_{\mathscr{L}\Lambda} \simeq_\beta u$. It follows that $(t_{\mathscr{L}\Lambda})x \simeq_\beta u$, and hence :
$\lambda x(t_{\mathscr{L}\Lambda})x \simeq_\beta \lambda x\, u = t$.
Now $t_{\mathscr{L}} = \lambda x\, u_{\mathscr{L}}$, and hence $CL \vdash (E)t_{\mathscr{L}} = t_{\mathscr{L}}$ (proposition 6.5).
Thus, by lemma 6.21, we have $(E)t_{\mathscr{L}\Lambda} \simeq_\beta t_{\mathscr{L}\Lambda}$.

On the other hand, by lemma 6.18, $(E)t_{\mathscr{L}\Lambda} \simeq_\beta \lambda x(t_{\mathscr{L}\Lambda})x$. It follows, finally, that $t_{\mathscr{L}\Lambda} \simeq_\beta \lambda x(t_{\mathscr{L}\Lambda})x \simeq_\beta t$.

Q.E.D.

Lemma 6.23. *For every term t of $\mathscr{L}$, $CL \vdash t_{\Lambda\mathscr{L}} = t$.*

The proof is by induction on t. This is immediate whenever t is a variable or $t = (u)v$. It remains to examine the cases where $t = K$ or $t = S$.
If $t = K$, then $K_\Lambda = \lambda x\lambda y\, x$ (λ-term), thus $K_{\Lambda\mathscr{L}} = \lambda x\lambda y\, x$ (term of $\mathscr{L}$).
Now $CL \vdash Kxy = x$ (axioms C_0), and hence (by weak extensionality) :

$$CL \vdash \lambda x\lambda y(K)xy = \lambda x\lambda y\, x = K_{\Lambda\mathscr{L}}.$$

Since $CL \vdash K = \lambda x\lambda y(K)xy$ (axioms C_1), it follows that $CL \vdash K = K_{\Lambda\mathscr{L}}$.
If $t = S$, then $S_{\Lambda\mathscr{L}} = \lambda x\lambda y\lambda z((x)z)(y)z$ (term of $\mathscr{L}$).
Now $CL \vdash (S)xyz = ((x)z)(y)z$, thus, by weak extensionality :

$$CL \vdash \lambda x\lambda y\lambda z(S)xyz = \lambda x\lambda y\lambda z((x)z)(y)z = S_{\Lambda\mathscr{L}}.$$

On the other hand :
$CL \vdash S = \lambda x\lambda y\lambda z(S)xyz$ (axioms C_1), and therefore $CL \vdash S = S_{\Lambda\mathscr{L}}$.

Q.E.D.

Lemma 6.24. *Let $t, u \in \Lambda$ and $v = u[t/x]$. Then $CL \vdash v_{\mathscr{L}} = u_{\mathscr{L}}[t_{\mathscr{L}}/x]$.*

The proof is by induction on u. This is immediate whenever u is a variable or $u = (u_1)u_2$. Suppose that $u = \lambda y\, u'$; then, we have $v = \lambda y\, v'$, where $v' = u'[t/x]$. Thus, by induction hypothesis, $CL \vdash v'_{\mathscr{L}} = u'_{\mathscr{L}}[t_{\mathscr{L}}/x]$. Now $v_{\mathscr{L}} = \lambda y\, v'_{\mathscr{L}}$ and hence :
$CL \vdash (v_{\mathscr{L}})y = u'_{\mathscr{L}}[t_{\mathscr{L}}/x]$ (proposition 6.1).
But we also have $u_{\mathscr{L}} = \lambda y\, u'_{\mathscr{L}}$, and therefore $CL \vdash (u_{\mathscr{L}})y = u'_{\mathscr{L}}$. It follows that $CL \vdash (u_{\mathscr{L}}[t_{\mathscr{L}}/x])y = u'_{\mathscr{L}}[t_{\mathscr{L}}/x]$, and hence $CL \vdash (u_{\mathscr{L}}[t_{\mathscr{L}}/x])y = (v_{\mathscr{L}})y$. By weak extensionality, we obtain $CL \vdash (E)v_{\mathscr{L}} = (E)u_{\mathscr{L}}[t_{\mathscr{L}}/x]$.
Now $v_{\mathscr{L}} = \lambda y\, v'_{\mathscr{L}}$, $u_{\mathscr{L}} = \lambda y\, u'_{\mathscr{L}}$, and therefore :

$CL \vdash (E)v_{\mathscr{L}} = v_{\mathscr{L}}$ and $CL \vdash (E)u_{\mathscr{L}} = u_{\mathscr{L}}$ (proposition 6.5) ;

thus $CL \vdash (E)u_{\mathscr{L}}[t_{\mathscr{L}}/x] = u_{\mathscr{L}}[t_{\mathscr{L}}/x]$. Finally, we have $CL \vdash v_{\mathscr{L}} = u_{\mathscr{L}}[t_{\mathscr{L}}/x]$.

Q.E.D.

Theorem 6.25. *Let t, u be two λ-terms. Then :*

i) $t \simeq_\beta u$ if and only if $CL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$.

ii) $t \simeq_{\beta\eta} u$ if and only if $ECL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$.

This theorem means that the β (resp. the $\beta\eta$)-equivalence is represented by the notion of consequence in CL (resp. ECL).

Proof of (i) : If $CL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$, then $t_{\mathscr{L}\Lambda} \simeq_\beta u_{\mathscr{L}\Lambda}$ by lemma 6.21, thus $t \simeq_\beta u$ by lemma 6.22.

Conversely, suppose that $t \simeq_\beta u$. To prove that $CL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$, we may suppose that $t\,\beta_0\,u$ (that is to say : u is obtained from t by contracting one redex). The proof is then by induction on t ; t may not be a variable (there is no redex in a variable).

If $t = \lambda x\, t'$, then $u = \lambda x\, u'$ with $t'\,\beta_0\,u'$. Thus $CL \vdash t'_{\mathscr{L}} = u'_{\mathscr{L}}$ (induction hypothesis) and, by weak extensionality, we have $CL \vdash \lambda x\, t'_{\mathscr{L}} = \lambda x\, u'_{\mathscr{L}}$, that is to say $CL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$.

If $t = (t')t''$, then there are three possible cases for u :

$u = (u')t''$, with $t'\,\beta_0\,u'$; then $CL \vdash t'_{\mathscr{L}} = u'_{\mathscr{L}}$ (induction hypothesis), and therefore $CL \vdash (t'_{\mathscr{L}})t''_{\mathscr{L}} = (u'_{\mathscr{L}})t''_{\mathscr{L}}$, that is to say $CL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$.

$u = (t')u''$, with $t''\,\beta_0\,u''$; same proof.

$t = (\lambda x\, t')t''$ and $u = t'[t''/x]$. By lemma 6.24, $CL \vdash u_{\mathscr{L}} = t'_{\mathscr{L}}[t''_{\mathscr{L}}/x]$; on the other hand, we have $t_{\mathscr{L}} = (\lambda x\, t'_{\mathscr{L}})t''_{\mathscr{L}}$ and hence $CL \vdash t_{\mathscr{L}} = t'_{\mathscr{L}}[t''_{\mathscr{L}}/x]$ (proposition 6.1). Thus $CL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$.

Proof of (ii) : If $ECL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$, then $t_{\mathscr{L}\Lambda} \simeq_{\beta\eta} u_{\mathscr{L}\Lambda}$ by lemma 6.21, thus $t \simeq_{\beta\eta} u$ by lemma 6.22.

Conversely, suppose that $t \simeq_{\beta\eta} u$. To prove that $ECL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$, we may suppose that $t\,\beta_0\,u$ or $t\,\eta_0\,u$. If $t\,\beta_0\,u$, we obtain the desired result in view of (i).

If $t\,\eta_0\,u$, the proof proceeds by induction on t (which may not be a variable) ; if $t = (t')t''$, then $u = (u')t''$ or $u = (t')u''$, with $t'\,\eta_0\,u'$ or $t''\,\eta_0\,u''$. Thus the result follows from the induction hypothesis.

If $t = \lambda x\, t'$, there are two possible cases for u :

$u = \lambda x\, u'$, with $t'\,\eta_0\,u'$. By induction hypothesis, $ECL \vdash t'_{\mathscr{L}} = u'_{\mathscr{L}}$;
it follows, by weak extensionality, that $ECL \vdash \lambda x\, t'_{\mathscr{L}} = \lambda x\, u'_{\mathscr{L}}$, that is to say $ECL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$.

$t = \lambda x(u)x$, x having no occurrence in u ; then $t_{\mathscr{L}} = \lambda x(u_{\mathscr{L}})x$, and therefore, by proposition 6.1, $ECL \vdash (t_{\mathscr{L}})x = (u_{\mathscr{L}})x$. Using extensionality (since x does not occur free in t, u), we conclude that $ECL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$.

Q.E.D.

There is a " canonical " method for constructing a model of CL (resp. ECL) : let $\mathscr{T}$ be the set of all terms of $\mathscr{L}$ (with variables). We define on $\mathscr{T}$ an equivalence relation $\sim_0$ (resp. $\sim_1$) by taking :

$t \sim_0 u \Leftrightarrow CL \vdash t = u$ (resp. $t \sim_1 u \Leftrightarrow ECL \vdash t = u$).

Then we have a model $\mathscr{N}_0$ of CL (resp. a model $\mathscr{N}_1$ of ECL) over the domain $\mathscr{T}/{\sim_0}$ (resp. $\mathscr{T}/{\sim_1}$) where the symbols K, S, Ap have obvious interpretations (take the canonical definition on the set of terms and then pass to the quotient set).

We now prove, for example, that $\mathscr{N}_0 \models CL$:

For those axioms of CL which are equations, the proof is immediate :

for instance, the axiom $(K)xy = x$ holds since, for all terms t, u of $\mathscr{L}$, we have

$CL \vdash (K)tu = t$, and thus, by definition of $\mathscr{N}_0$, $\mathscr{N}_0 \models (K)tu = t$.
It remains to check the weak extensionality axiom. Therefore, let $t, u \in \mathscr{N}_0$ be such that $\mathscr{N}_0 \models (t)x = (u)x$ for every $x \in \mathscr{N}_0$. Take x as a variable which does not occur in t, u. Then, by definition of $\mathscr{N}_0$:
$CL \vdash (t)x = (u)x$, thus $CL \vdash (E)t = (E)u$, and hence $\mathscr{N}_0 \models (E)t = (E)u$, which is the desired conclusion.

Proposition 6.26. *$\mathscr{M}_0$ and $\mathscr{N}_0$ (resp. $\mathscr{M}_1$ and $\mathscr{N}_1$) are isomorphic models.*

Consider the mapping $t \mapsto t_{\mathscr{L}}$ of Λ into $\mathscr{T}$; we know from theorem 6.25 that : $t \simeq_\beta u \Leftrightarrow CL \vdash t_{\mathscr{L}} = u_{\mathscr{L}}$; that is to say : $\mathscr{M}_0 \models t = u \Leftrightarrow \mathscr{N}_0 \models t_{\mathscr{L}} = u_{\mathscr{L}}$. Therefore, this mapping induces an isomorphism from $\mathscr{M}_0$ into $\mathscr{N}_0$.
Now consider the mapping $t \mapsto t_\Lambda$ of $\mathscr{T}$ into Λ. By lemma 6.21, we have : $\mathscr{N}_0 \models t = u \Rightarrow \mathscr{M}_0 \models t_\Lambda = u_\Lambda$. Therefore, this mapping induces a homomorphism from $\mathscr{N}_0$ into $\mathscr{M}_0$. According to lemmas 6.22 and 6.23, these are inverse homomorphisms.
The proof is similar for $\mathscr{M}_1$ and $\mathscr{N}_1$.

Q.E.D.

Theorem 6.27. *Let t, t' be two normalizable closed λ-terms which are not $\beta\eta$-equivalent. Then $ECL \vdash t_{\mathscr{L}} = t'_{\mathscr{L}} \leftrightarrow \mathbf{0} = \mathbf{1}$.*

In other words, the theory $ECL + t_{\mathscr{L}} = t'_{\mathscr{L}}$ has no other model than the trivial one.

The proof is as follows :
We have seen that $ECL \vdash \mathbf{0} = \mathbf{1} \to \forall x \forall y\{x = y\}$; thus $ECL \vdash \mathbf{0} = \mathbf{1} \to t_{\mathscr{L}} = t'_{\mathscr{L}}$.
Conversely, since t and t' are normalizable closed terms which are not $\beta\eta$-equivalent, in view of Böhm's theorem (theorem 5.2), there exist closed terms $t^1, \dots, t^n \in \Lambda$ such that $(t)t^1 \dots t^n \simeq_{\beta\eta} \mathbf{0}$ and $(t')t^1 \dots t^n \simeq_{\beta\eta} \mathbf{1}$. It then follows from theorem 6.25 that :
$ECL \vdash (t_{\mathscr{L}})t^1_{\mathscr{L}} \dots t^n_{\mathscr{L}} = \mathbf{0}$ and $ECL \vdash (t'_{\mathscr{L}})t^1_{\mathscr{L}} \dots t^n_{\mathscr{L}} = \mathbf{1}$. Therefore :
$ECL \vdash t_{\mathscr{L}} = t'_{\mathscr{L}} \to \mathbf{0} = \mathbf{1}$.

Q.E.D.

References for chapter 6

[Bar84], [Cur58], [Hin86].
(The references are in the bibliography at the end of the book).

Chapter 7

Models of lambda-calculus

1. Functional models

Given a set $\mathscr{D}$, let $\mathscr{F}(\mathscr{D})$ denote the set of all functions from $\mathscr{D}^{\mathbb{N}}$ into $\mathscr{D}$ which depend only on a finite number of coordinates ; for every $i \geq 0$, the i-th coordinate function will be denoted by x_{i+1}. Therefore, any member of $\mathscr{F}(\mathscr{D})$ may be denoted by $f(x_1,\ldots,x_n)$, for every large enough integer n.

For any two functions $f(x_1,\ldots,x_n)$, $g(x_1,\ldots,x_p)$ in $\mathscr{F}(\mathscr{D})$, we will denote the function $f(x_1,\ldots,x_{i-1},g(x_1,\ldots,x_p),x_{i+1},\ldots,x_n) \in \mathscr{F}(\mathscr{D})$ by $f[g/x_i]$.

Clearly, *if f does not depend on the coordinate x_i, then $f[g/x_i] = f$.*

Let us consider a subset $\mathscr{F}$ of $\mathscr{D}^{\mathscr{D}}$ and two functions $\Phi : \mathscr{D} \to \mathscr{F}$, and $\Psi : \mathscr{F} \to \mathscr{D}$.

For all $a, b \in \mathscr{D}$, define $(a)b$ to be $\Phi(a)(b)$ (so $\mathscr{D}$ is an applicative structure).

For every $f \in \mathscr{F}$, $\Psi(f)$ will also be denoted by $\lambda x\, f(x)$.

Let $f, g \in \mathscr{F}(\mathscr{D})$, $f = f(x_1,\ldots,x_n)$, and $g = g(x_1,\ldots,x_n)$. We define $(f)g \in \mathscr{F}(\mathscr{D})$, by taking $[(f)g](a_1,\ldots,a_n) = (f(a_1,\ldots,a_n))g(a_1,\ldots,a_n)$, for all $a_1,\ldots,a_n \in \mathscr{D}$.

We now consider a subset $\mathscr{F}^\infty$ of $\mathscr{F}(\mathscr{D})$ such that :

0. If $f \in \mathscr{F}^\infty$, $f = f(x_1,\ldots,x_n)$, and $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n \in \mathscr{D}$, then $f(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n) \in \mathscr{F}$ $(1 \leq i \leq n)$.

For each $f \in \mathscr{F}^\infty$, $f = f(x_1,\ldots,x_n)$, and for each coordinate x_i, let us define $\lambda x_i\, f \in \mathscr{F}(\mathscr{D})$ to be the function $g(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ such that :
$g(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n) = \lambda x\, f(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)$
for all $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n \in \mathscr{D}$.
Thus $\lambda x_i\, f$ does not depend on the coordinate x_i.

We now suppose that the following conditions hold :

1. Every coordinate function x_i is in $\mathscr{F}^\infty$;
2. If $f, g \in \mathscr{F}^\infty$, then $(f)g \in \mathscr{F}^\infty$.
3. If $f \in \mathscr{F}^\infty$, then $\lambda x_i\, f \in \mathscr{F}^\infty$ for every i.

Sets $\mathscr{D}, \mathscr{F}, \mathscr{F}^\infty$, and functions Φ and Ψ satisfying conditions 0, 1, 2, 3 form what we will call a *functional model of* λ*-calculus.*

Lemma 7.1.
Let $f, h \in \mathscr{F}^\infty$, x, y *be two distinct coordinate functions, and* $g = \lambda y\, h$. *Then* $\lambda y\, h[f/x] = g[f/x]$ *provided that* f *does not depend on* y.
In particular, $\lambda y\, h[z/x] = g[z/x]$ *for every coordinate* $z \neq y$.

Let $f = f(x_1, \dots, x_n, x)$, $h = h(x_1, \dots, x_n, x, y)$; for all $a_1, \dots, a_n, b \in \mathscr{D}$, we have $\lambda y\, h(a_1, \dots, a_n, b, y) = g(a_1, \dots, a_n, b)$.
In particular, if $b = f(a_1, \dots, a_n, a)$, this gives :
$\lambda y\, h(a_1, \dots, a_n, f(a_1, \dots, a_n, a), y) = g(a_1, \dots, a_n, f(a_1, \dots, a_n, a))$
which is the desired result.

Q.E.D.

Lemma 7.2. *Let* $f \in \mathscr{F}^\infty$, *and* x, y *be two distinct coordinates. If* f *does not depend on* y, *then* $\lambda y\, f[y/x] = \lambda x\, f$.

If $f = f[x_1, \dots, x_n, x]$, then $f[y/x] = f[x_1, \dots, x_n, y]$, which gives the result.

Q.E.D.

We now define a mapping of the set L of λ-terms into $\mathscr{F}^\infty$, denoted by $t \mapsto \|t\|$. We assume that the variables of the λ-calculus are $x_1, \dots, x_n, \dots$ The definition is by induction on t :

- if t is the variable x_i, then $\|t\|$ is the coordinate function x_i ;
- if $t = (u)v$, then $\|t\| = (\|u\|)\|v\|$;
- if $t = \lambda x\, u$, then $\|t\| = \lambda x \|u\|$.

Clearly, if the free variables of t are among $x_1, \dots, x_k$, then the function $\|t\| \in \mathscr{F}^\infty$ depends only on the coordinates $x_1, \dots, x_k$.

Lemma 7.3. *Let* t *be a* λ*-term, and* $f = \|t\|$; *then* $\|t<z/x>\| = f[z/x]$ *for all variables* z *except a finite number.*

From now on, we will use the expression : “ for almost all variables z ” as an abbreviation for : “ for all variables z except a finite number ”.

The proof is by induction on t ; the result is immediate if t is a variable, or $t = (u)v$, or $t = \lambda x\, u$.
Suppose $t = \lambda y\, u$, where $y \neq x$, and let $g = \|u\|$; then $f = \lambda y\, g$. Now
$\|t<z/x>\| = \|\lambda y\, u<z/x>\| = \lambda y \|u<z/x>\| = \lambda y\, g[z/x]$ for almost all variables z, by induction hypothesis. By lemma 7.1, we have $\lambda y\, g[z/x] = f[z/x]$ for almost all z ; this completes the proof.

Q.E.D.

Proposition 7.4.
Let t, t' be two λ-terms. If $t \equiv t'$ (t is α-equivalent to t'), then $\|t\| = \|t'\|$.

Proof by induction on t ; the result is immediate if t is a variable, or $t = (u)v$. If $t = \lambda x\, u$, then $t' = \lambda x' u'$ and $u<z/x> \equiv u'<z/x'>$ for almost all variables z. Hence, by induction hypothesis, $\|u<z/x>\| = \|u'<z/x'>\|$. Let $g = \|u\|$, $g' = \|u'\|$; then $\|t\| = \lambda x\, g$ and $\|t'\| = \lambda x' g'$.
By lemma 7.3, we have $\|u<z/x>\| = g[z/x]$ and $\|u'<z/x'>\| = g'[z/x']$ for almost all variables z. Thus $g[z/x] = g'[z/x']$, and therefore :
$\lambda z\, g[z/x] = \lambda z\, g'[z/x']$ for almost all variables z.
Hence, by lemma 7.2, $\lambda x\, g = \lambda x' g'$, that is $\|t\| = \|t'\|$.

Q.E.D.

Therefore, we may consider $t \mapsto \|t\|$ as a mapping of Λ into $\mathscr{F}^\infty$.

Proposition 7.5.
Let $t, u \in \Lambda$, and $f = \|t\|$, $g = \|u\|$. Then $\|u[t/x]\| = g[f/x]$.

Proof by induction on u ; this is immediate whenever u is a variable or $u = (v)w$. If $u = \lambda y\, v$, then take y not free in t (thus f does not depend on the coordinate y), and let $\|v\| = h$. Then $\|u[t/x]\| = \|\lambda y\, v[t/x]\| = \lambda y \|v[t/x]\| = \lambda y\, h[f/x]$ (by induction hypothesis). Now, by definition of $\|u\|$, we have $g = \lambda y\, h$. Therefore, by lemma 7.1, $\lambda y\, h[f/x] = g[f/x]$.

Q.E.D.

Now consider the following assumption :

(β) $\Phi \circ \Psi$ is the identity function on $\mathscr{F}$

in other words :

(β) $(\lambda x\, f(x))a = f(a)$ for all $a \in \mathscr{D}$ and $f \in \mathscr{F}$.

Under this assumption, $f \mapsto \lambda x\, f$ is obviously a one-one mapping of $\mathscr{F}$ into $\mathscr{D}$.
Any functional model satisfying (β) will be called a *functional β-model.*

Lemma 7.6. *In any β-model, we have $(\lambda x\, g)f = g[f/x]$, for every coordinate x and all $f, g \in \mathscr{F}^\infty$.*

Let $f = f[x_1,\ldots,x_n,x]$, $g = g[x_1,\ldots,x_n,x]$ and $\lambda x\, g = g'[x_1,\ldots,x_n]$. By ($\beta$), we have $(g'[a_1,\ldots,a_n])b = g[a_1,\ldots,a_n,b]$, for all $a_1,\ldots,a_n,b \in \mathscr{D}$.
Thus, by taking $b = f[a_1,\ldots,a_n,a]$, we obtain :

$$(g'[a_1,\ldots,a_n])f[a_1,\ldots,a_n,a] = g[a_1,\ldots,a_n,f[a_1,\ldots,a_n,a]]$$

which yields the result.

Q.E.D.

The following proposition explains the name “ β-model ”.

Proposition 7.7. *In any β-model, if $t, t' \in \Lambda$ and $t \simeq_\beta t'$, then $\|t\| = \|t'\|$.*

We may suppose that $t\,\beta_0\,t'$ (t' is obtained from t by one single β-reduction). The proof is by induction on t ; t is not a variable (if it were, no β-reduction could be made on it).
If $t = \lambda x\, u$, then $t' = \lambda x\, u'$, with $u\,\beta_0\,u'$; by induction hypothesis, $\|u\| = \|u'\|$, thus $\lambda x\|u\| = \lambda x\|u'\|$, that is to say $\|t\| = \|t'\|$.
If $t = (u)v$, then there are three possible cases for t' :

$t' = (u')v$ with $u\,\beta_0\,u'$; then $\|u\| = \|u'\|$, by induction hypothesis, and therefore $(\|u\|)\|v\| = (\|u'\|)\|v\|$, that is $\|t\| = \|t'\|$.

$t' = (u)v'$ with $v\,\beta_0\,v'$; same proof.

$t = (\lambda x\, v)u$ and $t' = v[u/x]$; let $f = \|u\|$, $g = \|v\|$; then $\|t\| = (\lambda x\, g)f$ and $\|t'\| = g[f/x]$ (proposition 7.5). Thus $\|t\| = \|t'\|$ by lemma 7.6.

Q.E.D.

Proposition 7.8. *Every β-model is a model of the Scott-Meyer axioms (and hence it provides a model of CL, see chapter 6, pages 98-99).*

We define a model of the Scott-Meyer axioms, where the domain is $\mathscr{D}$, Ap is the function $(a,b) \mapsto (a)b$ from $\mathscr{D} \times \mathscr{D}$ to $\mathscr{D}$, $e = \lambda x\lambda y(x)y$, $k = \lambda x\lambda y\, x$, and $s = \lambda x\lambda y\lambda z((x)z)(y)z$.
Indeed, it is obvious from condition β that $(k)xy = x$, $(s)xyz = (xz)yz$ and $(e)xy = (x)y$. In order to check the weak extensionality axiom, suppose that $(a)x = (b)x$ for all $x \in \mathscr{D}$; define $f[x,y] \in \mathscr{F}^\infty$ by taking $f[x,y] = (x)y$ (conditions 1, 2 of the definition of functional models). By definition of $\mathscr{F}$, both functions $x \mapsto (a)x$ and $x \mapsto (b)x$ are in $\mathscr{F}$; now they are assumed to be equal, and hence $\lambda x(a)x = \lambda x(b)x$. Moreover, by definition of e, according to condition β, we have $(e)a = \lambda x(a)x$, $(e)b = \lambda x(b)x$. Thus $(e)a = (e)b$.

Q.E.D.

A β-model is called *trivial* if it has only one element. A non-trivial β-model is necessarily infinite, since it is a model of the Scott-Meyer axioms, and hence a combinatory algebra (cf. proposition 6.2).

Remark. All functions of $\mathscr{F}^\infty$ used in the proof of proposition 7.8 have at most three arguments. Therefore, a model of the Scott-Meyer axioms can be obtained whenever the following elements are given :

- an applicative structure $\mathscr{D}$; thus we have a function $a, b \mapsto (a)b$ from $\mathscr{D} \times \mathscr{D}$ to $\mathscr{D}$.
- a set $\mathscr{F}^3$ of functions from $\mathscr{D} \times \mathscr{D} \times \mathscr{D}$ to $\mathscr{D}$, such that :
  the three coordinate functions are in $\mathscr{F}^3$;
  whenever $f, g \in \mathscr{F}^3$, then $(f)g \in \mathscr{F}^3$;
- a function $f \mapsto \lambda x\, f$ from $\mathscr{F}$ to $\mathscr{D}$ such that $(\lambda x\, f)a = f(a)$ for all $f \in \mathscr{F}$ and $a \in \mathscr{D}$; here $\mathscr{F}$ is defined as the set of functions from $\mathscr{D}$ to $\mathscr{D}$ obtained by replacing, in every function of $\mathscr{F}^3$, two of the three variables by arbitrary elements of $\mathscr{D}$;
- it is assumed that, whenever $f(x_1, x_2, x_3) \in \mathscr{F}^3$, then $\lambda x_i\, f \in \mathscr{F}^3$ ($i = 1, 2, 3$).

Consider a β-model $(\mathscr{D},\mathscr{F},\mathscr{F}^\infty)$; we may define another model of the Scott-Meyer axioms, over the domain $\mathscr{F}^\infty$, where Ap is the function $f,g \mapsto (f)g$, and $e = \lambda x\lambda y(x)y$, $k = \lambda x\lambda y\, x$, $s = \lambda x\lambda y\lambda z(xz)(y)z$.
Indeed, by lemma 7.6, we have : $(k)fg = f$, $(s)fgh = (fh)gh$, $(e)fg = (f)g$, for all $f,g,h \in \mathscr{F}^\infty$.
We now check the weak extensionality axiom : suppose that $(f)h = (g)h$ for all $h \in \mathscr{F}^\infty$; take h as any coordinate function x, on which f and g do not depend. Then we have $(f)x = (g)x$, thus $\lambda x(f)x = \lambda x(g)x$. It follows that $(e)f = (e)g$ because, from the definition of e and lemma 7.6, we have $(e)f = \lambda x(f)x$ and $(e)g = \lambda x(g)x$.

Proposition 7.9. *Let $(\mathscr{D},\mathscr{F},\mathscr{F}^\infty)$ be a β-model ; then the following conditions are equivalent :*
i) the extensionality axiom is satisfied in the model $\mathscr{D}$;
ii) $f \to \lambda x f$ is a mapping of $\mathscr{F}$ onto $\mathscr{D}$ (thus it is one-to-one) ;
iii) $\lambda x(a)x = a$ for every $a \in \mathscr{D}$.
iv) $\Psi \circ \Phi$ is the identity function on $\mathscr{D}$.
If these conditions hold, then the β-model under consideration is said to be extensional.

(iii) ⇒ (ii) is obvious.
(i) ⇒ (iii) : for every $b \in \mathscr{D}$, we have $(a)b = (a')b$, where $a' = \lambda x(a)x$ (by condition β). Therefore, $a = a'$ by extensionality.
(ii) ⇒ (i) : let $a,b \in \mathscr{D}$ be such that $(a)c = (b)c$ for every $c \in \mathscr{D}$; by hypothesis, there exist $f,g \in \mathscr{D}$ such that $a = \lambda x f$, $b = \lambda x g$. Therefore $(\lambda x f)c = (\lambda x g)c$, and hence $f(c) = g(c)$ (by β) for every $c \in \mathscr{D}$.
Thus $f = g$, and therefore $\lambda x f = \lambda x g$ and $a = b$.
Finally (ii) ⇔ (iv) : indeed, condition (ii) means that Ψ is one-to-one ; since we know that $\Phi \circ \Psi$ is the identity function on $\mathscr{F}$, we see that $\Psi \circ \Phi$ is the identity function on $\mathscr{D}$.

Q.E.D.

Remark. Conversely, every model $\mathscr{D}$ of the Scott-Meyer axioms can be obtained from a functional β-model : take $\mathscr{F}$ as the set of functions of the form $x \mapsto (a)x$, where $a \in \mathscr{D}$, and $\mathscr{F}^\infty$ as the set of functions of the form $t[x_1,\dots,x_k]$, where t is a term of $\mathscr{L}$ written with the indicated variables.
For all $a_2,\dots,a_k$, there exists $a \in \mathscr{D}$ such that $t[x,a_2,\dots,a_k] = (a)x$ (combinatory completeness of $\mathscr{D}$). Thus condition 0 of the definition of functional models is satisfied. Clearly, conditions 1 and 2 also hold.
Let $f \in \mathscr{F}$ be such that $f(x) = (a)x$; define $\lambda x f(x) = (e)a$. This is a correct definition : indeed, if $f(x) = (a')x$, then $(e)a = (e)a'$, by weak extensionality.
Condition β is satisfied : $(\lambda x f(x))c = (e)ac = (a)c = f(c)$.

Finally, we check that condition 3 is satisfied : let $f \in \mathscr{F}^\infty$ be defined by some term $t[x, x_1, \ldots, x_k]$ of $\mathscr{L}$; consider the term $u = \lambda x\, t$ (here, and only here, λx is taken in the sense of chapter 6), and let $g \in \mathscr{F}^\infty$ be the corresponding function. Then we have $(u)x = t$ in $\mathscr{D}$, and hence $(g)x = f$.
Thus $(g(a_1, \ldots, a_k))c = f(c, a_1, \ldots, a_k)$ for all $a_1, \ldots, a_k, c \in \mathscr{D}$; therefore, by definition :

$$\lambda x\, f(x, a_1, \ldots, a_k) = (e)g(a_1, \ldots, a_k).$$

Thus we have $\lambda x\, f(x, x_1, \ldots, x_k) = (e)g(x_1, \ldots, x_k)$ and this function is defined by the term $(e)u$, so it is in $\mathscr{F}^\infty$.

2. Spaces of continuous increasing functions

We will say that an ordered set $\mathscr{D}$ is *σ-complete* if every increasing sequence $a_n (n \in \mathbb{N})$ of elements of $\mathscr{D}$ has a least upper bound. This least upper bound will be denoted by $\sup_n a_n$.
Let $\mathscr{D}$, $\mathscr{D}'$ be two σ-complete ordered sets, and $f : \mathscr{D} \to \mathscr{D}'$ an increasing function. We will say that f is *σ-continuous increasing* (σ-c.i.) if, for every increasing sequence (a_n) in $\mathscr{D}$, we have $f(\sup_n a_n) = \sup_n f(a_n)$.
Let $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{E}$ be σ-complete ordered sets. We may define a structure of σ-complete ordered set on the cartesian product $\mathscr{D} \times \mathscr{D}'$, by putting :

$$(a, b) \le (a', b') \Leftrightarrow a \le a' \text{ and } b \le b'.$$

A function $f : \mathscr{D} \times \mathscr{D}' \to \mathscr{E}$ is σ-continuous increasing if and only if it is separately σ-continuous increasing (that is to say : for all $a \in \mathscr{D}$ and $a' \in \mathscr{D}'$, $f(x, a')$ and $f(a, x')$ are σ-c.i. functions).

The proof is immediate.

Let $\mathscr{D}$, $\mathscr{D}'$ be two σ-complete ordered sets. We may define a structure of σ-complete ordered set on the set $\mathscr{C}(\mathscr{D}, \mathscr{D}')$ of all σ-c.i. functions from $\mathscr{D}$ to $\mathscr{D}'$, by putting : $f \le g \Leftrightarrow f(a) \le g(a)$ for every $a \in \mathscr{D}$.

If $f_n (n \in \mathbb{N})$ is an increasing sequence in $\mathscr{C}(\mathscr{D}, \mathscr{D}')$, its least upper bound is the function $f : \mathscr{D} \to \mathscr{D}'$ defined by $f(a) = \sup_n f_n(a)$.
Indeed, f is clearly increasing ; we show that it is also σ-continuous :
let $a_k (k \in \mathbb{N})$ be an increasing sequence in $\mathscr{D}$, and $a = \sup_k a_k$.
Then $f(a) = \sup_n f_n(a) = \sup_n \sup_k f_n(a_k) = \sup_{n,k} f_n(a_k) = \sup_k \sup_n f_n(a_k) = \sup_k f(a_k)$.

The next proposition provides a very useful method for constructing functional β-models (and therefore models of combinatory logic).

Proposition 7.10.
The following data define a functional model of λ-calculus :
a σ-complete ordered set $\mathscr{D}$;

a σ-c.i. function $\Phi : \mathscr{D} \to \mathscr{C}(\mathscr{D},\mathscr{D})$;
a σ-c.i. function $\Psi : \mathscr{C}(\mathscr{D},\mathscr{D}) \to \mathscr{D}$.
This model is a β-model if and only if $\Phi \circ \Psi = Id$ (on $\mathscr{C}(\mathscr{D},\mathscr{D})$).
This β-model is extensional if and only if we have also : $\Psi \circ \Phi = Id$ (on $\mathscr{D}$). In this case, for all $a, b \in \mathscr{D}$, $a \le b$ if and only if $(a)c \le (b)c$ for every $c \in \mathscr{D}$.

For all $a, b \in \mathscr{D}$, define $(a)b = \Phi(a)(b)$; then $\mathscr{D}$ is an applicative structure, and the function $(a,b) \mapsto (a)b$ from $\mathscr{D} \times \mathscr{D}$ to $\mathscr{D}$ is σ-c.i. (obviously, it is separately σ-c.i.).
Let $\mathscr{F} = \mathscr{C}(\mathscr{D},\mathscr{D})$ and take $\mathscr{F}^\infty$ as the set of all σ-c.i. functions from $\mathscr{D}^{\mathbb{N}}$ to $\mathscr{D}$ which depend only on a finite number of coordinates. For every $f \in \mathscr{F}$, we put, by definition, $\lambda x\, f(x) = \Psi(f)$.
It remains to check conditions 1, 2, 3 of the definition of functional models.
It is obvious that each coordinate x_i is in $\mathscr{F}^\infty$. If $f, g \in \mathscr{F}^\infty$, then $(f)g$ is σ-c.i. (since $(a,b) \mapsto (a)b$ is σ-c.i.) and depends only on a finite number of coordinates ; thus $(f)g \in \mathscr{F}^\infty$. Finally, let $f(x, x_1, \ldots, x_k) \in \mathscr{F}^\infty$.
Then $(a_1, \ldots, a_k) \mapsto f(x, a_1, \ldots, a_k)$ is a σ-c.i. function from $\mathscr{D}^k$ to $\mathscr{F}$.
Hence $(a_1, \ldots, a_k) \mapsto \lambda x\, f(x, a_1, \ldots, a_k)$ is σ-c.i. from $\mathscr{D}^k$ to $\mathscr{D}$, which proves that $\lambda x\, f \in \mathscr{F}^\infty$.

The model obtained above is a β-model if and only if $\Phi \circ \Psi = Id$ on $\mathscr{C}(\mathscr{D},\mathscr{D})$ (by definition of β-models). This β-model is extensional if and only if we have, also : $\Psi \circ \Phi = Id$ on $\mathscr{D}$ (according to proposition 7.9.iv). Finally, if $(a)c \le (b)c$ for every $c \in \mathscr{D}$, then $\Phi(a) \le \Phi(b)$, thus $\Psi(\Phi(a)) \le \Psi(\Phi(b))$, since Ψ is increasing, and therefore $a \le b$.

Q.E.D.

3. Spaces of initial segments

Let D be a countable preordered set (recall that a preorder is a reflexive and transitive binary relation), the preorder on D being denoted by $\le$. A subset a of D will be called an *initial segment* if, for all $\alpha \in a$ and $\beta \le \alpha$, we have $\beta \in a$.
Let $a \subset D$; the least initial segment containing a is denoted by $\bar{a}$; it is the set of lower bounds of the elements of a.
We will denote by $\mathscr{S}(D)$ the space of initial segments of D ; the inclusion relation makes of $\mathscr{S}(D)$ a σ-complete ordered set. The set of finite subsets of D will be denoted by D^*.
On D^*, we define a preorder, still denoted by $\le$, by putting :

$a \le b \Leftrightarrow \bar{a} \subset \bar{b} \Leftrightarrow$ every member of a is a lower bound of an element of b.

Consider two countable preordered sets D and E ; let $\mathscr{D} = \mathscr{S}(D)$, $\mathscr{E} = \mathscr{S}(E)$.

For every $f \in \mathscr{C}(\mathscr{D},\mathscr{E})$, we define the *trace* of f, denoted by $\mathrm{tr}(f)$, which is a subset of $D^* \times E$:

$$\mathrm{tr}(f) = \{(a,\alpha) \in D^* \times E;\ \alpha \in f(\bar{a})\}.$$

Proposition 7.11.
The function tr is an isomorphism of ordered sets from $\mathscr{C}(\mathscr{D},\mathscr{E})$ onto the space $\mathscr{S}(D^* \times E)$ of initial segments of $D^* \times E$ with the product preorder :
$(a,\alpha) \le (b,\beta) \Leftrightarrow a \ge b$ and $\alpha \le \beta$.
For every $X \in \mathscr{S}(D^* \times E)$, we have $X = tr(f)$, where $f \in \mathscr{C}(\mathscr{D},\mathscr{E})$ is defined by : $f(u) = \{\beta \in E; (\exists a \in D^*)(a \subset u$ and $(a,\beta) \in X)\}$.

Let $f \in \mathscr{C}(\mathscr{D},\mathscr{E})$; then $\mathrm{tr}(f)$ is an initial segment of $D^* \times E$: indeed, if $(b,\beta) \in \mathrm{tr}(f)$ and $(a,\alpha) \le (b,\beta)$, then $\beta \in f(\bar{b})$, $\bar{a} \supset \bar{b}$ and $\alpha \le \beta$. Thus $\alpha \in f(\bar{b})$ (since $f(\bar{b})$ is an initial segment of E) and, since f is increasing, we have $f(\bar{b}) \subset f(\bar{a})$, and therefore $\alpha \in f(\bar{a})$.
Let $f, g \in \mathscr{C}(\mathscr{D},\mathscr{E})$; if $f \le g$, then $\mathrm{tr}(f) \subset \mathrm{tr}(g)$: indeed, if $(a,\alpha) \in \mathrm{tr}(f)$, then $\alpha \in f(\bar{a})$, and hence $\alpha \in g(\bar{a})$, since $f(\bar{a}) \subset g(\bar{a})$.
Conversely, we prove that $\mathrm{tr}(f) \subset \mathrm{tr}(g) \Rightarrow f \le g$: first, let a be a finite subset of D ; if $\alpha \in f(\bar{a})$, then $\alpha \in g(\bar{a})$, (since $\mathrm{tr}(f) \subset \mathrm{tr}(g)$) and hence $f(\bar{a}) \subset g(\bar{a})$.
Now let a be an initial segment of D ; since D is countable, we have, for instance $a = \{\alpha_0, \ldots, \alpha_n, \ldots\}$. Let $a_n = \{\alpha_0, \ldots, \alpha_n\} \in D^*$; $\bar{a}_n$ is an increasing sequence, the union of which is a. From what has just been proved, we deduce that $f(\bar{a}_n) \subset g(\bar{a}_n)$. Since both f and g are σ-c.i., we therefore have :
$f(a) = \cup_n f(\bar{a}_n) \subset \cup_n g(\bar{a}_n) = g(a)$.
Thus, tr is an isomorphism of ordered sets from $\mathscr{C}(\mathscr{D},\mathscr{E})$ into $\mathscr{S}(D^* \times E)$. It remains to prove that its image is the whole set $\mathscr{S}(D^* \times E)$.
Let $X \in \mathscr{S}(D^* \times E)$; we define $f : \mathscr{D} \to \mathscr{E}$ by taking $f(u) = \{\beta \in E; \exists a \in D^*, a \subset u, (a,\beta) \in X\}$ for every $u \in \mathscr{D}$. Indeed, $f(u)$ is an initial segment of E : if $\beta' \le \beta \in f(u)$, then there exists $a \in D^*$ such that $a \subset u$ and $(a,\beta) \in X$.
We have $(a,\beta') \le (a,\beta)$ in $D^* \times E$, thus $(a,\beta') \in X$, and hence $\beta' \in f(u)$.
Obviously, f is increasing ; it is also σ-continuous : indeed, let u_n be an increasing sequence in $\mathscr{D}$, and $u = \cup_n u_n$. We have $f(u_n) \subset f(u)$ for all n, thus $\cup_n f(u_n) \subset f(u)$. Conversely, if $\beta \in f(u)$, then there exists $a \in D^*$ such that $a \subset u$ and $(a,\beta) \in X$. Since a is finite, we have $a \subset u_n$ for some n, and therefore $\beta \in f(u_n)$. Thus $f(u) \subset \cup_n f(u_n)$.
Finally, we prove that $\mathrm{tr}(f) = X$: indeed, if $(a,\beta) \in X$, then, by definition of f, we have $\beta \in f(\bar{a})$ (since $a \subset \bar{a}$) ; thus $(a,\beta) \in \mathrm{tr}(f)$. Conversely, if $(a,\beta) \in \mathrm{tr}(f)$, then $\beta \in f(\bar{a})$, and hence, by definition of f, there exists $a' \in D^*$, $a' \subset \bar{a}$, such that $(a',\beta) \in X$. Since $a' \subset \bar{a}$, we have $a' \le a$, thus $(a,\beta) \le (a',\beta)$, and hence $(a,\beta) \in X$, since X is an initial segment.

Q.E.D.

We now consider a countable set D, and a function $i : D^* \times D \to D$.
If $a = \{\alpha_1, \dots, \alpha_n\} \in D^*$ and $\alpha \in D$, then $i(a, \alpha)$ will be denoted by $a \to \alpha$, or $\{\alpha_1, \dots, \alpha_n\} \to \alpha$.
We assume that a preorder is given on D ; we denote it by $\leq$ (as well as its extension to D^*, defined above). Let $\mathscr{D} = \mathscr{S}(D)$.
We wish to define two σ-c.i. functions :

$$\Phi : \mathscr{D} \to \mathscr{C}(\mathscr{D}, \mathscr{D}) \text{ and } \Psi : \mathscr{C}(\mathscr{D}, \mathscr{D}) \to \mathscr{D}.$$

This will be done as follows : there is a natural way of associating with the function $i : D^* \times D \to D$ two functions on the power sets denoted by i and i^{-1} :

$$i : \mathscr{P}(D^* \times D) \to \mathscr{P}(D) \text{ and } i^{-1} : \mathscr{P}(D) \to \mathscr{P}(D^* \times D).$$

Let $s : \mathscr{P}(D) \to \mathscr{S}(D)$ and $s' : \mathscr{P}(D^* \times D) \to \mathscr{S}(D^* \times D)$ be the functions defined by :

$s(X)$ (resp. $s'(X)$) $= \overline{X} =$ the least initial segment containing X

(X is any subset of D (resp. $D^* \times D$) and $\overline{X}$ is the set of lower bounds of elements of X). Thus we may define :
$\varphi = s' \circ i^{-1} : \mathscr{S}(D) \to \mathscr{S}(D^* \times D)$ and $\psi = s \circ i : \mathscr{S}(D^* \times D) \to \mathscr{S}(D)$.
Now, by proposition 7.11, tr is an isomorphism of ordered sets from $\mathscr{C}(\mathscr{D}, \mathscr{D})$ onto $\mathscr{S}(D^* \times D)$. Let $tr^{-1} : \mathscr{S}(D^* \times D) \to \mathscr{C}(\mathscr{D}, \mathscr{D})$ be the inverse function. Then, we may define :
$\Phi = tr^{-1} \circ \varphi : \mathscr{S}(D) \to \mathscr{C}(\mathscr{D}, \mathscr{D})$ and $\Psi = \psi \circ tr : \mathscr{C}(\mathscr{D}, \mathscr{D}) \to \mathscr{S}(D)$.
Since i, i^{-1}, s, s' are σ-c.i. functions, Φ and Ψ are also σ-c.i. Thus, by proposition 7.10, $(\mathscr{D}, \Phi, \Psi)$ defines a functional model of λ-calculus.

Lemma 7.12.

1. $u \supset \Psi \circ \Phi(u) (\equiv \lambda x(u)x)$ for every $u \in \mathscr{D}$ if and only if, for all $\alpha, \beta \in D$ and $a, b \in D^*$: $b \geq a$ and $\beta \leq \alpha \Rightarrow (b \to \beta) \leq (a \to \alpha)$.

2. $u \subset \Psi \circ \Phi(u)$ for every $u \in \mathscr{D}$ if and only if, for every $\gamma \in D$, there exist $\alpha, \beta \in D$ and $a, b \in D^*$ such that : $b \geq a$, $\beta \leq \alpha$ and $(a \to \alpha) \leq \gamma \leq (b \to \beta)$.
In particular, if i is onto, then $u \subset \Psi \circ \Phi(u)$ for every $u \in \mathscr{D}$.

Let $u \in \mathscr{D}$; then $\Psi \circ \Phi(u) = \psi \circ \varphi(u) = s \circ i \circ s' \circ i^{-1}(u)$; now
$s' \circ i^{-1}(u) =$

$$\{(b, \beta) \in D^* \times D \,;\, (\exists (a, \alpha) \in D^* \times D)\, (b, \beta) \leq (a, \alpha),\ i(a, \alpha) \in u\}.$$

Hence $\Psi \circ \Phi(u) =$

$$\{\gamma \in D \,;\, (\exists (a, \alpha), (b, \beta) \in D^* \times D)\ \gamma \leq i(b, \beta),\ (b, \beta) \leq (a, \alpha),\ i(a, \alpha) \in u\}.$$

1. Suppose that $(b, \beta) \leq (a, \alpha) \Rightarrow i(b, \beta) \leq i(a, \alpha)$ (i is a homomorphism with respect to $\leq$) ; then it is immediate that $\Psi \circ \Phi(u) \subset u$ for every $u \in \mathscr{D}$.
Conversely, suppose that $\Psi \circ \Phi(u) \subset u$ for every $u \in \mathscr{D}$, and let $\alpha, \beta \in D$ and $a, b \in D^*$ be such that $(b, \beta) \leq (a, \alpha)$. Take u as the set of lower bounds of $i(a, \alpha)$, and let $\gamma = i(b, \beta)$. It follows immediately that $\gamma \in \Psi \circ \Phi(u)$, thus $\gamma \in u$, and therefore $i(b, \beta) \leq i(a, \alpha)$.

2. Suppose that, for every $\gamma \in D$, there exist $\alpha, \beta \in D$, $a, b \in D^*$ such that : $(b, \beta) \le (a, \alpha)$ and $i(a, \alpha) \le \gamma \le i(b, \beta)$. If $\gamma \in u$, then $i(a, \alpha) \in u$ since u is an initial segment, thus $\gamma \in \Psi \circ \Phi(u)$.
Conversely, suppose $u \subset \Psi \circ \Phi(u)$ for every $u \in \mathscr{D}$. Let $\gamma \in D$, and take u as the set of lower bounds of γ. Then $\gamma \in \Psi \circ \Phi(u)$, and hence there exist $\alpha, \beta \in D$ and $a, b \in D^*$ such that : $\gamma \le i(b, \beta)$; $(b, \beta) \le (a, \alpha)$; $i(a, \alpha) \in u$. Therefore, $i(a, \alpha) \le \gamma$.
Q.E.D.

We may give explicit definitions of Ψ and Φ : let $f \in \mathscr{C}(\mathscr{D}, \mathscr{D})$; then $\Psi(f) = s \circ i(\mathrm{tr}(f))$, that is :

$$\Psi(f) = \{\beta \in D; (\exists \alpha \in D)(\exists a \in D^*)\ \beta \le i(a, \alpha) \text{ and } \alpha \in f(\bar{a})\}.$$

Now let $u, v \in \mathscr{D}$; then $\mathrm{tr}(\Phi(u)) = \varphi(u)$; thus, by proposition 7.11 (where we take $X = \varphi(u)$) :
$\Phi(u)(v) = \{\beta \in D\ ; \exists b \in D^*,\ b \subset v,\ (b, \beta) \in \varphi(u)\}$, that is to say
$\Phi(u)(v) = \{\beta \in D\ ; \exists a, b \in D^*, \exists \alpha \in D,\ b \subset v,\ (b, \beta) \le (a, \alpha),\ i(a, \alpha) \in u\}$.
Now condition $(b, \beta) \le (a, \alpha)$ may be written $\bar{b} \supset \bar{a}$ and $\beta \le \alpha$. Since v is an initial segment and $b \subset v$, we have $\bar{b} \subset v$, and hence $a \subset v$. Finally :

$$\Phi(u)(v) = \{\beta \in D; (\exists \alpha \in D)(\exists a \in D^*)\ a \subset v,\ \beta \le \alpha,\ i(a, \alpha) \in u\}.$$

The model defined by $(\mathscr{D}, \Phi, \Psi)$ is a functional β-model if and only if $\Phi \circ \Psi$ is the identity function on $\mathscr{C}(\mathscr{D}, \mathscr{D})$, or, equivalently, $\varphi \circ \psi$ is the identity function on $\mathscr{S}(D^* \times D)$ (since tr is an isomorphism).
Now, if $X \in \mathscr{S}(D^* \times D)$, then $\psi(X) = \{\beta\ ; (\exists (a, \alpha) \in X)\ \beta \le i(a, \alpha)\}$. Thus :
$\varphi \circ \psi(X) = \{(c, \gamma);$
$\qquad (\exists (a, \alpha), (b, \beta) \in D^* \times D)\ (c, \gamma) \le (b, \beta),\ i(b, \beta) \le i(a, \alpha) \text{ and } (a, \alpha) \in X\}$.
Clearly, $X \subset \varphi \circ \psi(X)$; $\varphi \circ \psi$ is the identity function if and only if, for every initial segment X of $D^* \times D$:
$(c, \gamma) \le (b, \beta)$, $i(b, \beta) \le i(a, \alpha)$, and $(a, \alpha) \in X \Rightarrow (c, \gamma) \in X$.
By taking $(c, \gamma) = (b, \beta)$, and X as the set of lower bounds of (a, α), we see that this condition can be written :

$$i(b, \beta) \le i(a, \alpha) \Rightarrow (b, \beta) \le (a, \alpha)$$

or, equivalently :

$$(b \to \beta) \le (a \to \alpha) \Rightarrow b \ge a \text{ and } \beta \le \alpha.$$

Let us notice that, if $D \ne \emptyset$, *the β-model $(\mathscr{D}, \Phi, \Psi)$ is non-trivial* : indeed, it has at least two elements, namely $\emptyset$ and D.

The model $(\mathscr{D}, \Phi, \Psi)$ is extensional if and only if we have, also, $\Psi \circ \Phi(u) = u$ for every $u \in \mathscr{D}$. By applying lemma 7.12, we obtain the following conditions :

$i(b, \beta) \le i(a, \alpha) \Leftrightarrow (b, \beta) \le (a, \alpha)$ for all $\alpha, \beta \in D$ and $a, b \in D^*$;

for every $\gamma \in D$, there exist $\alpha, \beta \in D$ and $a, b \in D^*$ such that
$i(a,\alpha) \leq \gamma \leq i(b,\beta)$ and $(b,\beta) \leq (a,\alpha)$.

Now, by the previous condition, we therefore have $i(b,\beta) \leq i(a,\alpha)$, and hence $\gamma \leq i(a,\alpha) \leq \gamma$. So we have proved :

Theorem 7.13. *Let D be a countable preordered set, and i a function from $D^* \times D$ into D. Define $\Phi : \mathscr{D} \to \mathscr{C}(\mathscr{D},\mathscr{D})$ and $\Psi : \mathscr{C}(\mathscr{D},\mathscr{D}) \to \mathscr{D}$ as follows :*
$\Phi(u)(v)$ (also denoted by $(u)v$)

$$= \{\beta \in D; (\exists \alpha \geq \beta)(\exists a \in D^*)\, a \subset v \text{ and } (a \to \alpha) \in u\} ;$$

$\Psi(f)$ (also denoted by $\lambda x\, f(x)$)

$$= \{\beta \in D; (\exists \alpha \in D)(\exists a \in D^*)\, \beta \leq (a \to \alpha) \text{ and } \alpha \in f(\bar{a})\}.$$

Then $(\mathscr{D},\Phi,\Psi)$ defines a functional model of λ-calculus, which is a β-model (necessarily non-trivial) if and only if :
$(b \to \beta) \leq (a \to \alpha) \Rightarrow b \geq a$ *and* $\beta \leq \alpha$ *for all* $\alpha, \beta \in D$ *and* $a, b \in D^*$.

$(\mathscr{D},\Phi,\Psi)$ is an extensional β-model if and only if :
1. *$(b \to \beta) \leq (a \to \alpha) \Leftrightarrow b \geq a$ and $\beta \leq \alpha$ for all $\alpha, \beta \in D$ and $a, b \in D^*$.*
2. *For every $\gamma \in D$, there exist $\alpha \in D$ and $a \in D^*$ such that $\gamma \leq (a \to \alpha) \leq \gamma$.*

In particular, if i is onto, and if condition 1 is satisfied, then $(\mathscr{D},\Phi,\Psi)$ is an extensional β-model.

Non-extensional models ($\mathscr{P}(\omega)$ and Engeler's model)

Here we take D as any countable set with the trivial preorder :
$\alpha \leq \beta \Leftrightarrow \alpha = \beta$. The induced preorder on D^* is : $a \leq b \Leftrightarrow a \subset b$.
We have $\bar{a} = a$ for every $a \in D^*$.
Any subset of D is an initial segment, thus $\mathscr{D} = \mathscr{P}(D)$.
We take i as any one-one function from $D^* \times D$ to D. Clearly, the following condition holds : $(b \to \beta) \leq (a \to \alpha) \Rightarrow b \geq a$ and $\beta \leq \alpha$. We therefore have a β-model of λ-calculus.
Note that, in this case, the definitions of Φ and Ψ are :

$(u)v = \{\alpha \in D\,; \exists a \in D^*,\, a \subset v$ and $(a \to \alpha) \in u\}$ for all $u, v \in \mathscr{D}$;
$\lambda x\, f(x) = \{a \to \alpha\,; \alpha \in D,\, a \in D^*$ and $\alpha \in f(a)\}$ for every $f \in \mathscr{C}(\mathscr{D},\mathscr{D})$.

By lemma 7.12(1), this model does not satisfy the condition $u \supset \Psi \circ \Phi(u)$, so it cannot be extensional ; indeed, this condition can be written :
$b \geq a$ and $\alpha \leq \beta \Rightarrow (b \to \beta) \leq (a \to \alpha)$, or equivalently :
$b \supset a$ and $\alpha = \beta \Rightarrow b = a$ and $\alpha = \beta$, which obviously does not hold.

We obtain *Plotkin and Scott's model* $\mathscr{P}(\omega)$ by taking $D = \mathbb{N}$; i is the " standard " one-to-one function from $D^* \times D$ onto D defined by :

$$i(e,n) = \frac{1}{2}(m+n)(m+n+1)+n, \text{ where } m = \sum_{k \in e} 2^k.$$

Engeler's model $\mathscr{D}_A$ is obtained as follows :
Let A be either a finite or a countable set, and D be the least set containing A such that $\alpha \in D$, $a \in D^* \Rightarrow (a,\alpha) \in D$ (it is assumed that none of the members of A are ordered pairs). The one-one function $i : D^* \times D \to D$ is defined by taking $i(a,\alpha) = (a,\alpha)$.

Extensional models

Theorem 7.14. *Let D be a countable set, i a one-to-one mapping of $D^* \times D$ into D, and $\leq_0$ a preorder on D such that :*

$$(b \to \beta) \leq_0 (a \to \alpha) \Rightarrow a \leq_0 b \text{ and } \beta \leq_0 \alpha.$$

Then, there exists a preorder on D, which we denote by $\leq$, as well as its extension to D^*, with the following properties :
i) $\beta \leq_0 \alpha \Rightarrow \beta \leq \alpha$.
ii) $(b \to \beta) \leq (a \to \alpha) \Leftrightarrow a \leq b$ *and* $\beta \leq \alpha$.

Remark. In view of theorem 7.13, we therefore obtain a non-trivial extensional β-model. We have the following definitions for functions Φ and Ψ in this β-model (u, v range in $\mathscr{D}$, while f ranges in $\mathscr{C}(\mathscr{D},\mathscr{D})$) :

$$\Phi(u)v \equiv (u)v = \{\alpha \in D;\ \exists a \in D^*,\ a \subset v \text{ and } (a \to \alpha) \in u\} ;$$
$$\Psi(f) \equiv \lambda x\, f(x) = \{a \to \alpha;\ \alpha \in D,\ a \in D^*,\ \alpha \in f(\bar{a})\}.$$

Indeed, if $\beta \leq a \to \alpha$ and $\alpha \in f(\bar{a})$, then $\beta = a' \to \alpha'$, with $a \leq a'$ (thus $\bar{a} \subset \bar{a}'$) and $\alpha' \leq \alpha$. Hence $\alpha' \in f(\bar{a})$ and finally $\alpha' \in f(\bar{a}')$.

Proof of the theorem : let R be a preorder on D ; the corresponding preorder on D^* will be denoted by R^*. Thus, by definition, for all $a, b \in D^*$:

$$a R^* b \Leftrightarrow (\forall \alpha \in a)(\exists \beta \in b)\alpha R \beta.$$

Consider the following condition, relative to the preorder R :
(C) $\quad a R^* b$ and $\beta R \alpha \Rightarrow (b \to \beta) R (a \to \alpha)$ for all $a, b \in D^*$ and $\alpha, \beta \in D$.
The intersection S of any set $\mathscr{R}$ of preorders which satisfy condition (C) still satisfies (C) : indeed, if $a S^* b$ and $\beta S \alpha$, then, clearly, $a R^* b$ and $\beta R \alpha$ for every $R \in \mathscr{R}$. Hence, $(b \to \beta) R (a \to \alpha)$, and therefore $(b \to \beta) S (a \to \alpha)$.
This allows us to define the least preorder R_0 on D which contains $\leq_0$ and satisfies condition (C) (R_0 is the intersection of all preorders which satisfy these conditions ; there exists at least one such preorder, namely $D \times D$). Now, since i is one-to-one, we can define a binary relation S_0 on D, by putting :

$(b \to \beta)\, S_0\, (a \to \alpha) \Leftrightarrow a\, R_0^*\, b$ and $\beta\, R_0\, \alpha$. Obviously, S_0 is a preorder, because i is one-one, and both R_0 and R_0^* are preorders.
Now $S_0 \subset R_0$, since R_0 satisfies condition (C).
It follows immediately that $S_0^* \subset R_0^*$. Let $a, b \in D^*$, $\alpha, \beta \in D$ be such that $a\, S_0^*\, b$ and $\beta\, S_0\, \alpha$; then we have $a\, R_0^*\, b$ and $\beta\, R_0\, \alpha$, and hence, by definition of S_0 :
$(b \to \beta)\, S_0\, (a \to \alpha)$. Thus S_0 satisfies condition (C).
Moreover, S_0 contains $\leq_0$:
Indeed, if $\beta \leq_0 \alpha$, then $\alpha = (a' \to \alpha')$ and $\beta = (b' \to \beta')$; by the hypothesis on the preorder $\leq_0$, we have $a' \leq_0 b'$ and $\beta' \leq_0 \alpha'$, and thus $(b' \to \beta')\, S_0\, (a' \to \alpha')$ by definition of S_0.
By the minimality of R_0, it follows that $R_0 \subset S_0$, and therefore $R_0 = S_0$.
Thus, by definition of S_0 :

$$(b \to \beta)\, R_0\, (a \to \alpha) \Leftrightarrow a\, R_0^*\, b \text{ and } \beta\, R_0\, \alpha.$$

So R_0 satisfies conditions (i) and (ii) of the theorem, and can be taken as the desired preorder $\leq$.

Proposition 7.15. *For all $\alpha, \beta \in D$, we have $\beta \leq \alpha$ if and only if there exist $k \geq 0$, $a_1, \ldots, a_k$, $b_1, \ldots, b_k \in D^*$ and $\alpha', \beta' \in D$, such that $a_i \leq b_i$ $(1 \leq i \leq k)$, $\beta' \leq_0 \alpha'$ and $\alpha = a_1, \ldots, a_k \to \alpha'$, $\beta = b_1, \ldots, b_k \to \beta'$.*

The notation $a_1, a_2, \ldots, a_k \to \alpha'$ stands for $a_1 \to (a_2 \to \ldots \to (a_k \to \alpha')\ldots)$.

Remark. In case $k = 0$, we understand that the condition means $\beta \leq_0 \alpha$.

Proof of the proposition : we still use the notation R_0 for the preorder $\leq$.
We define a binary relation R on D by :
$\beta\, R\, \alpha \Leftrightarrow$ there exist $k \geq 0$, $a_1, \ldots, a_k, b_1, \ldots, b_k \in D^*$ and $\alpha', \beta' \in D$, such that $a_i\, R_0^*\, b_i$ $(1 \leq i \leq k)$, $\beta' \leq_0 \alpha'$ and $\alpha = a_1, \ldots, a_k \to \alpha'$, $\beta = b_1, \ldots, b_k \to \beta'$.

We first prove that R is a preorder.
Let $\alpha, \beta, \gamma \in D$ be such that $\beta\, R\, \alpha$ and $\gamma\, R\, \beta$. Thus we have :
$\alpha = a_1, \ldots, a_k \to \alpha'$, $\beta = b_1, \ldots, b_k \to \beta'$, with $a_i\, R_0^*\, b_i$, $\beta' \leq_0 \alpha'$ and
$\beta = b'_1, \ldots, b'_l \to \beta''$, $\gamma = c_1, \ldots, c_l \to \gamma'$, with $b'_j\, R_0^*\, c_j$, $\gamma' \leq_0 \beta''$.
If $l \geq k$, then (using both expressions for β, and the fact that i is one-one) :
$b_1 = b'_1, \ldots, b_k = b'_k$ and $\beta' = b'_{k+1}, \ldots, b'_l \to \beta''$. Since $\beta' \leq_0 \alpha'$, we have, by the hypothesis on $\leq_0$ and the fact that i is onto :
$\alpha' = a'_{k+1}, \ldots, a'_l \to \alpha''$ with $\beta'' \leq_0 \alpha''$, and $a'_i \leq_0 b'_i$ $(k+1 \leq i \leq l)$; therefore $\gamma' \leq_0 \alpha''$ and $a'_i\, R_0^*\, c_i$ for $k+1 \leq i \leq l$.
Thus $\alpha = a_1, \ldots, a_k, a'_{k+1}, \ldots, a'_l \to \alpha''$ and $\gamma = c_1, \ldots, c_k, c_{k+1}, \ldots, c_l \to \gamma'$.
Now $b'_i\, R_0^*\, c_i\, (1 \leq i \leq l)$ and $a_i\, R_0^*\, b_i$, and thus $a_i\, R_0^*\, c_i$ for $1 \leq i \leq k$ (since $b'_i = b_i$).
It follows that $\gamma\, R\, \alpha$.
The proof is similar in case $k \geq l$.

We now prove that $R \subset R_0$; if $\beta\, R\, \alpha$, then we have :
$\alpha = a_1, \ldots, a_k \to \alpha'$, $\beta = b_1, \ldots, b_k \to \beta'$, with $a_i\, R_0^*\, b_i$ $(1 \leq i \leq k)$ and $\beta' \leq_0 \alpha'$.

We prove $\beta\, R_0\, \alpha$ by induction on k : this is obvious when $k = 0$. Assume the result for $k-1$; then $\beta''\, R_0\, \alpha''$, with $\alpha'' = a_2, \ldots, a_k \to \alpha'$, $\beta'' = b_2, \ldots, b_k \to \beta'$. Now $\alpha = a_1 \to \alpha''$, $\beta = b_1 \to \beta''$, and $a_1\, R_0^*\, b_1$, $\beta''\, R_0\, \alpha''$; thus $\beta\, R_0\, \alpha$.

Finally, we prove that R satisfies condition (C) :
Let $a, b \in D^*$, $\alpha, \beta \in D$ be such that $a\, R^*\, b$ and $\beta\, R\, \alpha$. Since $R \subset R_0$, it follows that $a\, R_0^*\, b$. Now $\beta\, R\, \alpha$ and therefore, by definition of R :
$\alpha = a_1, \ldots, a_k \to \alpha'$, $\beta = b_1, \ldots, b_k \to \beta'$, with $a_i\, R_0^*\, b_i$ $(1 \le i \le k)$ and $\beta' \le_0 \alpha'$. Thus :
$a \to \alpha = a, a_1, \ldots, a_k \to \alpha'$, and $b \to \beta = b, b_1, \ldots, b_k \to \beta'$. Now $a\, R_0^*\, b$, and hence $(b \to \beta)\, R\, (a \to \alpha)$.
Since $R \subset R_0$ and R satisfies (C), we see that $R_0 = R$: this is the expected conclusion.

Q.E.D.

Models over a set of atoms (Scott's model $\mathscr{D}^\infty$)

Let A be a finite or countable non-empty set, the elements of which will be called *atoms*. It is convenient to assume that no element of A is an ordered pair. We define, inductively, a set D of *formulas*, and a one-to-one function $i : D^* \times D \to D$ ($i(a, \alpha)$ will also be denoted by $a \to \alpha$) :

- Every atom α is a formula ;

Let α be a formula and a be a finite set of formulas ; then :

- If $\alpha \in A$ (α is an atom) and $a = \emptyset$, then we take $\emptyset \to \alpha = i(\emptyset, \alpha) = \alpha$.
- Otherwise, the ordered pair (a, α) is a formula, and we take :
$a \to \alpha = i(a, \alpha) = (a, \alpha)$.

It follows that the atoms are the only formulas which are not ordered pairs.

Clearly, i is onto ; it is also one-one : if $a \to \alpha = b \to \beta$, then, either the formula $a \to \alpha$ is an atom, and then $a = b = \emptyset$ and $\alpha = \beta$, or it is not an atom, and then $(a, \alpha) = (b, \beta)$.

Every formula α can be written in the form $\alpha = a_1, \ldots, a_k \to \alpha_0$, where $k \ge 0$, $\alpha_0 \in A$, $a_i \in D^*$. This expression is unique if we impose $a_k \ne \emptyset$, or $k = 0$. Thus the other possible expressions for α are :

$\alpha = a_1, \ldots, a_k, \emptyset, \ldots, \emptyset \to \alpha_0$.

The rank of a formula α, denoted by $rk(\alpha)$, is now defined by induction :
$rk(\alpha) = 0$ whenever α is an atom ;
$rk(a \to \alpha) = 1 + \sup(rk(\alpha), \sup\{rk(\xi); \xi \in a\})$ if $a \ne \emptyset$ or α is not an atom.

We consider a preorder on A, denoted by $\le$. We extend it to the whole set D by defining $\beta \le \alpha$ by induction on $rk(\alpha) + rk(\beta)$, as follows :

If $\alpha, \beta \in A$, then $\beta \le \alpha$ is already defined.

If $rk(\alpha)+rk(\beta)\geq 1$, then we write $\alpha = a \to \alpha'$, $\beta = b \to \beta'$, and we put $\beta \leq \alpha \Leftrightarrow \beta' \leq \alpha'$ and $b \geq a$ (every element of a is smaller than some element of b).

Note that $rk(\alpha')+rk(\beta') < rk(\alpha)+rk(\beta)$; also $b \geq a$ is already defined : indeed, if $\alpha_0 \in a$ and $\beta_0 \in b$, then $rk(\alpha_0)+rk(\beta_0) < rk(\alpha)+rk(\beta)$.

From this definition of the preorder $\leq$ on D, it follows that :

$$b \to \beta \leq a \to \alpha \Leftrightarrow b \geq a \text{ and } \beta \leq \alpha \ ;$$

this shows that we have defined an extensional β-model of λ-calculus (theorem 7.13).

Remark. This model could be obtained by using theorem 7.14 :
define $\alpha \leq_0 \beta$ for $\alpha, \beta \in D$, by $\alpha = \beta$ or ($\alpha, \beta \in A$ and $\alpha \leq \beta$). It is easy to check that $\leq_0$ is a preorder and that $(b \to \beta) \leq_0 (a \to \alpha) \Rightarrow a \leq_0 b$ and $\beta \leq_0 \alpha$.

4. Applications

In this section, we use the models over a set A of atoms defined page 124, taking the trivial preorder on A ($\alpha \leq \beta \Leftrightarrow \alpha = \beta$). In that case, the atoms are the maximal elements ; among the upper bounds of a given formula $a_1, \ldots, a_k \to \alpha_0$ ($\alpha_0 \in A$), there is one and only one atom which is α_0.
Let $\alpha, \beta \in D$; then α and β are not $\leq$-comparable unless there is an atom greater than α and β.
If $\alpha = a_1, \ldots, a_k \to \alpha_0$ with $\alpha_0 \in A, k \geq 0, a_k \neq \emptyset$, then $\alpha' \leq \alpha$ if and only if $\alpha' = a'_1, \ldots, a'_l \to \alpha_0$, with $l \geq k$ and $a_1 \leq a'_1, \ldots, a_k \leq a'_k$.

i) Embeddings of applicative structures

Theorem 7.16. *Every applicative structure may be embedded in a model of ECL (extensional combinatory logic, see page 99).*

Let A be an applicative structure (that is to say a set together with a binary function). We will assume that A is countable (the results below may be extended to the case where A is uncountable by means of the compactness theorem of predicate calculus). We consider the functional β-model constructed as above (page 124), with A as the set of atoms. We define $j : A \to D$ and $J : A \to \mathscr{D}$ by taking, for every $\alpha \in A$:
$j(\alpha) = \{o\} \to \alpha$ where o is some fixed element of A ;
$J(\alpha) = \{\delta \in D \,;\, (\exists k \geq 0)(\exists \alpha_1, \ldots, \alpha_k \in A)\ \delta \leq \{j(\alpha_1)\}, \ldots, \{j(\alpha_k)\} \to j(\alpha\alpha_1 \ldots \alpha_k)\}$.

Note that, if $\alpha, \alpha_1, \ldots, \alpha_k \in A$, then $\alpha\alpha_1 \ldots \alpha_k \in A$ (A is an applicative structure).

For every $\alpha \in A$, $J(\alpha)$ is clearly an initial segment of D.

We have seen that $\mathscr{D} = \mathscr{S}(D)$ is a model of ECL. Now we prove that J is the desired embedding of A into $\mathscr{D}$.

J is one-one : indeed, we have $j(\alpha) \in J(\alpha)$ for every $\alpha \in A$ (take $k = 0$ in the definition of $J(\alpha)$). Therefore, if $J(\alpha) = J(\alpha')$, then $j(\alpha') \in J(\alpha)$ that is :
$\{o\} \to \alpha' \leq \{j(\alpha_1)\}, \ldots, \{j(\alpha_k)\}, \{o\} \to \alpha\alpha_1 \ldots \alpha_k$.
Now, since α' and $\alpha\alpha_1 \ldots \alpha_k$ are atoms, we have necessarily $k = 0$ and $\alpha' = \alpha$.

$(J(\alpha))J(\alpha') \subset J(\alpha\alpha')$: let $\xi \in (J(\alpha))J(\alpha')$.
By theorem 7.13, there exists $d \subset J(\alpha')$ such that $d \to \xi \in J(\alpha)$, that is :
$d \to \xi \leq \{j(\alpha_1)\}, \ldots, \{j(\alpha_k)\} \to j(\alpha\alpha_1 \ldots \alpha_k)$.
If $k = 0$, then $d \to \xi \leq \{o\} \to \alpha$; then $o \in d$, which is impossible because $o \notin J(\alpha')$.
If $k \geq 1$, then $j(\alpha_1) \in \bar{d}$, thus $j(\alpha_1) \in J(\alpha')$, hence $\alpha_1 = \alpha'$ (see above).
Thus $\xi \leq \{j(\alpha_2)\}, \ldots, \{j(\alpha_k)\} \to j(\alpha\alpha'\alpha_2 \ldots \alpha_k)$, and therefore $\xi \in J(\alpha\alpha')$.

$J(\alpha\alpha') \subset (J(\alpha))J(\alpha')$:
If $\xi \in J(\alpha\alpha')$, then $\xi \leq \{j(\alpha_1)\}, \ldots, \{j(\alpha_k)\} \to j(\alpha\alpha'\alpha_1 \ldots \alpha_k)$.
Let $d = \{j(\alpha')\}$; then $d \subset J(\alpha')$. Moreover :
$d \to \xi \leq \{j(\alpha')\}, \{j(\alpha_1)\}, \ldots, \{j(\alpha_k)\} \to j(\alpha\alpha'\alpha_1 \ldots \alpha_k)$.
Therefore, $d \to \xi \in J(\alpha)$ and it follows that $\xi \in (J(\alpha))J(\alpha')$.

Q.E.D.

ii) Extensional combinatory logic with couple

Let $\mathscr{L}$ be the language of combinatory logic (see chapter 6), with additional constant symbols c, p_1, p_2. The term $(c)xy$ is called the *couple* (or *ordered pair*) x, y ; the term $(p_1)x$(resp. $(p_2)x$) is called the *first* (resp. the *second*) *projection of* x.

We denote by $ECLC$ (for *extensional combinatory logic with couple*) the following system of axioms, which an extension of ECL (*extensional combinatory logic*, see page 99) :

ECL, $(p_1)(c)xy = x$, $(p_2)(c)xy = y$, $((c)(p_1)x)(p_2)x = x$;
$(p_1)xy = (p_1)(x)y$, $(p_2)xy = (p_2)(x)y$.

The first three axioms mean that x (resp. y) is the first (resp. the second) projection of the couple $(c)xy$, and that each x is identical to the couple formed by p_1x, p_2x. The last two axioms mean that, for every x, the function defined by p_1x (resp. p_2x) is $p_1 \circ x$ (resp. $p_2 \circ x$).
As a consequence of these axioms, we have :

$$(c)xyz = ((c)(x)z)(y)z.$$

Indeed, according to the third axiom, it is sufficient to prove both :
$(p_1)(c)xyz = (p_1)((c)(x)z)(y)z$ and $(p_2)(c)xyz = (p_2)((c)(x)z)(y)z$.
Now we have : $(p_1)(c)xyz = ((p_1)(c)xy)z$ (4th axiom) $= (x)z$ (1st axiom)

$= (p_1)((c)(x)z)(y)z$ (1st axiom). Same proof for p_2.
We also deduce :

$$p_1 c = \mathbf{1},\ p_2 c = \mathbf{0},\ (c)p_1 p_2 = I.$$

Indeed $(p_1 c)xy = ((p_1)(c)x)y$ (4th axiom) $= (p_1)((c)x)y$ (4th axiom) $= x$ (1st axiom), and hence $p_1 c = \mathbf{1}$ by extensionality.
Moreover, $(cp_1 p_2)x = ((c)(p_1)x)(p_2)x$ (see above) $= x$ (3rd axiom), thus $cp_1 p_2 = I$ by extensionality.

Theorem 7.17.
ECLC has a non-trivial model (that is a model of cardinality > 1*).*

Consider an infinite countable set of atoms A, with the trivial preorder. Let $A = A_1 \cup A_2$ be some partition of A in two infinite subsets. Let D_i $(i = 1,2)$ be the set of lower bounds in D of the elements of A_i. Then $D = D_1 \cup D_2$ is a partition of D in two initial segments.
Let $\varphi_1 : A \to A_1$, $\varphi_2 : A \to A_2$ be two one-to-one mappings ; they can be extended to isomorphisms of ordered sets from D onto D_1, D_2 :
whenever $\alpha = a_1, \ldots, a_k \to \alpha_0$ $(\alpha_0 \in A)$, take $\varphi_1(\alpha) = a_1, \ldots, a_k \to \varphi_1(\alpha_0)$ and $\varphi_2(\alpha) = a_1, \ldots, a_k \to \varphi_2(\alpha_0)$.
Let $\mathscr{D} = \mathscr{S}(D)$; the function $\varphi_1^{-1} : \mathscr{P}(D) \to \mathscr{P}(D)$ maps $\mathscr{S}(D)$ into $\mathscr{S}(D)$, since φ_1 is an isomorphism from D onto D_1. Now this function is clearly σ-c.i., so there exists $p_1 \in \mathscr{D}$ such that $(p_1)u = \varphi_1^{-1}(u)$ for every $u \in \mathscr{D}$. Similarly, there is a $p_2 \in \mathscr{D}$ such that $(p_2)u = \varphi_2^{-1}(u)$.
Also, we may define $c \in \mathscr{D}$ such that $(c)uv = \varphi_1(u) \cup \varphi_2(v)$ for all $u, v \in \mathscr{D}$: indeed, since φ_1 and φ_2 are isomorphisms of ordered sets, $\varphi_1(u) \cup \varphi_2(v)$ is an initial segment of D whenever $u, v \in \mathscr{D}$. Thus, this function maps $\mathscr{D} \times \mathscr{D}$ into $\mathscr{D}$, and it is σ-c.i. : this yields the existence of c.
We therefore have :

$(p_1)(c)uv = \varphi_1^{-1}(\varphi_1(u) \cup \varphi_2(v)) = u$ and similarly $(p_2)(c)uv = v$.

Also, $((c)(p_1)u)(p_2)u = \varphi_1(\varphi_1^{-1}u) \cup \varphi_2(\varphi_2^{-1}u) = (u \cap D_1) \cup (u \cap D_2) = u$. Thus, the first three axioms of *ECLC* are satisfied in the model under consideration.
Moreover, we have $\alpha \in (p_1 u)v \Leftrightarrow (\exists a \subset v)(a \to \alpha) \in p_1 u$ (theorem 7.13) ; now, by definition of $p_1 u$, we have $\alpha \in (p_1 u)v \Leftrightarrow (\exists a \subset v)\varphi_1(a \to \alpha) \in u$; on the other hand, $\varphi_1(a \to \alpha) = a \to \varphi_1(\alpha)$ by definition of φ_1, and hence :
$\alpha \in (p_1 u)v \Leftrightarrow (\exists a \subset v) a \to \varphi_1(\alpha) \in u$;
therefore, we obtain $\alpha \in (p_1 u)v \Leftrightarrow \varphi_1(\alpha) \in (u)v$, i.e. $\alpha \in (p_1 u)v \Leftrightarrow \alpha \in \varphi_1^{-1}((u)v)$, and finally $(p_1)uv = (p_1)(u)v$. This proves the last two axioms.

Q.E.D.

We now give a set of equational formulas, denoted by $ECLC_=$, which axiomatize the universal consequences of *ECLC* :

$ECL_=$ (a set of equations which axiomatize the universal consequences of ECL, see chapter 6, page 104) ;
$\lambda x\lambda y(p_1)xy = \lambda x\lambda y\,x$; $\lambda x\lambda y(p_2)xy = \lambda x\lambda y\,y$; $\lambda x((c)(p_1)x)(p_2)x = \lambda x\,x$;
$\lambda x\lambda y(p_1)xy = \lambda x\lambda y(p_1)(x)y$; $\lambda x\lambda y(p_2)xy = \lambda x\lambda y(p_2)(x)y$.
Clearly, these formulas are universal consequences of $ECLC$. Conversely, let $\mathscr{M}$ be a model of these formulas : since $\mathscr{M}$ satisfies $ECL_=$, it can be embedded in a model of ECL, which satisfies the last five axioms (these are equations involving closed terms : since they hold in $\mathscr{M}$, they also hold in any extension of $\mathscr{M}$). Thus $\mathscr{M}$ is embedded in a model of $ECLC$, and therefore it satisfies all universal consequences of $ECLC$.

Theorem 7.18. *$ECLC$ is not equivalent to a system of universal axioms.*

It follows that neither CL nor ECL are equivalent to systems of universal axioms, since $ECLC$ is obtained by adding universal axioms either to CL or to ECL.

Proof : it suffices to exhibit a submodel of the above model of $ECLC$, in which the extensionality axiom fails.
With each formula $\alpha \in D$, we associate a value $|\alpha| \in \{0,1\}$, defined by induction on the rank of α, as follows :

if α is an atom, then $|\alpha| = 0$;

if $rk(\alpha) \geq 1$, say $\alpha = a \to \beta$, then we define $|a| = \inf\{|\gamma|; \gamma \in a\}$ (note that $|\gamma|$ is already defined since $rk(\gamma) < rk(\alpha)$; also, if $a = \emptyset$, then $|a| = 1$). Then we take $|\alpha| = |a| \to |\beta|$, where $\epsilon \to \epsilon'$ is defined in the usual way for $\epsilon, \epsilon' \in \{0,1\}$ ($|\beta|$ is already defined since $rk(\beta) < rk(\alpha)$).

For every subset u of D (particularly for $u \in \mathscr{D}$), we define $|u| = \inf\{|\alpha|; \alpha \in u\}$.

Lemma 7.19. *If $\alpha, \beta \in D$ and $\alpha \leq \beta$, then $|\alpha| \geq |\beta|$.*

The proof is by induction on $rk(\alpha) + rk(\beta)$.
If α, β are atoms, then $\alpha \leq \beta \Rightarrow \alpha = \beta$.
Otherwise, we have $\alpha = a \to \alpha'$, $\beta = b \to \beta'$.
Since $\alpha \leq \beta$, we have $a \geq b$ and $\alpha' \leq \beta'$. Suppose $|\alpha| < |\beta|$, that is $|\alpha| = 0$ and $|\beta| = 1$; thus $|a| = 1$ and $|\alpha'| = 0$. Since $a \geq b$, every element of b is smaller than some element of a ; therefore $|b| = 1$ (if $b = \emptyset$, this is obvious ; if $b \neq \emptyset$, it follows from the induction hypothesis). Since $|\beta| = |b| \to |\beta'| = 1$, it follows that $|\beta'| = 1$; since $\alpha' \leq \beta'$, we have, by induction hypothesis, $|\alpha'| \geq |\beta'|$, and hence $|\alpha'| = 1$, which is a contradiction.

Q.E.D.

Lemma 7.20. *Let $u \in \mathscr{D}$. Then $|u| = 1$ if and only if $|(u)v| = 1$ for every $v \in \mathscr{D}$ such that $|v| = 1$.*

Let $u, v \in \mathscr{D}$ be such that $|u| = |v| = 1$; we prove that $|(u)v| = 1$: if $\alpha \in uv$, then $a \subset v$ and $a \to \alpha \in u$; thus $|a| = |a \to \alpha| = 1$, and therefore $|\alpha| = 1$, by definition of $|a \to \alpha|$.
Conversely, suppose that $|u| = 0$; then there exists $\alpha \in u$ such that $|\alpha| = 0$. Since i is onto, we have $\alpha = b \to \beta$ for some $b \in D^*$ and $\beta \in D$. Thus $|b| = 1$ and $|\beta| = 0$. Let $v \in \mathscr{D}$ be the set of lower bounds of the elements of b. By lemma 7.19, we have $|v| = 1$; now $\beta \in (u)v$ since $b \subset v$ and $b \to \beta \in u$. Since $|\beta| = 0$, we have $|(u)v| = 0$.

Q.E.D.

Lemma 7.21. *Let $u \in \mathscr{D}$ and $k \in \mathbb{N}$. Then $|u| = 1$ if and only if $|(u)v_1 \dots v_k| = 1$ for all $v_1, \dots, v_k \in \mathscr{D}$ such that $|v_1| = \dots = |v_k| = 1$.*

This follows immediately from lemma 7.20, by induction on k.

Q.E.D.

Lemma 7.22. $|K| = |S| = |p_1| = |p_2| = |c| = 1$.

The considered model satisfies ECL, and therefore the axiom $(K)xy = x$. Thus $(K)uv = u$ for all $u, v \in \mathscr{D}$. Hence, $|u| = |v| = 1 \Rightarrow |(K)uv| = 1$; therefore, $|K| = 1$, by lemma 7.21.
Similarly, we have $(S)uvw = ((u)w)(v)w$ for all $u, v, w \in \mathscr{D}$. If $|u| = |v| = |w| = 1$, then $|((u)w)(v)w| = 1$ by lemma 7.20, and hence $|(S)uvw| = 1$.
Therefore, $|S| = 1$ (lemma 7.21).
Note that, for every formula $\alpha \in D$, we have $|\alpha| = |\varphi_1(\alpha)| = |\varphi_2(\alpha)|$: this is immediate from the definition of φ_1, φ_2, by induction on $rk(\alpha)$. Now, by definition of p_1, we have $\alpha \in (p_1)u \Leftrightarrow \varphi_1(\alpha) \in u$, for every $u \in \mathscr{D}$. Therefore, if $|u| = 1$, then $|\alpha| = 1$ for every $\alpha \in (p_1)u$, and hence $|(p_1)u| = 1$. It follows that $|p_1| = 1$ (lemma 7.21). Similarly, $|p_2| = 1$. Finally, for every formula $\alpha \in D$, and all $u, v \in \mathscr{D}$, we have $\alpha \in (c)uv \Leftrightarrow \alpha \in \varphi_1(u)$ or $\alpha \in \varphi_2(v)$. If $|u| = |v| = 1$, then $|\varphi_1(u)| = |\varphi_2(v)| = 1$, and hence $|\alpha| = 1$ for every $\alpha \in (c)uv$; thus $|(c)uv| = 1$, and therefore $|c| = 1$ by lemma 7.21.

Q.E.D.

It follows that $|t| = 1$ for every closed term t.

Let $D_0 = \{\alpha \in D\ ;\ |\alpha| = 1\}$; by lemma 7.19, D_0 is an initial segment of D.
Then we define $\mathscr{D}_0 \subset \mathscr{D}$ by taking $\mathscr{D}_0 = \{u \in \mathscr{D}\ ;\ |u| = 1\}$. So $\mathscr{D}_0$ is the set of initial segments of D_0. By lemma 7.20, $\mathscr{D}_0$ is closed under Ap ; by lemma 7.22, it contains K, S, p_1, p_2, c. Thus it is a submodel of $\mathscr{D}$. We will see that $\mathscr{D}_0$ is the desired submodel of $\mathscr{D}$.
We define a mapping $\varphi : \mathscr{D} \to \mathscr{D}$ by taking $\varphi(u) = u \cap D_0$ for every $u \in \mathscr{D}$. Clearly, φ is σ-c.i. ; let $f = \lambda x\, \varphi(x) \in \mathscr{D}$, therefore $(f)u = u \cap D_0$ for every $u \in \mathscr{D}$.

If $I = \lambda x\, x$, then $(f)u = (I)u = u$ for every $u \in \mathscr{D}_0$. By lemma 7.20, it follows that $|f| = |I| = 1$, and hence $f, I \in \mathscr{D}_0$.
Now $D \in \mathscr{D}$ (the whole set D is an initial segment), and $(f)D = D_0 \neq D = (I)D$ (indeed, D_0 contains no atom). Thus $f \neq I$, and therefore $\mathscr{D}_0$ does not satisfy the extensionality axiom.

Q.E.D.

In fact, $\mathscr{D}_0$ does not even satisfy the formula $\forall a(\forall x(ax = x) \to a = I)$. Therefore, we have proved the following strenghtening of theorem 7.18 :

Theorem 7.23. *The set of universal consequences of $ECLC$ (and also,* a fortiori, *of ECL) does not imply the formula $\forall a(\forall x(ax = x) \to a = I)$.*

Recall that the set of universal consequences of $ECLC$ (resp. ECL) is equivalent to the equations $ECLC_=$ (resp. $ECL_=$) given above, page 127 (resp. in chapter 6, page 104).

5. Retractions

Let $\mathscr{D} = \mathscr{S}(D)$ be a β-model of λ-calculus. Given $f, g \in \mathscr{D}$, we define :

$$f \circ g = \lambda x(f)(g)x \in \mathscr{D}.$$

Clearly, $\circ$ is an associative binary operation on $\mathscr{D}$. An element $\epsilon \in \mathscr{D}$ will be called a *retraction* if $\epsilon \circ \epsilon = \epsilon$. Then the image of ϵ, which will be called a *retract*, and will be denoted by $Im(\epsilon)$, is the set : $\{u \in \mathscr{D}\ ;\ (\epsilon)u = u\}$.

Remark. Since $\mathscr{S}(D)$ is a complete lattice and $Im(\epsilon)$ is the set of fixed points of ϵ (considered as a σ-c.i. function from $\mathscr{D}$ to $\mathscr{D}$), we see that every retract is a subset of $\mathscr{S}(D)$ which is a complete lattice ; this follows from a theorem due to Tarski, which claims that the set of fixed points of a monotone function on a complete lattice is a complete lattice [Tar55].

For every retraction ϵ, the retract $Im(\epsilon)$ is a σ-complete subspace of $\mathscr{D}$: let u_n ($n \in \mathbb{N}$) be an increasing sequence in $Im(\epsilon)$, and $u = \cup_n u_n$; then $u \in Im(\epsilon)$ (indeed, we have $(\epsilon)u = u$ since ϵ defines a σ-c.i. function on $\mathscr{D}$).
Moreover, it is easy to prove that, if ϵ_n ($n \in \mathbb{N}$) is an increasing sequence of retractions, then also $\epsilon = \cup_n \epsilon_n$ is a retraction (indeed, $(f, g) \mapsto f \circ g$ is a σ-c.i. function on $\mathscr{D} \times \mathscr{D}$).

Proposition 7.24. *If ϵ, ϵ' are retractions, then also*
$\epsilon \times \epsilon' = \lambda x \lambda f((f)(\epsilon)(x)1)(\epsilon')(x)0$ and $\epsilon \rightsquigarrow \epsilon' = \lambda y \lambda x(\epsilon')(y)(\epsilon)x = \lambda y\, \epsilon' \circ y \circ \epsilon$ are retractions.

Indeed, we have $(\epsilon \times \epsilon')(\epsilon \times \epsilon')u = \lambda f[(f)(\epsilon)(\epsilon \times \epsilon')u1](\epsilon')(\epsilon \times \epsilon')u0$.
Now $(\epsilon \times \epsilon')u1 = (\epsilon)(u)1$ and $(\epsilon \times \epsilon')u0 = (\epsilon')(u)0$; thus $(\epsilon \times \epsilon')(\epsilon \times \epsilon')u = (\epsilon \times \epsilon')u$

for every $u \in \mathscr{D}$. Therefore, $(\epsilon \times \epsilon') \circ (\epsilon \times \epsilon') = \lambda x(\epsilon \times \epsilon')(\epsilon \times \epsilon')x = \lambda x(\epsilon \times \epsilon')x$. Now $\lambda x(\epsilon \times \epsilon')x = \epsilon \times \epsilon'$ by definition of $\epsilon \times \epsilon'$.
On the other hand, we have, for every $v \in \mathscr{D}$:
$(\epsilon \rightsquigarrow \epsilon')(\epsilon \rightsquigarrow \epsilon')v = \lambda x(\epsilon')((\epsilon \rightsquigarrow \epsilon')v)(\epsilon)x$; now, for every $u \in \mathscr{D}$:
$((\epsilon \rightsquigarrow \epsilon')v)(\epsilon)u = (\epsilon')(v)(\epsilon)(\epsilon)u = (\epsilon')(v)(\epsilon)u$. Thus
$(\epsilon \rightsquigarrow \epsilon')(\epsilon \rightsquigarrow \epsilon')v = \lambda x(\epsilon')(\epsilon')(v)(\epsilon)x = \lambda x(\epsilon')(v)(\epsilon)x = (\epsilon \rightsquigarrow \epsilon')v$
for every $v \in \mathscr{D}$. It follows that :
$(\epsilon \rightsquigarrow \epsilon') \circ (\epsilon \rightsquigarrow \epsilon') = \lambda y(\epsilon \rightsquigarrow \epsilon')(\epsilon \rightsquigarrow \epsilon')y = \lambda y(\epsilon \rightsquigarrow \epsilon')y$.
Now, by definition of $\epsilon \rightsquigarrow \epsilon'$, we have $\lambda y(\epsilon \rightsquigarrow \epsilon')y = \epsilon \rightsquigarrow \epsilon'$.

Q.E.D.

The retract $Im(\epsilon \times \epsilon')$ is the set of all " couples " $\lambda f(f)aa'$ such that $a \in Im(\epsilon)$ and $a' \in Im(\epsilon')$.

Proposition 7.25.
The retract $Im(\epsilon \rightsquigarrow \epsilon')$ is canonically isomorphic with the space $\mathscr{C}(Im(\epsilon), Im(\epsilon'))$ of σ-c.i. functions from $Im(\epsilon)$ to $Im(\epsilon')$.

We now define two σ-c.i. functions :
$F : Im(\epsilon \rightsquigarrow \epsilon') \to \mathscr{C}(Im(\epsilon), Im(\epsilon'))$ and $G : \mathscr{C}(Im(\epsilon), Im(\epsilon')) \to Im(\epsilon \rightsquigarrow \epsilon')$.
Whenever $a \in Im(\epsilon \rightsquigarrow \epsilon')$, $F(a)$ is the σ-c.i. function defined on $Im(\epsilon)$ by :
$F(a)(u) = au$. We do have $au \in Im(\epsilon')$ since $a = (\epsilon \rightsquigarrow \epsilon')a = \epsilon' \circ a \circ \epsilon$ and hence $au = (\epsilon')(a)(\epsilon)u$. Clearly, F is σ-c.i.
Whenever $\varphi \in \mathscr{C}(Im(\epsilon), Im(\epsilon'))$, we define $\psi \in \mathscr{C}(\mathscr{D},\mathscr{D})$ by taking $\psi(x) = \varphi(\epsilon x)$.
Then we put $a_\varphi = \lambda x\psi(x)$ and $G(\varphi) = \epsilon' \circ a_\varphi \circ \epsilon$.
Thus $G(\varphi) = (\epsilon \rightsquigarrow \epsilon')a_\varphi$, and hence $G(\varphi) \in Im(\epsilon \rightsquigarrow \epsilon')$. Moreover, G is σ-c.i. since it is obtain by composing σ-c.i. functions.
We now prove that F and G are isomorphisms, each of them being the inverse of the other.

$G(F(a)) = a$ for every $a \in Im(\epsilon \rightsquigarrow \epsilon')$:
Let $F(a) = \varphi$; then $G(F(a)) = \epsilon' \circ a_\varphi \circ \epsilon$. Now $a_\varphi = \lambda x\varphi(\epsilon x) = \lambda x(a)(\epsilon)x$; on the other hand, $a = \epsilon' \circ a \circ \epsilon$ since $a \in Im(\epsilon \rightsquigarrow \epsilon')$; thus $(a)(\epsilon)x = (\epsilon')(a)(\epsilon)x$. It follows that $a_\varphi = \lambda x(\epsilon')(a)(\epsilon)x = \epsilon' \circ a \circ \epsilon = a$. Therefore $G(F(a)) = \epsilon' \circ a \circ \epsilon = a$.

$F(G(\varphi)) = \varphi$ for every $\varphi \in \mathscr{C}(Im(\epsilon), Im(\epsilon'))$:
Let $u \in Im(\epsilon)$. We have $G(\varphi) = \epsilon' \circ a_\varphi \circ \epsilon$, thus :
$F(G(\varphi))(u) = (\epsilon' \circ a_\varphi \circ \epsilon)u = (\epsilon')(a_\varphi)(\epsilon)u = (\epsilon')(a_\varphi)u$ since $(\epsilon)u = u$. Now :
$(a_\varphi)u = \varphi(\epsilon u)$ (by definition of a_φ) $= \varphi(u)$, and $(\epsilon')(a_\varphi)u = (\epsilon')\varphi(u) = \varphi(u)$ since $\varphi(u) \in Im(\epsilon')$. Thus $F(G(\varphi))(u) = \varphi(u)$ for every $u \in Im(\epsilon)$, and therefore $F(G(\varphi)) = \varphi$.

Q.E.D.

Extensional β-model constructed from a retraction

Let ϵ be a retraction $\neq \emptyset$, *such that* $\epsilon = \epsilon \rightsquigarrow \epsilon$; take $\mathscr{D}' = Im(\epsilon)$ and $\mathscr{F}' = \mathscr{C}(\mathscr{D}', \mathscr{D}')$. We shall define an extensional β-model by applying proposition 7.10. We first notice that $a, b \in \mathscr{D}' \Rightarrow (a)b \in \mathscr{D}'$. Indeed, $(\epsilon)ab = (\epsilon \rightsquigarrow \epsilon)ab = (\epsilon)(a)(\epsilon)b$; since $(\epsilon)a = a$ and $(\epsilon)b = b$, it follows that $ab = (\epsilon)(a)b$, and hence $ab \in \mathscr{D}'$.
We define $F : \mathscr{D}' \to \mathscr{F}'$ and $G : \mathscr{F}' \to \mathscr{D}'$ as in the proof of proposition 7.25, with $\epsilon = \epsilon' = \epsilon \rightsquigarrow \epsilon'$. We have $\mathscr{D}' = Im(\epsilon \rightsquigarrow \epsilon)$ and $\mathscr{F}' = \mathscr{C}(Im(\epsilon), Im(\epsilon))$. Thus $F(a)(b) = (a)b$ for all $a, b \in \mathscr{D}'$ and $G(\varphi) = \epsilon \circ a_\varphi \circ \epsilon = (\epsilon)a_\varphi$, where $a_\varphi = \lambda x\, \varphi(\epsilon x)$. We have seen that $F \circ G$ is the identity function on $\mathscr{C}(\mathscr{D}', \mathscr{D}')$ and that $G \circ F$ is the identity function on $\mathscr{D}'$. Thus, by proposition 7.10, we have an extensional β-model of λ-calculus.

In order to obtain a retraction ϵ with the required properties, it is enough to have a retraction $\epsilon_0 \neq \emptyset$, such that $\epsilon_0 \subset (\epsilon_0 \rightsquigarrow \epsilon_0)$.
Indeed, if $F = \lambda z(z \rightsquigarrow z) = \lambda z \lambda y \lambda x(z)(y)(z)x$, then $\epsilon_0 \subset (F)\epsilon_0$; then we define a sequence ϵ_n of retractions by taking $\epsilon_{n+1} = \epsilon_n \rightsquigarrow \epsilon_n = (F)\epsilon_n$. This is an increasing sequence (easy proof, by induction on n). Let $\epsilon = \cup_n \epsilon_n$; then ϵ is a retraction $\neq \emptyset$, and $\epsilon \rightsquigarrow \epsilon = (F)\epsilon = \cup_n (F)\epsilon_n = \cup_n \epsilon_{n+1} = \epsilon$.

Example. Obviously, $I = \lambda x\, x$ is a retraction ; we have $I \rightsquigarrow I = \lambda y \lambda x(I)(y)(I)x$, that is $I \rightsquigarrow I = \lambda y \lambda x(y)x$. Consider a non-extensional model $\mathscr{D} = \mathscr{S}(D)$ (so that $I \neq I \rightsquigarrow I$), in which the mapping $i : D^* \times D \to D$ is onto (for instance, the model $\mathscr{P}(\omega)$ defined above, page 121). Then, by lemma 7.12(2), we have $u \subset \lambda x(u)x$ for every $u \in \mathscr{D}$. Thus $\lambda y\, y \leq \lambda y \lambda x(y)x$ (since $\varphi \leq \psi \Rightarrow \lambda y \varphi(y) \leq \lambda y \psi(y)$ whenever $\varphi, \psi \in \mathscr{C}(\mathscr{D}, \mathscr{D})$). Therefore, $I \leq I \rightsquigarrow I$; this provides a retraction $\epsilon \geq I$ such that $\epsilon = \epsilon \rightsquigarrow \epsilon$. Thus $Im(\epsilon)$ is an extensional β-model of λ-calculus.

Models over a set of atoms

We consider an extensional model $\mathscr{D} = \mathscr{S}(D)$ constructed over a set A of atoms (see page 124). Let ϵ_0 be the initial segment of D generated by the set $\{\{\alpha\} \to \alpha\,;\, \alpha \in A\}$. If $\beta \in D$ and $u \in \mathscr{D}$, then :

$$\beta \in \epsilon_0 u \Leftrightarrow (\exists b \subset u)\; b \to \beta \in \epsilon_0 \Leftrightarrow (\exists b \subset u, \alpha \in A)\; \beta \leq \alpha \in b$$
$$\Leftrightarrow (\exists \alpha \in A \cap u)\; \beta \leq \alpha.$$

It follows that $\epsilon_0 u = \overline{A \cap u}$ (recall that this denotes the initial segment of D generated by $A \cap u$).
Let $\alpha \in A$; then $\alpha \in (\epsilon_0)(\epsilon_0)u \Leftrightarrow \alpha \in (\epsilon_0)u \Leftrightarrow \alpha \in u$. It follows that $(\epsilon_0)(\epsilon_0)u = (\epsilon_0)u$ and hence ϵ_0 is a retraction.
The retract $Im(\epsilon_0)$ is the set of all initial segments of D generated by the subsets of A ; this is a complete lattice which is isomorphic with the power set $\mathscr{P}(A)$.
Let $\epsilon_1 = \epsilon_0 \rightsquigarrow \epsilon_0$; we wish to prove that $\epsilon_0 \subset \epsilon_1$. To do so, it suffices to show that $\{\alpha\} \to \alpha \in \epsilon_1$ for every $\alpha \in A$. Let $a = \overline{\{\alpha\}}$; then $\alpha \in (\epsilon_0)a$ (since $\{\alpha\} \to \alpha \in \epsilon_0$) and

$a = (a)\emptyset$ (since $\alpha = \emptyset \to \alpha$) ; thus $a = (a)(\epsilon_0)\emptyset$. Finally, we have $\alpha \in (\epsilon_0)(a)(\epsilon_0)\emptyset$; now since $\epsilon_1 = \lambda y\lambda x(\epsilon_0)(y)(\epsilon_0)x$, we conclude that $\{\alpha\}, \emptyset \to \alpha \in \epsilon_1$, that is to say $\{\alpha\} \to \alpha \in \epsilon_1$.

Now, consider the increasing sequence ϵ_n of retractions and the retraction : $\epsilon = \cup_n \epsilon_n$ defined above. We therefore have $\epsilon = \epsilon \rightsquigarrow \epsilon$.
Clearly, $(\epsilon_0)u \subset u$ for every $u \in \mathscr{D}$, thus $\epsilon_0 \le I = \lambda x\, x$ (case of extensional models in proposition 7.10).
We prove, by induction on n, that $\epsilon_n \le I$ for every $n \in \mathbb{N}$: indeed, by induction hypothesis, $\epsilon_n \le I$; thus $\epsilon_{n+1} = \epsilon_n \rightsquigarrow \epsilon_n \le I \rightsquigarrow I = \lambda y\lambda x(y)x = I$ since $\mathscr{D}$ is an extensional model. Therefore $\epsilon_{n+1} \le I$. It follows that $\epsilon \le \lambda x\, x$.

Lemma 7.26.
i) If $\alpha \in D$ and $rk(\alpha) \le n$, then $(\{\alpha\} \to \alpha) \in \epsilon_n$;
ii) $\epsilon = \lambda x\, x$.

i) The proof is by induction on n.
If $rk(\alpha) = 0$, then $\alpha \in A$, and hence $\{\alpha\} \to \alpha \in \epsilon_0$.
Now let $\alpha \in D$ be such that $rk(\alpha) = n + 1$; we may write $\alpha = b \to \beta$, and put $a = \{\alpha\}$. We have $b = \{\beta_1, \dots, \beta_k\}$; by induction hypothesis, $\{\beta_i\} \to \beta_i \in \epsilon_n$ for $1 \le i \le k$; it follows that $(\epsilon_n)\bar{b} \supset b$, and hence $(\epsilon_n)\bar{b} \supset \bar{b}$; since $\epsilon_n \le \lambda x\, x$, we have $(\epsilon_n)\bar{b} = \bar{b}$. Now, clearly, $\beta \in (\bar{a})\bar{b}$, since $b \to \beta \in a$. By induction hypothesis, $\{\beta\} \to \beta \in \epsilon_n$, thus $\beta \in (\epsilon_n)(\bar{a})\bar{b} = (\epsilon_n)(\bar{a})(\epsilon_n)\bar{b}$. Therefore :
$(a, b \to \beta) \in \lambda y\lambda x(\epsilon_n)(y)(\epsilon_n)x = \epsilon_n \rightsquigarrow \epsilon_n = \epsilon_{n+1}$. Now $a, b \to \beta = \{\alpha\} \to \alpha$; this completes the inductive proof.
ii) Since $\lambda x\, x$ is the initial segment of D generated by the elements of the form $\{\alpha\} \to \alpha$, where $\alpha \in D$, we have $\epsilon \supset \lambda x\, x$, and therefore $\epsilon = \lambda x\, x$.
Q.E.D.

Lemma 7.27. $\epsilon_n \circ \epsilon_m = (\epsilon_{n+1})\epsilon_m = \epsilon_p$, *where* $p = \inf(m, n)$.

If $n \ge m$, then $(\epsilon_n)(\epsilon_m)u \ge (\epsilon_m)(\epsilon_m)u = (\epsilon_m)u$ since $\epsilon_n \ge \epsilon_m$. Now $\epsilon_n \le \lambda x\, x$, so we have $(\epsilon_n)(\epsilon_m)u = (\epsilon_m)u$. Thus, by extensionality, $\epsilon_n \circ \epsilon_m = \epsilon_m$. The case $n \le m$ is similar.
Now $\epsilon_{n+1} = \epsilon_n \rightsquigarrow \epsilon_n$, and hence $(\epsilon_{n+1})\epsilon_m u = (\epsilon_n)(\epsilon_m)(\epsilon_n)u$; we have just seen that the latter is equal to $(\epsilon_m)u$ if $n \ge m$ and to $(\epsilon_n)u$ if $n \le m$. The result follows, by extensionality.
Q.E.D.

Let $\mathscr{D}_n = Im(\epsilon_n) \subset \mathscr{D}$. By lemma 7.27, we have $m \ge n \Rightarrow (\epsilon_m)(\epsilon_n)u = (\epsilon_n)u$. Thus $\mathscr{D}_n$ is an increasing sequence of σ-complete ordered sets (since they are retracts). $\mathscr{D}_0$ is isomorphic with $\mathscr{P}(A)$ and $\mathscr{D}_{n+1}$ is isomorphic with $\mathscr{C}(\mathscr{D}_n, \mathscr{D}_n)$.
For every $u \in \mathscr{D}$, let $u_n = (\epsilon_n)u$. (u_n) is an increasing sequence, $u_n \in \mathscr{D}_n$, and $\sup_n u_n = u$ (we have $\sup_n \epsilon_n = \lambda x\, x$ by lemma 7.26).

Thus we have a structure in the model $\mathscr{D}$ which is similar to that of Scott's model $\mathscr{D}^\infty$ (see [Bar84], [Hin86]).

Now let $D_n = \{\alpha \in D;\ rk(\alpha) \le n\}$; then we have $D_0 = A$, (D_n) is an increasing sequence and $\cup_n D_n = D$.
The next proposition describes the structure of spaces $\mathscr{D}_n$.

Proposition 7.28. *i) $D_0 = A$; D_{n+1} (with the preorder induced by D) is isomorphic with $D_n^* \times D_n$;*
ii) ϵ_n is the initial segment of D generated by $\{\{\alpha\} \to \alpha\ ;\ rk(\alpha) \le n\}$;
iii) $\mathscr{D}_n$ is the set of all initial segments generated by the subsets of D_n ; it is isomorphic with $\mathscr{S}(D_n)$.

Proof of (i) : if $b \to \beta,\ c \to \gamma$ have rank $\le n+1$, then $b, c \in D_n^*$ and $\beta, \gamma \in D_n$; moreover, $(b \to \beta) \le (c \to \gamma) \Leftrightarrow b \ge c$ et $\beta \le \gamma \Leftrightarrow (b, \beta) \le (c, \gamma)$ in $D_n^* \times D_n$.
We prove (ii) by induction on n. This is obvious when $n = 0$, by definition of ϵ_0.
For all $\beta \in D,\ u \in \mathscr{D}$, we have : $\beta \in \epsilon_n u \Leftrightarrow \exists b \subset u,\ b \to \beta \in \epsilon_n$. By induction hypothesis, it follows that :
$\beta \in \epsilon_n u \Leftrightarrow \exists b \subset u, \exists \alpha \in D_n, b \to \beta \le \{\alpha\} \to \alpha \Leftrightarrow \exists \alpha \in D_n,\ \beta \le \alpha,\ \alpha \in u$ (indeed, $b \to \beta \le \{\alpha\} \to \alpha \Leftrightarrow \beta \le \alpha$ et $\alpha \in \bar{b}$). Therefore, $\epsilon_n u = \overline{D_n \cap u}$ (which proves part (iii) of the proposition).
Now let β be an arbitrary element of ϵ_{n+1} ; we are looking for some $\alpha \in D_{n+1}$ such that $\beta \le \{\alpha\} \to \alpha$. We may write $\beta = b, c \to \gamma$.
Since $\epsilon_{n+1} = \lambda y \lambda x(\epsilon_n)(y)(\epsilon_n)x$, we have $\gamma \in (\epsilon_n)(\bar{b})(\epsilon_n)\bar{c}$. Let $d' = (\epsilon_n)\bar{c} = \overline{D_n \cap \bar{c}}$.
Then $\gamma \in (\epsilon_n)(\bar{b})d'$, that is $\gamma \in \overline{D_n \cap \bar{b}d'}$.
Hence $\gamma \le \delta$ for some $\delta \in D_n \cap \bar{b}d'$. Therefore, there exists $d'' \subset d'$ such that $d'' \to \delta \in \bar{b}$. Now d'' is finite and $d'' \subset \overline{D_n \cap \bar{c}}$. Thus there exists some finite d such that $d \subset D_n \cap \bar{c}$ and $d'' \subset \bar{d}$. Since $\gamma \le \delta$ and $d \subset \bar{c}$, we have $c \to \gamma \le d \to \delta$; now $d \to \delta \le d'' \to \delta$, and hence $d \to \delta \in \bar{b}$.
It follows that $b, c \to \gamma \le \{d \to \delta\}, d \to \delta$. Take $\alpha = d \to \delta$; then $\alpha \in D_{n+1}$ (since $d \subset D_n$ and $\delta \in D_n$), and $b, c \to \gamma \le \{\alpha\} \to \alpha$.
This yields the result, since $\beta = b, c \to \gamma$.
Q.E.D.

6. Qualitative domains and stable functions

Let E be a countable set. A subset $\mathscr{D}$ of $\mathscr{P}(E)$ is called a *qualitative domain* if :
i) for every increasing sequence $u_n \in \mathscr{D}$ $(n \in \mathbb{N})$, we have $\cup_n u_n \in \mathscr{D}$;
ii) if $u \in \mathscr{D}$ and $v \subset u$, then $v \in \mathscr{D}$.

Let $\mathscr{D}_0$ be the set of finite elements of $\mathscr{D}$. Thus every element of $\mathscr{D}$ is the union of an increasing sequence of elements of $\mathscr{D}_0$.

We define the *web* D of $\mathscr{D}$ to be the union of all elements of $\mathscr{D}$: D is the least subset of E such that $\mathscr{D} \subset \mathscr{P}(D)$. We also have $D = \{\alpha \in E; \{\alpha\} \in \mathscr{D}\}$.
Let $\mathscr{D}, \mathscr{D}'$ be two qualitative domains, and D, D' their webs. Then $\mathscr{D} \times \mathscr{D}'$ is a qualitative domain (up to isomorphism), with web $D \oplus D'$ (the disjoint union of D and D', which can be represented by $(D \times \{0\}) \cup (D' \times \{1\})$).

Let $\mathscr{D}, \mathscr{D}'$ be two qualitative domains. A σ-c.i. function $f : \mathscr{D} \to \mathscr{D}'$ is said to be *stable* if and only if :
for every $u, v, w \in \mathscr{D}$ such that $u, v \subset w$, we have $f(u \cap v) = f(u) \cap f(v)$.

We will denote by $\mathscr{S}(\mathscr{D}, \mathscr{D}')$ the set of all stable functions from $\mathscr{D}$ to $\mathscr{D}'$.

Note that a σ-c.i. function f is stable if and only if :

$$u \cup v \in \mathscr{D} \Rightarrow f(u \cap v) \supset f(u) \cap f(v).$$

Let $\mathscr{D}_1, \ldots, \mathscr{D}_k, \mathscr{D}$ be qualitative domains, and $f : \mathscr{D}_1 \times \ldots \times \mathscr{D}_k \to \mathscr{D}$ a σ-c.i. function. Then f is stable (with respect to the above definition of the qualitative domain $\mathscr{D}_1 \times \ldots \times \mathscr{D}_k$) if and only if :

$$u_1 \cup v_1 \in \mathscr{D}_1, \ldots, u_k \cup v_k \in \mathscr{D}_k \Rightarrow$$
$$f(u_1 \cap v_1, \ldots, u_k \cap v_k) = f(u_1, \ldots, u_k) \cap f(v_1, \ldots, v_k).$$

Clearly, every projection function $p_i : \mathscr{D}_1 \times \ldots \times \mathscr{D}_k \to \mathscr{D}_i$, defined by :

$$p_i(u_1, \ldots u_k) = u_i$$

is stable.

Proposition 7.29.
i) Let $f_i : \mathscr{D} \to \mathscr{D}_i$ $(1 \le i \le k)$ be stable functions. Then the function :
$f : \mathscr{D} \to \mathscr{D}_1 \times \ldots \times \mathscr{D}_k$, defined by $f(u) = (f_1(u), \ldots, f_k(u))$ for every $u \in \mathscr{D}$, is stable.
ii) If $f : \mathscr{D} \to \mathscr{D}'$ and $g : \mathscr{D}' \to \mathscr{D}''$ are stable, then so is $g \circ f$.

i) Immediate, by definition of the qualitative domain $\mathscr{D}_1 \times \ldots \times \mathscr{D}_k$.
ii) If $u \cup v \in \mathscr{D}$, then $f(u \cap v) = f(u) \cap f(v)$; now $f(u), f(v) \subset f(u \cup v)$, and hence $g(f(u) \cap f(v)) = g(f(u)) \cap g(f(v))$. Therefore, $g(f(u \cap v)) = g(f(u)) \cap g(f(v))$.
Q.E.D.

It follows from this proposition that any composite function obtained from stable functions of several variables is also stable.

Proposition 7.30.
Let $\mathscr{D}, \mathscr{D}'$ be qualitative domains, D, D' their webs, and $f : \mathscr{D} \to \mathscr{D}'$ a σ-c.i. function. Then the following conditions are equivalent :
i) f is stable.
ii) If $u \in \mathscr{D}$, $\alpha \in D'$ and $\alpha \in f(u)$, then the set $\{v \subset u\ ;\ \alpha \in f(v)\}$ has a least element v_0.
iii) If $u \in \mathscr{D}$, $a \in \mathscr{D}'$, a is finite and $a \subset f(u)$, then $\{v \subset u\ ;\ a \subset f(v)\}$ has a least element v_0.
Moreover, if f is stable, then this least element v_0 is a finite set.

It is obvious that (iii) $\Rightarrow$ (ii). We now prove that (i) $\Rightarrow$ (iii) : let $f : \mathscr{D} \to \mathscr{D}'$ be a stable function, $u \in \mathscr{D}$, and a a finite subset of $f(u)$. Then there exists a finite subset v of u such that $a \subset f(v)$: indeed, u is the union of an increasing sequence (u_n) of finite sets, and $a \subset f(u) = \cup_n f(u_n)$, thus $a \subset f(u_n)$ for some n. On the other hand, if $v, w \subset u$ and $a \subset f(v), f(w)$, then $a \subset f(v) \cap f(w) = f(v \cap w)$. Therefore, the least element v_0 is the intersection of all finite subsets $v \subset u$ such that $a \subset f(v)$.
Proof of (ii) $\Rightarrow$ (i) : let $f : \mathscr{D} \to \mathscr{D}'$ be a σ-c.i. function satisfying condition (ii), and α, u, v be such that $u \cup v \in \mathscr{D}$ and $\alpha \in f(u) \cap f(v)$.
Let v_0 be the least element of $\{w \subset u \cup v\,;\, \alpha \in f(w)\}$. Since u and v are members of this set, we have $v_0 \subset u, v$, thus $v_0 \subset u \cap v$.
Since $\alpha \in f(v_0)$, we have $\alpha \in f(u \cap v)$, and therefore $f(u) \cap f(v) \subset f(u \cap v)$.

Q.E.D.

Let $\mathscr{D}, \mathscr{D}'$ be qualitative domains, D, D' their webs, and $f : \mathscr{D} \to \mathscr{D}'$ a stable function. The *trace* of f, denoted by tr(f), is a subset of $\mathscr{D}_0 \times D'$, defined as follows :

$\mathrm{tr}(f) = \{(a, \alpha) \in \mathscr{D}_0 \times D';\, \alpha \in f(a)$ and $\alpha \notin f(a')$, for every $a' \subset a,\ a' \neq a\}$.

If $u \in \mathscr{D}$ and $\alpha \in u$, then $\alpha \in f(u) \Leftrightarrow$ there exists $a \in \mathscr{D}_0$, such that $a \subset u$ and $(a, \alpha) \in \mathrm{tr}(f)$. Therefore, a stable function is completely determined by its trace.
We define a binary relation $\prec$ on $\mathscr{S}(\mathscr{D}, \mathscr{D}')$ by putting, for any two stable functions $f, g : \mathscr{D} \to \mathscr{D}'$, $f \prec g \Leftrightarrow f(u) = f(v) \cap g(u)$ for all $u, v \in \mathscr{D}$ such that $u \subset v$.
This relation is seen to be an order on $\mathscr{S}(\mathscr{D}, \mathscr{D}')$, known as the *Berry order*. Thus, if $f \prec g$, then $f(u) \subset g(u)$ for every $u \in \mathscr{D}$.

Proposition 7.31. *Let f, g be two stable functions from $\mathscr{D}$ to $\mathscr{D}'$. Then :*
$f \prec g \Leftrightarrow tr(f) \subset tr(g)$.

Suppose that $f \prec g$ and $(a, \alpha) \in \mathrm{tr}(f)$. Then $\alpha \in f(a) \subset g(a)$, and hence $\alpha \in g(a)$.
Thus there exists $a' \subset a$ such that $(a', \alpha) \in \mathrm{tr}(g)$.
Now $f(a') = f(a) \cap g(a')$, so $\alpha \in f(a')$, and hence $a' = a$, by definition of tr(f).
Thus $(a, \alpha) \in \mathrm{tr}(g)$ and therefore $\mathrm{tr}(f) \subset \mathrm{tr}(g)$.
Now suppose that $\mathrm{tr}(f) \subset \mathrm{tr}(g)$, and let $u, v \in \mathscr{D}$, $u \subset v$. If $\alpha \in f(u)$, then there exists $a \subset u$ such that $(a, \alpha) \in \mathrm{tr}(f)$. Thus $(a, \alpha) \in \mathrm{tr}(g)$ and $\alpha \in g(a) \subset g(u)$.
Therefore $\alpha \in f(v) \cap g(u)$.
Conversely, if $\alpha \in f(v) \cap g(u)$, then there exist $a \subset u$, $b \subset v$, such that :
$(a, \alpha) \in \mathrm{tr}(g)$, $(b, \alpha) \in \mathrm{tr}(f)$. Thus $(a, \alpha), (b, \alpha) \in \mathrm{tr}(g)$ and $a \cup b \subset v \in \mathscr{D}$.
It follows that $a = b$, hence $(a, \alpha) \in \mathrm{tr}(f)$, $\alpha \in f(a)$, and therefore $\alpha \in f(u)$.

Q.E.D.

Proposition 7.32. *Let us consider two qualitative domains $\mathscr{D}, \mathscr{D}'$, and their webs D, D'. Then the set of all traces of stable functions from $\mathscr{D}$ to $\mathscr{D}'$ is a qualitative domain with web $\mathscr{D}_0 \times D'$.*

Let f_n ($n \in \mathbb{N}$) be a sequence of stable functions, such that $\mathrm{tr}(f_n) \subset \mathrm{tr}(f_{n+1})$, and therefore $f_n \prec f_{n+1}$. Define $f : \mathscr{D} \to \mathscr{D}'$ by taking $f(u) = \cup_n f_n(u)$ for every $u \in \mathscr{D}$ (note that $f_n(u)$ is an increasing sequence in $\mathscr{D}$). Then f is stable :
indeed, if $u \cup v \in \mathscr{D}$, then $f(u \cap v) = \cup_n f_n(u \cap v) = \cup_n (f_n(u) \cap f_n(v)) = f(u) \cap f(v)$.
Moreover, $f_n \prec f$: if $u \subset v$, then $f_n(u) = f_n(v) \cap f_p(u)$ for every $p \geq n$, thus $f_n(u) = f_n(v) \cap \cup_p f_p(u) = f_n(v) \cap f(u)$. Therefore $\cup_n \mathrm{tr}(f_n) \subset \mathrm{tr}(f)$.
Conversely, if $(a,\alpha) \in \mathrm{tr}(f)$, then $\alpha \in f(a)$, and therefore there exists an integer n such that $\alpha \in f_n(a)$. Thus $(a',\alpha) \in \mathrm{tr}(f_n)$ for some $a' \subset a$. Since $\mathrm{tr}(f_n) \subset \mathrm{tr}(f)$, we have $(a',\alpha) \in \mathrm{tr}(f)$, and hence $a = a'$.
Thus $(a,\alpha) \in \mathrm{tr}(f_n)$ and therefore $\mathrm{tr}(f) \subset \cup_n \mathrm{tr}(f_n)$. Finally, $\mathrm{tr}(f) = \cup_n \mathrm{tr}(f_n)$.
Now let $f \in \mathscr{S}(\mathscr{D},\mathscr{D}')$ and $X \subset \mathrm{tr}(f)$. We prove that X is the trace of some stable function g, which we define by putting :
$\alpha \in g(u) \Leftrightarrow$ there exists $a \subset u$ such that $(a,\alpha) \in X$.
Using proposition 7.30(ii), we prove that g is stable : let $\alpha \in g(u)$; then $(a,\alpha) \in X$ for some $a \subset u$. If $v \subset u$ and $\alpha \in g(v)$, then $(b,\alpha) \in X$ for some $b \subset v$. Now $(a,\alpha),(b,\alpha) \in \mathrm{tr}(f)$, and $a,b \subset u$, thus $a = b$. Hence $a \subset v$, and a is the least element of the set $\{v \in \mathscr{D} \,;\, \alpha \in f(v)\}$.
We have $X = \mathrm{tr}(g)$: indeed, if $(a,\alpha) \in \mathrm{tr}(g)$, then $\alpha \in g(a)$, thus $(b,\alpha) \in X$ for some $b \subset a$. So $\alpha \in g(b)$, and hence $a = b$, by definition of $\mathrm{tr}(g)$. Therefore $(a,\alpha) \in X$.
Conversely, if $(a,\alpha) \in X$, then $\alpha \in g(a)$, thus $(b,\alpha) \in \mathrm{tr}(g)$ for some $b \subset a$.
It follows that $(b,\alpha) \in X$ (see above).
Hence $(a,\alpha),(b,\alpha) \in \mathrm{tr}(f)$, and therefore $a = b$, and $(a,\alpha) \in \mathrm{tr}(g)$.
Q.E.D.

In view of the previous proposition, the space $\mathscr{S}(\mathscr{D},\mathscr{D}')$ of all stable functions from $\mathscr{D}$ to $\mathscr{D}'$, equipped with the order $\prec$, may be identified with a qualitative domain with web $\mathscr{D}_0 \times D'$ (note that $\mathscr{D}_0 \times D'$ is countable).

Proposition 7.33. *Let us consider two qualitative domains $\mathscr{D}$, $\mathscr{D}'$, and their webs D, D'. Then the function Eval* $: \mathscr{S}(\mathscr{D},\mathscr{D}') \times \mathscr{D} \to \mathscr{D}'$*, defined by Eval*$(f,u) = f(u)$*, is stable.*

Let $u, v \in \mathscr{D}$, such that $u \cup v \in \mathscr{D}$, and $f, g, h \in \mathscr{S}(\mathscr{D},\mathscr{D}')$ such that :
$\mathrm{tr}(f) \cup \mathrm{tr}(g) = \mathrm{tr}(h)$. We need to prove $f(u) \cap g(v) \subset k(u \cap v)$, where $k \in \mathscr{S}(\mathscr{D},\mathscr{D}')$ is defined by $\mathrm{tr}(k) = \mathrm{tr}(f) \cap \mathrm{tr}(g)$.
Let $\alpha \in f(u), g(v)$. Then there exist $a \subset u$, $b \subset v$ such that :
$(a,\alpha) \in \mathrm{tr}(f)$, $(b,\alpha) \in \mathrm{tr}(g)$. Thus $(a,\alpha),(b,\alpha) \in \mathrm{tr}(h)$, and $a, b \subset u \cup v$.
It follows that $a = b \subset u \cap v$ and $(a,\alpha) \in \mathrm{tr}(f) \cap \mathrm{tr}(g) = \mathrm{tr}(k)$. Thus $\alpha \in k(u \cap v)$.
Q.E.D.

Proposition 7.34.
Let $\mathscr{D}$,$\mathscr{D}'$,$\mathscr{D}''$ be qualitative domains, and $f : \mathscr{D} \times \mathscr{D}' \to \mathscr{D}''$ a stable function.

Then the function $\text{Cur}_f : \mathscr{D} \to \mathscr{S}(\mathscr{D}', \mathscr{D}'')$, *defined by* $\text{Cur}_f(u)(u') = f(u, u')$ *for all* $u \in \mathscr{D}$, $u' \in \mathscr{D}'$, *is also stable.*

Remark. The operation $f \mapsto \text{Cur}_f$ is sometimes called " curryfication ".

We first prove that, if $u \subset v$, then $\text{Cur}_f(u) \prec \text{Cur}_f(v)$: let $u', v' \in \mathscr{D}'$ be such that $u' \subset v'$; since f is stable, we have :
$f(u, v') \cap f(v, u') = f(u \cap v, u' \cap v') = f(u, u')$. In other words :
$\text{Cur}_f(u)(v') \cap \text{Cur}_f(v)(u') = \text{Cur}_f(u)(u')$, which is the desired property.
Thus Cur_f is an increasing function ; it is also σ-continuous : let u_n ($n \in \mathbb{N}$) be an increasing sequence in $\mathscr{D}$, and $u = \cup_n u_n$.
We need to prove that $\text{Cur}_f(u)(u') = \cup_n \text{Cur}_f(u_n)(u')$ for every $u' \in \mathscr{D}'$, i.e. :
$f(u, u') = \cup_n f(u_n, u')$, which is clear, since f is σ-continuous.
Finally, we show that Cur_f is stable : let $u, v \in \mathscr{D}$ be such that $u \cup v \in \mathscr{D}$. We have to prove $\text{tr}(\text{Cur}_f(u \cap v)) \supset \text{tr}(\text{Cur}_f(u)) \cap \text{tr}(\text{Cur}_f(v))$.
Let $(a, \alpha) \in \text{tr}(\text{Cur}_f(u)) \cap \text{tr}(\text{Cur}_f(v))$; we have :
$\alpha \in \text{Cur}_f(u)(a) = f(u, a)$ and $\alpha \in f(v, a)$.
Since f is stable, $\alpha \in f(u \cap v, a) = \text{Cur}_f(u \cap v)(a)$. Thus there exists $b \subset a$ such that $(b, \alpha) \in \text{tr}(\text{Cur}_f(u \cap v)) \subset \text{tr}(\text{Cur}_f(u))$. Since $(a, \alpha) \in \text{tr}(\text{Cur}_f(u))$, we have $b = a$, and therefore $(a, \alpha) \in \text{tr}(\text{Cur}_f(u \cap v))$.

Q.E.D.

The next proposition provides a new method for constructing β-models :

Proposition 7.35. *Let* $\mathscr{D}$ *be a qualitative domain, D its web ; let :*
$\Phi : \mathscr{S}(\mathscr{D}, \mathscr{D}) \to \mathscr{D}$, $\Psi : \mathscr{D} \to \mathscr{S}(\mathscr{D}, \mathscr{D})$ *two stable functions.*
Then $\mathscr{D}$ *is a functional model of* λ*-calculus.* $\mathscr{D}$ *is a* β*-model provided that* $\Phi \circ \Psi$ *is the identity function on* $\mathscr{D}$ *; in that case, the* β*-model is extensional if and only if* $\Psi \circ \Phi$ *is the identity function on* $\mathscr{S}(\mathscr{D}, \mathscr{D})$.

In order to define the functional model, we take $\mathscr{F} = \mathscr{S}(\mathscr{D}, \mathscr{D})$, and we take $\mathscr{F}^\infty$ as the set of those stable functions from $\mathscr{D}^{\mathbb{N}}$ to $\mathscr{D}$ which depend only on a finite number of coordinates.

Remark. More precisely, let $f : \mathscr{D}^{\mathbb{N}} \to \mathscr{D}$ be a function which depends only on a finite number of coordinates. Thus, we may consider f as a function from $\mathscr{D}^n$ to $\mathscr{D}$ for some integer n ; we say that $f \in \mathscr{F}^\infty$ if, and only if this function is stable.

We put $(u)v = \Phi(u)(v)$ for all $u, v \in \mathscr{D}$, and $\lambda x f(x) = \Psi(f)$ for every $f \in \mathscr{F}$.
Let $Ap : \mathscr{D} \times \mathscr{D} \to \mathscr{D}$ be defined by $Ap(u, v) = (u)v$; it is a stable function : indeed, we have $Ap(u, v) = \text{Eval}(\Phi(u), v)$ (composition of stable functions Eval and Φ).
We now check conditions 1, 2, 3 of the definition of functional models of λ-calculus :
(1) Every coordinate function x_i is in $\mathscr{F}^\infty$: already seen, page 135.

(2) If $f,g \in \mathscr{F}^\infty$, then $(f)g \in \mathscr{F}^\infty$:
Indeed $(f)g$ is stable, since $(f)g = Ap(f,g)$ is given by composition of stable functions Ap, f, g.
(3) If $f(x_1,\dots,x_n) \in \mathscr{F}^\infty$, then $\lambda x_i f \in \mathscr{F}^\infty$:
For simpler notations, we suppose $i = n$ and we put :
$g(x_1,\dots,x_{n-1}) = \lambda x_n f(x_1,\dots,x_{n-1})$. We need to prove that g is stable. Now, if $u_1,\dots,u_{n-1} \in \mathscr{D}$, then $g(u_1,\dots,u_{n-1}) = \Psi(\mathrm{Cur}_f(u_1,\dots,u_{n-1}))$ (we consider f as a stable function from $\mathscr{D}^{n-1} \times \mathscr{D}$ to $\mathscr{D}$). Thus g is stable, since it is obtained by composing the stable functions Ψ and Cur_f.

Q.E.D.

Coherence spaces

A *coherence space* D is a finite or countable non-empty set, equipped with a *coherence relation* denoted by $\asymp$ (a reflexive and symmetric binary relation) ; $\alpha \asymp \beta$ should be read : " α is coherent with β ". If D, D' are two coherence spaces, then we can make of the product set $D \times D'$ a coherence space, by putting : $(\alpha,\alpha') \asymp (\beta,\beta') \Leftrightarrow \alpha \asymp \alpha'$ and $\beta \asymp \beta'$.
An *antichain* of D is a subset A of D such that $\alpha,\beta \in A$, $\alpha \asymp \beta \Rightarrow \alpha = \beta$.
The set of all antichains (resp. all finite antichains) of D is denoted by $\mathscr{A}(D)$ (resp. $\mathscr{A}_0(D)$).
The space $\mathscr{D} = \mathscr{A}(D)$ is a qualitative domain, with web D, called the *qualitative domain associated with the coherence space D.* The set $\mathscr{D}_0 = \mathscr{A}_0(D)$ of all finite antichains of D is a coherence space, the coherence relation being :
$a \asymp b \Leftrightarrow a \cup b \in \mathscr{A}_0(D)$, for all $a, b \in \mathscr{A}_0(D)$.
Let D, D' be two coherence spaces, and $\mathscr{D},\mathscr{D}'$ the associated qualitative domains. It follows from the above properties that $\mathscr{D}_0 \times D'$ can be considered as a coherence space.

A qualitative domain $\mathscr{D}$, with web D, is associated with a coherence space if and only if it satisfies the following property :
For every $u \subset D$, if every two-element subset of u is in $\mathscr{D}$, then u is in $\mathscr{D}$.
Indeed, if this property holds, then we may define a coherence relation on D by putting : $\alpha \asymp \alpha' \Leftrightarrow \alpha = \alpha'$ or $\{\alpha,\alpha'\} \notin \mathscr{D}$, for all $\alpha,\alpha' \in D$; then it can be seen easily that $\mathscr{D} = \mathscr{A}(D)$.

Proposition 7.36.
Let D, D' be two coherence spaces, $\mathscr{D} = \mathscr{A}(D)$, $\mathscr{D}' = \mathscr{A}(D')$ the corresponding qualitative domains. Then a subset X of the coherence space $\mathscr{D}_0 \times D'$ is an antichain if and only if it is the trace of some stable function from $\mathscr{D}$ to $\mathscr{D}'$.

Let X be an antichain in $\mathscr{D}_0 \times D'$. We define $f : \mathscr{D} \to \mathscr{D}'$ by taking :
$\alpha \in f(u) \Leftrightarrow$ there exists $a \subset u$ such that $(a,\alpha) \in X$ (for all $u \in \mathscr{D}$, $\alpha \in D'$).

Then $f(u)$ is an antichain of D' : if $\alpha, \beta \in f(u)$ and $\alpha \asymp \beta$, then there exist $a, b \subset u$ such that $(a, \alpha), (b, \beta) \in X$. Thus $(a, \alpha) \asymp (b, \beta)$, and, since X is an antichain, we have $\alpha = \beta$ (and $a = b$).
The function f is obviously σ-c.i. We now prove that f is stable : if $u \cup v \in \mathscr{D}$ and $\alpha \in f(u) \cap f(v)$, then there exist $a \subset u$, $b \subset v$ such that $(a, \alpha), (b, \alpha) \in X$. Now $(a, \alpha) \asymp (b, \alpha)$ since $a \cup b \in \mathscr{D}_0$. It follows that $a = b$, and hence $a \subset u \cap v$, and $\alpha \in f(u \cap v)$ by definition of f. Thus $f(u) \cap f(v) \subset f(u \cap v)$.
Finally, X is the trace of f : if $(a, \alpha) \in \mathrm{tr}(f)$, then $\alpha \in f(a)$, and hence $(b, \alpha) \in X$ for some $b \subset a$. Therefore, $\alpha \in f(b)$, by definition of f, so $b = a$ by definition of $\mathrm{tr}(f)$. Thus $(a, \alpha) \in X$.
Conversely, if $(a, \alpha) \in X$, then $\alpha \in f(a)$, and hence $(b, \alpha) \in \mathrm{tr}(f)$ for some $b \subset a$. Then $(b, \alpha) \in X$, as proved above. Since $(a, \alpha) \asymp (b, \alpha)$ and X is an antichain, it follows that $a = b$, and therefore $(a, \alpha) \in \mathrm{tr}(f)$.
Now let $f : \mathscr{D} \to \mathscr{D}'$ be a stable function. It remains to prove that $\mathrm{tr}(f)$ is an antichain in $\mathscr{D}_0 \times D'$. If $(a, \alpha) \asymp (b, \beta)$ and both are in $\mathrm{tr}(f)$, then $a \cup b \in \mathscr{D}$, and $\alpha \asymp \beta$. Now $\alpha \in f(a)$, $\beta \in f(b)$, and hence $\alpha, \beta \in f(a \cup b)$. Since $f(a \cup b)$ is an antichain in D', we have $\alpha = \beta$. Therefore, $(a, \alpha), (b, \alpha) \in \mathrm{tr}(f)$ and $a \cup b \in \mathscr{D}$. It then follows from the definition of $\mathrm{tr}(f)$ that $a = b$.

Q.E.D.

Therefore, for any two coherence spaces D, D', the space of all stable functions from $\mathscr{A}(D)$ to $\mathscr{A}(D')$ may be identified with $\mathscr{A}(\mathscr{D}_0 \times D')$, where $\mathscr{D}_0 = \mathscr{A}_0(D)$.

Proposition 7.37. *Let D be a coherence space, $\mathscr{D} = \mathscr{A}(D)$ the corresponding qualitative domain, and $\mathscr{D}_0 = \mathscr{A}_0(D)$. Let i be an isomorphism of coherence spaces from $\mathscr{D}_0 \times D$ onto D. Then, with the following definitions, $\mathscr{D}$ is an extensional β-model :*
$(u)v = \{\alpha \in D; (\exists a \subset v)\, i(a, \alpha) \in u\}$ for all $u, v \in \mathscr{D}$;
$\lambda x f(x) = \{i(a, \alpha);\ a \in \mathscr{A}_0(D),\ \alpha \in f(a)$ and $\alpha \notin f(a')$ for every $a' \subset a,\ a' \neq a\}$ for every $f \in \mathscr{S}(\mathscr{D}, \mathscr{D})$.

Define $\Phi : \mathscr{D} \to \mathscr{S}(\mathscr{D}, \mathscr{D})$ by taking, for every $u \in \mathscr{D}$, $\mathrm{tr}(\Phi(u)) = i^{-1}(u) = \{(a, \alpha) ;\ i(a, \alpha) \in u\}$ which is an antichain in $\mathscr{D}_0 \times D'$, and therefore the trace of some stable function from $\mathscr{D}$ to $\mathscr{D}$. Thus Φ is an isomorphism of qualitative domains from $\mathscr{D}$ onto $\mathscr{S}(\mathscr{D}, \mathscr{D})$. Now, define $\Psi : \mathscr{S}(\mathscr{D}, \mathscr{D}) \to \mathscr{D}$ by taking $\Psi(f) = i(\mathrm{tr}(f)) = \{i(a, \alpha) ;\ (a, \alpha) \in \mathrm{tr}(f)\}$ which is, indeed, an antichain in $\mathscr{D}$ (an isomorphism of coherence spaces takes antichains to antichains). Then Φ and Ψ are inverse isomorphisms, so they are stable ; thus, it follows from proposition 7.35 that $\mathscr{D}$ is an extensional β-model of λ-calculus. For all $u, v \in \mathscr{D}$, we have $(u)v = \Phi(u)(v) = \{\alpha \in D ;\ (a, \alpha) \in \mathrm{tr}(\Phi(u))$ for some $a \subset v\} = \{\alpha \in D ;\ i(a, \alpha) \in u$ for some $a \subset v\}$.

Q.E.D.

Models over a set of atoms

Let A be a finite or countable non-empty set ; the elements of A will be called *atoms*. We are going to repeat (see page 124) the construction of the set of "formulas" over A, already used in the definition of Scott's model. Here it will be denoted by Δ, D being used to denote the coherence space which will be defined after. So we suppose that none of the atoms are ordered pairs, and we give an inductive definition of Δ and the one-to-one function $i : \Delta^* \times \Delta \to \Delta$ ($i(a,\alpha)$ will be denoted by $a \to \alpha$) :

- every atom is a formula ;
- whenever a is a finite set of formulas and α is a formula, if $a \neq \emptyset$ or $\alpha \notin A$, then (a,α) is a formula and we take $a \to \alpha = i(a,\alpha) = (a,\alpha)$.
- if $\alpha \in A$, then we take $\emptyset \to \alpha = i(\emptyset,\alpha) = \alpha$.

As above (page 124), we define the rank of a formula $\alpha \in \Delta$, which is denoted by $rk(\alpha)$. Let Δ_n be the set of all formulas with rank $\leq n$.

We now consider a coherence relation, denoted by $\asymp$, on $A = \Delta_0$. Let D_0 be the coherence space therefore obtained. We define, by induction on n, a coherence space $D_n \subset \Delta_n$: if $\alpha \in \Delta_n$, then $\alpha \in D_n \Leftrightarrow$ there exist $\beta \in D_{n-1}$, $b \in \mathscr{A}_0(D_{n-1})$ such that $\alpha = (b \to \beta)$. Thus the restriction of i to $\mathscr{A}_0(D_{n-1}) \times D_{n-1}$ is a one-to-one mapping of $\mathscr{A}_0(D_{n-1}) \times D_{n-1}$ into D_n. We define the coherence relation on D_n in such a way as to make of this mapping an isomorphism of coherence spaces.

Now we prove, by induction on n, that D_n is a coherence subspace of D_{n+1}. If $n = 0$, then $A \subset D_1$, since $\alpha \in A \Rightarrow \alpha = (\emptyset \to \alpha)$. If $\alpha, \beta \in A$, then $\alpha \asymp \beta$ holds in D_0 if and only if $(\emptyset \to \alpha) \asymp (\emptyset \to \beta)$ holds in D_1. Thus D_0 is a coherence subspace of D_1.

Assume that D_{n-1} is a coherence subspace of D_n. Then $\mathscr{A}_0(D_{n-1}) \times D_{n-1}$ is a coherence subspace of $\mathscr{A}_0(D_n) \times D_n$. Since i is an isomorphism from $\mathscr{A}_0(D_n) \times D_n$ onto D_{n+1}, and also from $\mathscr{A}_0(D_{n-1}) \times D_{n-1}$ onto D_n, it follows that D_n is a coherence subspace of D_{n+1}.

Now we may define a coherence space D as the union of the D_n's ; i is therefore an isomorphism of coherence spaces from $\mathscr{A}_0(D) \times D$ onto D.

We will call D the coherence space constructed over the set of atoms $(A, \asymp)$. If the coherence relation on A is taken as the least one ($\alpha \asymp \beta \Leftrightarrow \alpha = \beta$), then D is called the coherence space constructed over A.

The qualitative domain $\mathscr{D} = \mathscr{A}(D)$ associated with D is therefore an extensional β-model of λ-calculus.

Universal retractions

Let $\mathscr{D}$ be a β-model of λ-calculus. Recall that by a *retraction* in $\mathscr{D}$, we mean an element ϵ such that $\epsilon \circ \epsilon = \epsilon$. The image of ϵ is called the *retract* associated with ϵ.
The coherence models constructed above have a *universal retraction* :
this means that the set of all retractions of the model is a retract.
This final section is devoted to the proof of :

Theorem 7.38. *Let ρ be a constant symbol added to the language of combinatory logic, and UR be the set of formulas :*
$\rho \circ \rho = \rho$; $\forall x[(\rho x) \circ (\rho x) = \rho x]$; $\forall x[x \circ x = x \to \rho x = x]$.
Then the system of axioms $ECL + UR$ has a non-trivial model.

We shall prove that this system of axioms is indeed satisfied in the model $\mathscr{D} = \mathscr{A}(D)$, where D is the coherence space constructed over a set of atoms. This result is due to S. Berardi [Bera91]. The proof below is Amadio's [Ama95]. See also [Berl92].
The first lemma is about a simple combinatorial property of any function $f : X \to X$, with finite range. The notation f^n will stand for $f \circ \ldots \circ f$ (f occurs n times) ; $f^0 = Id$.

Lemma 7.39. *Let $f : X \to X$ be a function with finite range. Then there is one and only one retraction in $\{f^n; n \geq 1\}$.*

Uniqueness : if both f^m and f^n are retractions, then $(f^m)^n = f^m$ (since $n \geq 1$), and $(f^n)^m = f^n$ (since $m \geq 1$). Thus $f^m = f^n$.

Existence : let X_n be the image of f^n. X_n ($n \geq 1$) is a decreasing sequence of finite sets, thus there exists an integer $k \geq 1$ such that $X_k = X_n$ for all $n \geq k$. Let f_k be the restriction of f to X_k. Then f_k is a permutation of X_k, and hence $(f_k)^N$ is the identity function on X_k if $N = (\mathrm{card}(X_k))!$. It follows that f^N is the identity on X_k, thus so is f^{Nk}. Now the image of $f^{Nk} = (f^k)^N$ is X_k and therefore f^{Nk} is a retraction from X into X_k.

Q.E.D.

Let $\mathscr{D}_0$ be the set of all finite elements of $\mathscr{D}$ (finite antichains of D). If $f \in \mathscr{D}_0$, then $\{f u; u \in \mathscr{D}\}$ is a finite set : indeed, if we put $K_f = \{\alpha \in D$; there exists $a \in \mathscr{D}_0$ such that $(a \to \alpha) \in f\}$, then K_f is clearly a finite subset of D and, for every $u \in \mathscr{D}$, $f u$ is an antichain of K_f.
By the previous lemma, we may associate with each $f \in \mathscr{D}_0$ a retraction $\rho_0(f) : \mathscr{D} \to \mathscr{D}$, with finite range. Since $\rho_0(f) = f^n$, we have $\rho_0(f) \in \mathscr{D}_0$, and therefore $\rho_0 : \mathscr{D}_0 \to \mathscr{D}_0$.
ρ_0 is an increasing function : let $f, g \in \mathscr{D}_0$, $f \subset g$; then $\rho_0(f) = f^m$, $\rho_0(g) = g^n$. Now $f^m = (f^m)^n \subset (g^m)^n = (g^n)^m = g^n$ since both f^m and g^n are retractions.

Now we may define $\rho : \mathscr{D} \to \mathscr{D}$ by taking $\rho(u) = \cup_i \rho_0(u_i)$, where u_i is any increasing sequence in $\mathscr{D}_0$ such that $u = \cup_i u_i$. In order to verify the soundness of this definition, let u'_i be any other such sequence ; then we have $u_i \subset u'_j$ for a suitable j (since u_i is finite), thus $\rho_0(u_i) \subset \cup_j \rho_0(u'_j)$, and hence $\cup_i \rho_0(u_i) \subset \cup_j \rho_0(u'_j)$. We also have the inverse inclusion, since u_i and u'_j play symmetric parts.

Obviously, $\rho : \mathscr{D} \to \mathscr{D}$ is an increasing function ; moreover, it is σ-continuous : indeed, if u_i ($i \in \mathbb{N}$) is an increasing sequence in $\mathscr{D}$, and $u = \cup_i u_i$, then we may take an increasing sequence v_i of finite sets such that $v_i \subset u_i$ and $u = \cup_i v_i$. Then we have $\rho(u) = \cup_i \rho_0(v_i)$, and hence $\rho(u) \subset \cup_i \rho_0(u_i)$. Since ρ is increasing, we obtain immediately the inverse inclusion.

Finally, ρ is a stable function from $\mathscr{D}$ to $\mathscr{D}$: indeed, consider first $f, g \in \mathscr{D}_0$ such that $f \cup g \in \mathscr{D}_0$. We have $\rho_0(f) = f^m$, $\rho_0(g) = g^n$ and $\rho_0(f \cap g) = (f \cap g)^p$. Since f^m, g^n and $(f \cap g)^p$ are retractions, and $x \to x^{mnp}$ is a stable function (all functions represented by a λ-term are stable), we obtain :

$(f \cap g)^p = (f \cap g)^{mnp} = f^{mnp} \cap g^{mnp}$. Now $f^{mnp} = f^m$ and $g^{mnp} = g^n$, thus $(f \cap g)^p = f^m \cap g^n$, that is to say $\rho_0(f \cap g) = \rho_0(f) \cap \rho_0(g)$.

Now, let $u, v \in \mathscr{D}$ be such that $u \cup v \in \mathscr{D}$. Let $u_i, v_i \in \mathscr{D}_0$ be two increasing sequences, such that $u = \cup_i u_i$, $v = \cup_i v_i$. Then we have $\rho(u \cap v) = \cup_i \rho_0(u_i \cap v_i) = \cup_i [\rho_0(u_i) \cap \rho_0(v_i)]$ (according to the property which was previously proved) $= \cup_i \rho_0(u_i) \cap \cup_i \rho_0(v_i) = \rho(u) \cap \rho(v)$.

Therefore, $\rho \in \mathscr{D}$. Now we will see that ρ is a universal retraction.

Lemma 7.40. *$\rho \circ \rho = \rho$; $(\rho u) \circ (\rho u) = \rho u$ for every $u \in \mathscr{D}$.*

It can be seen easily that $\rho_0 \circ \rho_0 = \rho_0$: if $f \in \mathscr{D}_0$, then $\rho_0(f) = f^m$ for the least $m \geq 1$ such that f^m is a retraction. Thus $\rho_0(f^m) = f^m$.

Now let $u \in \mathscr{D}$; we have $u = \cup_i u_i$, where u_i is an increasing sequence in $\mathscr{D}_0$. Therefore : $\rho(u) = \cup_i \rho_0(u_i) = \cup_i \rho_0(\rho_0(u_i)) = \rho \circ \rho(u)$, since $\rho_0(u_i)$ is an increasing sequence in $\mathscr{D}_0$ such that its union is $\rho(u)$.

The proof of $(\rho u) \circ (\rho u) = \rho u$ is immediate, since $(\rho_0 u_i) \circ (\rho_0 u_i) = \rho_0 u_i$, and $(x, y) \to x \circ y$ is a σ-c.i. function from $\mathscr{D} \times \mathscr{D}$ to $\mathscr{D}$.

Q.E.D.

We will now prove that $r \circ r = r \Rightarrow \rho r = r$, that is to say that the image of ρ contains all the retractions of $\mathscr{D}$. Let r be a retraction of $\mathscr{D}$, and $r_i \in \mathscr{D}_0$ an increasing sequence such that $r = \cup_i r_i$.

We have $\rho(r) \subset r$: indeed, $\rho_0(r_i) = r_i^{k_i} \subset r^{k_i} = r$. Thus $\rho(r) = \cup_i \rho_0(r_i) \subset r$.

So it remains to prove that $r \subset \rho(r)$.

Lemma 7.41. *Let $a, u, r \in \mathscr{D}$ be such that $r = r \circ r$, $a \subset r u$ and a is finite. Then there exists a finite $c \in \mathscr{D}$ such that $a \subset r c$, $c \subset r c$, $c \subset r u$.*

Since $r = r \circ r$, we have $a \subset rru$. According to proposition 7.30(iii), there exists a least finite c such that $a \subset rc$ and $c \subset ru$. Now, if we put $d = rc$, we have $rd = rrc = rc$, thus $a \subset rd$; on the other hand, $c \subset ru$, thus $rc \subset rru$, that is $d \subset ru$. Since c is the least element satisfying these properties, we have $c \subset d$, thus $c \subset rc$.

Q.E.D.

Lemma 7.42. *Let $a, r \in \mathscr{D}$ be such that $r \circ r = r$, $a \subset ra$ and a is finite. Then $ra = \rho(r)a$.*

We have $a \subset ra = \cup_i r_i a$, thus, for some i_0, $a \subset r_i a$ holds for every $i \geq i_0$. By applying r_i on both sides of this inclusion, we obtain :
$a \subset r_i a \subset r_i^2 a \subset \ldots \subset r_i^n a \subset \ldots$
Now $\rho_0(r_i) = r_i^{k_i}$ for some $k_i \geq 1$; thus $r_i a \subset \rho_0(r_i)a$ for every $i \geq i_0$. It suffices to take the limits to obtain $ra \subset \rho(r)a$. The inverse inclusion is immediate, since $\rho(r) \subset r$.

Q.E.D.

Now we are able to complete the proof of theorem 7.38.
Take $u \in \mathscr{D}$ and $a \in \mathscr{D}_0$ such that $a \subset ru$. By lemma 7.41, there exists $c \in \mathscr{D}_0$ such that $a \subset rc$, $c \subset rc$ and $c \subset ru$. By lemma 7.42, we have $rc = \rho(r)c$ and hence $a \subset \rho(r)c$.
Since c is finite and contained in ru and rc, there exists $i \geq 1$ such that $c \subset r_i u$, $c \subset r_i c$. By applying r_i on both sides, we obtain $c \subset r_i c \subset r_i^2 c \subset \ldots \subset r_i^n c \subset \ldots$ Now $\rho_0(r_i) = r_i^{k_i}$ for some $k_i \geq 1$. Since $c \subset r_i u$, we have $r_i^{k_i - 1} c \subset r_i^{k_i} u = \rho(r_i)u \subset \rho(r)u$. Thus $c \subset \rho(r)u$. Since $a \subset \rho(r)c$ and $\rho(r)$ is a retraction, we have $a \subset \rho(r) \circ \rho(r)u = \rho(r)u$. Now a is an arbitrary finite subset of ru, and hence we obtain $ru \subset \rho(r)u$. The inverse inclusion $\rho(r)u \subset ru$ follows from $\rho(r) \subset r$. Finally, $\rho(r)u = ru$, thus $\rho(r) = r$ since u is an arbitrary element in $\mathscr{D}$ and $\mathscr{D}$ is extensional.

References for chapter 7

[Ama95], [Bar84], [Bera91], [Berl92], [Berr78], [Cop84], [Gir86], [Gir89], [Eng81], [Hin86], [Lon83], [Mey82], [Plo74], [Plo78], [Sco73], [Sco76], [Sco80], [Sco82], [Sto77], [Tar55].
(The references are in the bibliography at the end of the book).

Chapter 8

System F

1. Definition of system $\mathscr{F}$ types

In this chapter, we deal with the *second order propositional calculus*, i.e. the set of formulas built up with :

- a countable set of variables X, Y,..., (called type variables or propositional variables)
- the connective $\to$ and the quantifier $\forall$.

Remark. We observe that the second order propositional calculus is exactly the same as the set L of λ-terms defined in chapter 1 (page 7), with simply a change of notation : $\to$ instead of application, $\forall$ instead of λ. Indeed, we could define inductively an isomorphism as follows (denoting by t_A the λ-term associated with the formula A) :

if X is a type variable, then t_X is X itself, considered as a λ-variable ;

if A, B are formulas, then $t_{A\to B}$ is $(t_A)t_B$ and $t_{\forall X A}$ is $\lambda X\, t_A$.

For instance, the λ-term which corresponds to the formula :
$\forall X \forall Y(X, Y \to X) \to \forall Z(Z \to Z)$ would be $(\lambda X \lambda Y(X)(Y)X)\lambda Z(Z)Z$.
In fact, we are not interested in the λ-term associated with a formula. We simply observe that this isomorphism allows us to define, for second order propositional calculus, all the notions defined in chapter 1 for the set L of λ-terms : simple substitution, α-equivalence, ...

Thus, let $F, A_1, \ldots, A_k$ be formulas and $X_1, \ldots, X_k$ distinct variables. The formula $F{<}A_1/X_1, \ldots, A_k/X_k{>}$, obtained by simple substitution, is defined as in chapter 1 (page 8), and has exactly the same properties.

We similarly define the α-equivalence of formulas, denoted by $F \equiv G$, by induction on F :

- if X is a propositional variable, then $X \equiv G$ if and only if $G = X$;
- if $F = A \to B$, then $F \equiv G$ if and only if $G = A' \to B'$, where $A \equiv A'$ and $B \equiv B'$;

• if $F = \forall X\, A$, then $F \equiv G$ if and only if $G = \forall Y\, B$ and $A{<}Z/X{>} \equiv B{<}Z/Y{>}$ for all variables Z but a finite number.

We shall identify α-equivalent formulas. Like in chapter 1, this allows the definition of substitution :
We define the formula $F[A_1/X_1, \ldots, A_k/X_k]$ as $F{<}A_1/X_1, \ldots, A_k/X_k{>}$, provided that we choose a representative of F, no bound variable of which occurs free in $A_1, \ldots, A_k$.
All the lemmas about substitution in chapter 1 still hold.

The types of system $\mathscr{F}$ are, by definition, the equivalence classes of formulas, relative to the α-equivalence.

2. Typing rules for system $\mathscr{F}$

We wish to build typings of the form $\Gamma \vdash_{\mathscr{F}} t : A$, where Γ is a context, that is an expression of the form $x_1{:}A_1, \ldots, x_k{:}A_k$, where $x_1, \ldots, x_k$ are distinct variables, $A_1, \ldots, A_k, A$ are types of system $\mathscr{F}$, and t is a λ-term. The typing rules are the following :
1. If x is a variable not declared in Γ, then $\Gamma, x{:}A \vdash_{\mathscr{F}} x{:}A$;
2. If $\Gamma, x{:}A \vdash_{\mathscr{F}} t{:}B$, then $\Gamma \vdash_{\mathscr{F}} \lambda x\, t : A \to B$;
3. If $\Gamma \vdash_{\mathscr{F}} t : A$ and $\Gamma \vdash_{\mathscr{F}} u : A \to B$, then $\Gamma \vdash_{\mathscr{F}} (u)t : B$;
4. If $\Gamma \vdash_{\mathscr{F}} t : \forall X\, A$, then $\Gamma \vdash_{\mathscr{F}} t : A[F/X]$ for every type F ;
5. If $\Gamma \vdash_{\mathscr{F}} t : A$, then $\Gamma \vdash_{\mathscr{F}} t : \forall X\, A$ for every variable X such that no type in Γ contains a free occurrence of X.

From now on, throughout this chapter, the notation $\Gamma \vdash t{:}A$ will stand for $\Gamma \vdash_{\mathscr{F}} t{:}A$.
Obviously, if $\Gamma \vdash t{:}A$, then all free variables of t are declared in the context Γ.

Proposition 8.1. *If* $\Gamma \vdash t{:}A$ *and* $\Gamma \subset \Gamma'$, *then* $\Gamma' \vdash t{:}A$.

Same proof as proposition 3.3.
Q.E.D.

Proposition 8.2.
Let Γ *be a context, and* $x_1, \ldots, x_k$ *be variables which are not declared in* Γ.
If $\Gamma \vdash t_i{:}A_i$ $(1 \le i \le k)$ *and* $\Gamma, x_1{:}A_1, \ldots, x_k{:}A_k \vdash u{:}B$, *then :*
$\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : B$.

In particular :

If $x_1, \ldots, x_k$ *do not occur free in* u, *and if* $\Gamma, x_1{:}A_1, \ldots, x_k{:}A_k \vdash u{:}B$, *then* $\Gamma \vdash u{:}B$.

The proof is by induction on the number of rules used to obtain the typing $\Gamma, x_1{:}A_1, \ldots, x_k{:}A_k \vdash u{:}B$. Consider the last one :

If it is rule 1, 2 or 3, the proof is the same as that of proposition 4.1.

If it is rule 4, then $B \equiv A[F/X]$, and the previous step was :
$\Gamma, x_1{:}A_1, \ldots, x_k{:}A_k \vdash u : \forall X\, A$. By induction hypothesis, we get :
$\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : \forall X\, A$, and therefore, by rule 4 :
$\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : A[F/X]$.

If it is rule 5, then $B \equiv \forall X\, A$, and $\Gamma, x_1{:}A_1, \ldots, x_k{:}A_k \vdash u{:}A$ is a previous typing such that X does not occur free in $\Gamma, A_1, \ldots, A_k$. By induction hypothesis, we get $\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : A$, and therefore, by rule 5, $\Gamma \vdash u[t_1/x_1, \ldots, t_k/x_k] : \forall X\, A$.

Q.E.D.

Lemma 8.3. *If* $\Gamma \vdash t : \forall X_1 \ldots \forall X_k A$*, then* $\Gamma \vdash t : A[B_1/X_1, \ldots, B_k/X_k]$.

Indeed, suppose that $X_1, \ldots, X_k$ have no occurrence in $B_1, \ldots, B_k$ (this is possible by taking a suitable representative of $\forall X_1 \ldots \forall X_k A$).
By rule 4, we get $\Gamma \vdash t : A[B_1/X_1] \ldots [B_k/X_k]$.
Now $A[B_1/X_1] \ldots [B_k/X_k] \equiv A[B_1/X_1, \ldots, B_k/X_k]$ by lemma 1.13.

Q.E.D.

The part of the quantifier $\forall$ in system $\mathscr{F}$ is similar to that of the connective $\wedge$ in system $\mathscr{D}$. The next proposition is the analogue of lemma 3.22 :

Proposition 8.4.
If $\Gamma, x : F[A_1/X_1, \ldots, A_k/X_k] \vdash t : B$*, then* $\Gamma, x : \forall X_1 \ldots \forall X_k F \vdash t : B$.

The proof is done by induction on the number of rules used to obtain :
$\Gamma, x : F[A_1/X_1, \ldots, A_k/X_k] \vdash t : B$.
Consider the last one ; the only non-trivial case is that of rule 1, when t is the variable x. Then $B \equiv F[A_1/X_1, \ldots, A_k/X_k]$ and the result follows from lemma 8.3.

Q.E.D.

Notation.
Let Γ be the context $x_1{:}A_1, \ldots, x_n{:}A_n$. We define $\Gamma[B_1/X_1, \ldots, B_k/X_k]$ as the context $x_1 : A_1[B_1/X_1, \ldots, B_k/X_k], \ldots, x_n : A_n[B_1/X_1, \ldots, B_k/X_k]$.

Proposition 8.5.
If $\Gamma \vdash t : A$*, then* $\Gamma[B_1/X_1, \ldots, B_k/X_k] \vdash t : A[B_1/X_1, \ldots, B_k/X_k]$.

By induction on the length of the proof of $\Gamma \vdash t : A$; we also prove that the length of the proof of $\Gamma[B_1/X_1, \ldots, B_k/X_k] \vdash t : A[B_1/X_1, \ldots, B_k/X_k]$ is the same as that of $\Gamma \vdash t : A$. Consider the last rule used.

The result is obvious whenever it is rule 1, 2 or 3.

If it is rule 4, then $A \equiv A'[C/Y]$ and we have a previous typing of the form $\Gamma \vdash t : \forall Y\, A'$. By induction hypothesis, we have :
$\Gamma[B_1/X_1, \ldots, B_k/X_k] \vdash t : \forall Y\, A'[B_1/X_1, \ldots, B_k/X_k]$ ($Y \neq X_1, \ldots, X_k$ and Y does

not occur free in $B_1,\dots,B_k$). Moreover, the length of the proof of this typing is the same as that of $\Gamma \vdash t : \forall Y\, A'$.
Thus, by rule 4, we have :
$\Gamma[B_1/X_1,\dots,B_k/X_k] \vdash t : A'[B_1/X_1,\dots,B_k/X_k][C'/Y]$
for any formula C'. Since Y does not occur free in $B_1,\dots,B_k$, by lemma1.13, this is equivalent to :
$\Gamma[B_1/X_1,\dots,B_k/X_k] \vdash t : A'[B_1/X_1,\dots,B_k/X_k, C'/Y]$.
Now take $C' \equiv C[B_1/X_1,\dots,B_k/X_k]$. Again by lemma 1.13, we have :
$A'[B_1/X_1,\dots,B_k/X_k, C'/Y] \equiv A'[C/Y][B_1/X_1,\dots,B_k/X_k] \equiv A[B_1/X_1,\dots,B_k/X_k]$.
Hence $\Gamma[B_1/X_1,\dots,B_k/X_k] \vdash t : A[B_1/X_1,\dots,B_k/X_k]$, and we obtain a proof of the same length as that of $\Gamma \vdash t : A$.

If it is rule 5, we have $\Gamma \vdash t : A'$ as a previous typing, and $A \equiv \forall Y\, A'$, where Y does not occur free in Γ. Take a variable $Z \neq X$, which does not occur in $\Gamma, A', B_1,\dots,B_k$. By induction hypothesis, we have :
$\Gamma[Z/Y] \vdash t : A''$, where $A'' \equiv A'[Z/Y]$. In other words, $\Gamma \vdash t : A''$ (since Y does not occur in Γ). Moreover, the length of the proof is the same, so we may use the induction hypothesis, and obtain :
$\Gamma[B_1/X_1,\dots,B_k/X_k] \vdash t : A''[B_1/X_1,\dots,B_k/X_k]$.
Since Z does not occur in $\Gamma, B_1,\dots,B_k$, it does not occur in $[B_1/X_1,\dots,B_k/X_k]$; therefore, by rule 5 :

$$\Gamma[B_1/X_1,\dots,B_k/X_k] \vdash t : \forall Z\, A''[B_1/X_1,\dots,B_k/X_k].$$

Now $\forall Z\, A'' \equiv \forall Y\, A'$ (lemma 1.10) $\equiv A$; hence :
$\forall Z\, A''[B_1/X_1,\dots,B_k/X_k] \equiv A[B_1/X_1,\dots,B_k/X_k]$, and therefore :
$\Gamma[B_1/X_1,\dots,B_k/X_k] \vdash t : A[B_1/X_1,\dots,B_k/X_k]$.
Q.E.D.

By an *open* formula, we mean a formula of which the first symbol is different from $\forall$; so it is either a type variable or a formula of the form $B \to C$.

For every formula A, we denote by A^0 the unique open formula such that :

$$A \equiv \forall X_1 \dots \forall X_n\, A^0 \ (n \in \mathbb{N}).$$

This formula A^0 will be called the *interior* of A.

Let Γ be a context (resp. F be a formula), $X_1,\dots,X_k$ type variables *with no free occurrence in* Γ (resp. F), and A a formula.
Any formula of the form $A[B_1/X_1,\dots,B_k/X_k]$ will be called a Γ*-instance* of A (resp. F*-instance* of A). Therefore :
If $A \equiv \forall X_1 \dots \forall X_k\, A^0$, then any formula of the form $A^0[B_1/X_1,\dots,B_k/X_k]$ is an A-instance of A^0.

The next lemma is the analogue of lemma 4.2.

Lemma 8.6. *Suppose that* $\Gamma \vdash t : A$, *where* A *is an open formula.*
i) if t *is a variable* x, *then* Γ *contains a declaration* $x : B$ *such that* A *is a* B*-instance of* B^0.

ii) if $t = \lambda x\, u$*, then* $A \equiv (B \to C)$*, and* $\Gamma, x : B \vdash u : C$*.*
iii) if $t = (u)v$*, then* $\Gamma \vdash u : C \to B, \Gamma \vdash v : C$*, where* B *is such that* A *is a* Γ*-instance of* B^0*.*

In the proof of $\Gamma \vdash t : A$, consider the first step at which one obtains $\Gamma \vdash t : B$, for some formula B such that A is a Γ-instance of B^0 (this happens at least once, for example with $B = A$). Examine the typing rule (page 146) used at that step.

It is not rule 4 : indeed, if it were, we would have obtained at the previous step $\Gamma \vdash t : \forall X C$, with $B = C[U/X]$. We may suppose that X does not occur in Γ.
We have $C = \forall X_1 \dots \forall X_n\; C^0$, where C^0 is an open formula ; thus C^0 is either a variable or a formula of the form $F \to G$.

If $C^0 = X$, then every formula (therefore particularly A) is a Γ-instance of C^0 ; this contradicts the definition of B.

If C^0 is a variable $Y \neq X$, then $B = C[U/X] = C$, so $B^0 = C^0$, and A is a Γ-instance of C^0 ; again, this contradicts the definition of B.

If $C^0 = F \to G$, then $B = \forall X_1 \dots \forall X_n\, C^0[U/X]$.
Now $C^0[U/X] = F' \to G'$ is an open formula. Thus $B^0 = C^0[U/X]$. Since A is a Γ-instance of B^0, we have, by lemma 1.13 :

$$A = B^0[U_1/Z_1,\dots,U_k/Z_k] = C^0[U/X][U_1/Z_1,\dots,U_k/Z_k]$$
$$= C^0[U_1/Z_1,\dots,U_k/Z_k,U'/X]$$

where $U' = U[U_1/Z_1,\dots,U_k/Z_k]$. Now, by hypothesis, $Z_1,\dots,Z_k$ are variables which do not occur in Γ, and neither does X. Thus A is a Γ-instance of C^0, contradicting the definition of B.

It is not rule 5 : suppose it were ; then $B = \forall X C$, and therefore $B^0 = C^0$. Hence $\Gamma \vdash t : C$ at the previous step, and A is a Γ-instance of C^0 ; this contradicts the definition of B.

Now we can prove the lemma :
In case (i), the rule applied at that step needs to be rule 1, since t is a variable x. Therefore Γ contains the declaration $x : B$, and A is a Γ-instance of B^0. Since the formula $B = \forall X_1 \dots \forall X_k\, B^0$ appears in the context Γ, the free variables of B^0 which do not occur free in Γ are $X_1,\dots,X_k$. Thus A is a B-instance of B^0.
In case (ii), the rule applied is rule 2. Thus :
$B = (C \to D)$, and $\Gamma, x : C \vdash u : D$.
Now B is an open formula, so A is a Γ-instance of $B^0 = B$.
Hence, we have $A = C' \to D'$, with :
$C' = C[U_1/X_1,\dots,U_k/X_k]$ and $D' = D[U_1/X_1,\dots,U_k/X_k]$.
By proposition 8.5, one deduces from $\Gamma, x : C \vdash u : D$ that :
$\Gamma[U_1/X_1,\dots,U_k/X_k],\ x : C[U_1/X_1,\dots,U_k/X_k] \vdash u : D[U_1/X_1,\dots,U_k/X_k]$. Since $X_1,\dots,X_k$ do not occur in Γ, we finally obtain $\Gamma, x : C' \vdash u : D'$ and $A = C' \to D'$.

In case (iii), the rule applied at that step is rule 3 since the term t is $(u)v$.
Hence $\Gamma \vdash u : C \to B$ and $\Gamma \vdash v : C$, so A is a Γ-instance of B^0.

Q.E.D.

Theorem 8.7. *If $\Gamma \vdash t : A$ and $t \,\beta\, t'$, then $\Gamma \vdash t' : A$.*

Recall that $t \,\beta\, t'$ means that t' is obtained from t by β-reduction.

It is sufficient to repeat the proof of proposition 4.3 (which is the corresponding statement for system $\mathscr{D}$), using lemma 8.6(ii) instead of lemma 4.2(ii) and proposition 8.2 instead of proposition 4.1.

Q.E.D.

Theorem 8.7 fails if one replaces the assumption $t \,\beta\, t'$ with $t \simeq_\beta t'$. Take for instance $t = \lambda x\, x$, $t' = \lambda x(\lambda y\, x)(x)x$; then $\vdash t : X \to X$, where X is a variable. Yet $\vdash t' : X \to X$ does not hold : indeed, by lemma 8.6, this would imply :
$x : X \vdash (\lambda y\, x)(x)x : X$, and therefore $x : X \vdash (x)x : A$ for some formula A, which is clearly impossible (again by lemma 8.6).

We shall denote by $\perp$ the formula $\forall X\, X$; thus we have $\Gamma, x : \perp \vdash x : A$ for every formula A (rules 1 and 4, page 146).
We define the connective $\neg$ by taking $\neg A \equiv A \to \perp$ for every formula A.

Proposition 8.8.
Every normal term t is typable in system $\mathscr{F}$, in the context $x_1 : \perp, \ldots, x_k : \perp$, where $x_1, \ldots, x_k$ are the free variables of t.

Proof by induction on the length of t. Let Γ be the context $x_1 : \perp, \ldots, x_k : \perp$, where $x_1, \ldots, x_k$ are the free variables of t.
If $t = \lambda x\, u$, then, by induction hypothesis, we have $\Gamma, x : \perp \vdash u : A$; thus :
$\Gamma \vdash \lambda x\, u : \perp \to A$.
If t does not start with λ, then $t = (x_1)t_1 \ldots t_n$.
By induction hypothesis, $\Gamma \vdash t_i : A_i$.
On the other hand, $\Gamma \vdash x_1 : \perp$, so $\Gamma \vdash x_1 : A_1, \ldots, A_n \to X$ (rule 4).
Therefore, $\Gamma \vdash t : X$.

Q.E.D.

Nevertheless, there are strongly normalizable closed terms which are not typable in system $\mathscr{F}$ (see [Gia88]).

3. The strong normalization theorem

In this section, we will prove the following theorem of J.-Y. Girard [Gir71] :

Theorem 8.9. *Every term which is typable in system $\mathscr{F}$ is strongly normalizable.*

We shall follow the proof of the corresponding theorem for system $\mathscr{D}$ (theorem 3.20). As there, $\mathscr{N}$ denotes the set of strongly normalizable terms and $\mathscr{N}_0$ the set of terms of the form $(x)t_1\dots t_n$, where x is a variable and $t_1,\dots,t_n \in \mathscr{N}$.
A subset $\mathscr{X}$ of Λ is $\mathscr{N}$*-saturated* if and only if :
$(\lambda x\, u)t\,t_1\dots t_n \in \mathscr{X}$ whenever $t \in \mathscr{N}$ and $(u[t/x])t_1\dots t_n \in \mathscr{X}$.
We proved in chapter 3 (page 53) that $(\mathscr{N}_0, \mathscr{N})$ is an adapted pair, that is :
i) $\mathscr{N}$ is $\mathscr{N}$-saturated ;
ii) $\mathscr{N}_0 \subset \mathscr{N}$; $\mathscr{N}_0 \subset (\mathscr{N} \to \mathscr{N}_0)$; $(\mathscr{N}_0 \to \mathscr{N}) \subset \mathscr{N}$.

An $\mathscr{N}$-interpretation $\mathscr{I}$ is a mapping $X \to |X|_{\mathscr{I}}$ of the set of type variables into the set of $\mathscr{N}$-saturated subsets of $\mathscr{N}$ which contain $\mathscr{N}_0$.

Let $\mathscr{I}$ be an $\mathscr{N}$-interpretation, X a type variable, and $\mathscr{X}$ an $\mathscr{N}$-saturated subset of Λ such that $\mathscr{N}_0 \subset \mathscr{X} \subset \mathscr{N}$. We define an $\mathscr{N}$-interpretation $\mathscr{J} = \mathscr{I}[X \leftarrow \mathscr{X}]$ by taking $|Y|_{\mathscr{J}} = |Y|_{\mathscr{I}}$ for every variable $Y \neq X$ and $|X|_{\mathscr{J}} = \mathscr{X}$.

For every type A, the value $|A|_{\mathscr{I}}$ of A in an $\mathscr{N}$-interpretation $\mathscr{I}$ is a set of terms defined as follows, by induction on A :

- if A is a type variable, then $|A|_{\mathscr{I}}$ is given with $\mathscr{I}$;
- $|A \to B|_{\mathscr{I}} = (|A|_{\mathscr{I}} \to |B|_{\mathscr{I}})$, in other words :

for every term t, $t \in |A \to B|_{\mathscr{I}}$ if and only if $(t)u \in |B|_{\mathscr{I}}$ for every $u \in |A|_{\mathscr{I}}$;

- $|\forall X\, A|_{\mathscr{I}} = \bigcap\{|A|_{\mathscr{I}[X\leftarrow\mathscr{X}]} ; \mathscr{X}$ is $\mathscr{N}$-saturated, $\mathscr{N}_0 \subset \mathscr{X} \subset \mathscr{N}\}$,

in other words : for every term t, $t \in |\forall X\, A|_{\mathscr{I}}$ if and only if $t \in |A|_{\mathscr{I}[X\leftarrow\mathscr{X}]}$ for every $\mathscr{N}$-saturated subset $\mathscr{X}$ of Λ such that $\mathscr{N}_0 \subset \mathscr{X} \subset \mathscr{N}$.

Clearly, the value $|A|_{\mathscr{I}}$ of a type A in an $\mathscr{N}$-interpretation $\mathscr{I}$ depends only on the values in $\mathscr{I}$ of the free variables of A. In particular, if A is a closed type, then $|A|_{\mathscr{I}}$ is independent of the interpretation $\mathscr{I}$.

Lemma 8.10. *For every type A and every $\mathscr{N}$-interpretation $\mathscr{I}$, the value $|A|_{\mathscr{I}}$ is an $\mathscr{N}$-saturated subset of $\mathscr{N}$ which contains $\mathscr{N}_0$.*

The proof is by induction on A :

If A is a type variable, this is obvious from the definition of $\mathscr{N}$-interpretations.

If $A = B \to C$, then, by induction hypothesis, $\mathscr{N}_0 \subset |B|_{\mathscr{I}}$ and $|C|_{\mathscr{I}} \subset \mathscr{N}$. Therefore, $|B \to C|_{\mathscr{I}} = |B|_{\mathscr{I}} \to |C|_{\mathscr{I}} \subset \mathscr{N}_0 \to \mathscr{N}$. Now $\mathscr{N}_0 \to \mathscr{N} \subset \mathscr{N}$ (definition of the adapted pairs) ; hence $|B \to C|_{\mathscr{I}} \subset \mathscr{N}$.
Also by induction hypothesis, we have $\mathscr{N}_0 \subset |C|_{\mathscr{I}}$ and $|B|_{\mathscr{I}} \subset \mathscr{N}$. It follows that $|B \to C|_{\mathscr{I}} = (|B|_{\mathscr{I}} \to |C|_{\mathscr{I}}) \supset \mathscr{N} \to \mathscr{N}_0$. Now $\mathscr{N} \to \mathscr{N}_0 \supset \mathscr{N}_0$, and therefore $|B \to C|_{\mathscr{I}} \supset \mathscr{N}_0$.
On the other hand, $|A|_{\mathscr{I}} = (|B|_{\mathscr{I}} \to |C|_{\mathscr{I}})$ is $\mathscr{N}$-saturated since $|C|_{\mathscr{I}}$ is (proposition 3.15).

If $A = \forall X B$, then $|\forall X\; B|_{\mathscr{I}} \subset |B|_{\mathscr{I}} \subset \mathscr{N}$ (by induction hypothesis) ; now $\mathscr{N}_0 \subset |B|_{\mathscr{J}}$ for any $\mathscr{N}$-interpretation $\mathscr{J}$ (induction hypothesis), and therefore

$\mathcal{N}_0 \subset |\forall X\ B|_{\mathcal{I}}$. Finally, $|\forall X\,B|_{\mathcal{I}}$ is $\mathcal{N}$-saturated, as the intersection of a set of $\mathcal{N}$-saturated subsets of Λ.

Q.E.D.

Lemma 8.11. *Let A, U be two types, X a variable, $\mathcal{I}$ an $\mathcal{N}$-interpretation and $\mathcal{X} = |U|_{\mathcal{I}}$. Then $|A[U/X]|_{\mathcal{I}} = |A|_{\mathcal{J}}$, where $\mathcal{J} = \mathcal{I}[X \leftarrow \mathcal{X}]$.*

Proof by induction on A.
This is obvious whenever A is a type variable or $A = B \to C$.
Suppose $A = \forall Y\,B$ ($Y \neq X$, and Y does not occur in U).
For each term $t \in \Lambda$, we have :
i) $t \in |\forall Y\,B[U/X]|_{\mathcal{I}}$ if and only if $t \in |B[U/X]|_{\mathcal{I}[Y \leftarrow \mathcal{Y}]}$ for every $\mathcal{N}$-saturated subset $\mathcal{Y}$ of Λ such that $\mathcal{N}_0 \subset \mathcal{Y} \subset \mathcal{N}$;
ii) $t \in |\forall Y\,B|_{\mathcal{J}}$ if and only if $t \in |B|_{\mathcal{J}[Y \leftarrow \mathcal{Y}]}$ for every $\mathcal{N}$-saturated subset $\mathcal{Y}$ of Λ such that $\mathcal{N}_0 \subset \mathcal{Y} \subset \mathcal{N}$.
Let $\mathcal{I}_0 = \mathcal{I}[Y \leftarrow \mathcal{Y}]$ and $\mathcal{J}_0 = \mathcal{J}[Y \leftarrow \mathcal{Y}]$; then $\mathcal{J}_0 = \mathcal{I}_0[X \leftarrow \mathcal{X}]$ since $Y \neq X$. On the other hand, $\mathcal{X} = |U|_{\mathcal{I}} = |U|_{\mathcal{I}_0}$ since Y is not a free variable in U. Hence, by induction hypothesis, $|B[U/X]|_{\mathcal{I}_0} = |B|_{\mathcal{J}_0}$. Thus, it follows from (i) and (ii) that $|\forall Y\,B[U/X]|_{\mathcal{I}} = |\forall Y\ B|_{\mathcal{J}}$.
Q.E.D.

Lemma 8.12 (Adequacy lemma)**.** *Let $\mathcal{I}$ be an $\mathcal{N}$-interpretation.*
If $x_1 : A_1, \ldots, x_k : A_k \vdash u : A$ and $t_i \in |A_i|_{\mathcal{I}}$ $(1 \leq i \leq k)$, then :
$u[t_1/x_1, \ldots, t_k/x_k] \in |A|_{\mathcal{I}}$.

The proof is by induction on the number of rules used to obtain the given typing $x_1 : A_1, \ldots, x_k : A_k \vdash u : A$. Consider the last one. If it is rule 1, 2 or 3, then the proof is the same as for the second adequacy lemma 3.16.

If it is rule 4, then $A = B[U/X]$, and we have :
$x_1 : A_1, \ldots, x_k : A_k \vdash u : \forall X\,B$ as a previous typing.
By induction hypothesis, $u[t_1/x_1, \ldots, t_k/x_k] \in |\forall X\ B|_{\mathcal{I}}$;
thus $u[t_1/x_1, \ldots, t_k/x_k] \in |B|_{\mathcal{J}}$, where $\mathcal{J} = \mathcal{I}[X \leftarrow \mathcal{X}]$, for every $\mathcal{N}$-saturated subset $\mathcal{X}$ of Λ such that $\mathcal{N}_0 \subset \mathcal{X} \subset \mathcal{N}$.
By taking $\mathcal{X} = |U|_{\mathcal{I}}$, we obtain $|B|_{\mathcal{J}} = |B[U/X]|_{\mathcal{I}}$, in view of lemma 8.11. Therefore $u[t_1/x_1, \ldots, t_k/x_k] \in |B[U/X]|_{\mathcal{I}}$.

If it is rule 5, then $A = \forall X\,B$, and we have a previous typing :
$x_1 : A_1, \ldots, x_k : A_k \vdash u : B$; moreover, X does not occur free in $A_1, \ldots, A_k$. Let $\mathcal{X}$ be an $\mathcal{N}$-saturated subset of Λ such that $\mathcal{N}_0 \subset \mathcal{X} \subset \mathcal{N}$, and let $\mathcal{J} = \mathcal{I}[X \leftarrow \mathcal{X}]$. Thus $|A_i|_{\mathcal{I}} = |A_i|_{\mathcal{J}}$, since X does not occur free in A_i. Hence $t_i \in |A_i|_{\mathcal{J}}$.
By induction hypothesis, we have $u[t_1/x_1, \ldots, t_k/x_k] \in |B|_{\mathcal{J}}$ and therefore :
$u[t_1/x_1, \ldots, t_k/x_k] \in |\forall X\ B|_{\mathcal{I}}$.

Q.E.D.

Now the proof of the strong normalization theorem easily follows :
Suppose $x_1 : A_1, \ldots, x_k : A_k \vdash t : A$ and consider the $\mathcal{N}$-interpretation $\mathcal{I}$ defined by taking $|X|_{\mathcal{I}} = \mathcal{N}$ for every variable X. By lemma 2, we have $\mathcal{N}_0 \subset |A_i|_{\mathcal{I}}$, so $x_i \in |A_i|_{\mathcal{I}}$. Thus, by the adequacy lemma 8.12, $t[x_1/x_1, \ldots, x_k/x_k] = t \in |A|_{\mathcal{I}}$. Now $|A|_{\mathcal{I}} \subset \mathcal{N}$ (by lemma 2), and therefore $t \in \mathcal{N}$.

Q.E.D.

4. Data types in system $\mathcal{F}$

Recall some definitions from chapter 3 :

A subset $\mathcal{X}$ of Λ is *saturated* if and only if $(\lambda x\ u)t t_1 \ldots t_n \in \mathcal{X}$ whenever $(u[t/x])t_1 \ldots t_n \in \mathcal{X}$.

An *interpretation* $\mathcal{I}$ is a mapping $X \to |X|_{\mathcal{I}}$ of the set of type variables into the set of saturated subsets of Λ.

Let $\mathcal{I}$ be an interpretation, X a type variable, and $\mathcal{X}$ a saturated subset of Λ. We define an interpretation $\mathcal{J} = \mathcal{I}[X \leftarrow \mathcal{X}]$ by taking $|Y|_{\mathcal{J}} = |Y|_{\mathcal{I}}$ for every variable $Y \neq X$ and $|X|_{\mathcal{J}} = \mathcal{X}$.

For every type A, the value $|A|_{\mathcal{I}}$ of A in an interpretation $\mathcal{I}$ is a set of terms defined as follows, by induction on A :

- if A is a type variable, then $|A|_{\mathcal{I}}$ is given with $\mathcal{I}$;
- $|A \to B|_{\mathcal{I}} = |A|_{\mathcal{I}} \to |B|_{\mathcal{I}}$, in other words :

for every term t, $t \in |A \to B|_{\mathcal{I}}$ if and only if $tu \in |B|_{\mathcal{I}}$ for every $u \in |A|_{\mathcal{I}}$;

- $|\forall X\, A|_{\mathcal{I}} = \bigcap\{|A|_{\mathcal{I}[X \leftarrow \mathcal{X}]}; \mathcal{X}$ is any saturated subset of $\Lambda\}$,

in other words : for every term t, $t \in |\forall X\, A|_{\mathcal{I}}$ if and only if $t \in |A|_{\mathcal{I}[X \leftarrow \mathcal{X}]}$ for every saturated subset $\mathcal{X}$ of Λ.

Lemma 8.13 (Adequacy lemma)**.** *Let $\mathcal{I}$ be an interpretation ;*
if $x_1 : A_1, \ldots, x_k : A_k \vdash u : A$ and $t_i \in |A_i|_{\mathcal{I}}$ $(1 \leq i \leq k)$, then :
$u[t_1/x_1, \ldots, t_k/x_k] \in |A|_{\mathcal{I}}$.

Same proof as above.

Q.E.D.

The value of a closed type A (that is a type with no free variables) is the same in all interpretations ; it will be denoted by $|A|$.

A closed type A will be called a *data type* if :
i) $|A| \neq \emptyset$;
ii) every term $t \in |A|$ is β-equivalent to a closed term.

Condition (ii) can also be stated this way :
ii') every term $t \in |A|$ can be transformed in a closed term by β-reduction.

Indeed, if (ii) holds, then $t \simeq_\beta u$ for some closed term u ; by the Church-Rosser theorem, t and u reduce to the same term v by β-reduction. Now β-reduction applied to a closed term produces only closed terms. Thus v is closed.

Proposition 8.14. *The types :*

$Id = \forall X(X \to X)$ *(identity type) ;*

$Bool = \forall X\{X, X \to X\}$ *(Booleans type) ;*

$Int = \forall X\{(X \to X) \to (X \to X)\}$ *(integers type)*

are data types. More precisely :

$t \in |Id| \Leftrightarrow t \simeq_\beta \lambda x\, x$ *;*

$t \in |Bool| \Leftrightarrow t \simeq_\beta \lambda x \lambda y\, x$ *or* $t \simeq_\beta \lambda x \lambda y\, y$ *;*

$t \in |Int| \Leftrightarrow t \simeq_\beta \lambda f \lambda x (f)^n x$ *for some integer* n *or* $t \simeq_\beta \lambda f\, f$*.*

Note that, in view of the adequacy lemma 8.13, we have the following consequences :

If $\vdash t : \mathrm{Id}$, then $t \simeq_\beta \lambda x\, x$.

If $\vdash t : \mathrm{Bool}$, then $t \simeq_\beta \lambda x \lambda y\, x$ or $t \simeq_\beta \lambda x \lambda y\, y$;

If $\vdash t : \mathrm{Int}$, then $t \simeq_\beta \lambda f \lambda x (f)^n x$ for some integer n or $t \simeq_\beta \lambda f\, f$.

Proof of the proposition : we first show the implications $\Rightarrow$.

1. Identity type :

Let $t \in |\mathrm{Id}|$ and x be a variable of the λ-calculus which does not occur in t ; we define an interpretation $\mathscr{I}$ by taking $|X|_{\mathscr{I}} = \{\tau \in \Lambda;\ \tau \simeq_\beta x\}$ for every type variable X. Since $t \in |\mathrm{Id}|$, we have $t \in |X \to X|$. Now $x \in |X|$, so $(t)x \in |X|$, and therefore $(t)x \simeq_\beta x$. Thus t is normalizable ($t \simeq_{\beta\eta} \lambda x\, x$). Let t' be its normal form ; then $t' = \lambda x_1 \ldots \lambda x_m (y) t_1 \ldots t_n$.

If $m = 0$, then $(t')x \simeq_\beta (y')u_1 \ldots u_n x$, where y' is a variable. This term cannot be equal to x, so we have a contradiction.

If $m \geq 1$, then we have $t' = \lambda x\, u$. So $(t')x \simeq_\beta u$; therefore $u \simeq_\beta x$, and $t' \simeq_\beta \lambda x\, x$. Since t' is normal, $t' = \lambda x\, x$.

2. Booleans type :

Let $t \in |\mathrm{Bool}|$ and x, y be variables of the λ-calculus which do not occur in t ; we define an interpretation $\mathscr{I}$ by taking $|X|_{\mathscr{I}} = \{\tau \in \Lambda;\ \tau \simeq_\beta x$ or $\tau \simeq_\beta y\}$. Since $t \in |\mathrm{Bool}|$, we have $t \in |X, X \to X|$. Now $x, y \in |X|$, so $(t)xy \in |X|$, that is, for instance, $(t)xy \simeq_\beta x$. Thus $t \simeq_{\beta\eta} \lambda x \lambda y\, x$, and t is normalizable. Let t' be its normal form ; then $t' = \lambda x_1 \ldots \lambda x_m (z) t_1 \ldots t_n$.

If $m = 0$ or 1, then $(t')xy \simeq_\beta (z')u_1 \ldots u_n xy$ or $(z')u_1 \ldots u_n y$, where z' is a variable. None of these terms can be equal to x, so we have a contradiction.

If $m \geq 2$, then we have $t' = \lambda x \lambda y\, u$, thus $(t')xy \simeq_\beta u$.

Therefore $u \simeq_\beta x$ and $t' \simeq_\beta \lambda x \lambda y\, x$. Since t' is normal, $t' = \lambda x \lambda y\, x$.

3. Integers type :
Let $t \in |\mathrm{Int}|$ and f, x be variables of the λ-calculus which do not occur in t ; we define an interpretation $\mathscr{I}$ by taking $|X|_{\mathscr{I}} = \{\tau \in \Lambda;\, \tau \simeq_\beta (f)^k x$ for some $k \geq 0\}$ for every type variable X. Thus $x \in |X|$ and $f \in |X \to X|$.
Since $t \in |\mathrm{Int}|$, we have $t \in |(X \to X), X \to X|$. Thus $(t)fx \in |X|$, and hence $(t)fx \simeq_\beta (f)^k x$. It follows that $t \simeq_{\beta\eta} \lambda f \lambda x (f)^k x$, so t is normalizable. Let t' be its normal form ; then $t' = \lambda x_1 \ldots \lambda x_m (y) t_1 \ldots t_n$.
If $m = 0$, then $(t')fx \simeq_\beta (y')u_1 \ldots u_n f x$, where y' is a variable. This term cannot be equal to $(f)^k x$, so we have a contradiction.
If $m = 1$, then we have $t' = \lambda f (y) t_1 \ldots t_n$. So $(t')fx \simeq_\beta (y) t_1 \ldots t_n x$. Since this term needs to be equal to $(f)^k x$, we necessarily have $y = f$ and $n = 0$; thus $t' = \lambda f f$.
If $m \geq 2$, then we have $t' = \lambda f \lambda x\, u$; so $(t')fx \simeq_\beta u$. Therefore $u \simeq_\beta (f)^k x$ and $t' \simeq_\beta \lambda f \lambda x (f)^k x$. Since t' is normal, we conclude that $t' = \lambda f \lambda x (f)^k x$.

Now we come to the implications $\Leftarrow$. We shall treat for instance the case of the type Int. Suppose $t \simeq_\beta \lambda f f$ or $t \simeq_\beta \lambda f \lambda x (f)^k x$ for some $k \geq 0$. In system $\mathscr{D}\Omega$, we have $\vdash_{\mathscr{D}\Omega} \lambda f f : (X \to X) \to (X \to X)$ and
$\vdash_{\mathscr{D}\Omega} \lambda f \lambda x (f)^k x : (X \to X) \to (X \to X)$.
Thus, by theorem 4.7, we have $\vdash_{\mathscr{D}\Omega} t : (X \to X) \to (X \to X)$. In view of the adequacy lemma for system $\mathscr{D}\Omega$ (lemma 3.5), we have :
$t \in |(X \to X) \to (X \to X)|_{\mathscr{I}}$ for every interpretation $\mathscr{I}$.
Hence $t \in |\forall X\{(X \to X) \to (X \to X)\}| = |\mathrm{Int}|$.

Q.E.D.

We can similarly define the type $\forall X\{(X \to X), (X \to X), X \to X\}$ of binary lists (finite sequences of 0's and 1's), the type $\forall X\{(X, X \to X), X \to X\}$ of binary trees, etc. All of them are data types.
In the next section, we give a syntactic condition which is sufficient in order that a formula be a data type (corollary 8.19).

The type Int → Int (of the functions from the integers to the integers) is not a data type.

Indeed, let $\xi = \lambda n\, n I 0 y$ where y is a variable and $I = \lambda x\, x$. Then ξ is a non-closed normal term, so it is not β-equivalent to any closed term.
Now $\xi \in |\mathrm{Int} \to \mathrm{Int}|$: suppose $\nu \in |\mathrm{Int}|$, then ν is β-equivalent to a Church numeral, and therefore $\xi \nu \simeq_\beta \lambda x\, x \in |\mathrm{Int}|$.
Indeed, even the type $\mathrm{Id} \to \mathrm{Id}$ is not a data type : apply the same method to $\xi' = \lambda f\, f 0 y$.

The next proposition shows that it is possible to obtain new data types from given ones :

Proposition 8.15. *Let A, B be two data types. Then the types :*

$A \wedge B : \forall X\{(A, B \to X) \to X\}$ (product of A and B) ;
$A \vee B : \forall X\{(A \to X), (B \to X) \to X\}$ (disjoint sum of A and B) ;
$L[A] : \forall X\{(A, X \to X), X \to X\}$ (type of the lists of objects of type A)
are data types. More precisely :
If $t \in |A \wedge B|$, then $t \simeq_\beta \lambda f(f)ab$, where $a \in |A|$ and $b \in |B|$.
If $t \in |A \vee B|$, then either $t \simeq_\beta \lambda f \lambda g(f)a$ for some $a \in |A|$ or $t \simeq_\beta \lambda f \lambda g(g)b$ for some $b \in |B|$.
If $t \in |L[A]|$, then either $t \simeq_\beta \lambda f \lambda x(f a_1)(f a_2)\ldots(f a_n)x$, where $n \geq 0$ and $a_i \in |A|$ for $1 \leq i \leq n$, or $t \simeq_\beta \lambda f(f)a$ for some $a \in |A|$.

Remark.
The term $\lambda f \lambda x(f a_1)(f a_2)\ldots(f a_n)x$ represents the n-tuple $(a_1, \ldots, a_n)$ in the λ-calculus ; if $n = 0$, this term is $\lambda f \lambda x\, x$ which represents the empty sequence ; if $n = 1$, the one element sequence (a) is represented either by $\lambda f \lambda x(f a)x$ or by $\lambda f(f)a$ which are η-equivalent.

Product type :
Let $t \in |A \wedge B|$ and f be a variable with no free occurrence in t. Define an interpretation $\mathscr{I}$ by : $|X|_{\mathscr{I}} = \{\tau \in \Lambda ; \tau \simeq_\beta (f)ab$ for some $a \in |A|$ and $b \in |B|\}$. Then $f \in |A, B \to X|_{\mathscr{I}}$; since $t \in |(A, B \to X) \to X|_{\mathscr{I}}$, we see that $(t)f \in |X|_{\mathscr{I}}$. Thus there exist $a \in |A|$, $b \in |B|$ such that $(t)f \simeq_\beta (f)ab$. It follows that t is solvable ; let t' be a head normal form of t.
If t' starts with λ, say $t' = \lambda f\, u$, then $(t)f \simeq_\beta (t')f \simeq_\beta u$, and therefore $u \simeq_\beta (f)ab$. Hence $t \simeq_\beta t' \simeq_\beta \lambda f(f)ab$, which is β-equivalent to a closed term since so are a and b, by hypothesis.
Otherwise, $t' = (x)t_1 \ldots t_n$, thus $(t')f \simeq_\beta (x)t_1 \ldots t_n f \simeq_\beta (t)f \simeq_\beta (f)ab$. Now $(x)t_1 \ldots t_n f \simeq_\beta (f)ab$, so we have $n = 1$ and $b \simeq_\beta f$. But this is impossible since b is β-equivalent to a closed term.

Disjoint sum type :
Let $t \in |A \vee B|$ and f, g be two distinct variables which do not occur free in t. Define an interpretation $\mathscr{I}$ by :
$|X|_{\mathscr{I}} = \{\tau \in \Lambda ; \tau \simeq_\beta (f)a$ for some $a \in |A|$ or $\tau \simeq_\beta (g)b$ for some $b \in |B|\}$;
then $f \in |A \to X|_{\mathscr{I}}$ and $g \in |B \to X|_{\mathscr{I}}$.
Since $t \in |(A \to X), (B \to X) \to X|_{\mathscr{I}}$, we can see that $(t)fg \in |X|_{\mathscr{I}}$. So we have, for instance, $(t)fg \simeq_\beta (f)a$ for some $a \in |A|$. It follows that t is solvable ; let t' be a head normal form of t.
If t' starts with at least two occurrences of λ, say $t' = \lambda f \lambda g\, u$, then we have $(t)fg \simeq_\beta (t')fg \simeq_\beta u$, and therefore $u \simeq_\beta (f)a$. Thus $t \simeq_\beta t' \simeq_\beta \lambda f \lambda g(f)a$, which is β-equivalent to a closed term since so is a, by hypothesis.
If t' starts with only one occurrence of λ, then $t' = \lambda f(x)t_1 \ldots t_n$ (x need not be distinct from f) ; thus $(t')fg \simeq_\beta (x)u_1 \ldots u_n g \simeq_\beta (t)fg \simeq_\beta (f)a$.

Now $(x)u_1 \dots u_n g \simeq_\beta (f)a$, so we have $n = 0$ and $a \simeq_\beta g$. But this is impossible since a is β-equivalent to a closed term.
If t' does not start with λ, then $t' = (x)t_1 \dots t_n$; so we have :
$(t')fg \simeq_\beta (x)t_1 \dots t_n fg \simeq_\beta (t)fg \simeq_\beta (f)a$.
It follows that $(x)t_1 \dots t_n fg \simeq_\beta (f)a$, but this is impossible : the head variable has at least two arguments in the first term, but only one in the second.

List type :
Let $t \in |L[A]|$ and f, x be two variables which do not occur free in t. Define an interpretation $\mathscr{I}$ by :
$|X|_\mathscr{I} = \{\tau \in \Lambda; \tau \simeq_\beta (fa_1)(fa_2)\dots(fa_n)x$, with $n \geq 0$ and $a_i \in |A|\}$.
Then $f \in |A, X \to X|_\mathscr{I}$ and $x \in |X|_\mathscr{I}$; since $t \in |(A, X \to X), X \to X|_\mathscr{I}$, we get $(t)fx \in |X|_\mathscr{I}$. So we have $(t)fx \simeq_\beta (fa_1)(fa_2)\dots(fa_n)x$. It follows that t is solvable ; let t' be a head normal form of t.
If t' starts with at least two occurrences of λ, say $t' = \lambda f \lambda x\, u$, then we have $(t)fx \simeq_\beta (t')fx \simeq_\beta u$, and therefore $u \simeq_\beta (fa_1)(fa_2)\dots(fa_n)x$.
Thus $t \simeq_\beta t' \simeq_\beta \lambda f \lambda x (fa_1)(fa_2)\dots(fa_n)x$, which is a closed term since so are the a_i's, by hypothesis.
If t' starts with only one occurrence of λ, then $t' = \lambda f(y)t_1 \dots t_n$ (y may be equal to f) ; thus :

$(t')fx \simeq_\beta (y)u_1 \dots u_n x \simeq_\beta (t)fx \simeq_\beta (fa_1)(fa_2)\dots(fa_n)x$.

So we have $(y)u_1 \dots u_n x \simeq_\beta (fa_1)(fa_2)\dots(fa_n)x$, and therefore $y = f$, $n = 1$ and $u_1 \simeq_\beta a_1$ (in both terms, the head variable is the same and its arguments are β-equivalent). It follows that $t \simeq_\beta t' \simeq_\beta \lambda f(f)a_1$.
If t' does not start with λ, then $t' = (y)t_1 \dots t_n$, so we have :
$(t')fx \simeq_\beta (y)t_1 \dots t_n fx \simeq_\beta (t)fx \simeq_\beta (fa_1)(fa_2)\dots(fa_n)x$. Therefore :
$(y)t_1 \dots t_n fx \simeq_\beta (fa_1)(fa_2)\dots(fa_n)x$; as before, it follows that $n = 0$, $y = f$, and $a_n = f$; but this is impossible since, by hypothesis, a_n is β-equivalent to a closed term.

Q.E.D.

Proposition 8.15 gives some particular cases of a general construction on data types, which will be developed in the next section (theorem 8.28). Let us, for the moment, consider one more instance.

Proposition 8.16.
For every data type A, the type $BT[A] = \forall X\{(A, X, X \to X), X \to X\}$ is also a data type, called the type of binary trees indexed by objects of type A.

Let $\mathscr{A} = \{t \in \Lambda$; there exists $a \in |A|$ such that $t \simeq_\beta a\}$. Thus $\mathscr{A} \neq \emptyset$ and every element of $\mathscr{A}$ is β-equivalent to a closed term.
We choose two distinct variables f, x, and we define $\mathscr{E}_{fx}$ as the least subset of Λ with the following properties :

$(\star)$ $\qquad x \in \mathscr{E}_{fx}$; if $a \in \mathscr{A}$ and $t, u \in \mathscr{E}_{fx}$, then $(f\,a)\,t\,u \in \mathscr{E}_{fx}$.

In other words, $\mathscr{E}_{fx}$ is the intersection of all subsets of Λ which have these properties. It follows that :

If $\tau \in \mathscr{E}_{fx}$, then

τ is β-equivalent to a term which has the only free variables f, x ;
if $\tau \neq x$, then f, x are free in τ ;
either $\tau = x$, or $\tau = (f\,a)\,t\,u$ with $a \in \mathscr{A}$ and $t, u \in \mathscr{E}_{fx}$;
if $\tau\,\beta\,\tau'$ then $\tau' \in \mathscr{E}_{fx}$.

Indeed, the set of λ-terms which have these properties has the properties $(\star)$.

Proposition 8.17 below shows, in particular, that every term in $|BT[A]|$ is β-equivalent to a closed term. This proves proposition 8.16.

Q.E.D.

Proposition 8.17. *If $t \in |BT[A]|$ and f, x are not free in t, then there is a $\tau \in \mathscr{E}_{fx}$ such that $t\,\beta\,\lambda f \lambda x\,\tau$.*

Remark. The terms of the form $\lambda f \lambda x\,\tau$, with $\tau \in \mathscr{E}_{fx}$, are exactly the λ-terms which represent binary trees indexed by elements of $\mathscr{A}$.

We define an interpretation $\mathscr{I}$ by setting, for every type variable X :
$|X|_{\mathscr{I}} = \{\xi \in \Lambda$; there exists $\tau \in \mathscr{E}_{fx}$ such that $\xi\,\beta\,\tau\}$.
Then, by definition of $\mathscr{E}_{fx}$, we have : $x \in |X|_{\mathscr{I}}$ and $f \in |A, X, X \to X|_{\mathscr{I}}$.
Since $t \in |(A, X, X \to X), X \to X|_{\mathscr{I}}$, we get $(t)\,f\,x \in |X|_{\mathscr{I}}$. In other words :

$$(t)\,f\,x\,\beta\,\tau \text{ for some } \tau \in \mathscr{E}_{fx}.$$

Since every element of $\mathscr{E}_{fx}$ is a head normal form, it follows that t is solvable ; thus, $t\,\beta\,t'$ where t' is a head normal form of t.
If t' starts with at least two occurrences of λ, say $t' = \lambda f \lambda x\,u$, then we have $(t)\,f\,x\,\beta\,(t')\,f\,x\,\beta\,u\,\beta\,\tau \in \mathscr{E}_{fx}$. Therefore, $t\,\beta\,t'\,\beta\,\lambda f \lambda x\,\tau$.

If t' starts with only one occurrence of λ, then $t' = \lambda f\,(y)\,t_1 \dots t_n$ for some variable y ; thus $(t)\,f\,x\,\beta\,(t')\,f\,x\,\beta\,(y)\,t_1 \dots t_n x\,\beta\,\tau \in \mathscr{E}_{fx}$.
Since $\tau \simeq_\beta (y)\,t_1 \dots t_n x$, we cannot have $\tau = x$. Therefore, $\tau = (f\,a)\,u\,v$ with $a \in \mathscr{A}$ and $u, v \in \mathscr{E}_{fx}$. Now, we have $(y)\,t_1 \dots t_n x\,\beta\,(f)\,a\,u\,v$ and therefore $y = f, n = 2, t_1\,\beta\,a, t_2\,\beta\,u$ and $v = x$. Thus, $t\,\beta\,t'\,\beta\,\lambda f\,(f)\,a\,u$ with $u \in \mathscr{E}_{fx}$. But x is free in $u \in \mathscr{E}_{fx}$, and therefore is also free in t, which is a contradiction.

If t' does not start with λ, then $t' = (y)\,t_1 \dots t_n$, so we have :
$(t)\,f\,x\,\beta\,(t')\,f\,x\,\beta\,(y)\,t_1 \dots t_n f\,x\,\beta\,\tau \in \mathscr{E}_{fx}$. Thus $\tau \neq x$, so that $\tau = (f\,a)\,u\,v$ with $a \in \mathscr{A}$ and $u, v \in \mathscr{E}_{fx}$. Therefore $y = f$ and it follows that f is free in t' ; thus, f is also free in t (because $t\,\beta\,t'$), which is a contradiction.

Q.E.D.

5. Positive second order quantifiers

We define formulas with positive (resp. negative) second order quantifiers, also called $\forall^+$-formulas (resp. $\forall^-$-formulas), by the following rules :

Every type variable is a $\forall^+$ and $\forall^-$-formula.
If A is a $\forall^+$-formula, then $\forall X\, A$ is also a $\forall^+$-formula.
If A is $\forall^-$ (resp. $\forall^+$) and B is $\forall^+$ (resp. $\forall^-$), then $A \to B$ is $\forall^+$ (resp. $\forall^-$).

Remark. Every quantifier free formula is $\forall^+$ and $\forall^-$.
There is no closed $\forall^-$-formula.

We shall now prove the following :

Theorem 8.18. *If A is a closed $\forall^+$-formula and $t \in |A|$, then t is β-equivalent to a normal closed λ-term.*

Corollary 8.19. *Every closed $\forall^+$-formula which is provable in system $\mathscr{F}$ is a data type.*

Let A be such a formula. By theorem 8.18, every term in $|A|$ is $\simeq_\beta$ to a closed term ; so we only need to prove that $|A| \neq \emptyset$. But, since A is provable in system $\mathscr{F}$, there is a λ-term t such that $\vdash t : A$. By the adequacy lemma 8.13, we deduce that $t \in |A|$.

Q.E.D.

In order to prove theorem 8.18, we need to generalize the notion of "value of a formula", defined page 153.
A *truth value set* is, by definition, a non empty set $\mathbb{V}$ of saturated subsets of Λ, which is closed by $\to$ and arbitrary intersection. In other words :

- $\mathbb{V} \neq \emptyset$; $\mathscr{X} \in \mathbb{V} \Rightarrow \mathscr{X}$ is a saturated subset of Λ ;
- the intersection of any non empty subset of $\mathbb{V}$ is in $\mathbb{V}$;
- $\mathscr{X}, \mathscr{Y} \in \mathbb{V} \Rightarrow (\mathscr{X} \to \mathscr{Y}) \in \mathbb{V}$.

For example, the set $\mathbb{V}_0$ of all saturated subsets of Λ is a truth value set ; other trivial examples are the two-elements set $\{\emptyset, \Lambda\}$ and the singleton $\{\Lambda\}$.

A $\mathbb{V}$-interpretation $\mathscr{I}$ is, by definition, a mapping $X \mapsto |X|^{\mathbb{V}}_{\mathscr{I}}$ of the set of type variables into $\mathbb{V}$.
Let $\mathscr{I}$ be a $\mathbb{V}$-interpretation, X a type variable and $\mathscr{X} \in \mathbb{V}$. We define a $\mathbb{V}$-interpretation $\mathscr{J} = \mathscr{I}[X \leftarrow \mathscr{X}]$ by taking $|Y|^{\mathbb{V}}_{\mathscr{J}} = |Y|^{\mathbb{V}}_{\mathscr{I}}$ for every type variable $Y \neq X$, and $|X|^{\mathbb{V}}_{\mathscr{J}} = \mathscr{X}$.

For every type A, the value $|A|^{\mathbb{V}}_{\mathscr{I}}$ of A in a $\mathbb{V}$-interpretation $\mathscr{I}$ is an element of $\mathbb{V}$ defined as follows, by induction on A :

- if A is a type variable, then $|A|^{\mathbb{V}}_{\mathscr{I}}$ is given with $\mathscr{I}$;

• $|A \to B|^{\mathbb{V}}_{\mathscr{I}} = |A|^{\mathbb{V}}_{\mathscr{I}} \to |B|^{\mathbb{V}}_{\mathscr{I}}$, in other words :
for every term t, $t \in |A \to B|^{\mathbb{V}}_{\mathscr{I}}$ if and only if $tu \in |B|^{\mathbb{V}}_{\mathscr{I}}$ for every $u \in |A|^{\mathbb{V}}_{\mathscr{I}}$;

• $|\forall X\, A|^{\mathbb{V}}_{\mathscr{I}} = \bigcap\{|A|^{\mathbb{V}}_{\mathscr{I}[X\leftarrow\mathscr{X}]};\, \mathscr{X} \in \mathbb{V}\}$, in other words :
for every term t, $t \in |\forall X\, A|^{\mathbb{V}}_{\mathscr{I}}$ if and only if $t \in |A|^{\mathbb{V}}_{\mathscr{I}[X\leftarrow\mathscr{X}]}$ for every $\mathscr{X} \in \mathbb{V}$.

Remarks.
i) The value $|A|_{\mathscr{I}}$ of a formula, defined page 153, is the particular case when the truth value set is the set $\mathbb{V}_0$ of all saturated subsets of Λ.
ii) The value $|A|^{\mathbb{V}}_{\mathscr{I}}$ does not really depends on the interpetation $\mathscr{I}$, but only on the restriction of $\mathscr{I}$ to the set of free variables of A. In particular, if A is a closed formula, this value does not depends on $\mathscr{I}$ at all and will be denoted $|A|^{\mathbb{V}}$.

Lemma 8.20. *Let $\mathbb{V} \subset \mathbb{W}$ be two truth value sets and $\mathscr{I}$ a $\mathbb{V}$-interpretation. If A is a $\forall^+$ (resp. a $\forall^-$)-formula then $|A|^{\mathbb{W}}_{\mathscr{I}} \subset |A|^{\mathbb{V}}_{\mathscr{I}}$ (resp. $|A|^{\mathbb{V}}_{\mathscr{I}} \subset |A|^{\mathbb{W}}_{\mathscr{I}}$).*

Proof by induction on the length of the formula A. The result is trivial if A is a variable, because we have $|A|^{\mathbb{V}}_{\mathscr{I}} = |A|^{\mathbb{W}}_{\mathscr{I}}$.
If $A \equiv B \to C$ and A is $\forall^+$, then B is $\forall^-$ and C is $\forall^+$. By induction hypothesis, we get $|B|^{\mathbb{V}}_{\mathscr{I}} \subset |B|^{\mathbb{W}}_{\mathscr{I}}$ and $|C|^{\mathbb{W}}_{\mathscr{I}} \subset |C|^{\mathbb{V}}_{\mathscr{I}}$.
It follows that $|B \to C|^{\mathbb{W}}_{\mathscr{I}} \subset |B \to C|^{\mathbb{V}}_{\mathscr{I}}$ which is the result.
If $A \equiv B \to C$ and A is $\forall^-$, the proof is the same.
If $A \equiv \forall X\, B$ and B is $\forall^+$, then $|A|^{\mathbb{V}}_{\mathscr{I}} = \bigcap\{|B|^{\mathbb{V}}_{\mathscr{I}[X\leftarrow\mathscr{X}]};\, \mathscr{X} \in \mathbb{V}\}$ and
$|A|^{\mathbb{W}}_{\mathscr{I}} = \bigcap\{|B|^{\mathbb{W}}_{\mathscr{I}[X\leftarrow\mathscr{X}]};\, \mathscr{X} \in \mathbb{W}\}$. By induction hypothesis, we have :
$|B|^{\mathbb{W}}_{\mathscr{I}[X\leftarrow\mathscr{X}]} \subset |B|^{\mathbb{V}}_{\mathscr{I}[X\leftarrow\mathscr{X}]}$; now, since $\mathbb{V} \subset \mathbb{W}$, it follows that $|A|^{\mathbb{W}}_{\mathscr{I}} \subset |A|^{\mathbb{V}}_{\mathscr{I}}$.
Q.E.D.

Corollary 8.21. *If A is a closed $\forall^+$-formula, then $|A| \subset |A|^{\mathbb{V}}$ for every truth value set $\mathbb{V}$.*

Immediate from lemma 8.20, since $|A| = |A|^{\mathbb{V}_0}$ and $\mathbb{V} \subset \mathbb{V}_0$ for every truth value set $\mathbb{V}$.
Q.E.D.

Consider now the pair $(\mathscr{N}_0, \mathscr{N})$ of subets of Λ defined page 47 :
$\mathscr{N}$ is the set of all terms which are normalizable by leftmost β-reduction ;
$\mathscr{N}_0 = \{(x)t_1 \ldots t_n;\, n \in \mathbb{N},\, t_1, \ldots, t_n \in \mathscr{N}\}$.
We put $\mathbb{V} = \{\mathscr{X} \subset \Lambda;\, \mathscr{X}$ is saturated, $\mathscr{N}_0 \subset \mathscr{X} \subset \mathscr{N}\}$.

Lemma 8.22. *$\mathbb{V}$ is a truth value set.*

$\mathbb{V}$ is obviously closed by arbitrary (non void) intersection. Now, if $\mathscr{X}, \mathscr{Y} \in \mathbb{V}$, we have $\mathscr{N}_0 \subset \mathscr{X}, \mathscr{Y} \subset \mathscr{N}$ and therefore :
$(\mathscr{N} \to \mathscr{N}_0) \subset (\mathscr{X} \to \mathscr{Y}) \subset (\mathscr{N}_0 \to \mathscr{N})$. But we have proved, page 47, that $(\mathscr{N}_0, \mathscr{N})$

is an *adapted pair*, and therefore that $\mathcal{N}_0 \subset (\mathcal{N} \to \mathcal{N}_0)$ and $(\mathcal{N}_0 \to \mathcal{N}) \subset \mathcal{N}$. It follows that $\mathcal{N}_0 \subset (\mathcal{X} \to \mathcal{Y}) \subset \mathcal{N}$.

Q.E.D.

We now choose a fixed λ-variable x ; let $\Lambda_x \subset \Lambda$ be the set of λ-terms the only free variable of which is x (every closed term is in Λ_x). We put :
$\mathcal{N}^x = \{t \in \Lambda; (\exists u \in \Lambda_x)\ t$ reduces to u by leftmost β-reduction$\}$
$\mathcal{N}_0^x = \{(x)t_1 \dots t_n;\ n \in \mathbb{N}, t_1, \dots, t_n \in \mathcal{N}\}$.

Lemma 8.23.
i) $\mathcal{N}_0^x \subset \mathcal{N}^x$ *; ii)* $\mathcal{N}_0^x \subset (\mathcal{N}^x \to \mathcal{N}_0^x)$ *; iii)* $(\mathcal{N}_0^x \to \mathcal{N}^x) \subset \mathcal{N}^x$.

Remark. This lemma means that the pair $(\mathcal{N}_0^x, \mathcal{N}^x)$ is an adapted pair, as defined page 46.

i) and ii) follow immediately from the definitions of $\mathcal{N}^x$ and $\mathcal{N}_0^x$.
iii) Let $t \in (\mathcal{N}_0^x \to \mathcal{N}^x)$; since $x \in \mathcal{N}_0^x$, we have $tx \in \mathcal{N}^x$, so that tx reduces to $u \in \Lambda_x$ by leftmost reduction. If this reduction takes place in t, then $u = vx$ and t reduces to $v \in \Lambda_x$ by leftmost reduction. Otherwise, t reduces to $\lambda y\, t'$ and $t'[x/y]$ reduces to u by leftmost reduction. Thus, there exists a λ-term u' with the only free variables x, y, such that t' reduces to u' by leftmost reduction. Therefore, by leftmost reduction, t reduces to $\lambda y\, t'$, then to $\lambda y\, u'$ and x is the only free variable of $\lambda y\, u'$.

Q.E.D.

Now, we define $\mathbb{V}_x = \{\mathcal{X}; \mathcal{X}$ is a saturated subset of Λ, $\mathcal{N}_0^x \subset \mathcal{X} \subset \mathcal{N}^x\}$.

Lemma 8.24. $\mathbb{V}_x$ *is a truth value set.*

We have only to check that $(\mathcal{X} \to \mathcal{Y}) \in \mathbb{V}_x$ if $\mathcal{X}, \mathcal{Y} \in \mathbb{V}_x$. By definition of $\mathbb{V}_x$, we have $\mathcal{N}_0^x \subset \mathcal{X}, \mathcal{Y} \subset \mathcal{N}^x$ and therefore :
$(\mathcal{N}^x \to \mathcal{N}_0^x) \subset (\mathcal{X} \to \mathcal{Y}) \subset (\mathcal{N}_0^x \to \mathcal{N}^x)$.
Using lemma 8.23, we get $\mathcal{N}_0^x \subset (\mathcal{X} \to \mathcal{Y}) \subset \mathcal{N}^x$.

Q.E.D.

We can now prove theorem 8.18. Let A be a closed $\forall^+$-formula and $t \in |A|$. By corollary 8.21 and lemma 8.22, we have $|A| \subset |A|^{\mathbb{V}} \subset \mathcal{N}$.
It follows that $t \in \mathcal{N}$, which means that t is normalizable.
Now, choose a λ-variable x which is not free in t.
By corollary 8.21 and lemma 8.24, we get $|A| \subset |A|^{\mathbb{V}_x} \subset \mathcal{N}^x$.
It follows that $t \in \mathcal{N}^x$, which means that t reduces, by leftmost reduction, to a term with the only free variable x. Since x is not free in t, this reduction gives a closed term.

Q.E.D.

The next theorem gives another interesting truth value set.

Theorem 8.25. *Let* $\mathscr{C} = \{t \in \Lambda$; *there exists a closed term* t' *such that* $t \,\beta\, t'\}$. *Then* $\{\mathscr{C}\}$ *is a (one-element) truth value set.*

Remark. By the Church-Rosser theorem 1.24, $\mathscr{C}$ is also the set of λ-terms which are β-equivalent to closed terms.

Lemma 8.26.
Let $\omega = (\lambda z\, zz)\lambda z\, zz$ *and* $t \in \Lambda$. *A step of* β*-reduction in* $t[\omega/x]$ *gives* $t'[\omega/x]$, *where* $t' = t$ *or* t' *is obtained by a step of* β*-reduction in* t.

Proof, by induction on the length of t. The result is immediate if t is a variable or if $t = \lambda x\, u$. If $t = uv$, then a redex in $t[\omega/x] = u[\omega/x]v[\omega/x]$ is either a redex in $u[\omega/x]$, or a redex in $v[\omega/x]$, or $t[\omega/x]$ itself. In the first two cases, we simply apply the induction hypothesis. In the last case, $u[\omega/x]$ begins with a λ and, therefore, $u = \lambda y\, u'$ and $t = (\lambda y\, u')v$. The redex we consider is $(\lambda y\, u'[\omega/x])v[\omega/x]$ and its reduction gives $u'[\omega/x][v[\omega/x]/y] = t'[\omega/x]$ with $t' = u'[v/y]$.

Q.E.D.

Lemma 8.27. *Let* $t \in \Lambda$ *; if there is a closed term* u *such that* $t[\omega/x]\,\beta\, u$, *then there is a term* u' *with the only free variable* x, *such that* $t \,\beta\, u'$.

Proof by induction on the length of the given β-reduction from $t[\omega/x]$ to u. If this length is 0, then $t[\omega/x]$ is closed and t has the only free variable x. Otherwise, by lemma 8.26, after one step of β-reduction, we get $t'[\omega/x]$ with $t\,\beta\, t'$. By the induction hypothesis, we have $t'\beta\, u'$ (u' has the only free variable x) and, therefore, $t\,\beta\, u'$.

Q.E.D.

We can now prove the theorem 8.25. It is clear that $\mathscr{C}$ is a saturated set ; thus, we only have to show : $\mathscr{C} = (\mathscr{C} \to \mathscr{C})$ and, in fact only : $(\mathscr{C} \to \mathscr{C}) \subset \mathscr{C}$, because the reverse inclusion is trivial.
Let $t \in (\mathscr{C} \to \mathscr{C})$, so that we have $t\omega \in \mathscr{C}$ and, therefore, $t\omega\,\beta\, u$ where u is closed. If this β-reduction takes place entirely in t, we have $t\,\beta\, t'$ and $t'\omega = u$; thus, t' is closed and $t \in \mathscr{C}$. Otherwise, we have $t\,\beta\,\lambda x\, t'$ and $t'[\omega/x]\,\beta\, u$. By lemma 8.27, we have $t'\beta\, u'$ (u' has the only free variable x) and, therefore, $t\,\beta\,\lambda x\, u'$. Since $\lambda x\, u'$ is closed, we get $t \in \mathscr{C}$.

Q.E.D.

This gives another proof of the second part of theorem 8.18 : if A is a closed $\forall^+$-formula, then by corollary 8.21 with $\mathbb{V} = \{\mathscr{C}\}$, we obtain $|A| \subset |A|^{\mathbb{V}} = \mathscr{C}$. This shows that every term in $|A|$ is β-equivalent to a closed term.

Consider a formula F and a type variable X ; for each free occurrence of X in F, we define its sign (*positive* or *negative*), inductively on the length of F :

- if $F \equiv X$, the occurrence of X is positive ;

• if $F \equiv (G \to H)$, the positive (resp. negative) free occurrences of X in F are the positive (resp. negative) free occurrences of X in H and the negative (resp. positive) free occurrences of X in G ;

• if $F \equiv \forall Y\, G$, with $Y \neq X$, the positive (resp. negative) free occurrences of X in F are the positive (resp. negative) free occurrences of X in G.

Theorem 8.28. *Suppose that $\forall X_1 \dots \forall X_k\, F$ is a closed $\forall^+$-formula which is provable in system $\mathscr{F}$, and that every free occurrence of $X_1,\dots,X_k$ in F is positive. If $A_1,\dots,A_k$ are data types, then $F[A_1/X_1,\dots,A_k/X_k]$ is a data type.*

Remark. In fact, we may suppose only that $|A_1|,\dots,|A_k| \subset \mathscr{C}$; the hypothesis $|A_i| \neq \emptyset$ is useless.

Lemma 8.29. *Let $X_1,\dots,X_k$ be distinct type variables, and $\mathscr{I},\mathscr{J}$ be two $\mathbb{V}$-interpretations such that : $|X_i|^{\mathbb{V}}_{\mathscr{I}} \supset |X_i|^{\mathbb{V}}_{\mathscr{J}}$ for $1 \leq i \leq k$ and $|X|^{\mathbb{V}}_{\mathscr{I}} = |X|^{\mathbb{V}}_{\mathscr{J}}$ for every type variable $X \neq X_1,\dots,X_k$.*
If $X_1,\dots,X_k$ have only positive (resp. negative) free occurrences in a formula F, then $|F|^{\mathbb{V}}_{\mathscr{I}} \supset |F|^{\mathbb{V}}_{\mathscr{J}}$ (resp. $|F|^{\mathbb{V}}_{\mathscr{I}} \subset |F|^{\mathbb{V}}_{\mathscr{J}}$).

Easy proof, by induction on the length of F.

Q.E.D.

Proof of theorem 8.28.
By hypothesis, we have $\vdash_{\mathscr{F}} t : \forall X_1 \dots \forall X_k\, F$ for some $t \in \Lambda$.
By the adequacy lemma 8.13, we deduce that $t \in |\forall X_1 \dots \forall X_k\, F|$ and, therefore $t \in |F[A_1/X_1,\dots,A_k/X_k]|$. This shows $|F[A_1/X_1,\dots,A_k/X_k]| \neq \emptyset$.

In lemma 8.20, we take $\mathbb{V} = \{\mathscr{C}\}$ and $\mathbb{W} = \mathbb{V}_0$ (the set of all saturated subsets of Λ) ; $\mathscr{I}$ is the single $\mathbb{V}$-interpretation, which is defined by $|X|_{\mathscr{I}} = \mathscr{C}$ for every type variable X. We apply this lemma to the $\forall^+$-formula F and we obtain : $|F|_{\mathscr{I}} = |F|^{\mathbb{W}}_{\mathscr{I}} \subset |F|^{\mathbb{V}}_{\mathscr{I}} = \mathscr{C}$.
We define an interpetation $\mathscr{J}$ as follows : $|X_i|_{\mathscr{J}} = |A_i|$ for $1 \leq i \leq k$ and $|X|_{\mathscr{J}} = \mathscr{C}$ for any type variable $X \neq X_1,\dots,X_k$.
Now, one hypothesis of the theorem is that $|A_1|,\dots,|A_k| \subset \mathscr{C}$. Moreover, the variables $X_1,\dots,X_k$ have only positive occurrences in the formula F. Therefore, the hypothesis of lemma 8.29 are fulfilled (the truth value set being $\mathbb{W} = \mathbb{V}_0$) and it follows that $|F|_{\mathscr{J}} \subset |F|_{\mathscr{I}}$; thus, $|F|_{\mathscr{J}} \subset \mathscr{C}$.
Now, $|F|_{\mathscr{J}}$ is the same as $|F[A_1/X_1,\dots,A_k/X_k]|$, and therefore we obtain the desired result : $|F[A_1/X_1,\dots,A_k/X_k]| \subset \mathscr{C}$.

Q.E.D.

References for chapter 8

[Boh85], [For83], [Gia88], [Gir71], [Gir72], [Gir86].
(The references are in the bibliography at the end of the book).

Chapter 9

Second order functional arithmetic

1. Second order predicate calculus

In this chapter, we will deal with the classical second order predicate calculus, with a syntax using the following symbols :

the logical symbols $\to$ and $\forall$ (and no other ones) ;

individual variables : $x, y, \ldots$ (also called first order variables) ;

n-ary relation variables ($n = 0, 1, \ldots$) : $X, Y, \ldots$ (also called second order variables) ;

n-ary function symbols ($n = 0, 1, \ldots$) (on individuals) ;

n-ary relation symbols ($n = 0, 1, \ldots$) (on individuals).

Each relation variable, each function or relation symbol, has a fixed arity $n \geq 0$. Function symbols of arity 0 are called *constant symbols*. Relation variables of arity 0 are also called *propositional variables*.

It is assumed that there are infinitely many individual variables and, for each $n \geq 0$, infinitely many n-ary relation variables.

The function and relation symbols determine what we call a *language* ; the other symbols are common to all languages.

Let $\mathscr{L}$ be a language.

The (individual) *terms* of $\mathscr{L}$ are built up in the usual way, that is by the following rules :

each individual variable, and each constant symbol, is a term ;

whenever f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, $f(t_1, \ldots, t_n)$ is a term.

The *atomic formulas* are the expressions of the form $A(t_1, \ldots, t_k)$, where A is a k-ary relation variable or symbol and $t_1, \ldots, t_k$ are terms.

The *formulas* are the expressions obtained by the following rules :

every atomic formula is a formula ;

whenever F, G are formulas, $(F \to G)$ is a formula ;

whenever F is a formula, x is an individual variable and X is a relation variable, $\forall x\, F$ and $\forall X\, F$ are formulas.

Definitions and notations

A *closed term* of $\mathscr{L}$ is a term which contains no variable. A *closed formula* is a formula in which no variable occurs free.

The *closure* of a formula F is the formula obtained by universal quantification of all the free variables of F.

A *universal* formula consists of a (finite) sequence of universal quantifiers followed by a quantifier free formula.

The formula $F_1 \to (F_2 \to (\ldots \to (F_n \to G)\ldots))$ will also be denoted by :
$F_1, F_2, \ldots, F_n \to G$.

Let X be a 0-ary relation variable, ξ any individual or relation variable which is $\neq X$, and F, G arbitrary formulas in which X does not occur free.

The formula $\forall X\, X$ is denoted by $\perp$ (read " false ").

The formula $F \to \perp$ is denoted by $\neg F$ (read " not F ").

The formula $\forall X[(F \to X), (G \to X) \to X]$ is denoted by $F \vee G$ (read " F or G ").

The formula $\forall X[(F, G \to X) \to X]$ is denoted by $F \wedge G$ (read " F and G ").

The formula $(F \to G) \wedge (G \to F)$ is denoted by $F \leftrightarrow G$
(read " F is equivalent to G ").

The formula $\forall X[\forall \xi (F \to X) \to X]$ is denoted by $\exists \xi\, F$
(read " there exists a ξ such that F ").

α-equivalent formulas and substitution

Let F be a formula, ξ a variable, and η the same sort of symbol as ξ (if ξ is an individual variable, then so is η ; if ξ is an n-ary relation variable, then η is an n-ary relation variable or symbol) ; we define the formula $F<\eta/\xi>$ by replacing in F all free occurrences of ξ by η.

We now define, by induction on F, the α-equivalence of two formulas F, G, denoted by $F \equiv G$:

- if F is an atomic formula, then $F \equiv G$ if and only if $F = G$;
- if $F = A \to B$, then $F \equiv G$ if and only if $G = A' \to B'$, where $A \equiv A'$ and $B \equiv B'$;
- if $F = \forall \xi\, A$, ξ being an individual or relation variable, then $F \equiv G$ if and only if $G = \forall \eta\, B$, where η is the same sort of variable as ξ, and $A<\zeta/\xi> \equiv B<\zeta/\eta>$ for all variables ζ of the same sort as ξ but a finite number.

From now on, we shall identify α-equivalent formulas.
If V is a finite set of variables (of any kind), and A is a formula, then there exists a formula $A' \equiv A$, such that no variable of V is bound in A'. A' has the same length as A (the only difference between A and A' is the name of the bound variables).

Let A be a formula, $x_1, \dots, x_k$ individual variables, and $t_1, \dots, t_k$ terms. The formula $A[t_1/x_1, \dots, t_k/x_k]$ is defined by choosing a representative of A such that none of its bound variables occur in the t_i's, and then by replacing in it each free occurrence of x_i by t_i $(1 \le i \le n)$.

Consider two formulas A and F, an n-ary relation variable X, and n individual variables $x_1, \dots, x_n$. We define the substitution of F to $X(x_1, \dots, x_n)$ in A : this produces a formula, denoted by $A[F/Xx_1 \dots x_n]$; the definition is by induction on A and requires a representative of A such that its bound variables do not occur in F :

- if A is an atomic formula of the form $X(t_1, \dots, t_n)$, then $A[F/Xx_1 \dots x_n]$ is the formula $F[t_1/x_1, \dots, t_n/x_n]$;
- if A is atomic and does not start with X, then $A[F/Xx_1 \dots x_n] = A$;
- if $A = B \to C$, then $A[F/Xx_1 \dots x_n] = B[F/Xx_1 \dots x_n] \to C[F/Xx_1 \dots x_n]$;
- if $A = \forall \xi\, B$, where ξ is an individual variable, or a relation variable different from X, then $A[F/Xx_1 \dots x_n] = \forall \xi\, B[F/Xx_1 \dots x_n]$;
- if $A = \forall X\, B$, then $A[F/Xx_1 \dots x_n] = A$.

Models

Recall briefly some classical definitions of model theory.
A *second order model* for the language $\mathscr{L}$ is a structure $\mathscr{M}$ consisting of :

- a domain $|\mathscr{M}|$ (the set of individuals, assumed non-empty) ;
- for each integer $n \ge 0$, a subset $|\mathscr{M}|_n$ of $\mathscr{P}(|\mathscr{M}|^n)$, which is the range for the values of the n-ary relation variables.
If $n = 0$, we assume that $|\mathscr{M}|_0 = \mathscr{P}(|\mathscr{M}|^0) = \{0, 1\}$;
- an interpretation, in $|\mathscr{M}|$, of the function and relation symbols of the language $\mathscr{L}$: namely, a mapping which associates with each n-ary function symbol f of $\mathscr{L}$, an n-ary function $f_{\mathscr{M}} : |\mathscr{M}|^n \to |\mathscr{M}|$, and with each n-ary relation symbol S of $\mathscr{L}$, an n-ary relation on $\mathscr{M}$, that is a subset

$S_{\mathscr{M}} \subset |\mathscr{M}|^n$. In particular, it associates with each constant symbol c an element $c_{\mathscr{M}} \in |\mathscr{M}|$.

We will say that an n-ary relation R on $|\mathscr{M}|$ (in other words a subset of $|\mathscr{M}|^n$) *is part of the model* $\mathscr{M}$ whenever $R \in |\mathscr{M}|_n$.
The elements of $|\mathscr{M}|_1$ are called the *classes* of $\mathscr{M}$.

The model $\mathscr{M}$ is called a *full model* if, for each $n \geq 0$, $|\mathscr{M}|_n = \mathscr{P}(|\mathscr{M}|^n)$ (that is to say : if all relations on $|\mathscr{M}|$ are part of the model $\mathscr{M}$).

Let $\mathscr{L}_{\mathscr{M}}$ denote the language obtained by adding to $\mathscr{L}$ every element of $|\mathscr{M}|$ as a constant symbol, and, for each $n \geq 0$, every element of $\mathscr{P}(|\mathscr{M}|^n)$ as an n-ary relation symbol (of course, we suppose that no symbol in $\mathscr{L}$ is an element of $|\mathscr{M}|$ or of $\mathscr{P}(|\mathscr{M}|^n)$).
The terms and formulas of $\mathscr{L}_{\mathscr{M}}$ are respectively called *terms* and *formulas of* $\mathscr{L}$ *with parameters in* $\mathscr{M}$.
There is an obvious way of extending the model $\mathscr{M}$ to a model for the language $\mathscr{L}_{\mathscr{M}}$: the new symbols of $\mathscr{L}_{\mathscr{M}}$ are their own interpretation.
With each closed term of $\mathscr{L}$, with parameters in $\mathscr{M}$, we associate its value $t_{\mathscr{M}} \in |\mathscr{M}|$, which is defined by induction on t :
if t is a constant symbol of $\mathscr{L}_{\mathscr{M}}$, then $t_{\mathscr{M}}$ is already defined ;
if $t = f(t^1, \ldots, t^n)$, then $t_{\mathscr{M}} = f_{\mathscr{M}}(t^1_{\mathscr{M}}, \ldots, t^n_{\mathscr{M}})$.
Let F be a closed formula of $\mathscr{L}$, with parameters in $\mathscr{M}$. We define, by induction on F, the expression $\mathscr{M}$ *satisfies* F, which is denoted by $\mathscr{M} \models F$:
if F is an atomic formula, say $R(t^1, \ldots, t^n)$, where R is an n-ary relation symbol of $\mathscr{L}_{\mathscr{M}}$, and $t^1, \ldots, t^n$ are closed terms of $\mathscr{L}_{\mathscr{M}}$, then $\mathscr{M} \models F$ if and only if $(t^1_{\mathscr{M}}, \ldots, t^n_{\mathscr{M}}) \in R_{\mathscr{M}}$.
if $F = G \to H$, then $\mathscr{M} \models F$ if and only if $\mathscr{M} \models G \Rightarrow \mathscr{M} \models H$.
if $F = \forall x\, G$, x being the only free variable in G, then $\mathscr{M} \models F$ if and only if $\mathscr{M} \models G<a/x>$ for every $a \in |\mathscr{M}|$.
if $F = \forall X\, G$, where the n-ary relation variable X is the only free variable in G, then $\mathscr{M} \models F$ if and only if $\mathscr{M} \models G<R/X>$ for every $R \in |\mathscr{M}|_n$.

Let $\mathscr{A}$ be a system of axioms of the language $\mathscr{L}$ (that is to say a set of closed formulas, also called a theory). By a model of $\mathscr{A}$, we mean a model which satisfies all formulas of $\mathscr{A}$. A closed formula F is said to be a *consequence* of $\mathscr{A}$ (which is denoted by $\mathscr{A} \vdash F$) if every model of $\mathscr{A}$ satisfies F. A closed formula F is said to be valid (we write $\vdash F$) if it is a consequence of $\emptyset$, in other words, if it is satisfied in every model.
Clearly, for every 0-ary relation variable X, no model satisfies the formula $\forall X\, X$. This is a justification for the definition of $\perp$.

Proposition 9.1. *Let A, F be two formulas with parameters in $\mathscr{M}$, such that the only free variable in A is an n-ary relation variable X, and all the free variables*

in F are among the individual variables $x_1, \dots, x_n$.
Let $\Phi = \{(a_1, \dots, a_n) \in |\mathscr{M}|^n ; \mathscr{M} \models F[a_1/x_1, \dots, a_n/x_n]\}$ (which is an n-ary relation on $|\mathscr{M}|$). Then $\mathscr{M} \models A[F/Xx_1 \dots x_n] \Leftrightarrow \mathscr{M} \models A{<}\Phi/X{>}$.

The proof is by induction on A.
If A is atomic and starts with X, then $A = Xt^1 \dots t^n$, so :
$\mathscr{M} \models A[F/Xx_1 \dots x_n] \Leftrightarrow \mathscr{M} \models F[t^1_{\mathscr{M}}/x_1, \dots, t^n_{\mathscr{M}}/x_n]$
$\Leftrightarrow \mathscr{M} \models \Phi(t^1_{\mathscr{M}}, \dots, t^n_{\mathscr{M}}) \Leftrightarrow \mathscr{M} \models A{<}\Phi/X{>}$.
If $A = \forall x B$, where x is an individual variable, then :
$\mathscr{M} \models \forall x B[F/Xx_1 \dots x_n] \Leftrightarrow (\forall a \in |\mathscr{M}|)\mathscr{M} \models B[F/Xx_1 \dots x_n]{<}a/x{>}$
$\Leftrightarrow (\forall a \in |\mathscr{M}|)\mathscr{M} \models B{<}a/x{>}[F/Xx_1 \dots x_n]$
$\Leftrightarrow (\forall a \in |\mathscr{M}|)\mathscr{M} \models B{<}a/x{>}{<}\Phi/X{>}$ (by induction hypothesis)
$\Leftrightarrow (\forall a \in |\mathscr{M}|)\mathscr{M} \models B{<}\Phi/X{>}{<}a/x{>} \Leftrightarrow \mathscr{M} \models \forall x B{<}\Phi/X{>}$.
Same proof when $A = \forall Y B$, for some relation variable $Y \neq X$.
The other cases of the inductive proof are trivial.

Q.E.D.

The comprehension axiom

This is an axiom scheme, denoted by CA ; it consists of the closure of all formulas of the following form :

$$(CA) \qquad \forall X A \to A[F/Xx_1 \dots x_n]$$

where A and F are arbitrary formulas, X is an n-ary relation variable ($n \geq 0$), and $x_1, \dots, x_n$ are n individual variables.

Proposition 9.2. *Every full model satisfies the comprehension axiom.*

Let $\mathscr{M}$ be a full model, X an n-ary relation variable, $x_1, \dots, x_n$, n individual variables, A a formula with parameters in $\mathscr{M}$ in which X is the only free variable, and F a formula with parameters in $\mathscr{M}$ in which all the free variables are among $x_1, \dots, x_n$. Suppose $\mathscr{M} \models \forall X A$, and let :
$\Phi = \{(a_1, \dots, a_n) \in |\mathscr{M}|^n ; \mathscr{M} \models F[a_1/x_1, \dots, a_n/x_n]\}$.
We have $\Phi \in \mathscr{P}(|\mathscr{M}|^n)$ and $\mathscr{M}$ is full : thus $\Phi \in |\mathscr{M}|_n$.
Since $\mathscr{M} \models \forall X A$, we have $\mathscr{M} \models A{<}\Phi/X{>}$;
therefore, by proposition 9.1, $\mathscr{M} \models A[F/Xx_1 \dots x_n]$.

Q.E.D.

Given a language $\mathscr{L}$, *the second order predicate calculus* on $\mathscr{L}$ is the theory consisting of all the axioms of the comprehension scheme.
Thus a model of the second order predicate calculus on the language $\mathscr{L}$ is a second order model $\mathscr{M}$ for $\mathscr{L}$ such that $\mathscr{M} \models CA$.

Proposition 9.3.
The comprehension axiom is equivalent to the following axiom scheme :
(CA′) $\exists Y \forall x_1 \ldots \forall x_n [Y(x_1, \ldots, x_n) \leftrightarrow F]$
where Y is an n-ary relation variable $(n \geq 0)$ and F an arbitrary formula.

(In fact, as above, we consider the closure of the formulas of CA').
Clearly, we have $\vdash \forall X\, A$, where A is the formula :
$\exists Y \forall x_1 \ldots \forall x_n [Y(x_1, \ldots, x_n) \leftrightarrow X(x_1, \ldots, x_n)]$.
Therefore $CA \vdash A[F/Xx_1 \ldots x_n]$, that is to say $CA \vdash CA'$.
Conversely, consider any model $\mathscr{M}$ of CA'. Suppose that $\mathscr{M} \models \forall X\, A$, where X is an n-ary relation variable, and A a formula with parameters in $\mathscr{M}$, where the only free variable is X. Let F be a formula with parameters in $\mathscr{M}$ and free variables among $x_1, \ldots, x_n$. Let $\Phi = \{(a_1, \ldots, a_n) \in |\mathscr{M}|^n\ ;\ \mathscr{M} \models F[a_1/x_1, \ldots, a_n/x_n]\}$.
We have $\mathscr{M} \models \exists Y \forall x_1 \ldots \forall x_n [Y(x_1, \ldots, x_n) \leftrightarrow F]$ by hypothesis.
Hence $\mathscr{M} \models \forall x_1 \ldots \forall x_n [\Psi(x_1, \ldots, x_n) \leftrightarrow F]$ for some $\Psi \in |\mathscr{M}|_n$.
Therefore : $\mathscr{M} \models \forall x_1 \ldots \forall x_n [\Psi(x_1, \ldots, x_n) \leftrightarrow \Phi(x_1, \ldots, x_n)]$. It follows that $\Phi = \Psi$, so $\Phi \in |\mathscr{M}|_n$. Since $\mathscr{M} \models \forall X\, A$, we have $\mathscr{M} \models A{<}\Phi/X{>}$; thus, by proposition 9.1, $\mathscr{M} \models A[F/Xx_1 \ldots x_n]$.
Q.E.D.

Equational formulas

We consider a second order language $\mathscr{L}$.
The formula $\forall X[X(x) \to X(y)]$ will be denoted by $x = y$. Obviously, we have $\vdash x = x$ and $\vdash x = y,\ y = z \to x = z$. Moreover, $CA \vdash x = y \to y = x$ (apply CA, taking A as the formula $X(x) \to X(y)$, then replace $X(y)$ with the formula $y = x$).
We also have, clearly, for every formula $F(x)$, $CA, x = y \vdash F(x) \to F(y)$.
It follows that, in every model $\mathscr{M}$ of the second order predicate calculus, the formula $x = y$ defines an equivalence relation which is compatible with the whole structure of the model. By taking the quotient, we thus obtain a model $\mathscr{M}'$ in which the interpretation of the formula $x = y$ is the identity relation. Such a model will be called an *identity model.*
Now it is clear that the models $\mathscr{M}$ and $\mathscr{M}'$ satisfy exactly the same formulas of $\mathscr{L}$. This allows us, when we deal with models of CA, to consider only identity models ; from now on, it is what we will do.

By an *equation* (or an *equational formula*), we mean the closure of any formula of the form $t = u$ (where t, u are terms). A set of equations will also be called a *system of equational axioms.*
A *particular case* of an equation $t = u$ is, by definition, a formula of one of two forms :
$t[v_1/x_1, \ldots, v_k/x_k] = u[v_1/x_1, \ldots, v_k/x_k]$ or

$u[v_1/x_1,\dots,v_k/x_k] = t[v_1/x_1,\dots,v_k/x_k]$,
where $v_1,\dots,v_k$ are terms.

Proposition 9.4. *Let $\mathscr{E}$ be a system of equational axioms in some language $\mathscr{L}$, and u,v two terms of $\mathscr{L}$.*
Then $CA+\mathscr{E} \vdash u = v$ if and only if the expression $\vdash_{\mathscr{E}} u = v$ can be obtained by means of the following rules :
i) if $u = v$ is a particular case of an axiom of $\mathscr{E}$, then $\vdash_{\mathscr{E}} u = v$;
ii) for all terms u,v,w of $\mathscr{L}$, we have $\vdash_{\mathscr{E}} u = u$;
if $\vdash_{\mathscr{E}} u = v$ and $\vdash_{\mathscr{E}} v = w$, then $\vdash_{\mathscr{E}} u = w$;
iii) if f is an n-ary function symbol of $\mathscr{L}$, and if $\vdash_{\mathscr{E}} u_1 = v_1,\dots,\vdash_{\mathscr{E}} u_n = v_n$, then $\vdash_{\mathscr{E}} f(u_1,\dots,u_n) = f(v_1,\dots,v_n)$.

Clearly, if one obtains $\vdash_{\mathscr{E}} u = v$ by these rules, then every model of $CA+\mathscr{E}$ satisfies $u = v$.
In order to prove the converse, we first show that $\vdash_{\mathscr{E}} u = v \Rightarrow \vdash_{\mathscr{E}} v = u$, by induction on the length of the derivation of $\vdash_{\mathscr{E}} u = v$ by rules (i), (ii), (iii).
Consider the last rule used. If it is rule (i), then the result is clear (if $u = v$ is a particular case of an axiom of $\mathscr{E}$, then so is $v = u$). If it is rule (ii), then $\vdash_{\mathscr{E}} u = w$ and $\vdash_{\mathscr{E}} w = v$ are already deduced ; thus, by induction hypothesis, $\vdash_{\mathscr{E}} w = u$ and $\vdash_{\mathscr{E}} v = w$; therefore $\vdash_{\mathscr{E}} v = u$.
The proof is similar in the case of rule (iii).
Thus the relation $\vdash_{\mathscr{E}} u = v$ defined by these rules is an equivalence relation on the set $\mathscr{T}$ of individual terms of $\mathscr{L}$: indeed, it is reflexive and transitive by rule (ii), and it has just been proved that it is symmetric. By rule (iii), it is compatible with the natural interpretation of the functional symbols of $\mathscr{L}$ in $\mathscr{T}$. It follows that the quotient set of $\mathscr{T}$ by this equivalence relation is a (first order) model $\mathscr{M}$ for the language $\mathscr{L}$. By rule (i), this model satisfies $\mathscr{E}$. By taking the full model over $\mathscr{M}$, we obtain a model of $CA+\mathscr{E}$.
Now let u,v be two terms of $\mathscr{L}$, such that $CA+\mathscr{E} \vdash u = v$; it is clear that the considered model satisfies $u = v$, which means that $\vdash_{\mathscr{E}} u = v$.
Q.E.D.

Notice that the system of axioms $CA+\mathscr{E}$ cannot be contradictory. Indeed, the full model over a one element set (with the unique possible interpretation of the function symbols) is clearly seen to satisfy $CA+\mathscr{E}$.

Deduction rules for the second order predicate calculus

Consider a second order language $\mathscr{L}$, and a system $\mathscr{E}$ of equational axioms of $\mathscr{L}$. Let A be a formula, and $\mathscr{A} = \{A_1,\dots,A_k\}$ a finite set of formulas of $\mathscr{L}$. By the completeness theorem of predicate calculus (applied to the system of axioms

$CA+\mathscr{E}$), A is a consequence of $CA+\mathscr{E}+\mathscr{A}$ if and only if the expression $\mathscr{A} \vdash_{\mathscr{E}} A$ can be obtained by means of the following " deduction rules " :

D0. For every formula A and every finite set of formulas $\mathscr{A}$: $\mathscr{A}, \neg\neg A \vdash_{\mathscr{E}} A$.
D1. For every formula A and every finite set of formulas $\mathscr{A}$: $\mathscr{A}, A \vdash_{\mathscr{E}} A$.
D2. If $\mathscr{A}, A \vdash_{\mathscr{E}} B$, then $\mathscr{A} \vdash_{\mathscr{E}} A \to B$.
D3. If $\mathscr{A} \vdash_{\mathscr{E}} A$ and $\mathscr{A} \vdash_{\mathscr{E}} A \to B$, then $\mathscr{A} \vdash_{\mathscr{E}} B$.
D4. If $\mathscr{A} \vdash_{\mathscr{E}} \forall x\, A$, then $\mathscr{A} \vdash_{\mathscr{E}} A[u/x]$ for every term u of $\mathscr{L}$.
D5. If $\mathscr{A} \vdash_{\mathscr{E}} A$ and if the individual variable x does not occur free in $\mathscr{A}$, then $\mathscr{A} \vdash_{\mathscr{E}} \forall x\, A$.
D6. If $\mathscr{A} \vdash_{\mathscr{E}} \forall X\, A$, where X is an n-ary relation variable, and if F is any formula of $\mathscr{L}$, then $\mathscr{A} \vdash_{\mathscr{E}} A[F/Xx_1 \ldots x_n]$.
D7. If $\mathscr{A} \vdash_{\mathscr{E}} A$ and if the n-ary relation variable X does not occur free in $\mathscr{A}$, then $\mathscr{A} \vdash_{\mathscr{E}} \forall X\, A$.
D8. Let A be a formula, x an individual variable and u, v two terms of $\mathscr{L}$ such that $u = v$ is a particular case of an axiom of $\mathscr{E}$.
If $\mathscr{A} \vdash_{\mathscr{E}} A[u/x]$, then $\mathscr{A} \vdash_{\mathscr{E}} A[v/x]$.

So the meaning of the expression $\mathscr{A} \vdash_{\mathscr{E}} A$ is : " A is a consequence of $\mathscr{A}$ with the system of equational axioms $\mathscr{E}$, in the *classical* second order predicate calculus ".
Similarly, we define the expression : " A is a consequence of $\mathscr{A}$ with the system of equational axioms $\mathscr{E}$, in the *intuitionistic* second order predicate calculus " ; this will be denoted by $\mathscr{A} \vdash^{i}_{\mathscr{E}} A$. The definition uses rules D1 through D8 above, but not D0.

2. System FA_2

We consider a second order language $\mathscr{L}$, and a system $\mathscr{E}$ of equational axioms of $\mathscr{L}$. We are going to describe a system of typed λ-calculus, called *second order functional arithmetic* (FA_2), where the types are the formulas of $\mathscr{L}$ (modulo α-equivalence). When writing the typed terms of this system, we will use the same symbols to denote the variables of the λ-calculus and the individual variables of the language $\mathscr{L}$.
A context Γ is a set of the form $x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k$, where $x_1, x_2, \ldots, x_k$ are distinct variables of the λ-calculus, and $A_1, A_2, \ldots, A_k$ are formulas of $\mathscr{L}$. We will say that an individual variable x (or a relation variable X) of $\mathscr{L}$ is not free in Γ if it does not occur free in $A_1, A_2, \ldots, A_k$.
The rules of typing are the following (t stands for a term of the λ-calculus) :

T1. $\Gamma, x : A \vdash_{\mathscr{E}} x : A$ whenever x is a variable of the λ-calculus which is not declared in Γ.

T2. If $\Gamma, x : A \vdash_{\mathscr{E}} t : B$, then $\Gamma \vdash_{\mathscr{E}} \lambda x\, t : A \to B$.
T3. If $\Gamma \vdash_{\mathscr{E}} t : A$ and $\Gamma \vdash_{\mathscr{E}} u : A \to B$, then $\Gamma \vdash_{\mathscr{E}} (u)t : B$.
T4. If $\Gamma \vdash_{\mathscr{E}} t : \forall x\, A$, and if u is a term of $\mathscr{L}$, then $\Gamma \vdash_{\mathscr{E}} t : A[u/x]$.
T5. If $\Gamma \vdash_{\mathscr{E}} t : A$, and if the individual variable x is not free in Γ, then $\Gamma \vdash_{\mathscr{E}} t : \forall x\, A$.
T6. If $\Gamma \vdash_{\mathscr{E}} t : \forall X\, A$, where X is an n-ary relation variable, then $\Gamma \vdash_{\mathscr{E}} t : A[F/Xx_1 \dots x_n]$ for every formula F of $\mathscr{L}$.
T7. If $\Gamma \vdash_{\mathscr{E}} t : A$, and if the relation variable X is not free in Γ, then $\Gamma \vdash_{\mathscr{E}} t : \forall X\, A$.
T8. Let u, v be two terms of $\mathscr{L}$, such that $u = v$ is a particular case of an axiom of $\mathscr{E}$, and A a formula of $\mathscr{L}$. If $\Gamma \vdash_{\mathscr{E}} t : A[u/x]$, then $\Gamma \vdash_{\mathscr{E}} t : A[v/x]$.

Whenever we obtain the typing $\Gamma \vdash_{\mathscr{E}} t : A$ by means of these rules, we will say that “ the λ-term t is of type A (or may be given type A) with the axioms of $\mathscr{E}$, in the context Γ ”.

Clearly, if $\Gamma \vdash_{\mathscr{E}} t : A$, then all the free variables of t are declared in Γ. Thus all terms which are typable in the empty context are closed.

The following statement, which is a form of the so called *Curry-Howard correspondence*, is an immediate consequence of the above definitions :

There exists a term which may be given type A with the equational system $\mathscr{E}$ in the context $x_1 : A_1, x_2 : A_2, \dots, x_k : A_k$ if and only if $A_1, A_2, \dots, A_k \vdash^i_{\mathscr{E}} A$.

Indeed, the constructions of typed terms by means of rules T1 through T8 correspond, in an obvious and canonical way, to the intuitionistic proofs with rules D1 through D8.

System $\mathscr{F}$ and the normalization theorem

The types of system $\mathscr{F}$ are, by definition (see chapter 8), the formulas built up with the logical symbols $\forall$, $\to$, and the 0-ary relation variables $X, Y, \dots$ (propositional variables). So these formulas are seen to appear in all second order languages.
The typing rules of system $\mathscr{F}$ form a subsystem of the above rules : they are rules T1, T2, T3, and T6, T7 restricted to the case $n = 0$.

Proposition 9.5. *Given a language $\mathscr{L}$ and a system $\mathscr{E}$ of equations of $\mathscr{L}$, a λ-term t is typable with $\mathscr{E}$ if and only if it is typable in system $\mathscr{F}$.*

The condition is obviously sufficient, since the typing rules of system $\mathscr{F}$ form a subsystem of rules T1, ..., T8.
To prove the converse, we associate with each formula A of $\mathscr{L}$, a formula A^- of system $\mathscr{F}$, obtained by “ forgetting in A the first order part ”. The definition of A^- is by induction on A :

if A is atomic, say $A = X(t_1,\dots,t_n)$(X being an n-ary relation variable or symbol), then $A^- = X$ (which is, here, a propositional variable) ;

if $A = B \to C$, then $A^- = B^- \to C^-$;

if $A = \forall x\, B$(x being an individual variable), then $A^- = B^-$.

if $A = \forall X\, B$(X being an n-ary relation variable), then $A^- = \forall X\, B^-$ (X being, here, a propositional variable).

Now consider a derivation of a typing $x_1 : A_1,\dots,x_k : A_k \vdash_{\mathscr{E}} t : A$, with the system of equations $\mathscr{E}$. In this derivation, replace each formula F of $\mathscr{L}$ by F^-. We therefore obtain a derivation, in system $\mathscr{F}$, of the typing :

$$x_1 : A_1^-, x_2 : A_2^-,\dots,x_k : A_k^- \vdash t : A^-.$$

Note that rules T4, T5 and T8 disappear after this transformation, since we have $(\forall x\, A)^- = A^-$ and $A[u/x]^- = A[v/x]^-$.

Q.E.D.

Theorem 9.6 (Normalization theorem)**.** *Let $\mathscr{L}$ be a second order language and $\mathscr{E}$ a system of equations of $\mathscr{L}$. Then, every term of the λ-calculus which is typable with $\mathscr{E}$ is strongly normalizable.*

By proposition 9.5, a λ-term which is typable with $\mathscr{E}$ is also typable in system $\mathscr{F}$, so the result follows from the normalization theorem for that system (theorem 8.9).

Q.E.D.

Derived rules for constructing typed terms

Let $\mathscr{L}$ be a second order language, and $\mathscr{E}$ a system of equations of $\mathscr{L}$.

Proposition 9.7. *If $\Gamma \vdash_{\mathscr{E}} t : A$ and $\Gamma \subset \Gamma'$, then $\Gamma' \vdash_{\mathscr{E}} t : A$.*

Immediate proof, by induction on the length of the derivation of $\Gamma \vdash_{\mathscr{E}} t : A$.

Q.E.D.

Proposition 9.8. *Let Γ be a context, and $x_1,\dots,x_k$ variables which are not declared in Γ. If $\Gamma \vdash_{\mathscr{E}} t_i : A_i$ ($1 \le i \le k$) and $\Gamma, x_1 : A_1,\dots,x_k : A_k \vdash_{\mathscr{E}} u : B$, then $\Gamma \vdash_{\mathscr{E}} u[t_1/x_1,\dots,t_k/x_k] : B$.*

In particular, if $x_1,\dots x_k$ do not occur free in u, and Γ, $x_1 : A_1,\dots,x_k : A_k \vdash_{\mathscr{E}} u : B$, then $\Gamma \vdash_{\mathscr{E}} u : B$.

The proof is the same as that of proposition 8.2.

Q.E.D.

Our purpose now is to prove :

Theorem 9.9.
Let t, t' be two λ-terms such that $t\,\beta\,t'$; if $\Gamma \vdash_{\mathscr{E}} t : A$, then $\Gamma \vdash_{\mathscr{E}} t' : A$.

Recall that $t\,\beta\,t'$ means that t' is obtained from t by β-reduction.

Lemma 9.10.
Let u be a term and x a variable of $\mathscr{L}$. If $\Gamma \vdash_{\mathscr{E}} \tau : A$, then $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A[u/x]$.

The proof is by induction on the length l of the derivation of $\Gamma \vdash_{\mathscr{E}} \tau : A$; in fact we will show that $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A[u/x]$ also has a derivation of length l.
Consider the last rule used. The result is immediate if it is T1,T2 or T3.
If it is T4, then we have $\Gamma \vdash_{\mathscr{E}} \tau : \forall y\, A'$ (as a previous typing), and $A = A'[v/y]$. By induction hypothesis, $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : \forall y\, A'[u/x]$ and therefore, by applying T4, $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A'[u/x][v'/y]$.
Take $v' = v[u/x]$; then $A'[u/x][v'/y] = A'[v/y][u/x]$ since y does not occur in u. Hence $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A[u/x]$.
If it is T5, then we have $\Gamma \vdash_{\mathscr{E}} \tau : A'$ (previous typing) and $A = \forall y\, A'$, where y is an individual variable which is not free in Γ. If we take a variable z with no occurrence in Γ, A', u, then, by induction hypothesis : $\Gamma[z/y] \vdash_{\mathscr{E}} \tau : A'[z/y]$, and the length of this derivation is l. Now $\Gamma[z/y]$ is identical to Γ. Let $A'' = A'[z/y]$; then $\Gamma \vdash_{\mathscr{E}} \tau : A''$, and therefore $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A''[u/x]$. Since z does not occur in $\Gamma[u/x]$, we may apply T5, so we obtain $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : \forall z\, A''[u/x]$.
Now $\forall z\, A'' \equiv \forall y\, A' \equiv A$; thus $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A[u/x]$.
If it is T6, then we have $\Gamma \vdash_{\mathscr{E}} \forall X\, A'$ (previous typing), X being an n-ary relation variable, and $A = A'[F/Xx_1 \ldots x_n]$.
By induction hypothesis, $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : \forall X\, A'[u/x]$; therefore, by applying T6, we obtain $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A'[u/x][F'/Xx_1 \ldots x_n]$. Take F' as $F[u/x]$; then :
$A'[u/x][F'/Xx_1 \ldots x_n] = A'[F/Xx_1 \ldots x_n][u/x]$
(since we may assume that $x_1, \ldots, x_n$ do not occur in u) $= A[u/x]$.
If it is T7, the proof is the same as for T5.
If it is T8, we have $\Gamma \vdash_{\mathscr{E}} \tau : A'[v/y]$ (previous typing) and $A = A'[w/y]$, $v = w$ being a particular case of $\mathscr{E}$. By induction hypothesis, $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A'[v/y][u/x]$; now, since we may assume that y does not occur in u, we also have :
$A'[v/y][u/x] = A'[u/x][v'/y]$, where $v' = v[u/x]$.
Thus $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A'[u/x][v'/y]$. Let $w' = w[u/x]$: we see that $v' = w'$ is a particular case of $\mathscr{E}$. By rule T8, we obtain $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A'[u/x][w'/y]$.
Now we have $A'[u/x][w'/y] = A'[w/y][u/x] = A[u/x]$. This yields the expected conclusion.

Q.E.D.

Lemma 9.11. *Let X be an n-ary relation variable of the language $\mathscr{L}$.*
If $\Gamma \vdash_{\mathscr{E}} \tau : A$, then $\Gamma[F/Xx_1 \ldots x_n] \vdash_{\mathscr{E}} \tau : A[F/Xx_1 \ldots x_n]$.

The proof of the previous lemma applies in cases 1, 2, 3, 4, 5, 7 and 8.
Suppose that the last rule applied is T6 ; then we have $\Gamma \vdash_{\mathscr{E}} \tau : \forall Y\, A'$ (as a previous typing) and $A = A'[G/Y y_1 \dots y_p]$.
By induction hypothesis, $\Gamma[F/X x_1 \dots x_n] \vdash_{\mathscr{E}} \tau : \forall Y\, A'[F/X x_1 \dots x_n]$; by applying T6, we obtain : $\Gamma[F/X x_1 \dots x_n] \vdash_{\mathscr{E}} \tau : A'[F/X x_1 \dots x_n][G'/Y y_1 \dots y_p]$; if we take G' as $G[F/X x_1 \dots x_n]$, we see that :

$A'[F/X x_1 \dots x_n][G'/Y y_1 \dots y_p] = A'[G/Y y_1 \dots y_p][F/X x_1 \dots x_n]$

(since Y does not occur in F) $= A[F/X x_1 \dots x_n]$; this ends the proof.

Q.E.D.

Lemma 9.12. *If $u = v$ is a particular case of $\mathscr{E}$ and $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A[u/x]$, then $\Gamma[v/x] \vdash_{\mathscr{E}} \tau : A[v/x]$.*

Let $\Gamma = x_1 : A_1, \dots, x_k : A_k$. By hypothesis, we have $\Gamma[u/x] \vdash_{\mathscr{E}} \tau : A[u/x]$, therefore, by rule T8, $x_1 : A_1[u/x], \dots, x_k : A_k[u/x] \vdash_{\mathscr{E}} \tau : A[v/x]$.
Now $\Gamma[v/x] \vdash_{\mathscr{E}} x_i : A_i[v/x]$ (rule T1) ; thus, by rule T8, $\Gamma[v/x] \vdash_{\mathscr{E}} x_i : A_i[u/x]$.
Then it follows from proposition 9.8 that $\Gamma[v/x] \vdash_{\mathscr{E}} \tau : A[v/x]$.

Q.E.D.

Let Γ be a context and A a formula. We define the class $\mathscr{C}_{\Gamma,A}$ of Γ-*instances of* A, which is the least class $\mathscr{C}$ of formulas of $\mathscr{L}$ which contains A and is such that :

if $B \in \mathscr{C}$, then $B[t/x] \in \mathscr{C}$ whenever x is an individual variable not free in Γ, and t is a term.

if $B \in \mathscr{C}$, then $B[F/X x_1 \dots x_n] \in \mathscr{C}$ whenever X is an n-ary relation variable not free in Γ, and F is a formula.

if $B[t/x] \in \mathscr{C}$, then $B[u/x] \in \mathscr{C}$ whenever $t = u$ is a particular case of $\mathscr{E}$.

A formula is said to be *open* if it does not start with $\forall$ (so it is either atomic or of the form $B \to C$). Every formula F can be written $\forall \xi_1 \dots \forall \xi_k F^0$ where F^0 is an open formula called *the interior of* F ($\xi_1, \dots, \xi_k$ are individual or relation variables).

Lemma 9.13. *If A' is an open formula, and if $\Gamma \vdash_{\mathscr{E}} t : A'$ can be deduced from $\Gamma \vdash_{\mathscr{E}} t : A$ using only rules T4 through T8, then A' is a Γ-instance of A^0.*

The proof is by induction on the number of steps in the deduction by means of rules T4 through T8. Consider the first rule used.
If it is T5 or T7, then the first step is to pass from $\Gamma \vdash_{\mathscr{E}} t : A$ to $\Gamma \vdash_{\mathscr{E}} t : \forall \xi\, A$; the result follows immediately, since A and $\forall \xi\, A$ have the same interior.
If it is T4, then A can be written $\forall x \forall \xi_1 \dots \forall \xi_k\, A^0$, and the first step of the derivation gives $\Gamma \vdash_{\mathscr{E}} t : \forall \xi_1 \dots \forall \xi_k\, A^0[u/x]$. By induction hypothesis A' is a Γ-instance of $A^0[u/x]$, and thus also of A^0.
If it is T6, then A can be written $\forall X \forall \xi_1 \dots \forall \xi_k\, A^0$, and the first step of the derivation gives $\Gamma \vdash_{\mathscr{E}} t : \forall \xi_1 \dots \forall \xi_k\, A^0[F/X x_1 \dots x_n]$. Now A^0 is an open formula :

If A^0 is either an atomic formula not beginning with X, or a formula of the form $B \to C$, then $A^0[F/Xx_1 \dots x_n]$ is of the same form, so it is open. By induction hypothesis, A' is a Γ-instance of $A^0[F/Xx_1 \dots x_n]$, thus also of A^0.
Otherwise, A^0 is of the form $X t_1 \dots t_n$; then :
$A^0[F/Xx_1 \dots x_n] \equiv F[t_1/x_1, \dots, t_n/x_n]$ and it follows from the induction hypothesis that A' is a Γ-instance of $F^0[t_1/x_1, \dots, t_n/x_n]$, in other words a Γ-instance of $A^0[F^0/Xx_1 \dots x_n]$, thus also of A^0.
If it is T8, then A is written $B[u/x]$, and the first step of the derivation gives $\Gamma \vdash_{\mathscr{E}} t : B[v/x]$, $u = v$ being a particular case of $\mathscr{E}$. We have $A^0 = B^0[u/x]$, and the interior of $B[v/x]$ is $B^0[v/x]$. By induction hypothesis, A' is a Γ-instance of $B^0[v/x]$, thus also of A^0.

Q.E.D.

Lemma 9.14. *Suppose that $\Gamma \vdash_{\mathscr{E}} t : A$, where A is an open formula.*
i) If t is some variable x, then Γ contains a declaration $x : B$, and A is a Γ-instance of B^0.
ii) If $t = \lambda x\, u$, then $A = B \to C$ and $\Gamma, x : B \vdash_{\mathscr{E}} u : C$.
iii) If $t = (v)u$, then $\Gamma \vdash_{\mathscr{E}} v : C \to B$ and $\Gamma \vdash_{\mathscr{E}} u : C$, and A is a Γ-instance of B^0.

Consider, in the derivation of $\Gamma \vdash_{\mathscr{E}} t : A$, the last step where rules T1, T2 or T3 occur. Suppose that the typing obtained at this step is $\Gamma \vdash_{\mathscr{E}} t : B$; we can then go on to $\Gamma \vdash_{\mathscr{E}} t : A$ using only rules T4, ..., T8. Therefore, by lemma 9.13, A is a Γ-instance of B^0.
If t is some variable x, the rule applied to obtain $\Gamma \vdash_{\mathscr{E}} t : B$ (which must be T1, T2 or T3) can only be T1. This proves case (i) of the lemma.
If $t = (v)u$, the rule applied to obtain $\Gamma \vdash_{\mathscr{E}} t : B$ can only be T3.
This proves case (ii).
If $t = \lambda x\, u$, the rule applied to obtain $\Gamma \vdash_{\mathscr{E}} t : B$ can only be T2.
Therefore $B \equiv C \to D$, and $\Gamma, x : C \vdash_{\mathscr{E}} u : D$.
Since B is open, A is a Γ-instance of B.
Let $\mathscr{C}$ be the class of formulas $P \to Q$ such that $\Gamma, x : P \vdash_{\mathscr{E}} u : Q$; clearly, this class contains B. We now prove that it contains the class $\mathscr{C}_{\Gamma,B}$ of Γ-instances of B (yielding case (ii) of the lemma, since $A \in \mathscr{C}_{\Gamma,B}$) ; so let $R \in \mathscr{C}$, $R \equiv P \to Q$.
If y is an individual variable not occurring in Γ, and v is a term, then :
$\Gamma, x : P \vdash_{\mathscr{E}} u : Q$ and therefore $\Gamma, x : P[v/y] \vdash_{\mathscr{E}} u : Q[v/y]$, by lemma 9.10.
Thus $R[v/y] \in \mathscr{C}$.
Similarly, we see, using lemma 9.11, that $R[F/Xx_1 \dots x_n] \in \mathscr{C}$ whenever X is a relation variable not occurring in Γ.
Now suppose that $R \equiv R'[v/y] \equiv P'[v/y] \to Q'[v/y]$, and $v = w$ is a particular case of $\mathscr{E}$. By hypothesis, we have $\Gamma, x : P'[v/y] \vdash_{\mathscr{E}} u : Q'[v/y]$; therefore,

by lemma 9.12, we also have $\Gamma,\ x : P'[w/y] \vdash_{\mathscr{E}} u : Q'[w/y]$, which proves that $R'[w/y] \in \mathscr{C}$.

Q.E.D.

Now we are able to prove theorem 9.9 : we simply repeat the proof of proposition 4.3 (which is the same statement for system $\mathscr{D}$), using proposition 9.8 instead of proposition 4.1, and lemma 9.14(ii) instead of lemma 4.2(ii).

Note the following derived rules :

Proposition 9.15.
If $\Gamma \vdash_{\mathscr{E}} t : A$ and A' is a Γ-instance of A, then $\Gamma \vdash_{\mathscr{E}} t : A'$.

Let $\mathscr{C}$ be the class of all formulas B such that $\Gamma \vdash_{\mathscr{E}} t : B$. We prove that $\mathscr{C}$ contains $\mathscr{C}_{\Gamma,A}$ (the class of Γ-instances of A). Clearly, $A \in \mathscr{C}$. Let $B \in \mathscr{C}$. If x is an individual variable not occurring in Γ, then $\Gamma \vdash_{\mathscr{E}} t : \forall x\, B$ (rule T5) ; thus $\Gamma \vdash_{\mathscr{E}} t : B[u/x]$ for every term u (rule T4) ; therefore $B[u/x] \in \mathscr{C}$.
Similarly, it can be seen that $B[F/Xx_1 \ldots x_n] \in \mathscr{C}$ whenever X is a relation variable with no occurrence in Γ (apply rule T7, then rule T6).
Finally, if $B = C[u/x]$ and $u = v$ is a particular case of $\mathscr{E}$, then, by applying rule T8 to $\Gamma \vdash_{\mathscr{E}} t : C[u/x]$, we obtain : $\Gamma \vdash_{\mathscr{E}} t : C[v/x]$, and therefore $C[v/x] \in \mathscr{C}$.

Q.E.D.

Proposition 9.16. *Let u, v be two terms such that $CA + \mathscr{E} \vdash u = v$.*
If $\Gamma \vdash_{\mathscr{E}} t : A[u/x]$, then $\Gamma \vdash_{\mathscr{E}} t : A[v/x]$.

The expression $\vdash_{\mathscr{E}} u = v$ can be obtained by applying rules (i), (ii), (iii) of proposition 9.4. We reason by induction on the number of steps in this derivation. Consider the last rule used :

if it is rule (i), then $u = v$ is a particular case of $\mathscr{E}$. Then, by rule T8, we obtain immediately $\Gamma \vdash_{\mathscr{E}} t : A[v/x]$.

if it is rule (ii), then either $u = v$ (in that case the result is trivial), or expressions of the form $\vdash_{\mathscr{E}} u = w$ and $\vdash_{\mathscr{E}} w = v$ are obtained at the previous step ; therefore, by induction hypothesis, we have, successively, $\Gamma \vdash_{\mathscr{E}} t : A[w/x]$ and $\Gamma \vdash_{\mathscr{E}} t : A[v/x]$.

if it is rule (iii), then we have obtained $\vdash_{\mathscr{E}} u_i = v_i$ $(1 \le i \le n)$ at the previous step, and we have $u = f(u_1, \ldots, u_n)$ and $v = f(v_1, \ldots, v_n)$.
By assumption, we have $\Gamma \vdash_{\mathscr{E}} t : A[f(u_1, \ldots, u_n)/x]$. Now we may apply, repeatedly (n times), the induction hypothesis ; thus we have successively :
$\Gamma \vdash_{\mathscr{E}} t : A[f(v_1, u_2, \ldots, u_n)/x]$, $\Gamma \vdash_{\mathscr{E}} t : A[f(v_1, v_2, u_3 \ldots u_n)/x]$, $\ldots$, and finally $\Gamma \vdash_{\mathscr{E}} t : A[f(v_1, \ldots v_n)/x]$.

Q.E.D.

3. Realizability

Let $\mathscr{L}$ be a second order language. With each n-ary relation variable X, we associate an $(n+1)$-ary relation variable X^+ (the mapping being one-one) ; with each n-ary relation symbol R, we associate a new $(n+1)$-ary relation symbol R^+ (not found in $\mathscr{L}$). Let $\mathscr{L}^+$ be the language obtained by adding to $\mathscr{L}$ these new relation symbols, as well as the constant symbols K, S and the binary function symbol Ap (in case they are not already in $\mathscr{L}$).

With each formula A of $\mathscr{L}$, we associate a formula A^+ of $\mathscr{L}^+$, also denoted by $x \Vdash A$, where x is an individual variable *not occurring in* A. $x \Vdash A$ should be read : x *realizes* A. It is defined, by induction on A, by the following conditions :

if A is atomic, say $A \equiv X(t_1,\ldots,t_n)$, where the t_i's are terms and X is an n-ary relation variable or symbol, then $x \Vdash A$ is $X^+(t_1,\ldots,t_n,x)$;

if $A \equiv B \to C$, then $x \Vdash A$ is $\forall y[y \Vdash B \to (x)y \Vdash C]$ (it is assumed that the individual variable y is distinct from x and does not occur free in A) ;

if $A \equiv \forall y B$, then $x \Vdash A$ is $\forall y(x \Vdash B)$ (the individual variable y is assumed $\neq x$) ;

if $A \equiv \forall X B$, then $x \Vdash A$ is $\forall X^+(x \Vdash B)$ (X is an n-ary relation variable).

Lemma 9.17.
Let A be a formula, $x, x_1,\ldots,x_k$ distinct individual variables, $t_1,\ldots,t_k$ terms, and $A^+ = x \Vdash A$. Then $x \Vdash A[t_1/x_1,\ldots,t_k/x_k]$ is the formula $A^+[t_1/x_1,\ldots,t_k/x_k]$.

This is immediate, by induction on the length of A.

Q.E.D.

Lemma 9.18. *Let A, F be two formulas, $x, x_1,\ldots,x_k$ distinct individual variables, X a k-ary relation variable, and $F^+ = x \Vdash F$. Then :*
$x \Vdash A[F/Xx_1\ldots x_k]$ *is the formula* $\{x \Vdash A\}[F^+/X^+x_1\ldots x_kx]$.

The proof is by induction on the length of A :

If A is atomic, then the result follows immediately from the previous lemma.

If $A \equiv B \to C$, then $x \Vdash A$ is $\forall y\{y \Vdash B \to (x)y \Vdash C\}$, thus
$\{x \Vdash A\}[F^+/X^+x_1\ldots x_kx]$ is
$\forall y(\{y \Vdash B\}[F^+/X^+x_1\ldots x_kx] \to \{(x)y \Vdash C\}[F^+/X^+x_1\ldots x_kx])$.

By induction hypothesis, this is :
$\forall y\{y \Vdash B[F/Xx_1\ldots x_k] \to (x)y \Vdash C[F/Xx_1\ldots x_k]\}$,
that is to say $x \Vdash A[F/Xx_1\ldots x_k]$.

The other cases of the induction are obvious.

Q.E.D.

Notation. We shall use the correspondence between λ-terms and terms of combinatory logic, as it was settled in chapter 6. Therefore, we use notations

from that chapter : with each λ-term t, we associate a term of $\mathscr{L}$, denoted by $t_{\mathscr{L}}$.
We shall also consider the system of equational axioms C_0 defined in chapter 6 :
(C_0) $\qquad (K)xy = x\,;\ (S)xyz = ((x)z)(y)z.$

Theorem 9.19. *Let $\mathscr{E}$ be a system of equational axioms of $\mathscr{L}$, and t a λ-term such that $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} t : A$. Then, we have :*
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : (t_{\mathscr{L}} \Vdash A)$, where $\mathscr{E}'$ is the equational system $\mathscr{E} + C_0$, and $t_{\mathscr{L}}$ the term of $\mathscr{L}$ which is associated with t.
In particular, $CA + C_0 + \mathscr{E} \vdash \forall x_1 \ldots \forall x_k \{x_1 \Vdash A_1, \ldots, x_k \Vdash A_k \to t_{\mathscr{L}} \Vdash A\}$.

In view of the Curry-Howard correspondence, the second part of the theorem easily follows from the first one. Indeed, if there exists a typing of the form :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : (t_{\mathscr{L}} \Vdash A)$, then $t_{\mathscr{L}} \Vdash A$ is an intuitionistic consequence of $CA, \mathscr{E}', x_1 \Vdash A_1, \ldots, x_k \Vdash A_k$; this yields the expected result.
The proof of the first part is by induction on the length of the derivation of the typing $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} t : A$. Consider the last rule used :

If it is T1, then the given typing can be written :
$x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} x_i : A_i$; it is then clear that
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} x_i : (x_i \Vdash A_i)$.

If it is T2, then we have $t = \lambda y\, u$, $A \equiv B \to C$
and $x_1 : A_1, \ldots, x_k : A_k, y : B \vdash_{\mathscr{E}} u : C$ was obtained as a previous typing. We may suppose that y does not occur in $A, A_1, \ldots, A_k$, and that $y \neq x_1, \ldots, x_k$. By induction hypothesis :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k),\ y : (y \Vdash B) \vdash_{\mathscr{E}'} u : (u_{\mathscr{L}} \Vdash C)$.
By rule T2, we obtain
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} \lambda y\, u : (y \Vdash B) \to (u_{\mathscr{L}} \Vdash C)$.
Since y does not occur free in the formulas $x_1 \Vdash A_1, \ldots, x_k \Vdash A_k$, we have, by rule T5 :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \forall y\{y \Vdash B \to u_{\mathscr{L}} \Vdash C\}$.
Now the equation $u_{\mathscr{L}} = t_{\mathscr{L}}\, y$ is a consequence of C_0, since $t = \lambda y\, u$. Thus, by rule T8, we obtain :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \forall y\{y \Vdash B \to t_{\mathscr{L}}\, y \Vdash C\}$,
that is to say $x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : t_{\mathscr{L}} \Vdash B \to C$.

If it is T3, then we have $t = uv$ and two previous typings :
$x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} u : B \to A$ and $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} v : B$. Therefore, by induction hypothesis :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} u : (u_{\mathscr{L}} \Vdash B \to A)$ and :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} v : (v_{\mathscr{L}} \Vdash B)$.
Now the formula $u_{\mathscr{L}} \Vdash B \to A$ is $\forall y[y \Vdash B \to u_{\mathscr{L}}\, y \Vdash A]$.
By applying rule T4, we obtain :

$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} u : v_{\mathscr{L}} \Vdash B \to u_{\mathscr{L}} v_{\mathscr{L}} \Vdash A$.
Finally, by rule T3, we deduce :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} uv : u_{\mathscr{L}} v_{\mathscr{L}} \Vdash A$.

If it is T4, then $A \equiv B[u/x]$, where u is some term of $\mathscr{L}$, and we have the previous typing $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} t : \forall x\, B$. The induction hypothesis implies that :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : (t_{\mathscr{L}} \Vdash \forall x\, B)$, that is to say :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \forall x(t_{\mathscr{L}} \Vdash B)$. By applying rule T4, we obtain $x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \{t_{\mathscr{L}} \Vdash B\}[u/x]$.
Now, by lemma 9.17, the formula $\{t_{\mathscr{L}} \Vdash B\}[u/x]$ is precisely $t_{\mathscr{L}} \Vdash B[u/x]$.

If it is T5, then $A \equiv \forall x\, B$, and we have the previous typing :
$x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} t : B$, where x does not occur free in $A_1, \ldots, A_k$. According to lemma 9.10, it can be assumed that $x \neq x_1, \ldots, x_k$ (otherwise, change the variable x : this does not modify $A_1, \ldots, A_k$) ; thus x does not occur free in t. By induction hypothesis, we have :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : (t_{\mathscr{L}} \Vdash B)$.
Since x has no occurrence in $x_i \Vdash A_i$, by applying rule T5, we obtain :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \forall x(t_{\mathscr{L}} \Vdash B)$.
Now x does not occur in $t_{\mathscr{L}}$; therefore, the formula $\forall x(t_{\mathscr{L}} \Vdash B)$ is identical to $t_{\mathscr{L}} \Vdash \forall x\, B$; this yields the result.

If it is T6, then $A \equiv B[F/Xx_1 \ldots x_n]$, and we have the previous typing :
$x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} t : \forall X\, B$, ($X$ being an n-ary relation variable). By induction hypothesis :

$$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \forall X^+(t_{\mathscr{L}} \Vdash B)\,;$$

therefore, by applying rule T6, we obtain

$$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \{t_{\mathscr{L}} \Vdash B\}[F^+/X^+x_1 \ldots x_n x],$$

F^+ being the formula $x \Vdash F$. Now, by lemma 9.18, the formula :
$\{t_{\mathscr{L}} \Vdash B\}[F^+/X^+x_1 \ldots x_n x]$ is precisely $t_{\mathscr{L}} \Vdash B[F/Xx_1 \ldots x_n]$.

If it is T7, then $A \equiv \forall X\, B$, and we have the previous typing :
$x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} t : B$, ($X$ having no free occurrence in $A_1, \ldots, A_k$). By induction hypothesis, we have :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : (t_{\mathscr{L}} \Vdash B)$. Since X^+ does not occur in $x_i \Vdash A_i$, by applying rule T7, we obtain :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \forall X^+(t_{\mathscr{L}} \Vdash B)$.
Now the formula $\forall X^+(t_{\mathscr{L}} \Vdash B)$ is identical to $t_{\mathscr{L}} \Vdash \forall X\, B$; this yields the result.

If it is T8, then $A \equiv B[v/x]$, and we have $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathscr{E}} t : B[u/x]$ as a previous typing, the equation $u = v$ being a particular case of $\mathscr{E}$. By induction hypothesis, we have :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : (t_{\mathscr{L}} \Vdash B[u/x])$;

now, by lemma 9.17, the formula $t_{\mathscr{L}} \Vdash B[u/x]$ is $\{t_{\mathscr{L}} \Vdash B\}[u/x]$.
Thus, by applying rule T8, we obtain :
$x_1 : (x_1 \Vdash A_1), \ldots, x_k : (x_k \Vdash A_k) \vdash_{\mathscr{E}'} t : \{t_{\mathscr{L}} \Vdash B\}[v/x]$, which is precisely the expected result, since $\{t_{\mathscr{L}} \Vdash B\}[v/x]$ is identical to $t_{\mathscr{L}} \Vdash B[v/x]$.

Q.E.D.

4. Data types

Let $\mathscr{L}$ be a second order language, and $\mathscr{L}^+$ the extended language defined in the beginning of the previous section, page 179 (so $\mathscr{L}^+$ contains the constant symbols K, S and the binary function symbol Ap).
We define a *standard model* of $\mathscr{L}^+$ as a full model such that its domain is $\Lambda/\simeq_{\beta\eta}$ (the set of λ-terms modulo $\beta\eta$-equivalence) and the interpretations of the symbols K, S and Ap are the standard ones.
In other words, we will say that a full model of $\mathscr{L}^+$ is standard if its restriction to the language of combinatory logic is the standard model of the extensional combinatory logic.

Let $\mathscr{M}$ be a standard model of $\mathscr{L}^+$, and $D[x]$ a formula of $\mathscr{L}$, where the individual variable x is the only free variable. We will say that $D[x]$ defines a *data type* in the model $\mathscr{M}$ if and only if the following conditions hold :

i) each $a \in |\mathscr{M}| = \Lambda/\simeq_{\beta\eta}$, such that $\mathscr{M} \models D[a]$, is a closed λ-term ;
ii) $\mathscr{M} \models \forall x \forall y\{y \Vdash D[x] \leftrightarrow x = y \wedge D[x]\}$.

We now give some basic examples of data types.

Booleans.

Consider two closed terms of $\mathscr{L}$, which we will denote by 0,1 (they may be constant symbols, terms of combinatory logic ...). Then :

Proposition 9.20. *The formula Bool*$[x] \equiv \forall X[X1, X0 \to Xx]$ *defines a data type in a standard model* $\mathscr{M}$ *if and only if the interpretation of* 1 *(resp.* 0*) in* $\mathscr{M}$ *is the Boolean* **1** *(resp.* **0***) of the* λ*-calculus.*

Indeed $y \Vdash \mathrm{Bool}[x]$ is the formula $\forall X \forall u \forall v[X(1,u), X(0,v) \to X(x,(y)uv)]$. It is equivalent to $\forall u \forall v[(x = 1 \wedge (y)uv = u) \vee (x = 0 \wedge (y)uv = v)]$.
Now, let $\mathscr{M}$ be a standard model, and y any element of $|\mathscr{M}| = \Lambda/\simeq_{\beta\eta}$. We can take u, v as two distinct variables of the λ-calculus, not occurring in y. Then $(y)uv = u$ (resp. v) if and only if $y = \mathbf{1}$ (resp. $\mathbf{0}$) (Booleans of the λ-calculus). Therefore :
$\mathscr{M} \models (y \Vdash \mathrm{Bool}[x]) \leftrightarrow (x = 0 \wedge y = \mathbf{0}) \vee (x = 1 \wedge y = \mathbf{1})$. Thus we see that Bool$[x]$ defines a data type if and only if $\mathscr{M}$ satisfies the formula :

$(x = 0 \land y = \mathbf{0}) \lor (x = 1 \land y = \mathbf{1}) \to x = y$. This completes the proof of our statement.

Q.E.D.

Integers.

Here we consider a closed term 0 and a term $s(x)$ of $\mathscr{L}$ having no variables but x. The integers type is then defined by the formula :

$$\text{Int}[x] \equiv \forall X[\forall y(Xy \to Xs(y)), X0 \to Xx].$$

If $\mathscr{M}$ is a standard model and $a \in |\mathscr{M}|$, then $\mathscr{M} \models \text{Int}[a]$ if and only if $\mathscr{M} \models a = s^n(0)$ for some $n \in \mathbb{N}$.

Proposition 9.21. *The formula Int$[x]$ defines a data type in a standard model $\mathscr{M}$ if and only if, for every integer n, the interpretation of the term $s^n(0)$ in $\mathscr{M}$ is Church numeral $\lambda f \lambda x(f)^n x$.*

Indeed, $y \Vdash \text{Int}[x]$ is the formula :
$\forall X \forall f \forall a\{\forall z \forall u[X(z,u) \to X(s(z),(f)u)], X(0,a) \to X(x,(y)fa)\}$.
Let $x_0, y_0 \in |\mathscr{M}| = \Lambda/\simeq_{\beta\eta}$; take f, a as two variables of the λ-calculus, not occurring in the terms x_0, y_0, and X as the following binary relation on $\mathscr{M}$:
$\{(s^n(0),(f)^n a)\ ;\ n \in \mathbb{N}\}$. With these interpretations of f, a, X, we clearly have :
$\mathscr{M} \models \forall z \forall u[X(z,u) \to X(s(z),(f)u)], X(0,a)$.
Therefore, if $\mathscr{M}$ satisfies $y_0 \Vdash \text{Int}[x_0]$, then :
$\mathscr{M} \models X(x_0,(y_0)fa))$, that is $x_0 = s^n(0)$ and $(y_0)fa = (f)^n a$ for some $n \in \mathbb{N}$. Now f, a are variables which do not occur in y_0. Hence $y_0 = \lambda f \lambda a(f)^n a$.
It follows that $\mathscr{M} \models y_0 \Vdash \text{Int}[x_0]$ if and only if $x_0 = s^n(0)$ and $y_0 = \lambda f \lambda a(f)^n a$ for some $n \in \mathbb{N}$.
Hence, if Int$[x]$ is a data type, then $\mathscr{M} \models (y_0 \Vdash \text{Int}[x_0]) \to x_0 = y_0$, and therefore $s^n(0) = \lambda f \lambda a(f)^n a$. Conversely, if $s^n(0) = \lambda f \lambda a(f)^n a$ for all $n \in \mathbb{N}$, we have, clearly, $\mathscr{M} \models \text{Int}[x_0] \land x_0 = y_0 \Leftrightarrow x_0 = y_0 = s^n(0)$ for some n, thus $x_0 = s^n(0)$ and $y_0 = \lambda f \lambda a(f)^n a$; therefore, $\mathscr{M} \models y_0 \Vdash \text{Int}[x_0]$.

Q.E.D.

Product of data types.

Let cpl(x,y) be a term of $\mathscr{L}$, with no variables but x, y, and $A[x]$, $B[y]$ two formulas which define data types in a standard model $\mathscr{M}$. We define the product type $(A \times B)[x]$ as the formula $\forall X\{\forall y \forall z(A[y], B[z] \to X\text{cpl}(y,z)) \to Xx\}$. If $c \in |\mathscr{M}|$, then $\mathscr{M} \models (A \times B)[c]$ if and only if $\mathscr{M} \models c = \text{cpl}(a,b)$, where $a, b \in |\mathscr{M}|$ and $\mathscr{M} \models A[a], B[b]$.

Proposition 9.22.
$(A \times B)[x]$ defines a data type in a standard model $\mathscr{M}$ if and only if, for every $a, b \in |\mathscr{M}|$ such that $\mathscr{M} \models A[a], B[b]$, the interpretation of cpl(a,b) in $\mathscr{M}$ is the ordered pair $\lambda f(f)ab$.

$u \Vdash (A \times B)[x]$ is the following formula :
$\forall X \forall f\{\forall y \forall z \forall v \forall w[v \Vdash A[y],\ w \Vdash B[z] \to X(\mathrm{cpl}(y,z),(f)vw)] \to X(x,uf)\}$.
Now the model $\mathscr{M}$ satisfies the formulas :
$(v \Vdash A[y]) \leftrightarrow A[y] \wedge (v = y)$ and $(w \Vdash B[z]) \leftrightarrow B[z] \wedge (w = z)$.
Thus, in $\mathscr{M}$, $u \Vdash (A \times B)[x]$ is equivalent to :
$\forall X \forall f\{\forall y \forall z(A[y], B[z] \to X(\mathrm{cpl}(y,z),(f)yz)) \to X(x,uf)\}$, and therefore to :
(i) $\forall f \exists y \exists z\{A[y] \wedge B[z] \wedge x = \mathrm{cpl}(y,z) \wedge uf = (f)yz\}$.
Suppose that : $\mathscr{M} \models A[a], B[b] \to \mathrm{cpl}(a,b) = \lambda f(f)ab$. Let $u_0, x_0 \in |\mathscr{M}|$ be such that $\mathscr{M} \models (u_0 \Vdash (A \times B)[x_0])$. Take any variable not occurring in u_0 as the interpretation of f. Then, by (i), there exist $a, b \in |\mathscr{M}|$ such that :
$\mathscr{M} \models A[a], B[b]$, $x_0 = \mathrm{cpl}(a,b)$ and $(u_0)f = (f)ab$.
Now a, b are closed terms, thus $u_0 = \lambda f(f)ab$. Hence $u_0 = x_0 = \mathrm{cpl}(a,b)$, and therefore $\mathscr{M} \models (A \times B)[x_0]$; it follows that $(A \times B)[x]$ defines a data type in $\mathscr{M}$.
Conversely, suppose that $(A \times B)[x]$ defines a data type in $\mathscr{M}$ and let $a, b \in |\mathscr{M}|$ be such that $\mathscr{M} \models A[a], B[b]$; take $x_0 = \mathrm{cpl}(a,b)$ and $u_0 = \lambda f(f)ab$. Then, by (i), $\mathscr{M}$ satisfies $u_0 \Vdash (A \times B)[x_0]$; therefore, $u_0 = x_0$, that is $\mathrm{cpl}(a,b) = \lambda f(f)ab$.
Q.E.D.

Direct sum of data types.

Let $i(x)$ and $j(x)$ be two terms of $\mathscr{L}$, where x is the only variable, and $A[x]$ and $B[y]$ two formulas which define data types in a standard model $\mathscr{M}$.
We define the direct sum type :
$(A+B)[x] \equiv \forall X\{\forall y(A[y] \to Xi(y)), \forall z(B[z] \to Xj(z)) \to Xx\}$.
If $c \in |\mathscr{M}|$, then $\mathscr{M} \models (A+B)[c]$ if and only if :
either $\mathscr{M} \models c = i(a)$ for some $a \in |\mathscr{M}|$ such that $\mathscr{M} \models A[a]$
or $\mathscr{M} \models c = j(b)$ for some $b \in |\mathscr{M}|$ such that $\mathscr{M} \models B[b]$.
We have the same proposition as in the previous case (with a similar proof) :

Proposition 9.23. *$(A+B)[x]$ defines a data type in a standard model $\mathscr{M}$ if and only if, for each a (resp. b) $\in |\mathscr{M}|$ such that $\mathscr{M} \models A[a]$ (resp. $B[b]$), the interpretation of $i(a)$ (resp. $j(b)$) in $\mathscr{M}$ is the term $\lambda f \lambda g(f)a$ (resp. $\lambda f \lambda g(g)b$).*

Lists of elements of a data type.

Let \$ be a closed term of $\mathscr{L}$ (for the empty list), and cons(x,y) a term of $\mathscr{L}$ where x, y are the only variables. Let $A[x]$ be a data type in a standard model $\mathscr{M}$. We define the type $LA[x]$ (the type of lists of objects of type A) as the following formula :

$$LA[x] \equiv \forall X\{\forall y \forall z(A[y], Xz \to X\mathrm{cons}(y,z)), X\$ \to Xx\}.$$

If $c \in |\mathcal{M}|$, then $\mathcal{M} \models LA[c]$ if and only if

$$\mathcal{M} \models c = \mathrm{cons}(a_1, \mathrm{cons}(a_2, \ldots, \mathrm{cons}(a_n, \$)\ldots))$$

where $\mathcal{M} \models A[a_i]$ $(1 \le i \le n)$.

Proposition 9.24. $LA[x]$ *defines a data type in a standard model* $\mathcal{M}$ *if and only if, for all* $a_1, \ldots, a_n \in |\mathcal{M}|$ *such that* $\mathcal{M} \models A[a_i]$ $(1 \le i \le n)$*, the interpretation of* $\mathit{cons}(a_1, \mathit{cons}(a_2, \ldots, \mathit{cons}(a_n, \$)\ldots))$ *(term of* $\mathcal{L}^+$*) in* $\mathcal{M}$ *is the* λ*-term :* $\lambda f \lambda x((f)a_1)((f)a_2 \ldots ((f)a_n)x.$

Indeed, $t \Vdash LA[x]$ is the formula :

$\forall X \forall f \forall a\{\forall y \forall z \forall u \forall v[u \Vdash A[y], X(z,v) \to X(\mathrm{cons}(y,z),(f)uv)],$

$$X(\$,a) \to X(x,(t)fa)\}.$$

Now $\mathcal{M}$ satisfies $u \Vdash A[y] \leftrightarrow A[y] \wedge u = y$; thus, in $\mathcal{M}$, $t \Vdash A[x]$ is equivalent to :

$\forall X \forall f \forall a\{\forall y \forall z \forall v[A[y], X(z,v) \to X(\mathrm{cons}(y,z),(f)yv],$

$$X(\$,a) \to X(x,(t)fa)\}.$$

Now this formula holds in the standard model $\mathcal{M}$ if and only if :

(ii) for all $f, a \in |\mathcal{M}|$, there exist $a_1, \ldots, a_n \in |\mathcal{M}|$ such that $\mathcal{M}$ satisfies $A[a_i]$, $x = \mathrm{cons}(a_1, \ldots, \mathrm{cons}(a_n, \$)\ldots)$, and $(t)fa = ((f)a_1)\ldots((f)a_n)a$.

Suppose that $\mathcal{M} \models \mathrm{cons}(a_1, \ldots, \mathrm{cons}(a_n, \$)\ldots) = \lambda f \lambda a((f)a_1)\ldots((f)a_n)a$ whenever $\mathcal{M} \models A[a_i]$. Let $t_0, x_0 \in |\mathcal{M}|$ be such that $\mathcal{M} \models (t_0 \Vdash LA[x_0])$. Take two variables not occurring in t_0 as the interpretations of f and a. Then, by (ii), there exist $a_1, \ldots, a_n \in |\mathcal{M}|$ such that $\mathcal{M} \models A[a_i]$, $x_0 = \mathrm{cons}(a_1, \ldots, \mathrm{cons}(a_n, \$)\ldots)$ and $(t_0)fa = ((f)a_1)\ldots((f)a_n)a$. Now, since A is a data type, the a_i's are closed terms ; thus $t_0 = \lambda f \lambda a((f)a_1)\ldots((f)a_n)a$. Therefore, $t_0 = x_0$ and $LA[x]$ defines a data type in $\mathcal{M}$.

Conversely, suppose that $LA[x]$ defines a data type in $\mathcal{M}$.

Let $a_1, \ldots, a_n \in |\mathcal{M}|$ be such that $\mathcal{M} \models A[a_i]$;

take $x_0 = \mathrm{cons}(a_1, \ldots, \mathrm{cons}(a_n, \$)\ldots)$, and $t_0 = \lambda f \lambda a((f)a_1)\ldots((f)a_n)a$.

Then, by (ii), $\mathcal{M}$ satisfies $t_0 \Vdash LA[x_0]$, and hence $t_0 = x_0$, that is :

$\mathrm{cons}(a_1, \ldots, \mathrm{cons}(a_n, \$)\ldots) = \lambda f \lambda a((f)a_1)\ldots((f)a_n)a.$

5. Programming in FA_2

We consider a standard model $\mathcal{M}$ of a second order language $\mathcal{L}$, and a system $\mathcal{E}$ of equations of $\mathcal{L}$ which is satisfied in $\mathcal{M}$. Let f be an n-ary function symbol of $\mathcal{L}$, and $D_1[x_1], \ldots, D_n[x_n]$, $E[y]$ formulas which define data types in $\mathcal{M}$. Let $D_1, \ldots, D_n$, $E \subset |\mathcal{M}|$ the sets of λ-terms defined in $\mathcal{M}$ by these formulas.

Then, for every λ-term t such that :

$$\vdash_{\mathscr{E}} t : \forall x_1 \ldots \forall x_n \{D_1[x_1], \ldots, D_n[x_n] \to E[f(x_1, \ldots, x_n)]\}$$

we have $\mathscr{M} \models (t)u_1 \ldots u_n = f(u_1, \ldots, u_n)$ for all $u_1 \in D_1, \ldots, u_n \in D_n$. In other words, the term t is a program for the function f on the domain $D_1 \times \ldots \times D_n$. Indeed, it then follows from theorem 9.19 that :
$CA + C_0 + \mathscr{E} \vdash t_{\mathscr{L}} \Vdash \forall x_1 \ldots \forall x_n \{D_1[x_1], \ldots, D_n[x_n] \to E[f(x_1, \ldots, x_n)]\}$
that is to say :
$CA + C_0 + \mathscr{E} \vdash \forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n \{y_1 \Vdash D_1[x_1], \ldots, y_n \Vdash D_n[x_n]$
$\to (t_{\mathscr{L}})y_1 \ldots y_n \Vdash E[f(x_1, \ldots, x_n)]\}$.
Therefore, this formula holds in $\mathscr{M}$. According to the definition of data types, we have $\mathscr{M} \models y_i \Vdash D_i[x_i] \leftrightarrow y_i = x_i \wedge D_i[x_i]$. Hence :
$\mathscr{M} \models \forall x_1 \ldots \forall x_n \{D_1[x_1], \ldots, D_n[x_n]$
$\to (E[f(x_1, \ldots, x_n)] \wedge (t_{\mathscr{L}})x_1 \ldots x_n = f(x_1, \ldots, x_n))\}$.
Now the interpretation of the term $t_{\mathscr{L}}$ in $\mathscr{M}$ is the λ-term t (lemma 6.22).

Thus we obtain a program for f, by proving :

$$D_1[x_1], \ldots, D_n[x_n] \vdash_{\mathscr{E}} E[f(x_1, \ldots, x_n)]$$

in second order intuitionistic logic, by means of rules D1 through D8.

Examples with integers

Let $\varphi_1, \ldots, \varphi_n$ be functions such that $\varphi_i : \mathbb{N}^{k_i} \to \mathbb{N}$; we wish to program φ_1, that is to say to obtain a λ-term t such that $(t)\underline{p}_1 \ldots \underline{p}_{k_1} \simeq_{\beta\eta} \varphi_1(p_1, \ldots, p_{k_1})$ for all Church numerals $\underline{p}_1, \ldots, \underline{p}_{k_1}$.
We consider a language $\mathscr{L}$ consisting only of functions symbols $f_1, \ldots, f_n$ (the arity of f_i being k_i), including 0 et s, which will be interpreted in $\mathbb{N}$ as the integer 0 and the successor function.
Let $\mathscr{E}$ be the set of those equational formulas of $\mathscr{L}$ which are satisfied in the following model $\mathscr{N}$: the domain is $\mathbb{N}$, and each symbol f_i is interpreted by the function φ_i.
We define a standard model $\mathscr{M}$ of $\mathscr{E}$, in which the interpretation of each symbol f_i is a function ψ_i which extends φ_i (thus ψ_i is a mapping of $|\mathscr{M}|^{k_i}$ into $|\mathscr{M}|$, where $|\mathscr{M}| = \Lambda/\simeq_{\beta\eta}$).
For that purpose, we consider the language $\mathscr{L}'$ obtained by adding to $\mathscr{L}$ an infinite sequence $c_0, \ldots, c_n, \ldots$ of constant symbols. Let $\mathscr{T}$ (resp. $\mathscr{T}'$) be the set of closed terms of $\mathscr{L}$ (resp. $\mathscr{L}'$).
We define an equivalence relation on $\mathscr{T}'$ by : $t \sim u \Leftrightarrow \mathscr{E} \vdash t = u$. Let $\mathscr{M}'$ be the model of $\mathscr{L}'$ such that its domain is $|\mathscr{M}'| = \mathscr{T}'/\!\sim$ and the function symbols are given their canonical interpretation. Then the restriction of $\mathscr{M}'$ to the subset $\mathscr{T}/\!\sim$ is a submodel $\mathscr{N}'$ which is obviously isomorphic to $\mathscr{N}$.

Moreover, $c_n \in |\mathscr{M}'| \setminus |\mathscr{N}'|$: otherwise, we would have $\mathscr{E} \vdash c_n = \tau$, for some closed term τ of $\mathscr{L}$, and therefore $\mathscr{E} \vdash \forall x(x = \tau)$, since c_n occurs neither in $\mathscr{E}$ nor in τ. Then $\mathscr{N}$ would contain only one element, but this is false (actually, $|\mathscr{N}|$ is an infinite countable set).
Also, $\mathscr{M}' \models c_m \neq c_n$ whenever $m \neq n$: otherwise, we would have :
$\mathscr{E} \vdash c_m = c_n$, thus $\mathscr{E} \vdash \forall x \forall y(x = y)$,
which would lead us to the same contradiction.
It follows that $|\mathscr{M}'| \setminus |\mathscr{N}'|$ is an infinite countable set.
Finally, $\mathscr{M}'$ satisfies $\mathscr{E}$: indeed, let $t = u$ be an equation of $\mathscr{E}$, where t and u are terms of $\mathscr{L}$, with variables $x_1, \ldots, x_n$, and let $\tau_1, \ldots, \tau_n \in \mathscr{T}'$. We need to prove that $\mathscr{M}' \models t[\tau_1/x_1, \ldots, \tau_n/x_n] = u[\tau_1/x_1, \ldots, \tau_n/x_n]$, that is to say :
$\mathscr{E} \vdash t[\tau_1/x_1, \ldots, \tau_n/x_n] = u[\tau_1/x_1, \ldots, \tau_n/x_n]$, which is clear.
Then the isomorphism from $\mathscr{N}'$ onto $\mathscr{N}$ can be extended to a one to one function from $|\mathscr{M}'|$ onto $\Lambda/\simeq_{\beta\eta}$: indeed, since $|\mathscr{N}|$ is the set of Church numerals, its complement in $\Lambda/\simeq_{\beta\eta}$ is countable. This allows us to transfer on $\Lambda/\simeq_{\beta\eta}$ the structure of $\mathscr{M}'$, defining therefore over $\Lambda/\simeq_{\beta\eta}$ a model $\mathscr{M}$ of $\mathscr{E}$ which is an extension of $\mathscr{N}$; this is what was expected.

Remark

The above method will be systematically used in the further examples of " programming " with various data types. It consists in extending, to the whole set $\Lambda/\simeq_{\beta\eta}$, functions which are defined only on data types, and preserving the equations which they satisfy. The above proof still applies, *provided that the data types under consideration do not consist of one single element.*

Thus we will take, as equational system $\mathscr{E}$, the set of all equational formulas satisfied by the functions to be programmed, on their domains, and we will be allowed to assume that $\mathscr{E}$ is satisfied on the whole standard model $\mathscr{M}$.

The formula $\mathrm{Int}[x] \equiv \forall X\{\forall y(Xy \to Xsy), X0 \to Xx\}$ is written in the language $\mathscr{L}$, using the functions symbols 0 and s. We proved above that this formula defines a data type. In order to program the function φ_1, it is thus sufficient to obtain an intuitionistic proof of :
$\forall x_1 \ldots \forall x_{k_1}\{\mathrm{Int}[x_1], \ldots, \mathrm{Int}[x_{k_1}] \to \mathrm{Int}[f_1(x_1, \ldots, x_{k_1})]\}$
by means of rules D1 through D8. In rule D8, we can use any equation satisfied in $\mathbb{N}$ by $\varphi_1, \ldots, \varphi_n$.

Consider, for instance, the language $\mathscr{L}$, consisting of the symbols 0, s, $+$, $\times$ and p (for the predecessor function). In order to program the successor function, we look for an intuitionistic proof of $\forall x\{\mathrm{Int}[x] \to \mathrm{Int}[s(x)]\}$, thus for a term of this type.
Now we have :

$\nu : \mathrm{Int}[x], f : \forall y(Xy \to Xsy), a : X0 \vdash (\nu)fa : Xx$

(by rules T1, T6, T4). Hence :

$v : \mathrm{Int}[x],\ f : \forall y(Xy \to Xsy),\ a : X0 \vdash (f)(v)fa : Xsx$;
therefore, by rule T2 :
$v : \mathrm{Int}[x] \vdash \lambda f \lambda a(f)(v)fa : \forall y(Xy \to Xsy), X0 \to Xsx$
and finally :
$\vdash suc : \mathrm{Int}[x] \to \mathrm{Int}[sx]$, where suc is defined as $\lambda v \lambda f \lambda a(f)(v)fa$.

We shall need below the derived rules stated in the next two propositions :

Proposition 9.25.
$x : A,\ y : B \vdash \lambda f(f)xy : A \wedge B$;
$x : A \wedge B \vdash (x)\mathbf{1} : A$; $x : A \wedge B \vdash (x)\mathbf{0} : B$;
$x : A \vdash \lambda f \lambda g(f)x : A \vee B$; $y : B \vdash \lambda f \lambda g(g)y : A \vee B$;
$a : A[t/x] \vdash \lambda f(f)a : \exists x A$;
$a : A[t/x] \to B \vdash \lambda z(a)z : \forall x A \to B$.

Notice that, using proposition 9.8, we obtain the following consequences :
if $\Gamma \vdash t : A$ and $\Gamma \vdash u : B$, then $\Gamma \vdash \lambda f(f)tu : A \wedge B$;
if $\Gamma \vdash t : A \wedge B$, then $\Gamma \vdash (t)1 : A$ and $\Gamma \vdash (t)0 : B$;
if $\Gamma \vdash t : A$, then $\Gamma \vdash \lambda f \lambda g(f)t : A \vee B$;
if $\Gamma \vdash u : B$, then $\Gamma \vdash \lambda f \lambda g(g)u : A \vee B$; etc.

Recall that $A \wedge B$, $A \vee B$, $\exists x A$ are, respectively, the following formulas :
$\forall X\{(A, B \to X) \to X\}$,
$\forall X\{(A \to X), (B \to X) \to X\}$,
$\forall X\{\forall x(A \to X) \to X\}$.

Proof of the proposition :
$x : A,\ y : B,\ f : A, B \to X \vdash (f)xy : X$ by rules T1 and T3 ;
therefore, $x : A,\ y : B \vdash \lambda f(f)xy : (A, B \to X) \to X$;
then, by T7, we obtain the first property.
$x : A \wedge B \vdash x : (A, B \to A) \to A$ by T1 and T6 ; now $\vdash \lambda x \lambda y\, x : A, B \to A$;
thus $x : A \wedge B \vdash (x)\mathbf{1} : A$.
$x : A,\ f : A \to X,\ g : B \to X \vdash (f)x : X$;
therefore $x : A \vdash \lambda f \lambda g(f)x : (A \to X), (B \to X) \to X$;
hence $x : A \vdash \lambda f \lambda g(f)x : A \vee B$.
$a : A[t/x],\ f : \forall x(A \to X) \vdash f : A[t/x] \to X$ by T1 and T6 ;
thus $a : A[t/x],\ f : \forall x(A \to X) \vdash (f)a : X$;
then $a : A[t/x] \vdash \lambda f(f)a : \forall x(A \to X) \to X$;
finally $a : A[t/x] \vdash \lambda f(f)a : \exists x A$.
$a : A[t/x] \to B,\ z : \forall x A \vdash z : A[t/x]$;
thus $a : A[t/x] \to B,\ z : \forall x A \vdash (a)z : B$;
finally $a : A[t/x] \to B \vdash \lambda z(a)z : \forall x A[x] \to B$.

Q.E.D.

Proposition 9.26 (Proofs by induction on $\mathbb{N}$)**.**
i) $\nu : Int[x], \varphi : \forall y(A[y] \to A[sy]), \alpha : A[0] \vdash (\nu)\varphi\alpha : A[x]$ *;*
ii) $\nu : Int[x], \varphi : \forall y(A[y] \to A[sy]), \alpha : A[0], \psi : \forall z(A[z], B[z] \to B[sz]), \beta : B[0] \vdash t : B[x]$*,*
where t can be taken either as :
$((\nu\lambda c\lambda f((f)(\varphi)(c)\mathbf{1})((\psi)(c)\mathbf{1})(c)\mathbf{0})\lambda g(g)\alpha\beta)\mathbf{0}$ *or as :*
$(\nu\lambda f\lambda a\lambda b((f)(\varphi)a)(\psi)ab)\mathbf{0}\alpha\beta$.

(i) is immediate since, by rules T1 and T6, we have :
$\nu : \mathrm{Int}[x] \vdash \nu : \forall y(A[y] \to A[sy]), A[0] \to A[x]$.

(ii) First proof : we prove $A[x] \wedge B[x]$ by induction (we mean : using (i)).
By proposition 9.25, we have : $\vdash \lambda g(g)\alpha\beta : A[0] \wedge B[0]$; on the other hand : $c : A[y] \wedge B[y] \vdash (c)\mathbf{1} : A[y], (c)\mathbf{0} : B[y]$;
thus $c : A[y] \wedge B[y] \vdash (\varphi)(c)\mathbf{1} : A[sy], ((\psi)(c)\mathbf{1})(c)\mathbf{0} : B[sy]$; therefore :
$c : A[y] \wedge B[y] \vdash \lambda f((f)(\varphi)(c)\mathbf{1})((\psi)(c)\mathbf{1})(c)\mathbf{0} : A[sy] \wedge B[sy]$; hence :
$\vdash \tau_0 : \forall y(A[y] \wedge B[y] \to A[sy] \wedge B[sy])$,
where $\tau_0 = \lambda c\lambda f((f)(\varphi)(c)\mathbf{1})((\psi)(c)\mathbf{1})(c)\mathbf{0}$.
It follows that : $\nu : \mathrm{Int}[x] \vdash (\nu\tau_0)\lambda g(g)\alpha\beta : A[x] \wedge B[x]$, and, finally :
$\nu : \mathrm{Int}[x] \vdash ((\nu\tau_0)\lambda g(g)\alpha\beta)\mathbf{0} : B[x]$.

Second proof : we prove $F[x] \equiv \forall y(A[y], B[y] \to B[x+y])$ by induction on x, using the following equations : $x+0=x$; $0+y=y$; $x+sy=sx+y$.
These equations are obviously satisfied in $\mathbb{N}$, so they also hold in the standard model, according to our remark page 187.
Clearly, $\vdash \mathbf{0} : F[0]$ (use rule T8 and the equation $0+y=y$).
On the other hand, we have :
$f : F[z], a : A[y], b : B[y] \vdash (\varphi)a : A[sy], (\psi)ab : B[sy]$, and therefore : $f : F[z], a : A[y], b : B[y] \vdash ((f)(\varphi)a)(\psi)ab : B[z+sy]$.
Then, using the equation $z+sy=sz+y$, we obtain :
$f : F[z] \vdash \lambda a\lambda b((f)(\varphi)a)(\psi)ab : A[y], B[y] \to B[sz+y]$.
Hence, $\vdash \tau_1 : F[z] \to F[sz]$, where $\tau_1 = \lambda f\lambda a\lambda b((f)(\varphi)a)(\psi)ab$.
According to (i) it follows that $\nu : \mathrm{Int}[x] \vdash (\nu)\tau_1\mathbf{0} : F[x]$.
Now, by rule T4, we obtain $\nu : \mathrm{Int}[x] \vdash (\nu)\tau_1\mathbf{0} : A[0], B[0] \to B[x+0]$.
Finally, using the equation $x+0=x$, we have :

$$\nu : \mathrm{Int}[x] \vdash (\nu)\tau_1\mathbf{0}\alpha\beta : B[x].$$

Q.E.D.

We obtain an alternative form of the inductive reasoning :

Corollary 9.27.
We have $\nu : Int[x], \psi : \forall y(Int[y], B[y] \to B[sy]), \beta : B[0] \vdash u : B[x]$*, where u is the term* $t[suc/\varphi, \underline{0}/\alpha]$*, and t is defined as in proposition 9.26.*

This is obvious from proposition 9.26, since $\vdash suc : \forall x(\mathrm{Int}[x] \to \mathrm{Int}[sx])$ and $\vdash \underline{0} : \mathrm{Int}[0]$.

Q.E.D.

To program the predecessor function on $\mathbb{N}$, we use the equations :
$p0 = 0$; $psx = x$ (and, if needed, the previous equations involving $+$).
By rules T1 and T8, we have :
$v : \mathrm{Int}[x], f : \forall y(Xy \to Xsy), a : X0 \vdash a : Xp0, \mathbf{1} : \forall y(Xy, Xpy \to Xpsy)$.
Then we apply proposition 9.26(ii), taking $A[x] \equiv Xx$, $B[x] \equiv Xpx$, $\varphi = f$, $\psi = \mathbf{1}$, $\alpha = \beta = a$. Thus we obtain a term u such that :
$v : \mathrm{Int}[x], f : \forall y(Xy \to Xsy), a : X0 \vdash u : Xpx$; therefore :
$v : \mathrm{Int}[x] \vdash \lambda f \lambda a\, u : \mathrm{Int}[px]$.
This provides the following term for the predecessor function :
$\lambda v \lambda f \lambda a(v \lambda g \lambda b \lambda c((g)(f)b)b)\mathbf{0}aa$.

The next proposition expresses the principle : every integer is either the successor of an integer or 0.

Proposition 9.28. *$v : Int[x] \vdash t : \forall X\{\forall y(Int[y] \to Xsy), X0 \to Xx\}$, where $t = (v\lambda h \lambda f \lambda a(f)((h)suc)\underline{0})\underline{0}$.*

Let $H[x]$ be the formula $\forall X\{\forall y(\mathrm{Int}[y] \to Xsy), X0 \to Xx\}$. It is proved by induction on x. Clearly, $\vdash \underline{0} : H[0]$. Moreover :

$$h : H[z] \vdash h : \forall y\{\mathrm{Int}[y] \to \mathrm{Int}[sy]\}, \mathrm{Int}[0] \to \mathrm{Int}[z]$$

(replace Xy with $\mathrm{Int}[y]$ in $H[z]$).
Since $\vdash suc : \forall y\{\mathrm{Int}[y] \to \mathrm{Int}[sy]\}$ and $\vdash \underline{0} : \mathrm{Int}[0]$, we may deduce that :
$h : H[z] \vdash ((h)suc)\underline{0} : \mathrm{Int}[z]$.
Thus $h : H[z], f : \forall y\{\mathrm{Int}[y] \to Xsy\}, a : X0 \vdash (f)((h)suc)\underline{0} : Xsz$.
Hence, $\vdash \lambda h \lambda f \lambda a(f)((h)suc)\underline{0} : \forall z(H[z] \to H[sz])$.
Finally, we get $v : \mathrm{Int}[x] \vdash t : H[x]$.

Q.E.D.

We therefore obtain another λ-term for the predecessor function on $\mathbb{N}$, using the same equations as above. With this aim, we replace Xx by $\mathrm{Int}[px]$ in proposition 9.28, which gives :
$v : \mathrm{Int}[x] \vdash t : \forall y(\mathrm{Int}[y] \to \mathrm{Int}[psy]), \mathrm{Int}[p0] \to \mathrm{Int}[px]$.
Now we have $psy = y$ and $p0 = 0$, thus $\vdash I : \forall y(\mathrm{Int}[y] \to \mathrm{Int}[psy])$ and $\vdash \underline{0} : \mathrm{Int}[p0]$. It follows that we may take $\lambda v(v\lambda h \lambda f \lambda a(f)((h)suc)\underline{0})\underline{0}I\underline{0}$ (where $I = \lambda x\, x$) as a term for the predecessor function.

Examples with lists

We add to $\mathscr{L}$ the constant symbol \$ and the binary function symbol cons. Let $A[x]$ be a data type ; then the type of the *lists of objects of A* is written :

$$LA[x] \equiv \forall X\{\forall y \forall z(A[y], Xz \to X\mathrm{cons}(y,z)), X\$ \to Xx\}.$$

Thus, \$ represents the empty list and $\mathrm{cons}(y,z)$ represents the list obtained by putting the data y in front of the list z.
For every formula F, we obviously have the following typing (inductive reasoning on lists) :
$\sigma : LA[x],\, \varphi : \forall y\forall z(A[y], F[z] \to F[\mathrm{cons}(y,z)]),\, \alpha : F[\$] \vdash (\sigma)\varphi\alpha : F[x]$.

Length of a list.
We use the equations : $l(\$) = 0$; $l(\mathrm{cons}(y,z)) = s(l(z))$.
In the context $\sigma : LA[x], f : \forall y(Xy \to Xsy), a : X0$, we prove $Xl(x)$ by induction on x. By the previous equations, we have :
$\sigma{:}\,LA[x], f{:}\,\forall y(Xy \to Xsy), a{:}\,X0 \vdash a{:}\,Xl(\$), f{:}\,Xl(z) \to Xl(\mathrm{cons}(y,z))$. Hence :
$\sigma{:}\,LA[x], f{:}\,\forall y(Xy \to Xsy), a{:}\,X0 \vdash \lambda x f{:}\,A[y], Xl(z) \to Xl(\mathrm{cons}(y,z))$.
It follows that $\sigma{:}\,LA[x], f{:}\,\forall y(Xy \to Xsy), a{:}\,X0 \vdash ((\sigma)\lambda x f)a{:}\,Xl(x)$
and therefore : $\vdash \lambda\sigma\lambda f\lambda a((\sigma)\lambda x f)a : \forall x(LA[x] \to \mathrm{Int}[l(x)])$, which provides a λ-term for the length of lists.

Reversal (or mirror) of a list.
We add to $\mathscr{L}$ function symbols mir (unary) and c (binary) ; $\mathrm{mir}(x)$ represents the reversal of the list x and $c(y,z)$ the list obtained by putting the data z at the end of the list y.
We will use the equations :

$c(\$,a) = \mathrm{cons}(a,\$)$; $c(\mathrm{cons}(b,x),a) = \mathrm{cons}(b, c(x,a))$;
$\mathrm{mir}(\$) = \$$; $\mathrm{mir}(\mathrm{cons}(a,x)) = c(\mathrm{mir}(x),a)$.

In the context $\sigma : LA[x]$, we prove $LA[\mathrm{mir}(x)]$ by induction on x.
First, we have $\vdash \mathbf{0} : LA[\mathrm{mir}(\$)]$.
Now we need a term of type $\forall y\forall z(A[y], LA[\mathrm{mir}(z)] \to LA[\mathrm{mir}(\mathrm{cons}(y,z))])$, that is to say $\forall y\forall z(A[y], LA[\mathrm{mir}(z)] \to LA[c(\mathrm{mir}(z),y)])$. It suffices to obtain a term of type : $\forall y\forall z(A[y], LA[z] \to LA[c(z,y)])$. Now we have :
$\alpha : A[y_0],\, \tau : LA[z_0],\, f : \forall y\forall z(A[y], Xz \to X\mathrm{cons}(y,z)),\, a : X\$$
$\vdash (f)\alpha a : X\mathrm{cons}(y_0,\$)$
and therefore $\vdash (f)\alpha a : Xc(\$,y_0)$.
On the other hand, the type $\forall y\forall z(A[y], Xc(z,y_0) \to Xc(\mathrm{cons}(y,z),y_0))$ can also be written: $\forall y\forall z(A[y], Xc(z,y_0) \to X\mathrm{cons}(y, c(z,y_0)))$.
To obtain a term of this type, it suffices to obtain one of type :

$\forall y\forall z(A[y], Xz \to X\mathrm{cons}(y,z))$;

therefore, we have :
$\alpha : A[y_0],\, \tau : LA[z_0],\, f : \forall y\forall z(A[y], Xz \to X\mathrm{cons}(y,z)),\, a : X\$$
$\vdash f : \forall y\forall z(A[y], Xc(z,y_0) \to Xc(\mathrm{cons}(y,z),y_0))$.
Finally :
$\alpha : A[y_0],\, \tau : LA[z_0],\, f : \forall y\forall z(A[y], Xz \to X\mathrm{cons}(y,z)),\, a : X\$$
$\vdash (\tau f)(f)\alpha a : Xc(z_0,y_0)$,
and therefore :

$\alpha : A[y_0], \tau : LA[z_0] \vdash \lambda f \lambda a(\tau f)(f)\alpha a : LA[c(z_0, y_0)]$, that is :
$\vdash \lambda\alpha\lambda\tau\lambda f \lambda a(\tau f)(f)\alpha a : \forall y \forall z(A[y], LA[z] \to LA[c(z, y)])$.
So we now have $\sigma : LA[x] \vdash ((\sigma)\lambda\alpha\lambda\tau\lambda f \lambda a(\tau f)(f)\alpha a)\mathbf{0} : LA[\mathrm{mir}(x)]$, which provides the term $\lambda\sigma((\sigma)\lambda\alpha\lambda\tau\lambda f \lambda a(\tau f)(f)\alpha a)\mathbf{0}$ as a reversal operator for lists.

References for chapter 9

[Kri87], [Kri90], [Lei83], [Par88].
(The references are in the bibliography at the end of the book).

Chapter 10

Representable functions in system F

We wish to give a characterization of the class of those recursive functions from $\mathbb{N}$ to $\mathbb{N}$ which are representable by a λ-term of type Int $\to$ Int in system $\mathscr{F}$ (in other words, the class of functions which can be " programmed " in system $\mathscr{F}$).

Our first remark is that this class does not contain all recursive functions ; this can be seen by the following simple diagonal argument :

Let $t_0, t_1, \ldots, t_n, \ldots$ be a recursive enumeration of the λ-terms of type Int $\to$ Int in system $\mathscr{F}$. We define a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ by taking, for every $n \in \mathbb{N}$, $\varphi(n) = 1$ (resp. $\varphi(n) = 0$) if the normal form of $(t_n)\underline{n}$ is $\underline{0}$ (resp. is $\neq \underline{0}$). If the function φ was represented by t_n for some integer n, then $(t_n)\underline{n}$ would be β-equivalent to the Church integer $\underline{\varphi(n)}$. This is false and, therefore, the recursive function φ is not in the class under consideration.

Consider the language $\mathscr{L}$ of combinatory logic, with the constant symbols K, S and the binary function symbol Ap. Recall that, with each λ-term t, we can associate a term $t_{\mathscr{L}}$ of $\mathscr{L}$, such that the interpretation of $t_{\mathscr{L}}$ in the standard model of $\mathscr{L}$ is t (lemma 6.22).

The λ-term $\lambda n \lambda f \lambda x (f)(n) f x$ is denoted by suc ; by abuse of notation, the terms $suc_{\mathscr{L}}$ and $0_{\mathscr{L}}$ (of $\mathscr{L}$) will still be denoted, respectively, by suc and 0. We define two formulas of $\mathscr{L}$:

Int $\equiv \forall X\{(X \to X) \to (X \to X)\}$ (where X is a propositional variable), and

Int$[x] \equiv \forall X\{\forall y(Xy \to X(suc)y), X0 \to Xx\}$.

In chapter 9, we have seen that the formula Int$[x]$ defines a data type in the standard model of $\mathscr{L}$, and therefore also in every standard model of any language $\mathscr{L}'$ which extends $\mathscr{L}$. Clearly, the interpretation of Int$[x]$ in any standard model is the set of Church numerals.

Let $\mathscr{T}$ be a theory (a system of axioms) in a language $\mathscr{L}(\mathscr{T}) \supset \mathscr{L}$, and $\varphi : \mathbb{N} \to \mathbb{N}$ a recursive function ; φ is said to be *provably total* in the theory $\mathscr{T}$ if there exists a term $t(x)$ of $\mathscr{L}(\mathscr{T})$, of which x is the only variable, such that :

- $\mathscr{T} \vdash \forall x\{\text{Int}[x] \to \text{Int}[t(x)]\}$ (in classical second order logic) ;

• There exists a standard model $\mathscr{M}$ of $\mathscr{T}$, in which the term $t(x)$ represents the function φ (in other words, for every Church numeral n, the interpretation of $t(n)$ in $\mathscr{M}$ is the Church numeral $\varphi(n)$).

Proposition 10.1. *We have the following typings :*
i) $v:(x \Vdash Int) \vdash v: Int[((x)suc)0]$ *;*
ii) $v: Int[x] \vdash_{C_0} ((v)suc)0:(x \Vdash Int)$.

Recall that the system of axioms C_0 consists of both equations $(K)xy = x$ and $(S)xyz = ((x)z)(y)z$.
i) The formula $x \Vdash \text{Int}$ can be written
$$\forall X \forall f \forall a\{\forall y(Xy \to X(f)y), Xa \to X(x)fa\}.$$
Therefore, by the typing rules T1 and T4 (replace f by suc and a by 0), we immediately obtain :
$v:(x \Vdash \text{Int}) \vdash v: \forall X\{\forall y(Xy \to X(suc)y), X0 \to X((x)suc)0\}$, that is :
$v:(x \Vdash \text{Int}) \vdash v: \text{Int}[((x)suc)0]$.
ii) We prove $x \Vdash \text{Int}$ by induction on x ; $0 \Vdash \text{Int}$ is the formula :
$\forall X \forall f \forall a\{\forall y(Xy \to X(f)y), Xa \to X(0)fa\}$.
Now $C_0 \vdash (0)fa = a$, and we have, trivially :
$\vdash 0: \forall X \forall f \forall a\{\forall y(Xy \to X(f)y), Xa \to Xa\}$.
Hence $\vdash_{C_0} 0:(0 \Vdash \text{Int})$ (rule T8).
We now look for a term of type $x \Vdash \text{Int} \to (suc)x \Vdash \text{Int}$. We have :
$v:(x \Vdash \text{Int}), \varphi: \forall y(Xy \to X(f)y), \alpha: Xa \vdash (v)\varphi\alpha: X(x)fa$, therefore :
$v:(x \Vdash \text{Int}), \varphi: \forall y(Xy \to X(f)y), \alpha: Xa \vdash (\varphi)(v)\varphi\alpha: X(f)(x)fa$. Now :
$C_0 \vdash (suc)xfa = (f)(x)fa$. By rule T8, we obtain :
$v:(x \Vdash \text{Int}), \varphi: \forall y(Xy \to X(f)y), \alpha: Xa \vdash_{C_0} (\varphi)(v)\varphi\alpha: X(suc)xfa$
and therefore, by T2 :
$v:(x \Vdash \text{Int}) \vdash_{C_0} \lambda\varphi\lambda\alpha(\varphi)(v)\varphi\alpha:((suc)x \Vdash \text{Int})$. Hence :
$\vdash_{C_0} suc: \forall x\{x \Vdash \text{Int} \to (suc)x \Vdash \text{Int}\}$.
We have proved $0 \Vdash \text{Int}$ and $\forall x\{x \Vdash \text{Int} \to (suc)x \Vdash \text{Int}\}$; it follows that :
$v: \text{Int}[x] \vdash_{C_0} ((v)suc)0:(x \Vdash \text{Int})$.
Q.E.D.

Proposition 10.2. *Let t be a λ-term such that $\vdash t: Int \to Int$ is a typing in system $\mathscr{F}$. Then $\vdash_{C_0} \lambda n(t)(n)suc\underline{0}: \forall x\{Int[x] \to Int[(t_{\mathscr{L}})x\,suc\,0]\}$ is a typing in system FA_2, with the equational axioms C_0.*

By theorem 9.19, we have $\vdash_{C_0} t: t_{\mathscr{L}} \Vdash \text{Int} \to \text{Int}$, that is :
(*) $\quad\vdash_{C_0} t: \forall x\{x \Vdash \text{Int} \to (t_{\mathscr{L}})x \Vdash \text{Int}\}$.
By proposition 10.1(ii), $n: \text{Int}[x] \vdash_{C_0} (n)suc\,0: x \Vdash \text{Int}$, and therefore, by (*) and rule T3, we have $n: \text{Int}[x] \vdash_{C_0} (t)(n)suc\,0:(t_{\mathscr{L}})x \Vdash \text{Int}$. Then it follows from proposition 10.1(i) that :

$n : \mathrm{Int}[x] \vdash_{C_0} (t)(n)suc\,0 : \mathrm{Int}[(t_{\mathscr{L}})x\ suc\ 0]$, hence :
$\vdash_{C_0} \lambda n(t)(n)suc\,0 : \mathrm{Int}[x] \to \mathrm{Int}[(t_{\mathscr{L}})x\ suc\ 0]$.
Q.E.D.

Theorem 10.3. *Let t be a λ-term such that $\vdash t : Int \to Int$ is a typing in system $\mathscr{F}$. Then t represents a function from $\mathbb{N}$ to $\mathbb{N}$ which is provably total in the theory $CA + C_0$.*

Using proposition 10.2 and the Curry-Howard correspondence (as stated chapter 9, page 173), we get $CA + C_0 \vdash \forall x\{\mathrm{Int}[x] \to \mathrm{Int}[(t_{\mathscr{L}})x\,suc\,0]\}$. Thus the term $(t_{\mathscr{L}})x\,suc\,0$ represents a function $\psi : \mathbb{N} \to \mathbb{N}$, which is provably total in the theory $CA + C_0$.
The term t represents a function $\varphi : \mathbb{N} \to \mathbb{N}$: indeed, if n is a Church numeral, then, in system $\mathscr{F}$, we have $\vdash n : \mathrm{Int}$, and therefore $\vdash (t)n : \mathrm{Int}$. It follows (by the adequacy lemma 8.13 and proposition 8.14) that $(t)n$ is β-equivalent to a Church numeral.
Then it is enough to prove that $\varphi = \psi$. The interpretation of $t_{\mathscr{L}}$ in the standard model is $t_{\mathscr{L}\Lambda} \simeq_\beta t$ (lemma 6.22). Consequently, for every Church numeral n, the interpretation of $(t_{\mathscr{L}})n\ suc\ 0$ in the standard model is $(t)n\ suc\ 0$. Now $(t)n\ suc\ 0 \simeq_\beta (t)n$, since $(t)n$ is a Church numeral. Hence $\psi(n) = \varphi(n)$.
Q.E.D.

The next theorem is a strengthened converse of theorem 10.3.

Theorem 10.4. *Let $\mathscr{E}$ be a system of equations in a language $\mathscr{L}(\mathscr{E}) \supset \mathscr{L}$, and $\varphi : \mathbb{N} \to \mathbb{N}$ a function which is provably total in $CA + \mathscr{E}$. Then there exists a λ-term t, of type $Int \to Int$ in system $\mathscr{F}$, which represents the function φ.*

By hypothesis, there exist a term $u(x)$ of $\mathscr{L}(\mathscr{E})$, the only variable of which is x, and a standard model $\mathscr{M}$ of $\mathscr{E}$, such that :
i) $CA + \mathscr{E} \vdash \forall x\{\mathrm{Int}[x] \to \mathrm{Int}[u(x)]\}$ and
ii) $\mathscr{M} \models u(n) = \varphi(n)$ for every Church numeral n.
According to (i), the expression $\vdash_{\mathscr{E}} \mathrm{Int}[x] \to \mathrm{Int}[u(x)]$ can be obtained by means of the deduction rules D0 through D8 of chapter 9, page 172 (completeness theorem for the classical second order predicate calculus). In view of theorem 10.5 below, *there also exists an intuitionistic proof for this expression*, that is a proof only involving rules D1 through D8. Now, by the Curry-Howard correspondence (chapter 9, page 173), such a proof provides a λ-term t such that $\vdash_{\mathscr{E}} t : \mathrm{Int}[x] \to \mathrm{Int}[u(x)]$ (a typed term in system FA_2 with the equational axioms $\mathscr{E}$).
The term t represents the function φ ; indeed, by theorem 9.19, we have :
$CA + \mathscr{E} + C_0 \vdash (t_{\mathscr{L}} \Vdash \mathrm{Int}[x] \to \mathrm{Int}[u(x)])$, that is :
$CA + \mathscr{E} + C_0 \vdash \forall x \forall y\{y \Vdash \mathrm{Int}[x] \to (t_{\mathscr{L}})y \Vdash \mathrm{Int}[u(x)]\}$.

Thus the standard model $\mathscr{M}$ satisfies the formula :
$\forall x \forall y\{y \Vdash \mathrm{Int}[x] \to (t_{\mathscr{L}})y \Vdash \mathrm{Int}[u(x)]\}$. Now the formula $\mathrm{Int}[x]$ defines a data type, in the standard model $\mathscr{M}$. Hence :
$\mathscr{M} \models \forall x \forall y\{y \Vdash \mathrm{Int}[x] \leftrightarrow \mathrm{Int}[x] \wedge x = y\}$, and therefore :
$\mathscr{M} \models \forall x\{\mathrm{Int}[x] \to (t_{\mathscr{L}})x = u(x)\}$. In other words, the term $(t_{\mathscr{L}})x$ represents the same function as $u(x)$, that is φ. Since the interpretation of $t_{\mathscr{L}}$ in the standard model $\mathscr{M}$ is t (lemma 6.22), we see that t represents φ.
Finally, the term t is of type $\mathrm{Int} \to \mathrm{Int}$ in system $\mathscr{F}$. Indeed, we have the typing $\vdash_{\mathscr{E}} t : \mathrm{Int}[x] \to \mathrm{Int}[u(x)]$ in system FA_2. Thus we also have :
$\vdash t : \mathrm{Int}[x]^- \to \mathrm{Int}[u(x)]^-$ as a typing in system $\mathscr{F}$ (see the proof of the normalization theorem 9.6 for FA_2).
Now this typing is simply $\vdash t : \mathrm{Int} \to \mathrm{Int}$.

Q.E.D.

Gödel's ¬-translation

Theorem 10.5. *Let $\mathscr{E}$ be a system of equations in a language $\mathscr{L}(\mathscr{E}) \supset \mathscr{L}$, and σ, τ two terms of $\mathscr{L}(\mathscr{E})$. If the expression $\vdash_{\mathscr{E}} \mathrm{Int}[\sigma] \to \mathrm{Int}[\tau]$ can be proved in classical second order logic (that is with rules D0 through D8, page 172), then it can also be proved in intuitionistic second order logic (in other words, without using rule D0).*

We add to the language $\mathscr{L}(\mathscr{E})$ a propositional constant O (that is a 0-ary relation symbol); whenever A is a formula, we will denote the formula $A \to O$ by $\neg_0 A$.

For every formula A, we define a formula A^*, by induction, by the following conditions :

if A is atomic, then A^* is $\neg_0 A$;
$(A \to B)^*$ is $A^* \to B^*$;
$(\forall \xi\, A)^*$ is $\forall \xi\, A^*$ whenever ξ is an individual variable or a relation variable.

So the formula A^* is obtained by putting $\neg_0$ before every atomic subformula of A. A^* will be called the *Gödel translation* of A.

Remark. This is not exactly the classical definition of the Gödel translation of A, according to which one should put $\neg_0\neg_0$ before every atomic subformula of A.

Lemma 10.6. *i)* $\neg_0\neg_0\neg_0 A \vdash^i \neg_0 A$ *;*
ii) $\neg_0\neg_0(A \to B) \vdash^i \neg_0\neg_0 A \to \neg_0\neg_0 B$ *;*
iii) $\neg_0\neg_0\forall\xi\, A \vdash^i \forall\xi\neg_0\neg_0 A$ *whenever ξ is a first or second order variable.*

The notation $A_1, \ldots, A_k \vdash^i A$ means that A is an intuitionistic consequence of $A_1, \ldots, A_k$, that is to say that the expression $A_1, \ldots, A_k \vdash A$ can be obtained by means of the rules D1 through D8 of chapter 9 (page 172).

i) Remark that, if $X \vdash^i Y$, then $\neg_0 Y \vdash^i \neg_0 X$; indeed, if Y is deduced from X, then O is deduced from X and $Y \to O$.
Now, clearly, $A \vdash^i \neg_0\neg_0 A$. Therefore, by the previous remark, we have :
$\neg_0\neg_0\neg_0 A \vdash^i \neg_0 A$.
ii) With the premises $((A \to B) \to O) \to O$, $(A \to O) \to O$, $B \to O$, we have to deduce O. From $B \to O$, we deduce $(A \to B) \to (A \to O)$; with $(A \to O) \to O$, we obtain $(A \to B) \to O$.
From this and $((A \to B) \to O) \to O$, we deduce O.
iii) We wish to show $((\forall\xi\, A) \to O) \to O \vdash^i (A \to O) \to O$; so with the premises $((\forall\xi\, A) \to O) \to O$ and $A \to O$, we have to deduce O. Now we know $\forall\xi\, A \vdash^i A$; with $A \to O$, we deduce $\forall\xi\, A \to O$; from this and $((\forall\xi\, A) \to O) \to O$, we obtain O.

Q.E.D.

Lemma 10.7. $\neg_0\neg_0 A^* \vdash^i A^*$ *for every formula* A.

The proof is by induction on the length of the formula A.
If A is atomic, what we have to prove is $\neg_0\neg_0\neg_0 A \vdash^i \neg_0 A$: this is precisely lemma 10.6(i).
If A is $B \to C$, $\neg_0\neg_0 A^*$ is $\neg_0\neg_0(B^* \to C^*)$; by lemma 10.6(ii), we have :
$\neg_0\neg_0 A^* \vdash^i \neg_0\neg_0 B^* \to \neg_0\neg_0 C^*$.
Now $B^* \vdash^i \neg_0\neg_0 B^*$ (obvious), and $\neg_0\neg_0 C^* \vdash^i C^*$ (induction hypothesis).
Hence $\neg_0\neg_0 A^* \vdash^i B^* \to C^*$, that is $\neg_0\neg_0 A^* \vdash^i A^*$.
If A is $\forall\xi\, B$, where ξ is a first order or second order variable, then $\neg_0\neg_0 A^*$ is $\neg_0\neg_0\forall\xi\, B^*$. By lemma 10.6(iii), we have $\neg_0\neg_0 A^* \vdash^i \forall\xi\neg_0\neg_0 B^*$ and therefore $\neg_0\neg_0 A^* \vdash^i \neg_0\neg_0 B^*$. Now, by the induction hypothesis, $\neg_0\neg_0 B^* \vdash^i B^*$. Thus $\neg_0\neg_0 A^* \vdash^i B^*$, and since ξ does not occur free in $\neg_0\neg_0 A^*$, we have :
$\neg_0\neg_0 A^* \vdash^i \forall\xi\, B^*$, that is $\neg_0\neg_0 A^* \vdash^i A^*$.

Q.E.D.

Lemma 10.8. $(\neg\neg A)^* \vdash^i A^*$ *for every formula* A.

Since $\bot$ is the formula $\forall X\, X$, $\bot^*$ is $\forall X \neg_0 X$, that is $\forall X(X \to O)$. Therefore $O \vdash^i \bot^*$ (obvious) and $\bot^* \vdash^i O$ (replace X by $O \to O$ in the previous formula). Thus $\bot^*$ is equivalent to O in intuitionistic logic.
$(\neg\neg A)^*$ is the formula $((A \to \bot) \to \bot)^*$, that is $(A^* \to \bot^*) \to \bot^*$. Thus $(\neg\neg A)^* \vdash^i (A^* \to O) \to O$, or equivalently $(\neg\neg A)^* \vdash^i \neg_0\neg_0 A^*$. Then the conclusion follows from lemma 10.7.

Q.E.D.

Lemma 10.9. *Let A, B be two formulas, and X a k-ary relation variable. Then :*
$\{A[B/Xx_1 \dots x_k]\}^* \vdash^i A^*[\neg_0 B^*/Xx_1 \dots x_k]$ *and*
$A^*[\neg_0 B^*/Xx_1 \dots x_k] \vdash^i \{A[B/Xx_1 \dots x_k]\}^*$.

The proof is by induction on the length of A. If A is atomic and its first symbol is X, say $A \equiv X t_1 \dots t_k$, then :

$A^*[\neg_0 B^*/X x_1 \dots x_k] \equiv \neg_0\neg_0 B^*[t_1/x_1, \dots, t_k/x_k]$ and

$\{A[B/X x_1 \dots x_k]\}^* \equiv B^*[t_1/x_1, \dots, t_k/x_k]$.

Then the result follows from lemma 10.7. The other cases of the inductive proof are trivial.

Q.E.D.

Theorem 10.10. *Let $\mathscr{E}$ be a system of equations in a language $\mathscr{L}(\mathscr{E}) \supset \mathscr{L}$, let $\mathscr{A}$ be a finite set of formulas of $\mathscr{L}(\mathscr{E})$, and $\mathscr{A}^* = \{F^*; F \in \mathscr{A}\}$. If one can obtain $\mathscr{A} \vdash_{\mathscr{E}} A$ by rules D0 through D8, page 172, then one can obtain $\mathscr{A}^* \vdash^i_{\mathscr{E}} A^*$ by rules D1 through D8 only.*

The theorem means that if $\mathscr{A} \vdash_{\mathscr{E}} A$ can be proved in *classical* second order logic, then the Gödel translation $\mathscr{A}^* \vdash^i_{\mathscr{E}} A^*$ can be proved in *intuitionistic* second order logic.

We shall prove it by induction on the length of the derivation of $\mathscr{A} \vdash_{\mathscr{E}} A$ with rules D0, ..., D8. Consider the last rule used.

If it is D0, then $\mathscr{A} \vdash_{\mathscr{E}} A$ can be written : $\mathscr{B}, \neg\neg A \vdash_{\mathscr{E}} A$. It is enough to show that $(\neg\neg A)^* \vdash^i A^*$: this was done in lemma 10.8.

If it is D1, D2, D3, D5 or D7, the result is obvious from the definition of A^*.

If it is D4 or D8, we obtain the result by proving that $\{A[t/x]\}^* \equiv A^*[t/x]$ for every term t and every formula A of $\mathscr{L}$ (this is immediate, by induction on A).

If it is D6, then $A \equiv B[C/X x_1 \dots x_k]$; by the induction hypothesis, the expression $\mathscr{A}^* \vdash^i \forall X\, B^*$ was previously deduced ; so we also obtain :

$\mathscr{A}^* \vdash^i B^*[\neg_0 C^*/X x_1 \dots x_k]$.

By lemma 10.9, we finally deduce $\mathscr{A}^* \vdash^i \{B[C/X x_1 \dots x_k]\}^*$.

Q.E.D.

Proposition 10.11. *Let U, V be two formulas of $\mathscr{L}(\mathscr{E})$ such that $U \vdash^i_{\mathscr{E}} U^*$ and $V^* \vdash^i_{\mathscr{E}} \neg_0\neg_0 V$. If one can obtain $U \vdash_{\mathscr{E}} V$ by rules D0 through D8, then one can obtain $U \vdash^i_{\mathscr{E}} V$ by rules D1 through D8 only.*

By theorem 10.10, $U^* \vdash^i_{\mathscr{E}} V^*$ can be obtained by rules D1 through D8. The hypotheses about the formulas U, V show that one can also deduce $U \vdash^i_{\mathscr{E}} \neg_0\neg_0 V$ by means of these rules, that is : $U \vdash^i_{\mathscr{E}} (V \to O) \to O$. Now O is a propositional constant which does not occur in U. Thus it suffices to replace O by V to obtain the desired result : $U \vdash^i_{\mathscr{E}} V$.

Q.E.D.

Any type $U[x]$ such that $U[x] \vdash^i U^*[x]$ will be called an *input type*, while a type $V[x]$ such that $V^*[x] \vdash^i \neg_0\neg_0 V[x]$ will be called an *output type*.

Proposition 10.12. *The type* $\mathrm{Int}[x]$ *is an input-output one, that is to say that we have :* $\mathrm{Int}[x] \vdash^i \mathrm{Int}^*[x]$*, and* $\mathrm{Int}^*[x] \vdash^i \neg_0\neg_0\mathrm{Int}[x]$*.*

$\mathrm{Int}[x]$ is the formula : $\forall X\{\forall y(Xy \to X(suc)y), X0 \to Xx\}$. By replacing X with $\neg_0 X$, we immediately obtain $\mathrm{Int}^*[x]$, which is :
$\forall X\{\forall y(\neg_0 Xy \to \neg_0 X(suc)y), \neg_0 X0 \to \neg_0 Xx\}$.
Now in the formula $\mathrm{Int}^*[x]$, replace Xx with $\neg_0\mathrm{Int}[x]$; the result is :
$\forall y(\neg_0\neg_0\mathrm{Int}[y] \to \neg_0\neg_0\mathrm{Int}[(suc)y]), \neg_0\neg_0\mathrm{Int}[0] \to \neg_0\neg_0\mathrm{Int}[x]$.
Now it can be seen easily that $\vdash^i \mathrm{Int}[y] \to \mathrm{Int}[(suc)y]$, so that :
$\vdash^i \neg_0\neg_0\mathrm{Int}[y] \to \neg_0\neg_0\mathrm{Int}[(suc)y]$. We also have $\vdash^i \mathrm{Int}[0]$, and therefore :
$\vdash^i \neg_0\neg_0\mathrm{Int}[0]$. Finally, $\mathrm{Int}^*[x] \vdash^i \neg_0\neg_0\mathrm{Int}[x]$.

Q.E.D.

Now we are able to prove theorem 10.5 : suppose that $\mathrm{Int}[\sigma] \vdash_{\mathscr{E}} \mathrm{Int}[\tau]$ have been obtained by means of rules D0, D1, ..., D8. By proposition 10.12, we have $\mathrm{Int}[\sigma] \vdash^i \mathrm{Int}^*[\sigma]$ and $\mathrm{Int}^*[\tau] \vdash^i \neg_0\neg_0\mathrm{Int}[\tau]$. Therefore, by proposition 10.11, we can obtain $\mathrm{Int}[\sigma] \vdash^i_{\mathscr{E}} \mathrm{Int}[\tau]$ by rules D1, ..., D8 only.

Q.E.D.

Theorems 10.3 and 10.4 provide a characterization of the class of those recursive functions from $\mathbb{N}$ to $\mathbb{N}$ which are represented by a λ-term of type $\mathrm{Int} \to \mathrm{Int}$ in system $\mathscr{F}$ (and therefore also of the class of those recursive functions which are represented by a typed λ-term in FA_2, of type $\mathrm{Int}[x] \to \mathrm{Int}[t(x)]$, with an arbitrary equational system $\mathscr{E}$, in a language $\mathscr{L}(\mathscr{E}) \supset \mathscr{L}$, $t(x)$ being a term of $\mathscr{L}(\mathscr{E})$). This is the class of functions which are provably total in the theory $CA + C_0$; it is also the class of functions which are provably total in the theory $CA + \mathscr{E}$, where $\mathscr{E}$ is any equational system containing C_0.

Undecidability of strong normalization

As an application of the above results (namely theorems 8.9 and 10.4), we will now show :

Theorem 10.13. *The set of strongly normalizable λ-terms is not recursive.*

The argument is a modification of [Urz03]. We first prove :

Theorem 10.14.
Let $f : \mathbb{N}^2 \to \{0,1\}$ be representable by a λ-term of type Int, Int → Bool in system $\mathscr{F}$. Then, there exists a λ-term Φ, with the only free variable x, such that, for all $m \in \mathbb{N}$:

i) $\Phi[\hat{m}/x]$ *is solvable* $\Rightarrow$ $(\exists n \in \mathbb{N})\, f(m,n) = 1$.

ii) $(\exists n \in \mathbb{N})\, f(m,n) = 1$ $\Rightarrow$ $\Phi[\hat{m}/x]$ *is strongly normalizable.*

Remark. Recall that :
$\mathrm{Int} \equiv \forall X((X \to X), X \to X)$ and $\mathrm{Bool} \equiv \forall X(X, X \to X)$;
if $m \in \mathbb{N}$, then $\hat{m} = (suc)^m \underline{0}$; $suc = \lambda n \lambda f \lambda x (f)(n) f x$ is a λ-term for the successor ;
$\underline{0} = \mathbf{0} = \lambda x \lambda y\, y$, $\mathbf{1} = \lambda x \lambda y\, x$.

Let ϕ be a λ-term which represents f, such that:

$$\vdash_{\mathscr{F}} \phi : \mathrm{Int}, \mathrm{Int} \to \mathrm{Bool}$$

Consider the following λ-term, with a free variable x :

$$W = \lambda y (\phi x y \mathbf{0}) \lambda w (w) y^+ w, \text{ with } y^+ = (suc) y.$$

We define $\Phi = (W)\underline{0}W$ and we show that Φ has the desired property.

For each integer m, we put : $W^m = W[\hat{m}/x] = \lambda y (\phi \hat{m} y \mathbf{0}) \lambda w (w) y^+ w$.

Proof of (i)

Let m be a fixed integer such that $f(m, n) = 0$ for all $n \in \mathbb{N}$. We have :

$$W^m \hat{n} W^m \succ_w ((\phi \hat{m} \hat{n} \mathbf{0}) \lambda w (w) \hat{n}^+ w) W^m.$$

Recall that $\succ_w$ denotes the weak head reduction (see page 30).

Since ϕ represents f, we have $\phi \hat{m} \hat{n} \simeq_\beta \mathbf{0}$ for all $n \in \mathbb{N}$. Therefore, by lemma 2.12, we have :

$$((\phi \hat{m} \hat{n} \mathbf{0}) \lambda w (w) \hat{n}^+ w) W^m \succ_w (\lambda w (w) \hat{n}^+ w) W^m \succ_w W^m \hat{n}^+ W^m.$$

We have shown that $W^m \hat{n} W^m \succ_w W^m \hat{n}^+ W^m$ for all n. But $\hat{n}^+ = (suc)\hat{n} = \hat{p}$ with $p = n + 1$. It follows that :
$\Phi[\hat{m}/x] = W^m \hat{0} W^m \succ_w W^m \hat{1} W^m \succ_w \cdots \succ_w W^m \hat{n} W^m \succ_w \cdots$
This infinite weak head reduction shows that $\Phi[\hat{m}/x]$ is not solvable (theorem 4.9).

Proof of (ii)

Let $A = \mathrm{Int} \to \forall X(X \to \mathrm{Id})$ where $\mathrm{Id} = \forall X(X \to X)$. We first show that :

$$\vdash_{\mathscr{F}} W^m : \mathrm{Int}, A \to \mathrm{Id} \text{ for every } m \in \mathbb{N}.$$

Indeed, we have :
$y : \mathrm{Int} \vdash_{\mathscr{F}} y^+ : \mathrm{Int}$ because $\vdash_{\mathscr{F}} suc : \mathrm{Int} \to \mathrm{Int}$.
$y : \mathrm{Int}, w : A \vdash_{\mathscr{F}} w y^+ : \forall X(X \to \mathrm{Id})$ and therefore :
$y : \mathrm{Int}, w : A \vdash_{\mathscr{F}} w y^+ w : \mathrm{Id}$. It follows that :
$y : \mathrm{Int} \vdash_{\mathscr{F}} \lambda w\, w y^+ w : A \to \mathrm{Id}$. Now, since $\mathbf{0} = \lambda x \lambda y\, y$, we have trivially :
$y : \mathrm{Int} \vdash_{\mathscr{F}} \mathbf{0} : A \to \mathrm{Id}$.
But, by hypothesis, $x : \mathrm{Int}, y : \mathrm{Int} \vdash_{\mathscr{F}} \phi x y : \mathrm{Bool}$, and therefore :
$y : \mathrm{Int} \vdash_{\mathscr{F}} (\phi \hat{m} y \mathbf{0}) \lambda w\, w y^+ w : A \to \mathrm{Id}$
(note that $\vdash_{\mathscr{F}} \hat{m} : \mathrm{Int}$, because $\vdash_{\mathscr{F}} \underline{0} : \mathrm{Int}$ and $\vdash_{\mathscr{F}} suc : \mathrm{Int} \to \mathrm{Int}$).
Thus, we get $\vdash_{\mathscr{F}} \lambda y (\phi \hat{m} y \mathbf{0}) \lambda w\, w y^+ w : \mathrm{Int}, A \to \mathrm{Id}$ which is the result.

If $p \in \mathbb{N}$, then we have $\vdash_{\mathscr{F}} \hat{p} : \mathrm{Int}$. It follows that :

$$\vdash_{\mathscr{F}} W^m \hat{p} : A \to \mathrm{Id} \text{ for every } m, p \in \mathbb{N}.$$

In particular, W^m and $W^m \hat{p}$ are strongly normalizable (theorem 8.9).

Lemma 10.15. *Let $t, t^*, t_1, \ldots, t_k \in \Lambda$ such that $t \succ_w t^*$ (t^* is obtained from t by weak head reduction). If t and $t^* t_1 \ldots t_k$ are strongly normalizable, then $t t_1 \ldots t_k$ is strongly normalizable.*

Proof by induction on the length of the weak head reduction from t to t^*. If this length is 0, the result is obvious, since $t = t^*$. Otherwise, we have :
$t = (\lambda x\, u) v u_1 \ldots u_l$ and we put $t' = u[v/x] u_1 \ldots u_l$. By the induction hypothesis, we see that $t' t_1 \ldots t_k = u[v/x] u_1 \ldots u_l t_1 \ldots t_k$ is strongly normalizable. But v is also strongly normalizable, since t is. Therefore, by lemma 4.27 :
$(\lambda x\, u) v u_1 \ldots u_l t_1 \ldots t_k = t t_1 \ldots t_k$ is strongly normalizable.

Q.E.D.

We now consider a fixed integer m such that $f(m, p) = 1$ for some p. Let n be the first such p. We have to show that $W^m \hat{0} W^m$ is strongly normalizable. In fact we show, by a backward recursion from n to 0, that $W^m \hat{p} W^m$ is strongly normalizable, for $0 \le p \le n$. With this aim in view, we apply lemma 10.15, with $t = W^m \hat{p}$, $k = 1$, $t_1 = W^m$. We have already proved that t and t_1 are strongly normalizable. We have :
$t = (\lambda y (\phi \hat{m} y \mathbf{0}) \lambda w(w) y^+ w) \hat{p} \succ_w (\phi \hat{m} \hat{p} \mathbf{0}) \lambda w(w) \hat{q} w$ with $q = p + 1$,
since $(suc)\hat{p} = \hat{q}$.

Consider first the case $p = n$; by hypothesis, we have $\phi \hat{m} \hat{n} \simeq_\beta \mathbf{1}$. Therefore, by lemma 2.12, we have $(\phi \hat{m} \hat{n} \mathbf{0}) \lambda w(w) \hat{q} w \succ_w \mathbf{0}$.
It follows that $t = W^m \hat{n} \succ_w \mathbf{0}$ and we can take $t^* = \mathbf{0}$.
We have to show that $t^* t_1$, i.e. $\mathbf{0} W^m$, is strongly normalizable, which is trivial, since W^m is.

Consider now the case $p < n$; by hypothesis, we have $\phi \hat{m} \hat{p} \simeq_\beta \mathbf{0}$. Therefore, by lemma 2.12, we have $(\phi \hat{m} \hat{p} \mathbf{0}) \lambda w(w) \hat{q} w \succ_w \lambda w(w) \hat{q} w$.
It follows that $t = W^m \hat{p} \succ_w \lambda w(w) \hat{q} w$ and we can take $t^* = \lambda w(w) \hat{q} w$.
We have to show that $t^* t_1$, i.e. $(\lambda w(w) \hat{q} w) W^m$, is strongly normalizable.
By lemma 4.27, it suffices to show that W^m and $W^m \hat{q} W^m$ are strongly normalizable. This is already known for W^m, and for $W^m \hat{q} W^m$, this follows from the induction hypothesis, since $q = p + 1$ (we are doing a backward induction).

Q.E.D.

We shall now assume the following results from recursivity theory :

(1) For every recursively enumerable set $E \subset \mathbb{N}$, there exists a primitive recursive function $f : \mathbb{N}^2 \to \{0, 1\}$ such that :

$$E = \{m \in \mathbb{N}; (\exists n \in \mathbb{N})\, f(m, n) = 1\}.$$

In other words, every recursively enumerable set of integers is the projection of a subset of $\mathbb{N}^2$, the characteristic function of which is primitive recursive.

(2) Every primitive recursive function is provably total in the theory $CA + \mathscr{E}$ for some set $\mathscr{E}$ of equations.

Remark. Given a primitive recursive function, the idea is simply to write down the equations defining it and to prove with them, in classical second order logic, that this function sends integers into integers. The details will be written in a future version of this book.

We can now prove theorem 10.13. More precisely, we show :

Theorem 10.16. *The set of strongly normalizable terms and the set of unsolvable terms are recursively inseparable. In other words, a recursive set which contains every strongly normalizable term must contain an unsolvable term.*

Let $\mathscr{R}$ be a recursive set which contains every strongly normalizable term and no unsolvable term. We choose a recursively enumerable set $E \subset \mathbb{N}$ which is not recursive. Let f be a primitive recursive function, obtained by (1). By means of (2) and theorem 10.4, we see that f is representable, in system $\mathscr{F}$, by a λ-term of type $\mathrm{Int}, \mathrm{Int} \to \mathrm{Bool}$. By theorem 10.14, we get a λ-term Φ such that, for all $m \in \mathbb{N}$:

$\Phi[\hat{m}/x]$ is solvable $\Rightarrow$ $m \in E$;

$m \in E \Rightarrow \Phi[\hat{m}/x]$ is strongly normalizable.

By hypothesis on $\mathscr{R}$, this gives : $\Phi[\hat{m}/x] \in \mathscr{R} \Leftrightarrow m \in E$.

This is a contradiction, because $\mathscr{R}$ is recursive and E is not.

Q.E.D.

References for chapter 10

[Fri77], [Gir71], [Gir72], [Urz03].

(The references are in the bibliography at the end of the book).

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