On the structure of classical realizability models of ZF

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Introduction

In [6, 7, 9], we have introduced the technique of *classical realizability*, which permits to extend the Curry-Howard correspondence between proofs and programs [5], to Zermelo-Fraenkel set theory. The models of ZF we obtain in this way are called *realizability models* ; this technique is an extension of the method of forcing, in which the ordered sets (sets of *conditions*) are replaced with more complex first order structures called *realizability algebras*. These structures are refinements of the well known *combinatory algebras* [3], with the `call/cc` instruction of [4].

We show here that every realizability model $\mathcal{N}$ of ZF contains a transitive submodel, which has the same ordinals as $\mathcal{N}$, and which is an elementary extension of the ground model. It follows that the constructible universe of a realizability model is an elementary extension of the constructible universe of the ground model (a trivial fact in the particular case of forcing, since these classes are *identical*).

We obtain this result by showing the existence of an ultrafilter on the *characteristic Boolean algebra* $\gimel 2$ of the realizability model, which is defined in [7, 9].

From this result, it follows that the *Shoenfield absoluteness theorem* applies to realizability models and therefore that :

Every closed Σ^1_3 formula which is true in the ground model is realized by a closed λ_c-term.

Another application is given in [8] : the *bar-recursion operator* was defined and studied in [1, 2, 11] where it is shown that it realizes the *axiom of dependent choice*.

In [8] it is shown, by means of the results of the present paper, that every closed formula of analysis (i.e. Σ^1_n or Π^1_n) which is true in the ground model, is realized by a closed λ_c-term containing this operator ; and that the same is true for the axiom : $\mathbb{R}$ *is well-ordered.*

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Background and notations

We use here the basic notions and notations of the theory of *classical realizability*, which was developed in [6, 7, 9].[1]
We consider a model $\mathcal{M}$ of ZF + V = L, which we call the *ground model*[2] and, in $\mathcal{M}$, a *realizability algebra* $\mathcal{A} = (\Lambda, \Pi, \Lambda \star \Pi, \text{QP}, \perp\!\!\!\perp)$.
Λ is the set of *terms*, Π is the set of *stacks*, $\Lambda \star \Pi$ is the set of *processes*, $\text{QP} \subset \Lambda$ is the set of *proof-like terms*, and $\perp\!\!\!\perp$ is a distinguished subset of $\Lambda \star \Pi$.
They satisfy the axioms of *realizability algebra*, which are given in [6] or [9].
In the model $\mathcal{M}$, we use the language of ZF with the binary relation symbols $\notin, \subset$ and function symbols, which we shall define when needed, by means of formulas of ZF.
We can now build (see [6]) the *realizability model* $\mathcal{N}$, which has the same set of individuals as $\mathcal{M}$, the truth value set of which is $\mathscr{P}(\Pi)$, endowed with a suitable Boolean algebra structure (not the usual one for the powerset).
The language of this model has three binary relation symbols $\not\varepsilon, \notin, \subset$, and the same function symbols as the model $\mathcal{M}$, with the same interpretation.

The *formulas* are built as usual, from atomic formulas, *with the only logical symbols* $\perp, \to, \forall$.

ε is called the *strong membership relation* ; $\in$ is called the *weak* or *extensional membership relation*.

The formula $\forall z(x \not\varepsilon z \to y \not\varepsilon z)$ is written $x = y$; it is the *strong* or *Leibniz equality*.
The formula $x \subset y \land y \subset x$ is written $x \simeq y$; it is the *weak* or *extensional equality*.

Notations. We shall write :
$\neg F$ for $F \to \perp$; $F_1, \ldots, F_n \to F$ for $F_1 \to (\ldots \to (F_n \to F)\ldots)$;
$\exists x\, F$ for $\neg\forall x \neg F$; $\exists x\{F_1, \ldots, F_n\}$ for $\neg\forall x(F_1, \ldots, F_n \to \perp)$.
We shall often use the notation $\vec{x}$ for a finite sequence $x_1, \ldots, x_n$; for instance, we shall write $F[\vec{x}]$ for $F[x_1, \ldots, x_n]$.

By means of the completeness theorem, we obtain from $\mathcal{N}$ an ordinary model $\mathcal{N}'$, with truth values in $\{0,1\}$. The set of individuals of $\mathcal{N}'$ generally *strictly contains* $\mathcal{N}$.
The elements of $\mathcal{N}'$ are called *individuals of* $\mathcal{N}'$ or even *individuals of* $\mathcal{N}$. The individuals are generally denoted by $a, b, c, \ldots, a_0, a_1, \ldots$

In [6] or [7], we define a theory ZF_ε, written in this language. The axioms for ε are essentially the same as the axioms for $\in$ in ZF (sometimes in an unusual form), *without extensionality*.
For instance, the infinity axiom is the following scheme :

$$\forall \vec{z} \forall a \exists b \big\{ a \,\varepsilon\, b, (\forall x \,\varepsilon\, b)(\exists y\, F[x, y, \vec{z}] \to (\exists y \,\varepsilon\, b) F[x, y, \vec{z}]) \big\}$$

for every formula $F[x, y, z_1, \ldots, z_n]$.

The axioms for $\in, \subset$ are a kind of coinductive definition from ε :

$$\forall x \forall y (x \in y \leftrightarrow (\exists z \,\varepsilon\, y) x \simeq z) \;;\; \forall x \forall y (x \subseteq y \leftrightarrow (\forall z \,\varepsilon\, x) z \in y).$$

We show that ZF_ε is a *conservative extension* of ZF, and that the model $\mathcal{N}$ *satisfies the axioms of* ZF_ε, which means that each one of these axioms is *realized by a proof-like term*.

[1] The papers [6, 7, 9, 8] are available at `http://www.pps.univ-paris-diderot.fr/~krivine/`

[2] In fact, it suffices that $\mathcal{M}$ satisfy the *choice principle CP*, which is written as follows, in the language of ZF with a new binary relation symbol $\lhd$: *“$\lhd$ is a well ordering relation on $\mathcal{M}$”*.
It is well known that, in every countable model of ZFC, we can define such a binary symbol, so as to get a model of ZF + CP. Thus, ZF + CP is a *conservative extension* of ZFC.

Given a term $\xi \in \Lambda$ and a closed formula $F[a_1,\dots,a_n]$ in the language of ZF_ε, with parameters $a_1,\dots,a_n$ in $\mathcal{N}$ (or, which is the same, in $\mathcal{M}$), we shall write :
$\xi \Vdash F[a_1,\dots,a_n]$ in order to say that the term ξ realizes $F[a_1,\dots,a_n]$.
The truth value of this formula is a subset of Π, denoted by $\|F[a_1,\dots,a_n]\|$.
We write $\Vdash F$ in order to say that F is realized by some proof-like term.

Thus, the model $\mathcal{N}'$ satisfies ZF_ε ; therefore, in $\mathcal{N}'$, we can define a model of ZF, denoted $\mathcal{N}'_\in$, in which equality is interpreted by extensional equivalence.

The general properties of the realizability models are described in [9] ; we shall use the definitions and notations of this paper.

In what follows, unless otherwise stated, each formula of ZF_ε must be interpreted in $\mathcal{N}$ (its truth value is a subset of Π) or, if one prefers, in $\mathcal{N}'$ (then its truth value is 0 or 1). If the formula must be interpreted in $\mathcal{M}$, (in that case, it does not contains the symbol $\not\varepsilon$) it will be explicitly stated.

Function symbols

Notations. The formula $\forall z(z \not\varepsilon y \to z \not\varepsilon x)$ is denoted by $x \subseteq y$ (*strong inclusion*) ;
the formula $x \subseteq y \land y \subseteq x$ is denoted by $x \cong y$ (*strong extensional equivalence*).
We recall that $\subset$ and $\simeq$ are the symbols of inclusion and of extensional equivalence of ZF :
$x \subset y \equiv \forall z(z \notin y \to z \not\varepsilon x)$; $x \simeq y \equiv (x \subset y \land y \subset x)$.

Function symbols associated with axioms of ZF_ε

In this section, we define a function symbol for each of the following axioms of ZF_ε :

comprehension, pairing, union, power set and collection.

Comprehension.
For each formula $F[y,\vec{z}]$ of ZF_ε, (where $\vec{z}$ is a finite sequence of variables $z_1,\dots,z_n$) we define, in $\mathcal{M}$, a symbol of function of arity $n+1$, denoted provisionally by $\mathrm{Compr}_F(x,\vec{z})$, (Compr is an abbreviation for *Comprehension*) by setting :
$\mathrm{Compr}_F(a,\vec{c}) = \{(b,\xi \centerdot \pi)\,;\,(b,\pi) \in a,\ \xi \Vdash F[b,\vec{c}]\}$.
It was shown in [9] (and it is easily checked) that we have :
$\|b \not\varepsilon \mathrm{Compr}_F(a,\vec{c})\| = \|F[b,\vec{c}] \to b \not\varepsilon a\|$. Thus, we have :
$\mathsf{I} \Vdash \forall x \forall y \forall \vec{z}(y \not\varepsilon \mathrm{Compr}_F(x,\vec{z}) \to (F[y,\vec{z}] \to y \not\varepsilon x))$;
$\mathsf{I} \Vdash \forall x \forall y \forall \vec{z}((F[y,\vec{z}] \to y \not\varepsilon x) \to y \not\varepsilon \mathrm{Compr}_F(x,\vec{z}))$.
Therefore, instead of $\mathrm{Compr}_F(x,\vec{z})$, we shall use for this function symbol, the more intuitive notation $\{y \,\varepsilon\, x\,;\, F[y,\vec{z}]\}$, in which y is a *bound variable.*

Pairing.
We define the following binary function symbol :
$$\mathrm{pair}(x,y) = \{z \,\varepsilon\, \{x,y\} \times \Pi\,;\,(z = x) \lor (z = y)\}.$$
It is easily checked that we have the desired property :
$$\Vdash \forall x \forall y \forall z(z \,\varepsilon\, \mathrm{pair}(x,y) \leftrightarrow z = x \lor z = y).$$
Remark. We could also define a symbol $\mathrm{pair}(x,y)$, with this property, directly in $\mathcal{M}$, as follows :
$$\mathrm{pair}(x,y) = \{(x,\underline{1} \centerdot \pi)\,;\,\pi \in \Pi\} \cup \{(y,\underline{0} \centerdot \pi)\,;\,\pi \in \Pi\}.$$

In the sequel, when working in $\mathcal{N}$, we shall use the (natural) abbreviations :
$\{x,y\}$ for $\mathrm{pair}(x,y)$; (x,y) for $\mathrm{pair}(\mathrm{pair}(x,x),\mathrm{pair}(x,y))$.

Union and power set.

We define below two unary function symbols $\overline{\bigcup}x$ and $\overline{\mathscr{P}}(x)$, such that :

$\Vdash \forall x\forall z(z\,\varepsilon\,\overline{\bigcup}x \leftrightarrow (\exists y\,\varepsilon\,x)\,z\,\varepsilon\,y)$.

$\Vdash \forall x(\forall y\,\varepsilon\,\overline{\mathscr{P}}(x))(\forall z\,\varepsilon\,y)(z\,\varepsilon\,x)$; $\Vdash \forall x\forall y(\exists y'\,\varepsilon\,\overline{\mathscr{P}}(x))\forall z(z\,\varepsilon\,y' \leftrightarrow z\,\varepsilon\,x \wedge z\,\varepsilon\,y)$.

Theorem 1. *Let $\mathscr{V},\mathscr{Q}$ be the unary function symbols defined in $\mathscr{M}$ as follows :*
$\mathscr{V}(a) = Cl(a)\times\Pi$ and $\mathscr{Q}(a) = \mathscr{P}(Cl(a)\times\Pi)\times\Pi$
where $Cl(a)$ is the transitive closure of a. Then, we have :
i) $\mathsf{I} \Vdash \forall x\forall y\forall z(z\,\varepsilon\,y, z\not\varepsilon\,\mathscr{V}(x) \to y\not\varepsilon\,x)$.
ii) $\mathsf{I} \Vdash \forall x\forall\vec{z}\big(\{y\,\varepsilon\,x\,;\,F[y,\vec{z}]\}\,\varepsilon\,\mathscr{Q}(x)\big)$ for every formula $F[x,\vec{z}]$ of ZF_ε.

i) Let a,b,c be individuals in $\mathscr{M}$, $\xi,\eta\in\Lambda$ and $\pi\in\Pi$ such that :
$\xi \Vdash c\,\varepsilon\,b$, $\eta \Vdash c\not\varepsilon\,\mathscr{V}(a)$ and $\pi\in\|b\not\varepsilon\,a\|$; we have therefore $(b,\pi)\in a$.
We must show $\xi\star\eta\scriptstyle\bullet\displaystyle\pi\in\perp\!\!\!\perp$.
We show that $\|c\not\varepsilon\,b\|\subset\|c\not\varepsilon\,\mathscr{V}(a)\|$: indeed, if $\rho\in\|c\not\varepsilon\,b\|$, then we have $(c,\rho)\in b$. But we have $(b,\pi)\in a$ and thus $c\in\mathrm{Cl}(a)$ and it follows that $\|c\not\varepsilon\,\mathscr{V}(a)\| = \Pi$.
Therefore, $\eta\Vdash c\not\varepsilon\,b$; by hypothesis on ξ, we have $\xi\star\eta\scriptstyle\bullet\displaystyle\pi\in\perp\!\!\!\perp$.

ii) Let $a,\vec{c}$ be individuals in $\mathscr{M}$; we must show $\mathsf{I}\Vdash A\,\varepsilon\,\mathscr{Q}(a)$, where $A=\{y\,\varepsilon\,a\,;\,F[y,\vec{c}]\}$.
We have $A=\{(b,\xi\scriptstyle\bullet\displaystyle\pi)\,;\,(b,\pi)\in a,\ \xi\Vdash F[b,\vec{c}]\}$ and therefore $A\subset\mathrm{Cl}(a)\times\Pi$. But we have :
$\|A\not\varepsilon\,\mathscr{Q}(a)\| = \{\pi\in\Pi\,;\,(A,\pi)\in\mathscr{Q}(a)\} = \Pi$ and therefore $\mathsf{I}\Vdash A\,\varepsilon\,\mathscr{Q}(a)$.

Q.E.D.

We can now define the function symbols $\overline{\bigcup}$ and $\overline{\mathscr{P}}$ by setting :

$$\overline{\bigcup}x = \{z\,\varepsilon\,\mathscr{V}(x)\,;\,(\exists y\,\varepsilon\,x)\,z\,\varepsilon\,y\}\ ;\ \overline{\mathscr{P}}(x) = \{y\,\varepsilon\,\mathscr{Q}(x)\,;\,y\subseteq x\}.$$

Collection.

We shall use in the following, function symbols associated with a strong form of the *collection scheme.*

In order to define these function symbols, it is convenient to decompose them, which is done in theorems 2, 3 and 4.

Theorem 2. *For each formula $F(x,\vec{z})$ of ZF_ε, we have :*

$$\Vdash \forall\vec{z}\big(\exists x\,F(x,\vec{z}) \to (\exists x\,\varepsilon\,\phi_F(\vec{z}))F(x,\vec{z})\big)\ ;\ \Vdash \forall\vec{z}(\forall x\,\varepsilon\,\phi_F(\vec{z}))F(x,\vec{z})$$

where ϕ_F is a function symbol defined in $\mathscr{M}$.

We show $\lambda x(x)\mathsf{I}\Vdash\forall x(x\,\varepsilon\,\Phi_F(\vec{z})\to F(x,\vec{z}))\to\forall x\,F(x,\vec{z})$ where the function symbol Φ_F is defined as follows :

By means of the collection scheme in $\mathscr{M}$, we define a function symbol $\Psi(\vec{z})$ such that :
$\|\forall x\,F(x,\vec{z})\| = \bigcup_{x\in\Psi(\vec{z})}\|F(x,\vec{z})\|$ and we set $\Phi_F(\vec{z}) = \Psi(\vec{z})\times\Pi$.
Let $\xi\Vdash\forall x(x\,\varepsilon\,\Phi_F(\vec{z})\to F(x,\vec{z}))$ and $\pi\in\|\forall x\,F(x,\vec{z})\|$.
Then $\pi\in\|F(x,\vec{z})\|$ for some $x\in\Psi(\vec{z})$, and therefore $\mathsf{I}\Vdash x\,\varepsilon\,\Phi_F(\vec{z})$ and $\xi\star\mathsf{I}\scriptstyle\bullet\displaystyle\pi\in\perp\!\!\!\perp$.

Therefore, by replacing F with $\neg F$, we have $\Vdash\exists x\,F(x,\vec{z})\to(\exists x\,\varepsilon\,\Phi_{\neg F}(\vec{z}))\,F(x,\vec{z})$.

Thus, we only need to set $\phi_F(\vec{z}) = \{x\,\varepsilon\,\Phi_{\neg F}(\vec{z})\,;\,F(x,\vec{z})\}$.

Q.E.D.

Theorem 3. *For every formula $F(y,\vec{z})$ of ZF_ε, we have :*

$$\Vdash \forall\vec{z}\big(\exists x\forall y(F(y,\vec{z})\to y\,\varepsilon\,x)\to\forall y(F(y,\vec{z})\leftrightarrow y\,\varepsilon\,\gamma_F(\vec{z}))\big)$$

where γ_F is a function symbol defined in $\mathcal{M}$.

By theorem 2, we have :

$$\Vdash \forall\vec{z}\big(\exists x\forall y(F(y,\vec{z})\to y\,\varepsilon\,x)\to(\exists x\,\varepsilon\,\phi(\vec{z}))\forall y(F(y,\vec{z})\to y\,\varepsilon\,x)\big)$$

where ϕ is a function symbol. Therefore we have, by definition of $\overline{\bigcup}\phi(\vec{z})$:

$$\Vdash \forall\vec{z}\Big(\exists x\forall y(F(y,\vec{z})\to y\,\varepsilon\,x)\to\forall y(F(y,\vec{z})\to y\,\varepsilon\overline{\bigcup}\phi(\vec{z}))\Big).$$

Now, we only need to set $\gamma_F(\vec{z})=\{y\,\varepsilon\overline{\bigcup}\phi(\vec{z})\,;\,F(y,\vec{z})\}$ (comprehension scheme).

Q.E.D.

When the hypothesis $\exists x\forall y(F(y,\vec{z})\to y\,\varepsilon\,x)$ is satisfied, we say that *the formula $F(y,\vec{z})$ defines a set.*
For the function symbol $\gamma_F(\vec{z})$, we shall use the more intuitive notation $\{y\,;\,F(y,\vec{z})\}$, where y is a bound variable.

Theorem 4.
Let $f(x,\vec{z})$ be a $(n+1)$-ary function symbol (defined in $\mathcal{M}$). Then, we have :

$$\Vdash \forall a\forall y\forall\vec{z}\big(y\,\varepsilon\,\phi_f(a,\vec{z})\leftrightarrow(\exists x\,\varepsilon\,a)(y=f(x,\vec{z}))\big)$$

where ϕ_f is a $(n+1)$-ary function symbol.

We define, in $\mathcal{M}$, the symbol ϕ_f as follows :
Let $a_0,y_0,\vec{z}_0$ be fixed individuals in $\mathcal{M}$; we set $\phi_f(a_0,\vec{z}_0)=\{(f(x,\vec{z}_0),\pi)\,;\,(x,\pi)\in a_0\}$.
Then, we have immediately $\|y_0\not\varepsilon\,\phi_f(a_0,\vec{z}_0)\|=\|\forall x(y_0=f(x,\vec{z}_0)\hookrightarrow x\not\varepsilon\,a_0)\|$. Therefore :
$\Vdash\forall x(y_0=f(x,\vec{z}_0)\hookrightarrow x\not\varepsilon\,a_0)\leftrightarrow y_0\not\varepsilon\,\phi_f(a_0,\vec{z}_0)$ which gives the desired result.

Q.E.D.

Remark. The *connective* $\hookrightarrow$ is defined in [7, 9]. It is equivalent to $\to$ but simpler to realize. Its hypothesis must be a strong equality.

For the function symbol $\phi_f(a,\vec{z})$, we shall use the more intuitive notation $\{f(x,\vec{z})\,;\,x\,\varepsilon\,a\}$, where x is a bound variable. We call it *image of a by the function $f(x)$*.

Miscellaneous symbols

In the following, we shall use some function symbols, the definition and properties of which are given in [9]. We simply recall their definition below.

- The unary function symbol $\gimel$, defined in $\mathcal{M}$ by $\gimel x=x\times\Pi$.
  For any individual E of $\mathcal{M}$, the *restricted quantifier* $\forall x^{\gimel E}$ is defined in [7] or [9] by :
  $\|\forall x^{\gimel E}F[x]\|=\bigcup_{x\in E}\|F[x]\|$ and we have $\Vdash\forall x^{\gimel E}F[x]\leftrightarrow\forall x(x\,\varepsilon\gimel E\to F[x])$.
  In the realizability model $\mathcal{N}$, the formula $x\,\varepsilon\gimel E$ may be intuitively understood as *"x is of type E"*. For instance, $\gimel 2$ may be considered as *the type of booleans* and $\gimel\mathbb{N}$ as *the type of integers.*

- the function symbols $\wedge,\vee,\neg$, with domains $\{0,1\}\times\{0,1\}$ and $\{0,1\}$, and values in $\{0,1\}$, are defined in $\mathcal{M}$ by means of the usual truth tables.
  These functions define, in $\mathcal{N}$, a structure of Boolean algebra on $\gimel 2$.
  We call it the *characteristic Boolean algebra* of the realizability model $\mathcal{N}$.

- a binary function symbol with domain $\{0,1\}\times\mathcal{M}$, denoted by $(\alpha,x)\mapsto\alpha x$, by setting :
  $$0x=\emptyset\ ;\ 1x=x.$$
  In the model $\mathcal{N}$, the domain of this function is $\daleth 2\times\mathcal{N}$.

- a binary function symbol $\sqcup$ with domain $\mathcal{M}\times\mathcal{M}$, by setting $x\sqcup y=x\cup y$.
  Remark. The extension of this function to the model $\mathcal{N}$ *is not* the union $\cup$, which explains the use of another symbol.

Lemma 5 (Linearity)**.**
Let f be a binary function symbol, defined in $\mathcal{M}$. Then, we have :
i) $\mathsf{I} \Vdash \forall\alpha^{\daleth 2}\forall x\forall y(\alpha f(x,y)=\alpha f(\alpha x,y))$.
ii) Moreover, if $f(\emptyset,\emptyset)=\emptyset$, then :
$\mathsf{I} \Vdash \forall\alpha^{\daleth 2}\forall\alpha'^{\daleth 2}\forall x\forall y\forall x'\forall y'\big(\alpha\wedge\alpha'=0\hookrightarrow f(\alpha x\sqcup\alpha' x',\alpha y\sqcup\alpha' y')=\alpha f(x,y)\sqcup\alpha' f(x',y')\big)$.

It suffices to check :
for (i) the two cases $\alpha=0,1$;
for (ii) the three cases $(\alpha,\alpha')=(0,0),(0,1),(1,0)$;
which is is trivial.

Q.E.D.

Symbols for characteristic functions

Let $R(x_1,\dots,x_n)$ be an n-ary relation defined in $\mathcal{M}$. Its *characteristic function*, with values in $\{0,1\}$, will be denoted by $\langle R(x_1,\dots,x_n)\rangle$. Therefore, we have : $\mathcal{M}\models\forall\vec{x}(R(\vec{x})\leftrightarrow\langle R(\vec{x})\rangle=1)$.
Therefore, in the realizability model $\mathcal{N}$, the function symbol $\langle R(\vec{x})\rangle$ takes its values in $\daleth 2$.

The theorem 8 below shows that, if a binary relation $y\prec x$ is well founded in $\mathcal{M}$, then the relation $\langle y\prec x\rangle=1$ is well founded in $\mathcal{N}$.

Well founded relations

In this section, we study properties of well founded relations in $\mathcal{N}$. All the results obtained here are, of course, trivial in ZF. The difficulties come from the fact that the relation ε of strong membership does not satisfy extensionality.

Given a binary relation $\prec$, an individual a is said *minimal for* $\prec$ if we have $\forall x\neg(x\prec a)$.
The binary relation $\prec$ is called *well founded* if we have :
$$\forall X\big(\forall x(\forall y(y\prec x\to y\not\varepsilon X)\to x\not\varepsilon X)\to\forall x(x\not\varepsilon X)\big).$$
The intuitive meaning is that each non empty individual X has an ε-element minimal for $\prec$.
Theorem 6 shows that this also true for non empty classes.

Theorem 6.
If the relation $x\prec y$ is well founded then, for every formula $F[x,\vec{z}]$ of ZF_ε, we have :
$$\forall\vec{z}\big(\forall x(\forall y(y\prec x\to F[y,\vec{z}])\to F[x,\vec{z}])\to\forall x\,F[x,\vec{z}]\big).$$

Proof by contradiction ; we consider, in $\mathcal{N}$, an individual a and a formula $G[x]$ such that :
(1) $$G[a]\ ;\ \forall x\big(G[x]\to\exists y\{G[y],y\prec x\}\big).$$

We apply the axiom scheme of infinity of ZF_ε :

(2) $$\exists b\big\{a\,\varepsilon\, b, (\forall x\,\varepsilon\, b)\,\big(\exists y\, H(x,y) \to (\exists y\,\varepsilon\, b) H(x,y)\big)\big\}$$

by setting $H(x,y) \equiv G[x] \wedge G[y] \wedge y \prec x$. Let $X = \{x\,\varepsilon\, b\,;\, G(x)\}$; by (1) and (2), we get $a\,\varepsilon\, X$.
We obtain a contradiction with the hypothesis, by showing $(\forall x\,\varepsilon\, X)(\exists y\,\varepsilon\, X)(y \prec x)$:
suppose $x\,\varepsilon\, b$ and $G[x]$; by (2), we have :

$$\exists y\{G[x], G[y], y \prec x\} \to (\exists y\,\varepsilon\, b)\{G[x], G[y], y \prec x\}.$$

By $G[x]$ and (1), we have $\exists y\{G[x], G[y], y \prec x\}$.
Therefore, we have $(\exists y\,\varepsilon\, b)\{G[y], y \prec x\}$, hence the result.

Q.E.D.

Therefore, in order to show $\forall x\, F[x]$, it suffices to show $\forall x\big(\forall y(y \prec x \to F[y]) \to F[x]\big)$.
Then, we say that we have shown $\forall x\, F[x]$ *by induction on x, following the well founded relation $\prec$.*

Theorem 7. *The binary relation $x \in y$ is well founded.*

We must show $\forall x(\forall y(y \in x \to y \not\varepsilon X) \to x \not\varepsilon X) \to \forall x(x \not\varepsilon X)$.
We apply theorem 6 to the well founded relation $x\,\varepsilon\, y$ and the formula $F[x] \equiv x \notin X$.
This gives : $\forall x(\forall y(y\,\varepsilon\, x \to y \notin X) \to x \notin X) \to \forall x(x \notin X)$.
Now, we have immediately $\Vdash x \notin X \to x \not\varepsilon X$. Thus, it remains to show :
$\Vdash \forall x(\forall y(y \in x \to y \not\varepsilon X) \to x \not\varepsilon X) \to \forall x(\forall y(y\,\varepsilon\, x \to y \notin X) \to x \notin X)$.
But we have $x \notin X \equiv \forall x'(x' \simeq x \to x' \not\varepsilon X)$. Therefore, we need to show :
$\Vdash \forall x(\forall y(y \in x \to y \not\varepsilon X) \to x \not\varepsilon X), \forall y(y\,\varepsilon\, x \to y \notin X), x' \simeq x \to x' \not\varepsilon X$; it is enough to show :
$\Vdash \forall y(y\,\varepsilon\, x \to y \notin X), x' \simeq x \to \forall y(y \in x' \to y \not\varepsilon X)$.
Now, from $x' \simeq x, y \in x'$, we deduce $y \in x$. Thus, there is some $y' \simeq y$ such that $y'\,\varepsilon\, x$.
Then, from $\forall y(y\,\varepsilon\, x \to y \notin X)$, we deduce $y' \notin X$, and therefore $y \not\varepsilon X$.

Q.E.D.

For instance, in the following, we shall use the fact that, if there is an ordinal ρ such that $F[\rho]$, then there exists a least such ordinal, for any formula $F[\rho]$ *written in the language of ZF_ε.* This follows from theorem 7.

Preservation of well-foundedness

Theorem 8. *Let $\prec$ be a well founded binary relation defined in the ground model $\mathscr{M}$. Then, the relation $\langle y \prec x\rangle = 1$ is well founded in $\mathscr{N}$. In fact, we have :*

$$\mathsf{Y} \Vdash \forall X\big(\forall x(\forall y(\langle y \prec x\rangle = 1 \hookrightarrow y \not\varepsilon X) \to x \not\varepsilon X) \to \forall x(x \not\varepsilon X)\big)$$

where $\mathsf{Y} = (\lambda x\lambda f(f)(x)xf)\lambda x\lambda f(f)(x)xf$ (Turing fixpoint combinator).

Let $\xi \in \Lambda$ be such that $\xi \Vdash \forall x(\forall y(\langle y \prec x\rangle = 1 \hookrightarrow y \not\varepsilon X_0) \to x \not\varepsilon X_0)$, X_0 being any individual in $\mathscr{M}$. We set $F[x] \equiv (\forall \pi \in \|x \not\varepsilon X_0\|)(\mathsf{Y} \star \xi \,.\, \pi \in \perp\!\!\!\perp)$, and we have to show $\forall x\, F[x]$.
Since $\prec$ is a well founded relation, it suffices to show $\forall x\big(\forall y(y \prec x \to F[y]) \to F[x]\big)$, or equivalently $\neg F[x_0] \to (\exists y \prec x_0)\neg F[y]$, for any individual x_0.
By the hypothesis $\neg F[x_0]$, there exists $\pi_0 \in \|x_0 \not\varepsilon X_0\|$ such that $\mathsf{Y} \star \xi \,.\, \pi_0 \notin \perp\!\!\!\perp$ and therefore, we have $\xi \star \mathsf{Y}\xi \,.\, \pi_0 \notin \perp\!\!\!\perp$.
By hypothesis on ξ, we deduce $\mathsf{Y}\xi \not\Vdash \forall y(\langle y \prec x_0\rangle = 1 \hookrightarrow y \not\varepsilon X_0)$.
Thus, there exists $y_0 \prec x_0$ such that $\mathsf{Y}\xi \not\Vdash y_0 \not\varepsilon X_0$.
Therefore, we have $(\exists \pi \in \|y_0 \not\varepsilon X_0\|)(\mathsf{Y} \star \xi \,.\, \pi \notin \perp\!\!\!\perp)$, that is $\neg F[y_0]$.

Q.E.D.

Definition of a rank function

Definition. A *function with domain D* is an individual ϕ such that :
$(\forall z \,\varepsilon\, \phi)(\exists x \,\varepsilon\, D)\exists y(z = (x,y))$; $(\forall x \,\varepsilon\, D)\exists y((x,y) \,\varepsilon\, \phi)$;
$\forall x \forall y \forall y'((x,y) \,\varepsilon\, \phi, (x,y') \,\varepsilon\, \phi \to y = y')$.

Let ϕ be a function with domain D and $F[y,\vec{z}]$ a formula of ZF_ε. Then, the formula :
$\exists y\{(x,y) \,\varepsilon\, \phi, F[y,\vec{z}]\}$ is denoted by $F[\phi(x),\vec{z}]$.

Remark. Beware, despite the same notation $\phi(x)$, it is not a function symbol.

By means of theorem 3, we define the binary function symbol Im by setting :

$$\mathrm{Im}(\phi, D) = \{y \,;\, (\exists x \,\varepsilon\, D)\,(x,y) \,\varepsilon\, \phi\}.$$

When ϕ is a function with domain D, we shall use, for $\mathrm{Im}(\phi, D)$, the more intuitive notation $\{\phi(x) \,;\, x \,\varepsilon\, D\}$, which we call *image of the function* ϕ.

Let $D' \subseteq D$, that is $\forall x(x \not\varepsilon\, D \to x \not\varepsilon\, D')$; *a restriction of* ϕ *to* D' is, by definition, a function ϕ' with domain D' such that $\phi' \subseteq \phi$.
For instance, $\{z \,\varepsilon\, \phi \,;\, (\exists x \,\varepsilon\, D')\exists y(z = (x,y))\}$ is a restriction of ϕ to D'.
If ϕ'_0, ϕ'_1 are both restrictions of ϕ to D', then $\phi'_0 \cong \phi'_1$.

Definition.
A binary relation $\prec$ is called *ranked*, if we have $\forall x \exists y \forall z(z \prec x \to z \,\varepsilon\, y)$, in other words : the minorants of any individual form a set.
By theorem 3, if the relation $\prec$ is ranked and defined by a formula $P[x,y,\vec{u}]$ of ZF_ε with parameters $\vec{u}$ in $\mathcal{N}$, we have :
$\mathcal{N} \models \forall x \forall y(x \prec y \leftrightarrow x \,\varepsilon\, f(y,\vec{u}))$, *for some symbol of function* f*, defined in* $\mathcal{M}$.

In what follows, we suppose that $\prec$ is a ranked *transitive* binary relation.

A function ϕ with domain $\{x \,;\, x \prec a\}$ will be called a*-inductive for* $\prec$, if we have :
$\phi(x) \simeq \{\phi(y) \,;\, y \prec x\}$ for every $x \prec a$. In other words :
$(\forall x \prec a)(\forall y \prec x)\,\phi(y) \in \phi(x)$; $(\forall x \prec a)(\forall z \,\varepsilon\, \phi(x))(\exists y \prec x)\, z \simeq \phi(y)$.

If ϕ is a-inductive for $\prec$, we set $\mathsf{O}(\phi, a) = \{\phi(x) \,;\, x \prec a\}$ (image of ϕ).

Lemma 9. *Let* ϕ, ϕ' *be two functions,* a*-inductive for* $\prec$*. Then :*
i) $\phi(x) \simeq \phi'(x)$ *for every* $x \prec a$.
ii) $\mathsf{O}(\phi, a) \simeq \mathsf{O}(\phi', a)$.
iii) $(\forall x \prec a)\mathrm{On}(\phi(x))$ *;* $\mathsf{O}(\phi, a)$ *is an ordinal, called* ordinal of ϕ.

i) Proof by induction on $\phi(x)$, following $\in$: if $u \,\varepsilon\, \phi(x)$, then $u \simeq \phi(y)$ with $y \prec x$.
Since $\phi(y) \in \phi(x)$, we have $\phi(y) \simeq \phi'(y)$ by the induction hypothesis ;
therefore $\phi(y) \in \phi'(x)$ and $\phi(x) \subset \phi'(x)$.
Conversely, if $u \,\varepsilon\, \phi'(x)$, then $u \simeq \phi'(y)$ with $y \prec x$. Thus, we have $\phi(y) \in \phi(x)$, and therefore $\phi(y) \simeq \phi'(y)$ by the induction hypothesis ; therefore $u \in \phi(x)$ and $\phi'(x) \subset \phi(x)$.

ii) Immediate, by (i).

iii) We show $\mathrm{On}(\phi(x))$ by induction on $\phi(x)$, for the well founded relation $\in$:
If $u \,\varepsilon\, \phi(x)$, we have $u \simeq \phi(y)$ with $y \prec x$; therefore, we have $\mathrm{On}(u)$ by the induction hypothesis. If $v \,\varepsilon\, u$, then $v \,\varepsilon\, \phi(y)$, therefore $v \simeq \phi(z)$ with $z \prec y$; therefore $v \in \phi(x)$.
It follows that $\phi(x)$ is a transitive set of ordinals, thus an ordinal.
Then, $\mathsf{O}(\phi, a)$ is also a transitive set of ordinals, and therefore an ordinal.

Q.E.D.

Lemma 10. *If ϕ is a-inductive for $\prec$, and if $b \prec a$, then every restriction ψ of ϕ to the domain $\{x\,;\ x \prec b\}$ is a b-inductive function for $\prec$.*

Indeed, we have, $\psi(x) = \phi(x) \simeq \{\phi(y)\,;\ y \prec x\} \simeq \{\psi(y)\,;\ y \prec x\}$.

Q.E.D.

By means of theorem 2, we define a unary function symbol Φ, such that :

$\forall x(\forall f\,\varepsilon\,\Phi(x))(f$ is a x-inductive function) ;

$\forall x \forall f\Big(f$ is a x-inductive function $\to \exists f(f\,\varepsilon\,\Phi(x))\Big)$.

In other words, $\Phi(x)$ is a set of x-inductive functions, which is non void if there exists at least one such function.

Finally, we define the unary function symbol Rk, using theorem 4, by setting :

$$\mathrm{Rk}(x) = \overline{\bigcup}\{\mathsf{O}(f,x)\,;\ f\,\varepsilon\,\Phi(x)\}$$

(the symbol $\overline{\bigcup}$ is defined after theorem 1).

Therefore, $\mathrm{Rk}(x)$ is the union of the ordinals of the x-inductive functions in the set $\Phi(x)$. Since all these ordinals are extensionally equivalent, by lemma 9(ii), their union $\mathrm{Rk}(x)$ is also an equivalent ordinal.

Remarks.

If there exists no x-inductive function, then $\mathrm{Rk}(x)$ is void.

The function symbols $\mathsf{O}, \Phi, \mathrm{Rk}$ have additional arguments, which are the parameters $\vec{u}$ of the formula $P[x, y, \vec{u}]$ which defines the relation $y \prec x$.

We suppose now that $\prec$ is a ranked transitive relation, which is *well founded*. It is therefore a *strict ordering*.

Lemma 11. *Every restriction of* Rk *to the domain* $\{x\,;\ x \prec a\}$ *is an a-inductive function for* $\prec$.

Proof by induction on a, following $\prec$.

Let f be a restriction of Rk to the domain $\{x\,;\ x \prec a\}$ and let $x \prec a$. We must show that $f(x) \simeq \{f(y)\,;\ y \prec x\}$, in other words, that we have :

$$\mathrm{Rk}(x) \simeq \{\mathrm{Rk}(y)\,;\ y \prec x\}.$$

Let ψ be any restriction of Rk to the domain $\{y\,;\ y \prec x\}$. By the induction hypothesis, ψ is a x-inductive function for $\prec$.

We now show that $\mathrm{Rk}(x) \simeq \{\mathrm{Rk}(y)\,;\ y \prec x\}$:

i) If $u\,\varepsilon\,\mathrm{Rk}(x)$, then $u\,\varepsilon\,\mathsf{O}(\phi, x)$ for some function ϕ which is x-inductive for $\prec$, *provided that there exists such a function.* Now, there exists effectively one, otherwise $\mathrm{Rk}(x)$ would be void. Therefore, by definition of $\mathsf{O}(\phi, x)$, we have $u = \phi(y)$ with $y \prec x$. But $\mathrm{Rk}(y) \simeq \phi(y)$, since ϕ, ψ are both x-inductive functions for $\prec$, and $\psi(y) = \mathrm{Rk}(y)$ (lemma 9(i)).

Therefore, we have $u \simeq \mathrm{Rk}(y)$, with $y \prec x$.

ii) Conversely, if $y \prec x$, then $\mathrm{Rk}(y) = \psi(y)$. Let $\phi\,\varepsilon\,\Phi(x)$; then ϕ, ψ are x-inductive for $\prec$; therefore $\phi(y) \simeq \psi(y)$ (lemma 9(i)).

Now $\phi(y)\,\varepsilon\,\mathsf{O}(\phi, x)$, and therefore $\phi(y)\,\varepsilon\,\mathrm{Rk}(x)$ by definition of $\mathrm{Rk}(x)$.

It follows that $\mathrm{Rk}(y) = \psi(y) \in \mathrm{Rk}(x)$.

Q.E.D.

Theorem 12. *We have* $\mathrm{Rk}(x) \simeq \{\mathrm{Rk}(y)\,;\ y \prec x\}$ *for every x.*

Proof by induction on x, following $\prec$; let ψ be any restriction of Rk to the domain $\{y\,;\ y \prec x\}$. By lemma 11, ψ is a x-inductive function for $\prec$.
Then, we finish the proof, by repeating paragraphs (i) and (ii) of the proof of lemma 11.

Q.E.D.

Rk is called the *rank function* of the ranked, well founded and transitive relation $\prec$.
$\mathrm{Rk}(x)$ is, for every x, a representative of the ordinal of any x-inductive function for $\prec$.

The values of the rank function Rk form an initial segment of On, which we shall call *the image of Rk*. It is therefore, *either an ordinal, or the whole of On.*

Proposition 13. *Let* $\prec_0, \prec_1$ *be two ranked transitive well founded relations, and* f *a function such that* $\forall x \forall y (x \prec_0 y \to f(x) \prec_1 f(y))$.
If $\mathrm{Rk}_0, \mathrm{Rk}_1$ *are their rank functions, then we have* $\forall x \big(\mathrm{Rk}_0(x) \le \mathrm{Rk}_1(f(x))\big)$, *and the image of* Rk_0 *is an initial segment of the image of* Rk_1.

We show immediately $\forall x \big(\mathrm{Rk}_0(x) \le \mathrm{Rk}_1(f(x))\big)$ by induction following $\prec_0$. Hence the result, since the image of a rank function is an initial segment of On.

Q.E.D.

An ultrafilter on $\beth 2$

In all of the following, we write $y < x$ for $y \in \mathrm{Cl}(x)$ in $\mathcal{M}$, where $\mathrm{Cl}(x)$ denotes the transitive closure of x. It is a strict well founded ordering (many other such orderings would do the job, for instance the relation $\mathrm{rank}(y) < \mathrm{rank}(x)$).
The binary function symbol $\langle y < x \rangle$ is therefore defined in $\mathcal{N}$, with values in $\beth 2$.
By theorem 8, the binary relation $\langle y < x \rangle = 1$ is well founded in $\mathcal{N}$.

Theorem 14. $\Vdash$ *There exists an ultrafilter* $\mathcal{D}$ *on* $\beth 2$*, which is defined as follows :*
$\mathcal{D} = \{\alpha\,\varepsilon\,\beth 2\,;$ *the relation* $\langle y < x \rangle \ge \alpha$ *is well founded*$\}$.

Remark. By lemma 5, the formula $\langle y < x \rangle \ge \alpha$ may be written $\langle \alpha y < \alpha x \rangle = \alpha$.

The formula $\alpha\,\varepsilon\,\mathcal{D}$, which we shall also write $\mathcal{D}[\alpha]$, is therefore :

$$\mathcal{D}[\alpha] \equiv \forall X \big(\forall x (\forall y (\langle y < x \rangle \ge \alpha \hookrightarrow y \not\varepsilon X) \to x \not\varepsilon X) \to \forall x (x \not\varepsilon X)\big)$$

Remark. We have :
$\mathcal{D}[1] \equiv \forall X \big(\forall x (\forall y (\langle y < x \rangle = 1 \hookrightarrow y \not\varepsilon X) \to x \not\varepsilon X) \to \forall x (x \not\varepsilon X)\big)$.
$\mathcal{D}[0] \equiv \forall X ((\emptyset \not\varepsilon X \to \emptyset \not\varepsilon X) \to \emptyset \not\varepsilon X)$.

We have immediately : $\lambda x\, x\, \mathsf{I} \Vdash \neg \mathcal{D}[0]$; $\mathsf{Y} \Vdash \mathcal{D}[1]$;
$\mathsf{I} \Vdash \forall \alpha^{\beth 2} \forall \beta^{\beth 2} \big(\alpha \le \beta \hookrightarrow (\mathcal{D}[\alpha] \to \mathcal{D}[\beta])\big)$ (more precisely : $\|\mathcal{D}[1]\| \subset \|\mathcal{D}[0]\|$).

Therefore, in order to prove theorem 14, it suffices to show :
$\Vdash \forall \alpha^{\beth 2} \forall \beta^{\beth 2} \big(\alpha \wedge \beta = 0 \hookrightarrow (\mathcal{D}[\alpha \vee \beta] \to \mathcal{D}[\alpha] \vee \mathcal{D}[\beta])\big)$; see theorem 15 ;
$\Vdash \forall \alpha^{\beth 2} \forall \beta^{\beth 2} \big(\alpha \wedge \beta = 0 \hookrightarrow (\mathcal{D}[\alpha], \mathcal{D}[\beta] \to \bot)\big)$; or even only :
$\Vdash \forall \alpha^{\beth 2} (\mathcal{D}[\alpha], \mathcal{D}[\neg \alpha] \to \bot)$; see theorem 22.

Q.E.D.

Notation. For $\alpha\,\varepsilon\,\beth 2$, we shall write $x <_\alpha y$ for $\langle x < y \rangle \ge \alpha$.

Theorem 15.
i) $\Vdash \forall \alpha^{\gimel 2} \forall \beta^{\gimel 2} \big(\alpha \wedge \beta = 0 \hookrightarrow (\mathscr{D}[\alpha \vee \beta] \to \mathscr{D}[\alpha] \vee \mathscr{D}[\beta])\big)$.
ii) $\Vdash \forall \alpha^{\gimel 2} \forall \beta^{\gimel 2} \big(\mathscr{D}[\alpha \vee \beta] \to \mathscr{D}[\alpha] \vee \mathscr{D}[\beta]\big)$.

i) Let $\alpha, \beta \,\varepsilon\, \gimel 2$ be such that $\alpha \wedge \beta = 0, \neg \mathscr{D}[\alpha], \neg \mathscr{D}[\beta]$. We have to show $\neg \mathscr{D}[\alpha \vee \beta]$.
By hypothesis on α and β, there exists individuals a_0, A (resp. b_0, B) such that $a_0 \,\varepsilon\, A$ (resp. $b_0 \,\varepsilon\, B$) and A (resp. B) has no minimal ε-element for $<_\alpha$ (resp. for $<_\beta$). We set :

$$c_0 = \alpha a_0 \sqcup \beta b_0 \ \text{ and } C = \{\alpha x \sqcup \beta y \,;\, x \,\varepsilon\, A, y \,\varepsilon\, B\}.$$

Therefore, we have $c_0 \,\varepsilon\, C$; it suffices to show that C has no minimal ε-element for $<_{\alpha \vee \beta}$. Let $c \,\varepsilon\, C, c = \alpha a \sqcup \beta b$, with $a \,\varepsilon\, A, b \,\varepsilon\, B$. By hypothesis on A, B, there exists $a' \,\varepsilon\, A$ and $b' \,\varepsilon\, B$ such that $a' <_\alpha a$, $b' <_\beta b$. If we set $c' = \alpha a' \sqcup \beta b'$, we have $c' \,\varepsilon\, C$, as needed. We also have :
$\langle c' = a' \rangle \ge \alpha, \langle a' < a \rangle \ge \alpha, \langle c = a \rangle \ge \alpha$; it follows that $\langle c' < c \rangle \ge \alpha$.
In the same way, we have $\langle c' < c \rangle \ge \beta$ and therefore, finally, $\langle c' < c \rangle \ge \alpha \vee \beta$.

ii) We set $\beta' = \beta \wedge (\neg \alpha)$; we have $\alpha \wedge \beta' = 0$ and $\alpha \vee \beta' = \alpha \vee \beta$. Therefore, we have :
$\mathscr{D}[\alpha \vee \beta] \to \mathscr{D}[\alpha] \vee \mathscr{D}[\beta']$. Now, we have $\beta' \le \beta$ and therefore $\mathscr{D}[\beta'] \to \mathscr{D}[\beta]$.

Q.E.D.

Lemma 16.
i) $\mathsf{I} \Vdash \forall x \forall y (\langle x < y \rangle \ne 1 \to x \not\varepsilon\, y)$.
ii) If $\mathscr{M} \models u \in v$, *then* $\mathsf{I} \Vdash u \,\varepsilon\, \gimel v$.
iii) $\mathsf{I} \Vdash \forall x \forall y \forall \alpha^{\gimel 2} \big(\langle x < y \rangle \ge \alpha \hookrightarrow \alpha x \,\varepsilon\, \gimel \mathrm{Cl}(\{y\})\big)$.
iv) $\Vdash \forall x \forall y \big(\langle x < y \rangle = 1 \leftrightarrow x \,\varepsilon\, \gimel \mathrm{Cl}(y)\big)$.

i) Let a, b be two individuals. Let $\xi \Vdash \langle a < b \rangle \ne 1$, $\pi \in \| a \not\varepsilon\, b \|$; then $(a, \pi) \in b$ and therefore $\langle a < b \rangle = 1$ and $\xi \Vdash \perp$; thus $\xi \star \pi \in \perp\!\!\!\perp$.

ii) Indeed, we have $\| u \not\varepsilon\, \gimel v \| = \{\pi \in \Pi \,;\, (u, \pi) \in v \times \Pi\} = \Pi$.

iii) Let $\alpha \in \{0, 1\}$ and $a, b \in \mathscr{M}$ such that $\langle a < b \rangle \ge \alpha$.
If $\alpha = 0$, we must show $\mathsf{I} \Vdash \emptyset \,\varepsilon\, \gimel \mathrm{Cl}(\{y\})$ which follows from (ii).
If $\alpha = 1$, then $\langle a < b \rangle = 1$, that is $a \in \mathrm{Cl}(b)$, therefore $a \in \mathrm{Cl}(\{b\})$.
From (ii), it follows that $\mathsf{I} \Vdash a \,\varepsilon\, \gimel \mathrm{Cl}(\{b\})$.

iv) Indeed, if a, b are individuals of $\mathscr{M}$, we have trivially :
$\| \langle a < b \rangle \ne 1 \| = \| a \not\varepsilon\, \gimel \mathrm{Cl}(b) \|$.

Q.E.D.

Lemma 17. *The well founded relation* $\langle x < y \rangle = 1$ *is ranked, and its rank function* R *has for image the whole of* On.

Lemma 16(iv) shows that this relation is ranked.
Let ρ be an ordinal and r an individual $\simeq \rho$. We show, by induction on ρ, that $\mathsf{R}(r) \ge \rho$.
Indeed, for every $\rho' \in \rho$, there exists $r' \,\varepsilon\, r$ such that $r' \simeq \rho'$. We have $\mathsf{R}(r') \ge \rho'$ by induction hypothesis, and $\langle r' < r \rangle = 1$ from lemma 16(i). Therefore, we have $\rho' \in \mathsf{R}(r)$ by definition of R, and finally $\mathsf{R}(r) \ge \rho$. This shows that the image of R is not bounded in On. Since it is an initial segment, it is the whole of On.

Q.E.D.

Theorem 18. *Let* $F(x, y)$ *be a formula of* ZF_ε*, with parameters. Then, we have :*
$\mathsf{I} \Vdash \forall x \forall y \big(\forall \varpi^{\gimel \Pi} F(x, f(x, \varpi)) \to F(x, y)\big)$
for some function symbol f*, defined dans* $\mathscr{M}$*, with domain* $\mathscr{M} \times \Pi$.

Since the ground model $\mathcal{M}$ satisfies V = L (or only the *choice principle*), we can define, in $\mathcal{M}$, a function symbol f such that :

$$\forall x \forall y (\forall \varpi \in \Pi) \big(\varpi \in \|F(x,y)\| \to \varpi \in \|F(x, f(x,\varpi))\|\big).$$

Let a, b be individuals, $\xi \Vdash \forall \varpi^{\beth\Pi} F(a, f(a,\varpi))$ and $\pi \in \|F(a,b)\|$.
Thus, we have $\pi \in \|F(a, f(a,\pi))\|$, and therefore $\xi \star \pi \in \perp\!\!\!\perp$.

Q.E.D.

Definitions. Let a be any individual of $\mathcal{N}$ and κ an ordinal (therefore, κ is not an individual of $\mathcal{N}$, but an equivalence class for $\simeq$).
A *function* or *application* from κ into a is, by definition, a binary relation $R(\rho, x)$ such that : $\forall x \forall x' (\forall \rho, \rho' \in \kappa) \big(R(\rho,x), R(\rho',x'), \rho \simeq \rho' \to x = x'\big)$; $(\forall \rho \in \kappa)(\exists x \,\varepsilon\, a) R(\rho, x)$.
It is an *injection* if we have $\forall x (\forall \rho, \rho' \in \kappa) \big(R(\rho,x), R(\rho',x) \to \rho \simeq \rho'\big)$.
A *surjection* from a onto κ is a function f of domain a such that :
$(\forall \rho \in \kappa)(\exists x \,\varepsilon\, a) f(x) \simeq \rho$.

Theorem 19.
For any individual a, there exists an ordinal κ, such that there is no surjection from a onto κ.

Let f be a surjection from a onto an ordinal ρ. We define a strict ordering relation $\prec_f$ by setting $x \prec_f y \Leftrightarrow x \,\varepsilon\, a \wedge y \,\varepsilon\, a \wedge f(x) < f(y)$. It is clear that this relation is well founded, that f is an a-inductive function, and that $\mathsf{O}(f,a) \simeq \rho$.
We may consider this relation as a subset of $a \times a$.
By means of the axioms of union, power set and collection given above (theorems 1 to 4), we define an ordinal κ_0, which is the union of the $\mathsf{O}(f,a)$ for all the functions f which are a-inductive for some well founded strict ordering relation on a.
In fact, we consider the set :

$$\mathcal{B}(a) = \{X \,\varepsilon\, \overline{\mathcal{P}}(a \times a) \,;\, X \text{ is a well founded strict ordering relation on } a\}.$$

Then, we set $\kappa_0 = \overline{\bigcup}\{\mathsf{O}(f,a) \,;\, X \,\varepsilon\, \mathcal{B}(a), f \,\varepsilon\, \Phi(X,a)\}$.
In this definition, we use the function symbol Φ, defined after lemma 10, which associates with each well founded strict ordering relation X on a, a *non void* set of a-inductive functions for this relation.

Then, there exists no surjection from a onto $\kappa_0 + 1$.

Q.E.D.

We denote by Δ the first ordinal of $\mathcal{N}$ such that there is no surjection from $\beth\Pi$ onto Δ : for every function ϕ, there exists $\delta \in \Delta$ such that $\forall x^{\beth\Pi}(\phi(x) \neq \delta)$.

For each $\alpha \,\varepsilon\, \beth 2$, we denote by $\mathcal{N}_\alpha$ the class defined by the formula $x = \alpha x$.

Lemma 20. *Let $\alpha_0, \alpha_1 \,\varepsilon\, \beth 2$, $\alpha_0 \wedge \alpha_1 = 0$ and R_0 (resp. R_1) be a functional relation of domain $\mathcal{N}_{\alpha_0}$ (resp. $\mathcal{N}_{\alpha_1}$) with values in* On. *Then, either R_0, or R_1, is not surjective onto* Δ.

Proof by contradiction : we suppose that R_0 and R_1 are both surjective onto Δ.
We apply theorem 18 to the formula $F(x_0, x_1) \equiv \neg(R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1))$, and we get :

$$\forall x_0 \big(\exists x_1 (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1)) \to \exists \varpi^{\beth\Pi} (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 f(x_0, \varpi)))\big)$$

where f is a suitable function symbol (therefore *defined in* $\mathcal{M}$).
Replacing x_0 with $\alpha_0 x_0$, we obtain :

$$\forall x_0 \big(\exists x_1 (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1)) \to \exists \varpi^{\beth\Pi} (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 f(\alpha_0 x_0, \varpi)))\big).$$

But, by lemma 5(i), we have $\alpha_1 f(\alpha_0 x, \varpi) = \alpha_1 f(\alpha_1\alpha_0 x, \varpi) = \alpha_1 f(\emptyset, \varpi)$. It follows that :

$$\forall x_0 \big(\exists x_1 (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1)) \to \exists \varpi^{ℷ\Pi}(R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 f(\emptyset,\varpi)))\big).$$

By hypothesis, we have $(\forall \rho \in \Delta)\exists x_0 \exists x_1 (\rho \simeq R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1))$. It follows that :
$(\forall \rho \in \Delta)\exists x_0 \exists \varpi^{ℷ\Pi}\big(\rho \simeq R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 f(\emptyset,\varpi))\big)$; therefore, we have :
$(\forall \rho \in \Delta)\exists \varpi^{ℷ\Pi}\big(\rho \simeq R_1(\alpha_1 f(\emptyset,\varpi))\big)$.

Therefore, the function $\varpi \mapsto R_1(\alpha_1 f(\emptyset,\varpi))$ is a surjection from $ℷ\Pi$ onto Δ. But this is a contradiction with the definition de Δ.

Remark. We should write $f(\alpha_0,\alpha_1,x_0,\varpi)$ instead of $f(x_0,\varpi)$, since the function symbol f depends on the four variables $\alpha_0,\alpha_1,x_0,\varpi$. In fact, it depends also on the parameters which appear in R_0, R_1. The proof does not change.

Q.E.D.

Corollary 21. *Let $\alpha_0, \alpha_1 \,\varepsilon\, ℷ2$, $\alpha_0 \wedge \alpha_1 = 0$, and $\prec_0, \prec_1$ be two well founded ranked strict ordering relations with respective domains $\mathscr{N}_{\alpha_0}, \mathscr{N}_{\alpha_1}$. Let* Rk_0, Rk_1 *be their rank functions. Then, either the image of* Rk_0, *or that of* Rk_1 *is an ordinal* $< \Delta$.

In order to be able to define the rank functions Rk_0, Rk_1, we consider the relations $\prec'_0, \prec'_1$, with domain the whole of $\mathscr{N}$, defined by $x \prec'_i y \equiv (x = \alpha_i x) \wedge (y = \alpha_i y) \wedge (x \prec_i y)$ for $i = 0, 1$.
These strict ordering relations are well founded and ranked.
Their rank functions Rk'_0, Rk'_1 take the value 0 outside $\mathscr{N}_{\alpha_0}, \mathscr{N}_{\alpha_1}$ respectively : indeed, all the individuals outside $\mathscr{N}_{\alpha_i}$ are minimal for $\prec'_i$.
By lemma 20, one of them, Rk'_0 for instance, is not surjective onto Δ.
Since the image of any rank function is an initial segment of On, the image of Rk_0 is an ordinal $< \Delta$.

Q.E.D.

Theorem 22.
i) $\Vdash \forall \alpha_0^{ℷ2} \forall \alpha_1^{ℷ2} (\alpha_0 \wedge \alpha_1 = 0 \hookrightarrow (\mathscr{D}[\alpha_0], \mathscr{D}[\alpha_1] \to \bot))$.
ii) $\Vdash \forall \alpha_0^{ℷ2} \forall \alpha_1^{ℷ2} (\mathscr{D}[\alpha_0], \mathscr{D}[\alpha_1] \to \mathscr{D}[\alpha_0 \wedge \alpha_1])$.

i) In $\mathscr{N}$, let $\alpha_0, \alpha_1 \,\varepsilon\, ℷ2$ be such that $\alpha_0 \wedge \alpha_1 = 0$ and the relations $\langle x < y \rangle \geq \alpha_0$, $\langle x < y \rangle \geq \alpha_1$ be well founded. Therefore, we have $\alpha_0, \alpha_1 \neq 0$ (and thus, $\alpha_0, \alpha_1 \neq 1$).
Therefore, the relations $x \prec_i y \equiv (x = \alpha_i x) \wedge (y = \alpha_i y) \wedge (\langle x < y \rangle = \alpha_i)$ for $i = 0, 1$, are well founded strict orderings.
From lemma 16(iii), it follows that these relations are *ranked.*
Now, by lemma 5, we have : $\Vdash \forall x \forall y \forall \alpha^{ℷ2} (\langle x < y \rangle = 1 \to \langle \alpha x < \alpha y \rangle = \alpha)$.
But, by lemma 17, the rank function of the well founded relation $\langle x < y \rangle = 1$ has for image the whole of On. Therefore, by proposition 13, the same is true for the rank functions of the well founded strict order relations $x \prec_0 y$ and $x \prec_1 y$.
But this contradicts corollary 21.

ii) We have $\alpha_0 \leq (\alpha_0 \wedge \alpha_1) \vee (\neg \alpha_1)$. Therefore, by $\mathscr{D}[\alpha_0]$ and theorem 15, we have $\mathscr{D}[\alpha_0 \wedge \alpha_1]$ or $\mathscr{D}[\neg \alpha_1]$. But $\mathscr{D}[\neg \alpha_1]$ is impossible, by $\mathscr{D}[\alpha_1]$ and (i).

Q.E.D.

Corollary 23. *$\mathscr{D}[\alpha]$ is equivalent with each one of the following propositions :*
i) There exists a well founded ranked strict ordering relation $\prec$ with domain $\mathscr{N}_\alpha$, the rank

function of which has an image $\geq \Delta$.
ii) There exists a function with domain $\mathcal{N}_\alpha$ *which is surjective onto* Δ.

$\mathscr{D}[\alpha] \Rightarrow$ (i) :
By definition of $\mathscr{D}[\alpha]$, the binary relation $(x = \alpha x) \wedge (y = \alpha y) \wedge (\langle x < y\rangle = \alpha)$ is well founded. By lemma 16(iii), this relation is ranked. We have seen, in the proof of theorem 22, that the image of its rank function is the whole of On.

(i) $\Rightarrow$ (ii) : obvious.

(ii) $\Rightarrow \mathscr{D}[\alpha]$:
Since $\mathscr{D}$ is an ultrafilter, it suffices to show $\neg\mathscr{D}[\neg\alpha]$. But, (ii) and $\mathscr{D}[\neg\alpha]$ contradict lemma 20.
Q.E.D.

Theorem 24.
If $\daleth 2$ *is non trivial, there exists no set, which is totally ordered by* ε*, the ordinal of which is* $\geq \Delta$.

Let $\alpha \,\varepsilon\, \daleth 2, \alpha \neq 0, 1$ and X be a set which is totally ordered by ε, and equipotent with Δ.
Then, we show that the application $x \mapsto \alpha x$ is an injection from X into $\mathcal{N}_\alpha$:
Indeed, by lemma 16(i), we have $x \,\varepsilon\, y \to \langle x < y\rangle = 1$ and, by lemma 5, we have :
$\langle x < y\rangle = 1 \to \langle \alpha x < \alpha y\rangle = \alpha$. Therefore, if $x, y \,\varepsilon\, X$ and $x \neq y$, we have, for instance $x \,\varepsilon\, y$, therefore $\langle \alpha x < \alpha y\rangle = \alpha$ and therefore $\alpha x \neq \alpha y$ since $\alpha \neq 0$.

Thus, there exists a function with domain $\mathcal{N}_\alpha$ which is surjective onto Δ. The same reasoning, applied to $\neg\alpha$ gives the same result for $\neg\alpha$. But this contradicts lemma 20.
Q.E.D.

Remark. Theorem 24 shows that it is impossible to define Von Neumann ordinals in $\mathcal{N}$, with ε instead of $\in$, unless $\daleth 2$ is trivial, i.e. the realizability model is, in fact, a forcing model.

The model $\mathscr{M}_{\mathscr{D}}$

For each formula $F[x_1, \ldots, x_n]$ of ZF, we have defined, in the ground model $\mathscr{M}$, an n-ary function symbol with values in $\{0, 1\}$, denoted by $\langle F[x_1, \ldots, x_n]\rangle$, by setting, for any individuals $a_1, \ldots, a_n$ of $\mathscr{M}$: $\langle F[a_1, \ldots, a_n]\rangle = 1 \Leftrightarrow \mathscr{M} \models F[a_1, \ldots, a_n]$.
In $\mathcal{N}$, the function symbol $\langle F[x_1, \ldots, x_n]\rangle$ takes its values in the Boolean algebra $\daleth 2$.

We define, in $\mathcal{N}$, two binary relations $\in_{\mathscr{D}}$ and $=_{\mathscr{D}}$, by setting :

$$(x \in_{\mathscr{D}} y) \equiv \mathscr{D}[\langle x \in y\rangle] \,;\; (x =_{\mathscr{D}} y) \equiv \mathscr{D}[\langle x = y\rangle].$$

The class $\mathcal{N}$, equipped with these relations, will be denoted $\mathscr{M}_{\mathscr{D}}$.

For each formula $F[\vec{x}, y]$ of ZF, with $n+1$ free variables $x_1, \ldots, x_n, y$, we can define, by means of the *choice principle* in $\mathscr{M}$, an n-ary function symbol f_F, such that :

$$\mathscr{M} \models \forall \vec{x}\big(F[\vec{x}, f_F(\vec{x})] \to \forall y\, F[\vec{x}, y]\big) \,;$$

f_F is called the *Skolem function* of the formula $F[\vec{x}, y]$.

Lemma 25.
i) $\mathsf{I} \Vdash \forall \vec{x} \forall y \big(\langle \forall y\, F[\vec{x}, y]\rangle \leq \langle F[\vec{x}, y]\rangle\big)$
ii) $\mathsf{I} \Vdash \forall \vec{x} \forall y \big(\langle \forall y\, F[\vec{x}, y]\rangle = \langle F[\vec{x}, f_F(\vec{x})]\rangle\big)$.

Trivial.

Q.E.D.

For each formula $F[\vec{x}]$ of ZF, we define, by recurrence on F, *a formula of* ZF_ε, which has the same free variables, and that we denote $\mathscr{M}_\mathscr{D} \models F[\vec{x}]$ (read : $\mathscr{M}_\mathscr{D}$ *satisfies* $F[\vec{x}]$).

- F is atomic :

$(\mathscr{M}_\mathscr{D} \models x_1 \in x_2)$ is $x_1 \in_\mathscr{D} x_2$; $(\mathscr{M}_\mathscr{D} \models x_1 = x_2)$ is $x_1 =_\mathscr{D} x_2$; $(\mathscr{M}_\mathscr{D} \models \bot)$ is $\bot$.

- $F \equiv F_0 \to F_1$: then $(\mathscr{M}_\mathscr{D} \models F)$ is the formula $(\mathscr{M}_\mathscr{D} \models F_0) \to (\mathscr{M}_\mathscr{D} \models F_1)$.
- $F[\vec{x}] \equiv \forall y\, G[\vec{x}, y]$: then $(\mathscr{M}_\mathscr{D} \models F[\vec{x}])$ is the formula $\forall y(\mathscr{M}_\mathscr{D} \models G[\vec{x}, y])$.

Lemma 26. *For each formula* $F[\vec{x}]$ *of ZF, we have* $\Vdash \forall \vec{x}\Big((\mathscr{M}_\mathscr{D} \models F[\vec{x}]) \leftrightarrow \mathscr{D}\langle F[\vec{x}]\rangle\Big)$.

Proof by recurrence on the length of F. If F is atomic, we have :

$\mathsf{I} \Vdash \forall \vec{x}\Big((\mathscr{M}_\mathscr{D} \models F[\vec{x}]) \to \mathscr{D}\langle F[\vec{x}]\rangle\Big)$ and $\mathsf{I} \Vdash \forall \vec{x}\Big(\mathscr{D}\langle F[\vec{x}]\rangle \to (\mathscr{M}_\mathscr{D} \models F[\vec{x}])\Big)$
because $(\mathscr{M}_\mathscr{D} \models F[\vec{x}])$ is identical with $\mathscr{D}\langle F[\vec{x}]\rangle$.

If $F \equiv F_0 \to F_1$, the formula $(\mathscr{M}_\mathscr{D} \models F) \leftrightarrow \mathscr{D}\langle F\rangle$ is :
$((\mathscr{M}_\mathscr{D} \models F_0) \to (\mathscr{M}_\mathscr{D} \models F_1)) \leftrightarrow \mathscr{D}\langle F_0 \to F_1\rangle$.
Since $\mathscr{D}$ is an ultrafilter, this formula is equivalent with :
$((\mathscr{M}_\mathscr{D} \models F_0) \to (\mathscr{M}_\mathscr{D} \models F_1)) \leftrightarrow (\mathscr{D}\langle F_0\rangle \to \mathscr{D}\langle F_1\rangle)$, which is a logical consequence of :
$(\mathscr{M}_\mathscr{D} \models F_0) \leftrightarrow \mathscr{D}\langle F_0\rangle$ and $(\mathscr{M}_\mathscr{D} \models F_1) \leftrightarrow \mathscr{D}\langle F_1\rangle$.
Hence the result, by the recurrence hypothesis.

If $F[\vec{x}] \equiv \forall y\, G[\vec{x}, y]$, let $f_G(\vec{x})$ be the Skolem function of G.
Then, we have $(\mathscr{M}_\mathscr{D} \models \forall y\, G[\vec{x}, y]) \equiv \forall y(\mathscr{M}_\mathscr{D} \models G[\vec{x}, y])$, and therefore :
$\mathsf{I} \Vdash (\mathscr{M}_\mathscr{D} \models \forall y\, G[\vec{x}, y]) \to (\mathscr{M}_\mathscr{D} \models G[\vec{x}, f_G(\vec{x})])$.
Therefore, by the recurrence hypothesis, we have :
$\Vdash (\mathscr{M}_\mathscr{D} \models \forall y\, G[\vec{x}, y]) \to \mathscr{D}\langle G[\vec{x}, f_G(\vec{x})]\rangle$.
Applying lemma 25(ii), we obtain $\Vdash (\mathscr{M}_\mathscr{D} \models \forall y\, G[\vec{x}, y]) \to \mathscr{D}\langle \forall y\, G[\vec{x}, y]\rangle$.
Conversely, by lemma 25(i), we have $\Vdash \forall y\big(\mathscr{D}\langle \forall y\, G[\vec{x}, y]\rangle \to \mathscr{D}\langle G[\vec{x}, y]\rangle\big)$.
Therefore, applying the recurrence hypothesis, we obtain :
$\Vdash \mathscr{D}\langle \forall y\, G[\vec{x}, y]\rangle \to \forall y(\mathscr{M}_\mathscr{D} \models G[\vec{x}, y])$, and thus, by definition of $(\mathscr{M}_\mathscr{D} \models \forall y\, G[\vec{x}, y])$:
$\Vdash \mathscr{D}\langle \forall y\, G[\vec{x}, y]\rangle \to (\mathscr{M}_\mathscr{D} \models \forall y\, G[\vec{x}, y])$.

We can manage the case $F[\vec{x}] \equiv \forall y\, G[\vec{x}, y]$ without using Skolem functions, and therefore without assuming that $\mathscr{M} \models AC$. We have to show $\Vdash \forall y(\mathscr{M}_\mathscr{D} \models G[\vec{x}, y]) \leftrightarrow \mathscr{D}\langle \forall y\, G[\vec{x}, y]\rangle$.
By the recurrence hypothesis, we have $\Vdash \forall y(\mathscr{M}_\mathscr{D} \models G[\vec{x}, y] \leftrightarrow \mathscr{D}\langle G[\vec{x}, y]\rangle)$.
Thus it suffices to show $\Vdash \forall \vec{x}(\forall y\, \mathscr{D}\langle G[\vec{x}, y]\rangle \leftrightarrow \mathscr{D}\langle \forall y\, G[\vec{x}, y]\rangle)$. In fact, we show :
$\|\forall y\, \mathscr{D}\langle G[\vec{a}, y]\rangle\| = \|\mathscr{D}\langle \forall y\, G[\vec{a}, y]\rangle\|$ for every $\vec{a} \in \mathscr{M}$.
If $\mathscr{M} \models \forall y\, G[\vec{a}, y]$, we have $\langle \forall y\, G[\vec{a}, y]\rangle = 1$ and $\langle G[\vec{a}, b]\rangle = 1$ for every $b \in \mathscr{M}$.
Thus $\|\forall y\, \mathscr{D}\langle G[\vec{a}, y]\rangle\| = \|\mathscr{D}\langle \forall y\, G[\vec{a}, y]\rangle\| = \|\mathscr{D}(1)\|$.
If $\mathscr{M} \models \neg\forall y\, G[\vec{a}, y]$, we have $\langle \forall y\, G[\vec{a}, y]\rangle = 0$ and $\langle G[\vec{a}, b]\rangle = 0$ for some $b \in \mathscr{M}$. Moreover $\langle G[\vec{a}, c]\rangle = 0$ or 1 for any $c \in \mathscr{M}$. Therefore, we have :
i) $\|\mathscr{D}\langle \forall y\, G[\vec{a}, y]\rangle\| = \|\mathscr{D}(0)\|$;
ii) $\|\forall y\, \mathscr{D}\langle G[\vec{a}, y]\rangle\| = \bigcup_c \|\mathscr{D}\langle G[\vec{a}, c]\rangle\|$ and thus $\|\mathscr{D}(0)\| \subset \|\forall y\, \mathscr{D}\langle G[\vec{a}, y]\rangle\| \subset \|\mathscr{D}(0)\| \cup \|\mathscr{D}(1)\|$.
It follows that $\|\forall y\, \mathscr{D}\langle G[\vec{a}, y]\rangle\| = \|\mathscr{D}(0)\|$ because $\|\mathscr{D}(1)\| \subset \|\mathscr{D}(0)\|$.

Q.E.D.

Theorem 27. *$\mathscr{M}_{\mathscr{D}}$ is an elementary extension of the ground model $\mathscr{M}$.*

Remark. Theorem 27 is, in fact, true for *any* ultrafilter on $\daleth 2$, with the same proof, *when $\mathscr{M}$ satisfies the principle of choice.*

Let $F[\vec{a}]$ be a closed formula of ZF, with parameters $a_1, \ldots, a_n$ in $\mathscr{M}$.
If $\mathscr{M} \models F[\vec{a}]$, we have $\langle F[\vec{a}]\rangle = 1$ (by definition), and therefore, of course, $\Vdash \mathscr{D}\langle F[\vec{a}]\rangle$.
Therefore, by lemma 26, we have $\Vdash (\mathscr{M}_{\mathscr{D}} \models F[\vec{a}])$.
If $\mathscr{M} \not\models F[\vec{a}]$, then $\mathscr{M} \models \neg F[\vec{a}]$; therefore, we have $\Vdash (\mathscr{M}_{\mathscr{D}} \models \neg F[\vec{a}])$.
Q.E.D.

Theorem 28. *Let $\sqsubset$ be a well founded binary relation, defined in the ground model $\mathscr{M}$. Then the relation $\mathscr{D}\langle x \sqsubset y\rangle$ is well founded in the realizability model $\mathscr{N}$.*

Remark. Theorem 28 is an improvement on theorem 8.

Notations. We shall write $x \sqsubset_{\mathscr{D}} y$ for $\langle x \sqsubset y\rangle \,\varepsilon\, \mathscr{D}$.
Recall that $x < y$ means $x \in \mathrm{Cl}(y)$; and that $x <_\alpha y$ means $\langle x < y\rangle \geq \alpha$, for $\alpha \,\varepsilon\, \daleth 2$.

We define, in the model $\mathscr{M}$, a binary relation $\sqsubset\!\!\sqsubset$ on the class $\{0,1\} \times \mathscr{M}$ by setting, for any $\alpha, \alpha' \in \{0,1\}$ and a, a' in $\mathscr{M}$:

$$(\alpha', a') \sqsubset\!\!\sqsubset (\alpha, a) \Leftrightarrow (\alpha' < \alpha) \vee (\alpha = \alpha' = 0 \wedge a' < a) \vee (\alpha = \alpha' = 1 \wedge a' \sqsubset a).$$

The relation $\sqsubset\!\!\sqsubset$ is the *ordered direct sum* of the relations $\sqsubset, <$.
It is easily shown that it is *well founded in $\mathscr{M}$.*

The binary function symbol associated with this relation, of domain $\{0,1\} \times \mathscr{M}$ and values in $\{0,1\}$, is given by :

$$\langle(\alpha', a') \sqsubset\!\!\sqsubset (\alpha, a)\rangle = (\neg\alpha' \wedge \alpha) \vee (\neg\alpha' \wedge \neg\alpha \wedge \langle a' < a\rangle) \vee (\alpha' \wedge \alpha \wedge \langle a' \sqsubset a\rangle).$$

This definition gives, in $\mathscr{N}$, a binary function symbol with arguments in $\daleth 2 \times \mathscr{N}$, and values in $\daleth 2$.
By theorem 8, *the binary relation $\langle(\alpha', a') \sqsubset\!\!\sqsubset (\alpha, a)\rangle = 1$ is well founded in $\mathscr{N}$.*

Proof of theorem 28.
Proof by contradiction : we assume that the binary relation $\sqsubset_{\mathscr{D}}$ is not well founded.
Thus, there exists a_0, A_0 such that $a_0 \,\varepsilon\, A_0$ and A_0 has no minimal ε-element for $\sqsubset_{\mathscr{D}}$.
We define, in $\mathscr{N}$, the class $\mathscr{X}$ of ordered pairs (α, x), such that :
There exists X such that $x \,\varepsilon\, X$ and X has no minimal ε-element, neither for $\sqsubset_{\mathscr{D}}$ nor for $<_{\neg\alpha}$.
Therefore, the formula $\mathscr{X}(\alpha, x)$ is :

$$\alpha \,\varepsilon\, \daleth 2 \wedge \exists X\Big\{x \,\varepsilon\, X, (\forall u \,\varepsilon\, X)\{(\exists v \,\varepsilon\, X)(v \sqsubset_{\mathscr{D}} u), (\exists w \,\varepsilon\, X)(w <_{\neg\alpha} u)\}\Big\}.$$

If (α, x) is in $\mathscr{X}$, then we have $\mathscr{D}(\alpha)$: indeed, the set X is non void and has no minimal ε-element for $<_{\neg\alpha}$. Therefore, we have $\neg\mathscr{D}(\neg\alpha)$, and thus $\mathscr{D}(\alpha)$, since $\mathscr{D}$ is an ultrafilter.

We obtain the desired contradiction by showing that the class $\mathscr{X}$ is non void and has no minimal element for the binary relation $\langle(\alpha', x') \sqsubset\!\!\sqsubset (\alpha, x)\rangle = 1$.

The ordered pair $(1, a_0)$ is in $\mathscr{X}$: indeed, we have $x <_0 x$ for every x, and therefore A_0 has no minimal ε-element for $<_0$.

Now let (α, a) be in $\mathscr{X}$; we search for (α', a') in $\mathscr{X}$ such that $\langle(\alpha', a') \sqsubset\!\!\sqsubset (\alpha, a)\rangle = 1$.

By hypothesis on (α, a), there exists A such that $a \,\varepsilon\, A$ and A has no minimal ε-element, neither for $\sqsubset_{\mathscr{D}}$ nor for $<_{\neg\alpha}$. Thus, there exists $a^0, a^1 \,\varepsilon\, A$ such that we have $\mathscr{D}\langle a^0 \sqsubset a\rangle$ and

$a^1 <_{\neg\alpha} a$.
We set $\alpha' = (\alpha \wedge \langle a^0 \sqsubset a \rangle)$ and therefore, we have $\mathscr{D}(\alpha')$. We set $\beta = \neg\alpha' \wedge \alpha$; therefore $\alpha', \neg\alpha, \beta$ form a partition of 1 in the Boolean algebra $\beth 2$.
We have $\neg\mathscr{D}(\beta)$; therefore, by definition of $\mathscr{D}$, the relation $<_\beta$ is not well founded. Thus, there exists b, B such that $b \,\varepsilon\, B$ and B has no minimal ε-element for $<_\beta$. Then, we set :
$a' = \alpha' a^0 \sqcup (\neg\alpha) a^1 \sqcup \beta b$ and $A' = \{\alpha' x \sqcup (\neg\alpha) y \sqcup \beta z \,;\, x, y \,\varepsilon\, A, z \,\varepsilon\, B\}$.
Therefore, we have $a' \,\varepsilon\, A'$, as needed ; moreover :
$\neg\alpha' \wedge \neg\alpha \wedge \langle a' < a \rangle = \neg\alpha$, since $\neg\alpha' \geq \neg\alpha$ and $\langle a' < a \rangle \geq \neg\alpha \wedge \langle a^1 < a \rangle = \neg\alpha$;
$\alpha' \wedge \alpha \wedge \langle a' \sqsubset a \rangle = \alpha' \wedge \langle a' \sqsubset a \rangle = \alpha' \wedge \langle a^0 \sqsubset a \rangle = \alpha'$.
By definition of $\langle (\alpha', a') \sqsubset\!\!\sqsubset (\alpha, a) \rangle$, it follows that $\langle (\alpha', a') \sqsubset\!\!\sqsubset (\alpha, a) \rangle = \beta \vee \neg\alpha \vee \alpha' = 1$.
It remains to show that A' has no minimal ε-element for $\sqsubset_{\mathscr{D}}$ and for $<_{\neg\alpha'}$.
Therefore, let $u \,\varepsilon\, A'$, thus $u = \alpha' x \sqcup (\neg\alpha) y \sqcup \beta z$ with $x, y \,\varepsilon\, A$ and $z \,\varepsilon\, B$.
By hypothesis on A, B, there exists $x', y' \,\varepsilon\, A$, $x' \sqsubset_{\mathscr{D}} x$, $y' <_{\neg\alpha} y$ and $z' \,\varepsilon\, B$, $z' <_\beta z$.
Then, if we set $u' = \alpha' x' \sqcup (\neg\alpha) y' \sqcup \beta z'$, we have $u' \,\varepsilon\, A'$.
Moreover, we have $\langle u' \sqsubset u \rangle \geq \alpha' \wedge \langle x' \sqsubset x \rangle$, and therefore $\mathscr{D}\langle u' \sqsubset u \rangle$, that is $u' \sqsubset_{\mathscr{D}} u$.
Finally, $\langle u' < u \rangle \geq (\neg\alpha \wedge \langle y' < y \rangle) \vee (\beta \wedge \langle z' < z \rangle) = \neg\alpha \vee \beta = \neg\alpha'$; therefore, we have $u' <_{\neg\alpha'} u$.

Q.E.D.

Theorem 29. *$\mathscr{M}_{\mathscr{D}}$ is well founded, and therefore has the same ordinals as $\mathscr{N}'_\in$.*

We apply theorem 28 to the binary relation $\in$ which is well founded in $\mathscr{M}$. We deduce that the relation $\mathscr{D}\langle x \in y \rangle$, that is $x \in_{\mathscr{D}} y$, is well founded in $\mathscr{N}$.

Q.E.D.

The relation $\in_{\mathscr{D}}$ is well founded *and extensional*, which means that we have, in $\mathscr{N}$:

$$\forall x \forall y \big(\forall z (z \in_{\mathscr{D}} x \leftrightarrow z \in_{\mathscr{D}} y) \to \forall z (x \in_{\mathscr{D}} z \to y \in_{\mathscr{D}} z)\big).$$

It follows that we can define a *collapsing*, by means of a function symbol Φ, which is an *isomorphism* of $(\mathscr{M}_{\mathscr{D}}, \in_{\mathscr{D}})$ on a transitive class in the model $\mathscr{N}_\in$ of ZF, which contains the ordinals. This means that we have :

$$\forall x \forall y (y \in_{\mathscr{D}} x \to \Phi(y) \in \Phi(x)) \; ; \; \forall x (\forall z \in \Phi(x))(\exists y \in_{\mathscr{D}} x)\, z \simeq \Phi(y).$$

The definition of Φ is analogous with that of the *rank function* already defined for a *transitive* well founded relation. The details will be given in a later version of this paper.
Il follows that :

Theorem 30. *The realizability model $\mathscr{N}_\in$ contains a transitive class, which contains the ordinals and is an elementary extension of the ground model $\mathscr{M}$.*

Corollary 31. *The class $L^{\mathscr{M}}$ of constructible sets in $\mathscr{M}$ is an elementary submodel of $L^{\mathscr{N}}$.*

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