Some properties of realizability models

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Outline

We give some general properties of classical realizability
and we look at some particular models :

- True *arithmetical formulas*, and even true Π^1_1 *formulas* are realized ;

thus, realizability models cannot give indecidability results in arithmetic.

- A model is given by forcing iff its Boolean algebra $\gimel 2$ *is trivial.*
- We build models in which $\gimel 2$ *is non trivial* and *finite.*
- Following T. Ehrhard and T. Streicher, the usual models of lambda-calculus

have, in fact, a structure of realizability algebra.
Therefore, they give rise to realizability models of ZF.
We study a simple case, in which $\gimel 2$ *is non trivial* and *integers are preserved.*

A game on first order formulas

We consider first order formulas written with :

$\to$, $\forall$, $\top$, $\bot$, $\neq$, predicate constants, function symbols for recursive functions.

A 1st order formula has the form $\forall \vec{x}[\Phi_1, \ldots, \Phi_n \to A]$ where $\Phi_1, \ldots, \Phi_n$ are 1st order formulas and A is atomic (i.e. $R t_1 \ldots t_k$ or $t_0 \neq t_1$ or $\top$ or $\bot$).

In the following, we only consider *closed* 1st order formulas.

The atomic closed formula $t_0 \neq t_1$ is interpreted as $\top$ (resp. $\bot$) if it is true (resp. false) in $\mathbb{N}$.

We define a game with two players : $\exists$ (the *client*) and $\forall$ (the *server*).

At each step, the *position* is a sequent $\mathscr{U} \vdash \mathscr{A}$ with closed 1st order formulas ; the formulas of $\mathscr{A}$ are atomic and $\bot \in \mathscr{A}$; $\mathscr{U}$ and $\mathscr{A}$ increase at each step.

The game starts with a sequent $\mathscr{U}_0 \vdash \mathscr{A}_0$.

A move in this game is as follows :

Player $\exists$ chooses $\Psi \in \mathscr{U}$, $\Psi = \forall \vec{y}[\Phi_1(\vec{y}), \ldots, \Phi_n(\vec{y}) \to B(\vec{y})]$

and $\vec{j} \in \mathbb{N}^l$ such that $B(\vec{j}) \in \mathscr{A}$ (if this is impossible, then $\exists$ has lost).

Player $\forall$ chooses a formula $\Phi \in \mathscr{V} = \{\Phi_1(\vec{j}), \ldots, \Phi_n(\vec{j})\}$,

$\Phi \equiv \forall \vec{x}[\Psi_1(\vec{x}), \ldots, \Psi_m(\vec{x}) \to A(\vec{x})]$; $\forall$ chooses also $\vec{i} \in \mathbb{N}^k$.

The atomic formula $A(\vec{i})$ must not be $\top$ (otherwise, $\forall$ has lost).

Then $\Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$ *are added* to $\mathscr{U}$ and $A(\vec{i})$ *is added* to $\mathscr{A}$.

$\exists$ wins iff $\forall$ cannot play at some step

(every formula of $\mathscr{V}$ ends with $\top$, in particular if $\mathscr{V} = \emptyset$).

In fact, player $\forall$ tries to build a model over $\mathbb{N}$ in which

the formula $\mathscr{V}_0 = \bigwedge \mathscr{U}_0 \to \bigvee \mathscr{A}_0$ is false, and $\exists$ tries to avoid this :

Theorem. i) Any model $\mathscr{M}$ over $\mathbb{N}$ s.t. $\mathscr{M} \not\models \mathscr{V}_0$ gives a winning strategy for $\forall$.
ii) There exists a "trivial" strategy for the player $\exists$ such that each play $\exists$ loses using it, gives a model $\mathscr{M}$ over $\mathbb{N}$, $\mathscr{M} \not\models \mathscr{V}_0$.

i) We define a strategy for $\forall$ such that, at each step :
every formula of $\mathscr{U}$ (resp. $\mathscr{A}$) is true (resp. false) in $\mathscr{M}$.
This is true at the beginning of the game.
Then $\exists$ chooses $\Psi \in \mathscr{U}$, $\Psi = \forall \vec{y}[\Phi_1(\vec{y}), \ldots, \Phi_n(\vec{y}) \to B(\vec{y})]$ and $\vec{j} \in \mathbb{N}^l$
such that $B(\vec{j}) \in \mathscr{A}$. Therefore, $\mathscr{M} \models \neg B(\vec{j})$ and $\mathscr{M} \models \Psi$.
Thus, $\forall$ can choose $\Phi \in \mathscr{V} = \{\Phi_1(\vec{j}), \ldots, \Phi_n(\vec{j})\}$ s.t. $\mathscr{M} \models \neg\Phi$.
Let $\Phi = \forall \vec{x}[\Psi_1(\vec{x}), \ldots, \Psi_m(\vec{x}) \to A(\vec{x})]$.
Then $\forall$ can choose $\vec{i} \in \mathbb{N}^k$ s.t. $\mathscr{M} \models \Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$ and $\neg A(\vec{i})$.
Finally $\Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$ are added to $\mathscr{U}$ and $A(\vec{i})$ to $\mathscr{A}$.
Thus $\mathscr{U}$ and the negation of formulas of $\mathscr{A}$ remain true in $\mathscr{M}$.

ii) Here is the “trivial” strategy for $\exists$:

fix an enumeration of all ordered pairs $<\Psi, \vec{j}>$ *(*Ψ *is a closed formula,* $\vec{j} \in \mathbb{N}^l$*).*

At each step, $\exists$ *chooses the first allowed pair* $<\Psi, \vec{j}>$*, not chosen before.*

Suppose $\exists$ loses some play with this strategy. Let $\mathscr{M}$ be the model which satisfies exactly the closed atomic formulas never put in $\mathscr{A}$ during this play.

A pair $<\Psi, \vec{j}>$ is called *acceptable* if Ψ is put in $\mathscr{U}$ and $B(\vec{j})$ in $\mathscr{A}$ at some step (not necessarily the same) where $B(\vec{y})$ is the final atom of Ψ.

Every acceptable pair is effectively played by $\exists$ at some step :

namely when every acceptable pair strictly less than it has been played.

We prove, by induction, that $\mathscr{M}$ satisfies *every formula* Ψ *which is put in* $\mathscr{U}$ and the negation of *every formula* Φ *chosen by* $\forall$ during the play.

Proof for Ψ. The result is clear if Ψ is atomic because, if Ψ is both in $\mathscr{U}$ and $\mathscr{A}$ then $<\Psi, \emptyset>$ is acceptable and thus will be chosen by $\exists$; then $\exists$ wins.
Otherwise, let $\Psi = \forall \vec{y}[\Phi_1(\vec{y}), \ldots, \Phi_n(\vec{y}) \to B(\vec{y})]$. We must show that $\mathscr{M} \models \Phi_1(\vec{j}), \ldots, \Phi_n(\vec{j}) \to B(\vec{j})$ for every $\vec{j} \in \mathbb{N}^k$.
This is clear if $B(\vec{j})$ is never put in $\mathscr{A}$, because $\mathscr{M} \models B(\vec{j})$.
Otherwise, $<\Psi, \vec{j}>$ is acceptable and is chosen by $\exists$ at some step.
Then $\mathscr{V} = \{\Phi_1(\vec{j}), \ldots, \Phi_n(\vec{j})\}$ and $\Phi_1(\vec{j})$, for instance, is chosen by $\forall$.
By induction hypothesis, we have $\mathscr{M} \models \neg\Phi_1(\vec{j})$, which gives the result.
Proof for Φ. Let $\Phi = \forall \vec{x}[\Psi_1(\vec{x}), \ldots, \Psi_m(\vec{x}) \to A(\vec{x})]$; $\forall$ chooses $\vec{i}$ and puts $A(\vec{i})$ in $\mathscr{A}$ and $\Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$ in $\mathscr{U}$. By induction hypothesis, $\mathscr{M} \models \Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$; and, by definition, $\mathscr{M} \not\models A(\vec{i})$. Thus $\mathscr{M} \models \neg\Phi$.
It follows that $\mathscr{M} \not\models \mathscr{V}_0$ since $\mathscr{M} \models \mathscr{U}_0$ and $\mathscr{M} \models \neg A$ for $A \in \mathscr{A}_0$. QED

Well founded recursive relations

Let $f : \mathbb{N}^2 \to \{0,1\}$ be arbitrary. The predicate $f(x,y) = 1$ is well founded iff the formula $\forall X \forall z\{\forall x[\forall y(f(x,y) = 1 \to Xy) \to Xx] \to Xz\}$ is true in $\mathbb{N}$.

We show that, in this case, this formula is even *realized*.

Theorem. If the predicate $f(x,y) = 1$ is well founded, then
$\mathsf{Y} \Vdash \forall X \forall z\{\forall x[\forall y(f(x,y) = 1 \mapsto Xy) \to Xx] \to Xz\}$.

Let $t \Vdash \forall x[\forall y(f(x,y) = 1 \mapsto Xy) \to Xx]$ and $n \in \mathbb{N}$; we show by induction on n, following the well founded predicate " $f(x,y) = 1$ ", that $\mathsf{Y}t \Vdash Xn$.

Since $\mathsf{Y}t \star \pi \succ t \star \mathsf{Y}t \,.\, \pi$, it suffices to show that $\mathsf{Y}t \Vdash \forall y(f(n,y) = 1 \mapsto Xy)$
i.e. $\mathsf{Y}t \Vdash f(n,p) = 1 \mapsto Xp$. This is trivial if $f(n,p) \neq 1$
and this follows from the induction hypothesis if $f(n,p) = 1$.

Thus, if $\pi \in \|Xn\|$, we have $t \star \mathsf{Y}t \,.\, \pi \in \perp\!\!\!\perp$ and therefore $\mathsf{Y} \star t \,.\, \pi \in \perp\!\!\!\perp$. QED

This shows that a *recursive* well founded predicate on integers is also well founded *in every realisability model*.

True Π^1_1 formulas

A Π^1_1 formula is of the form $F \equiv \forall \vec{X} \Phi[\vec{X}]$ where Φ is a 1st order formula written with the function symbols $0, 1, +, \times$ and the predicate symbols $\neq, \vec{X}$.

Theorem. If F is a true Π^1_1 formula, then F^{int} is realized.

This shows, in particular, that the integers of any realizability model are *elementary equivalent* to the integers of the ground model.

It is not possible to show the independence of some *arithmetical* (and even Π^1_1) formula by means of realizability models.

Open problems : What about Σ^1_1 (or higher) formulas ?

Are the *constructible universes* of the ground model and the realizability model elementarily equivalent ? This is (trivially) true in the case of forcing.

Proof. Fix a recursive enumeration of closed formulas and also of sequents $\mathscr{U} \vdash \mathscr{A}$.
Let $F \equiv \forall \vec{X} \neg \Phi[\vec{X}]$ be a *true* Π^1_1 formula.
The meaning of F is that the 1st order formula $\Phi \to \perp$ has no model.
Thus, *the "trivial" strategy for $\exists$ is winning*
in the game which starts with the sequent $\Phi \vdash \perp$.
Now, let $f(x,y) = 1$ be the recursive predicate which says that
x, y are (numbers of) successive positions chosen by $\forall$ such that, between them,
$\exists$ has applied (once) the trivial strategy.
This strategy is winning for $\exists$ iff each play is finite, i.e. iff
the predicate $f(x,y) = 1$ is well founded.
Now, by the above theorem, we obtain :
$\mathsf{Y} \Vdash \forall X\{\forall x[\forall y(f(x,y) = 1 \mapsto Xy) \to Xx] \to \forall x\, Xx\}$.
But we have just proved that : "$f(x,y) = 1$ is well founded" $\to F$.
Let θ be a proof-like term associated with this proof. Then $\theta\mathsf{Y} \Vdash F$. QED

The case of arithmetical formulas

An arithmetical formula is of the form

$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n (f(x_1, y_1, \dots, x_n, y_n) \neq 0)$

where $f : \mathbb{N}^{2n} \to \{0,1\}$ is recursive.

Theorem. Let $f : \mathbb{N}^{2n} \to \{0,1\}$ be an arbitrary function, such that $\forall x_1 \exists y_1 \dots \forall x_n \exists y_n (f(x_1, y_1, \dots, x_n, y_n) \neq 0)$ is true in $\mathbb{N}$. Then $\forall x_1 \exists y_1^{\mathrm{int}} \dots \forall x_n \exists y_n^{\mathrm{int}} (f(x_1, y_1, \dots, x_n, y_n) \neq 0)$ is realized by a proof-like term that depends only on n.

This theorem shows once again that any true arithmetical formula is realized.

For $n = 1$, the proof is very simple :

Theorem. Let $\theta \in \mathrm{QP}$ be such that $\theta \star \underline{n}.\xi.\pi \succ \xi \star \underline{n}.\theta\underline{n}^{+}\xi.\pi$ with $\underline{n}^{+} = (s)\underline{n}$.
Then $\theta\underline{0} \Vdash \forall x\Big(\forall y^{\mathrm{int}}(f(x,y) \neq 0 \to \bot) \to \bot\Big)$
for every $f : \mathbb{N}^2 \to 2$ such that $\mathbb{N} \models \forall x \exists y (f(x,y) = 1)$.

We simply need to prove $\theta\underline{0} \Vdash \forall y^{\mathrm{int}}(f(y) \neq 0 \to \bot) \to \bot$
for every $f : \mathbb{N} \to 2$ such that $\mathbb{N} \models \exists y (f(y) = 1)$.

Lemma. Let $\xi \Vdash \forall y^{\mathrm{int}}(f(y) \neq 0 \to \bot)$; if $\theta\underline{n}\xi \not\Vdash \bot$, then $f(n) = 0$ and $\theta\underline{n}^{+}\xi \not\Vdash \bot$.

We have $\theta \star \underline{n}.\xi.\pi \notin \perp\!\!\!\perp$, thus $\xi \star \underline{n}.\theta\underline{n}^{+}\xi.\pi \notin \perp\!\!\!\perp$;

therefore $\theta\underline{n}^{+}\xi \not\Vdash f(n) \neq 0$ hence the result. QED

Suppose $\theta \star \underline{0}.\xi.\pi \notin \perp\!\!\!\perp$; the lemma gives $f(n) = 0$ for all $n \in \mathbb{N}$, a contradiction. QED

We consider now the case $n=2$, which is typical for the general case.

Theorem. Let $\theta = \lambda x \lambda t \lambda \sigma \lambda m \lambda n (x m) \lambda y (H \sigma m y n)((t)(\Sigma)\sigma m y) m' n'$ where H, Σ are closed λ-terms defined below ; $<m', n'>$ is the successor of $<m, n>$ in $\mathbb{N}^2$.
Then, for every $f : \mathbb{N}^3 \to \{0,1\}$, there exists $\phi : \mathbb{N} \to \mathbb{N}$ such that :
$\lambda x((\mathsf{Y})(\theta)x)000 \Vdash \forall x^{\mathsf{int}} \exists y \forall z^{\mathsf{int}} (f[x,y,z]=1) \to \forall x \forall z (f[x, \phi x, z] \neq 0)$.

Definition of H, Σ. The variables m, n represent integers ; η an arbitrary term ;
the variable σ represents a finite sequence of ordered pairs $<m, \eta>$.
If no pair $<m, \bullet>$ is in σ, set $\Sigma \sigma m \eta = \sigma \smile <m, \eta>$, $H \sigma m \eta = \eta$.
Else, set $\Sigma \sigma m \eta = \sigma$; $H \sigma m \eta = \zeta$ for the first $<m, \zeta>$ appearing in σ.

Proof by contradiction. Suppose $\xi \Vdash \forall x^{\mathsf{int}} \neg \forall y^{\daleth\mathbb{N}} \neg \forall z^{\mathsf{int}} (f[x,y,z]=1)$;
$((\mathsf{Y})(\theta)\xi)000 \not\Vdash \perp$ and $f[x_0, \phi x_0, z_0] = 0$.
We show, by recurrence on $<m, n> \leq <x_0, z_0>$, that $((\mathsf{Y})(\theta)\xi)\sigma_{mn} m n \not\Vdash \perp$,
with $\sigma_{mn}, \eta_{mn}, b_{mn}$ defined by recurrence ; it's true for $\sigma_{00} = 0$. If it's true for $<m, n>$
we have $((\mathsf{Y})(\theta)\xi)\sigma_{mn} m n \star \pi \notin \perp\!\!\!\perp$, i.e. $\theta \xi \star (\mathsf{Y})(\theta)\xi \,.\, \sigma_{mn} \,.\, m \,.\, n \,.\, \pi \notin \perp\!\!\!\perp$, or else :

$\xi \star m\,.\,\lambda y(H\sigma_{mn}myn)(((\curlyvee)(\theta)\xi)(\Sigma)\sigma_{mn}my)m'n'\,.\,\pi \notin \perp\!\!\!\perp$. Thus, there exists b_{mn} s.t. : $\lambda y(H\sigma_{mn}myn)(((\curlyvee)(\theta)\xi)(\Sigma)\sigma_{mn}my)m'n' \not\Vdash \neg\forall z^{\text{int}}(f[m,b_{mn},z]=1)$ and thus there exists $\eta_{mn} \Vdash \forall z^{\text{int}}(f(m,b_{mn},z)=1)$ such that

(*) $\qquad H\sigma_{mn}m\eta_{mn} \star n\,.\,(((\curlyvee)(\theta)\xi)(\Sigma)\sigma_{mn}m\eta_{mn})m'n'\,.\,\pi \notin \perp\!\!\!\perp$.

Definition of ϕm : i) if no pair $<m,\,.>$ appears in σ_{mn} then set $\phi[m]=b_{mn}$;
ii) else, let $<m,\eta_{mq}>$ be the first (indeed only) pair $<m,\,.>$ appearing in σ_{mn} ;
then, set $\phi[m]=b_{mq}$. Now, $H\sigma_{mn}m\eta_{mn} \Vdash \forall z^{\text{int}}(f(m,\phi m,z)=1)$ because :
in case (i) $H\sigma_{mn}m\eta_{mn}=\eta_{mn}$ and $\phi m=b_{mn}$; in case (ii), by induction on $<m,n>$ since $H\sigma_{mn}m\eta_{mn}=\eta_{mq}$ with $<m,q>$ strictly before $<m,n>$.

Thus $\qquad H\sigma_{mn}m\eta_{mn}n \Vdash f(m,\phi m,n)\neq 1 \to \perp$.

Now, we set $\qquad \sigma_{m'n'}=(\Sigma)\sigma_{mn}m\eta_{mn}$.

Thus, by (*), we have $\qquad ((\curlyvee)(\theta)\xi)\sigma_{m'n'}m'n' \not\Vdash f(m,\phi m,n)\neq 1$.

therefore $f[m,\phi m,n]=1$ and $((\curlyvee)(\theta)\xi)\sigma_{m'n'}m'n' \not\Vdash \perp$.

Since $f[x_0,\phi x_0,z_0]=0$, we have a contradiction if $<m,n>=<x_0,z_0>$.

Else, we have done the recurrence step. QED

Consider now a function $f : \mathbb{N}^4 \to \{0,1\}$ s.t. $\mathbb{N} \models \forall u \exists x \forall y \exists z (f(u,x,y,z) = 0)$.

This gives $\forall u \big(\forall x \exists y \forall z (f(u,x,y,z) \neq 0) \to \bot\big)$.

Thus, for every $u \in \mathbb{N}$ and $\phi : \mathbb{N} \to \mathbb{N}$ we get :

$$\|\forall x \forall z (f(u,x,\phi x,z) \neq 0)\| = \|\bot\| = \Pi.$$

It follows from the previous theorem that

$$\lambda x((\mathsf{Y})(\theta)x)000 \Vdash \forall u \neg \forall x^{\mathrm{int}} \exists y \forall z^{\mathrm{int}} (f(u,x,y,z) = 1)$$

which is the case $n = 2$ for arithmetical formulas.

$\gimel 2$ trivial

Let δ be a proof-like term s.t. $\delta \Vdash \forall x^{\gimel 2}(x \neq 0, x \neq 1 \to \perp)$ (i.e. $\gimel 2$ is trivial).

We have $\delta \in |\top, \perp \to \perp| \cap |\perp, \top \to \perp|$. Let $\delta' = \lambda x \lambda y \mathsf{cc} \lambda k(\delta)((k)x)(k)y$; then

$$\xi \star \pi \in \perp\!\!\!\perp \text{ or } \eta \star \pi \in \perp\!\!\!\perp \Rightarrow \delta' \star \xi . \eta . \pi \in \perp\!\!\!\perp$$

Thus, $\delta' \Vdash X, Y \to X$ and $\delta' \Vdash X, Y \to Y$ for every truth values X, Y.

Theorem. $(\exists \Phi \in \mathrm{QP})(\forall \theta \in \mathrm{QP})(\forall X \subset \Pi)(\theta \Vdash X \Rightarrow \Phi \Vdash X)$.

Define e (read *eval*) by the following program :

$\mathsf{e}\,\underline{0} = \mathsf{B}$, $\mathsf{e}\,\underline{1} = \mathsf{C}$, $\mathsf{e}\,\underline{2} = \mathsf{E}$, $\mathsf{e}\,\underline{3} = \mathsf{I}$, $\mathsf{e}\,\underline{4} = \mathsf{K}$, $\mathsf{e}\,\underline{5} = \mathsf{W}$, $\mathsf{e}\,\underline{6} = \mathsf{cc}$, $\mathsf{e}\,\underline{7} = \delta$;

$\mathsf{e}\,\underline{n+8} = ((\mathsf{e})(\mathsf{p}_0)\underline{n})(\mathsf{e})(\mathsf{p}_1)\underline{n}$;

where $\mathsf{p}_0, \mathsf{p}_1$ define a recursive bijection from $\mathbb{N}$ onto $\mathbb{N}^2$.

For every $\theta \in \mathrm{QP}$, there is an integer n s.t. $\mathsf{e}\underline{n} \succ \theta$.

Now define ϕ by : $\phi \star n . \pi \succ \delta' \star \mathsf{e}n . (\phi)(s)n . \pi$. Finally Φ is $\phi\underline{0}$.

Let $\theta \in \mathrm{QP}$ s.t. $\theta \Vdash X$; thus, we have $\phi\underline{n} \Vdash X$ for some n,

then $\phi\underline{n-1} \Vdash X, \ldots$; eventually $\phi\underline{0} \Vdash X$. QED

$\beth 2$ trivial

Let $\mathscr{B} = \mathscr{P}(\Pi)$ be the Boolean algebra of truth values.

The order is defined by $A \le B \Leftrightarrow (\exists\theta \in \mathrm{QP})(\theta \Vdash A \to B)$.

Thus, the order on $\mathscr{B}$ is defined by $A \le B \Leftrightarrow \Phi \Vdash A \to B$.

Theorem. $\mathscr{B}$ is a complete Boolean algebra :

If $B_i (i \in I)$ is a family of truth values, then $\inf_{i \in I} B_i = \bigcup_{i \in I} B_i$.

Let $A \le B_i$ for $i \in I$. Then $\Phi \Vdash A \to B_i$, thus $\Phi \Vdash A \to \bigcup_{i \in I} B_i$.

Conversely $I \Vdash \bigcup_{i \in I} B_i \to B_{i_0}$. QED

Thus, the realizability model is, in fact, a *forcing model*.

The converse is also true : in the case of forcing, the realizability algebra is a commutative idempotent monoid with a unity $\mathbf{1}$; then $\mathrm{QP} = \{\mathbf{1}\}$.

We have $\mathbf{1} \Vdash X, Y \to X$ and $X, Y \to Y$; thus $\beth 2$ is trivial.

$\beth 2$ with 4 elements

Theorem. Let d be a term such that :

If two out of three processes $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in $⫫$, then $\mathsf{d} \star \xi . \eta . \zeta . \pi \in ⫫$.

Then d $\Vdash$ "$\beth 2$ has at most 4 elements".

We have $\mathsf{d} \in |\top, \bot, \bot \to \bot| \cap |\bot, \top, \bot \to \bot| \cap |\bot, \bot, \top \to \bot|$.

Thus $\mathsf{d} \Vdash \forall x^{\beth 2} \forall y^{\beth 2} (x \neq 0, y \neq 1, x \neq y \to xy \neq x)$ QED

We now build a model in which $\beth 2$ has exactly 4 elements.

The only term constants are the elementary combinators, cc and a new constant d.

There are two stack constants π^0, π^1. Let $\omega = (WI)(W)I = (\lambda x\, xx)\lambda x\, xx$.

For $i \in \{0, 1\}$, let Λ^i (resp. Π^i) be the set of terms (resp. stacks)

which contain the only stack constant π^i.

$\beth 2$ with 4 elements

For $i, j \in \{0,1\}$, define $\perp\!\!\!\perp^i_j$ as the least set $P \subset \Lambda^i \star \Pi^i$ of processes such that :

1. $\omega \star \underline{j} \cdot \pi \in P$ for every $\pi \in \Pi^i$;
2. $\xi \star \pi \in \Lambda^i \star \Pi^i, \xi \star \pi \succ \xi' \star \pi' \in P \Rightarrow \xi \star \pi \in P$ (P is saturated in $\Lambda^i \star \Pi^i$) ;
3. if 2 out of 3 processes $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in P, then $\mathsf{d} \star \xi \cdot \eta \cdot \zeta \cdot \pi \in P$.

We define $\perp\!\!\!\perp$ by : $\Lambda \star \Pi \setminus \perp\!\!\!\perp = \bigcup_{i \in \{0,1\}} (\Lambda^i \star \Pi^i \setminus \perp\!\!\!\perp^i_i)$

In other words, a process is in $\perp\!\!\!\perp$ iff

either it is in $\perp\!\!\!\perp^0_0 \cup \perp\!\!\!\perp^1_1$ or it contains both stack constants π^0, π^1.

Lemma. If $\xi \star \pi \in \perp\!\!\!\perp^i_j$ and $\xi \star \pi \succ \xi' \star \pi'$ then $\xi' \star \pi' \in \perp\!\!\!\perp^i_j$ (closure by reduction).

Suppose $\xi_0 \star \pi_0 \succ \xi'_0 \star \pi'_0$; $\xi_0 \star \pi_0 \in \perp\!\!\!\perp^i_j$; $\xi'_0 \star \pi'_0 \notin \perp\!\!\!\perp^i_j$;

We may suppose that $\xi_0 \star \pi_0 \succ \xi'_0 \star \pi'_0$ is exactly one step of execution.

Then $\perp\!\!\!\perp^i_j \setminus \{\xi_0 \star \pi_0\}$ has properties 1,2,3 defining $\perp\!\!\!\perp^i_j$; contradiction. QED

$\beth_2$ with 4 elements

Lemma. $⫫_0^i \cap ⫫_1^i = \emptyset$.

We prove that $\Lambda^i \star \Pi^i \setminus ⫫_1^i \supset ⫫_0^i$ by showing properties 1, 2, 3.

1. $\omega \star \underline{0} . \pi^i \notin ⫫_1^i$ because $⫫_1^i \setminus \{\omega \star \underline{0} . \pi^i\}$ has properties 1, 2, 3 defining $⫫_1^i$.
2. Follows from previous lemma.
3. Suppose $\xi \star \pi, \eta \star \pi \notin ⫫_1^i$; then $\mathsf{d} \star \xi . \eta . \zeta . \pi \notin ⫫_1^i$ because $⫫_1^i \setminus \{\mathsf{d} \star \xi . \eta . \zeta . \pi\}$ has properties 1, 2, 3 defining $⫫_1^i$. QED

Theorem. This realizability model is coherent.

Let $\theta \in$QP s.t. $\theta \star \pi^0 \in ⫫_0^0$ and $\theta \star \pi^1 \in ⫫_1^1$. Then $\theta \star \pi^0 \in ⫫_0^0 \cap ⫫_1^0$. QED

Remark. *If* $\pi \in \Pi \setminus (\Pi_0 \cup \Pi_1)$*, then* $\xi \star \pi \in ⫫$ *for every term* ξ.

Thus, we can remove these stacks and consider only $\Pi^0 \cup \Pi^1$.

$\beth 2$ with 4 elements

We define two individuals in this realizability model :

$\gamma_0 = (\{0\} \times \Pi^0) \cup (\{1\} \times \Pi^1)$; $\gamma_1 = (\{1\} \times \Pi^0) \cup (\{0\} \times \Pi^1)$.

Obviously, $\gamma_0, \gamma_1 \subset \beth 2 = \{0,1\} \times \Pi$. Now we have :

$\|\forall x(x \not\varepsilon \gamma_0)\| = \Pi^0 \cup \Pi^1 = \|\bot\|$; $\omega \underline{0} \Vdash 0 \not\varepsilon \gamma_0$ et $\omega \underline{1} \Vdash 1 \not\varepsilon \gamma_0$.

It follows that γ_0 is not ε-empty and that every ε-element of γ_0 is $\neq 0, 1$.

Thus the Boolean algebra $\beth 2$ *is not trivial and has exactly 4 ε-elements.*

We have $\xi \Vdash \forall x^{\beth 2}(x \varepsilon \gamma_0, x \varepsilon \gamma_1 \to \bot)$ for *every* term ξ :

Indeed, $|i \varepsilon \gamma_0| = \{\mathsf{k}_\pi \,;\, \pi \in \Pi^i\}$ for $i = 0, 1$ and $\xi \star \mathsf{k}_{\rho_0} . \mathsf{k}_{\rho_1} . \pi \in \perp\!\!\!\perp$ if $\rho_i \in \Pi^i$.

It follows that γ_0, γ_1 are the singletons of the ε-elements $\neq 0, 1$ of $\beth 2$.

Remark. *We can easily modify this construction in order to obtain for $\beth 2$ any finite Boolean algebra.*

Denotational semantics

T. Ehrhard has found a method which converts usual models of λ-calculus into realizability algebras, by defining stacks, cc and k_π in such models.

The construction of stacks was also given by T. Streicher.

We need to avoid *parallel or*, because we don't want to get *forcing models*.

Thus, our example will be the simplest *coherent model of λ-calculus*.

Let us recall (one of) its construction.

Let o be a fixed set which is not an ordered pair.

The set V of *formulas* is the smallest set such that :

$\mathrm{o} \in V$; if $\alpha \in V$, $a \in \mathscr{P}_f(V)$ and $<a, \alpha> \neq <\emptyset, \mathrm{o}>$ then $<a, \alpha> \in V$

($\mathscr{P}_f(V)$ is the set of finite subsets of V).

If $a \in \mathscr{P}_f(V)$ and $\alpha \in V$, we set $a \to \alpha = <a, \alpha>$ except that $(\emptyset \to \mathrm{o}) = \mathrm{o}$.

Every element of V except o is an ordered pair.

If $\alpha \in V$, its rank $r(\alpha)$ is the total number of $\to$ in α.

Each $\alpha \in V$ has a unique normal form $\alpha = (a_1, \ldots, a_k \to \mathsf{o})$
with $k \in \mathbb{N}, a_1, \ldots, a_k \in \mathscr{P}_f(V)$ and $a_k \neq \emptyset$. Then $\alpha = (a_1, \ldots, a_k, \emptyset, \ldots, \emptyset \to \mathsf{o})$.
The *truth value* $|\alpha| \in \{0,1\}$ of a formula α is defined by induction :
$|\mathsf{o}| = 0$; $|a_1, \ldots, a_k \to \mathsf{o}| = 1$ iff $(\exists \beta \in a_1 \cup \ldots \cup a_k)(|\beta| = 0)$.
If $\alpha = (a_1, \ldots, a_k \to \mathsf{o}), \beta = (b_1, \ldots, b_k \to \mathsf{o})$ we define

$$\alpha \sqcap \beta = (a_1 \cup b_1, \ldots, a_k \cup b_k \to \mathsf{o}).$$

This operation is associative, commutative and idempotent ; o is neutral ;
it defines an order relation : $\alpha \leq \beta \Leftrightarrow b_1 \subset a_1, \ldots, b_k \subset a_k$.
Define a subset D of V (the *web*) by induction on the rank :
$(a_1, \ldots, a_k \to \mathsf{o}) \in D$ iff, for $1 \leq i \leq k$,
$a_i \subset D$ and $(\forall \beta, \gamma \in a_i)(\beta \neq \gamma \Rightarrow \beta \sqcap \gamma \notin D)$ (a_i is an *antichain* of D).
D is a final segment of V : let $\alpha = (a_1, \ldots, a_k \to \mathsf{o}), \beta = (b_1, \ldots, b_k \to \mathsf{o})$,
$\alpha \in D, \alpha \leq \beta$. Then $b_i \subset a_i$ and a_i is an antichain of D, thus so is b_i.
$\alpha, \beta \in D$ are called *compatible* if $\alpha \sqcap \beta \in D$; in symbols $\alpha \asymp \beta$.
If $\alpha_1, \ldots, \alpha_n$ are pairwise compatible, then $\alpha_1 \sqcap \ldots \sqcap \alpha_n \in D$.

The realizability algebra

Λ_D is the set $\mathscr{A}(D)$ of antichains of D, i.e. $t \subset D$ is a term iff $(\forall \alpha, \beta \in t)(\alpha \sqcap \beta \in D \to \alpha = \beta)$.

Π_D is the set $\mathscr{S}(D)$ of filters of D, i.e. $\pi \subset D$ is a stack iff $(\forall \alpha, \beta \in \pi)\, \alpha \sqcap \beta \in \pi$; $\forall \alpha \forall \beta (\alpha \in \pi, \alpha \le \beta \to \beta \in \pi)$; $\mathsf{O} \in \pi$.

Remark. Π_D *can be identified with* $\Lambda_D^{\mathbb{N}}$ *: a sequence of terms* $t_n (n \in \mathbb{N})$ *corresponds with the filter* $\{(a_0, \ldots, a_k \to \mathsf{O})\ ;\ k \in \mathbb{N},\ a_0 \subset t_0, \ldots, a_k \subset t_k\}$.

$\Lambda_D \star \Pi_D$ is $\{0, 1\}$ and $\perp\!\!\!\perp$ is $\{1\}$.

If $t \in \Lambda_D, \pi \in \Pi_D$ then $t \star \pi \in \perp\!\!\!\perp$ iff $t \cap \pi \neq \emptyset$ (i.e. $t \cap \pi$ is a singleton).

$t \centerdot \pi = \{a \to \alpha\ ;\ a \subset t,\ \alpha \in \pi\}$;

$tu = \{\alpha \in D;\ (\exists a \subset u)(a \to \alpha) \in t\}$;

K is the set of all formulas : $\{\alpha\}, \emptyset \to \alpha$ for $\alpha \in D$.

S is the set of all formulas :

$\{a_0, \{\alpha_1, \ldots, \alpha_k\} \to \beta\}, \{a_1 \to \alpha_1, \ldots, a_k \to \alpha_k\}, a_0 \cup a_1 \cup \ldots \cup a_k \to \beta$

with $\{\alpha_1, \ldots, \alpha_k\} \in \mathscr{A}(D)$ and $a_0 \cup a_1 \cup \ldots \cup a_k \in \mathscr{A}(D)$.

k_π is the set of formulas : $(\{\alpha\} \to \mathrm{o})$ for $\alpha \in \pi$;
cc is the set of all formulas :
$\{a \to \alpha\} \to \alpha \sqcap \alpha_1 \sqcap \ldots \sqcap \alpha_k$ with $a = \{\{\alpha_1\} \to \mathrm{o}, \ldots, \{\alpha_k\} \to \mathrm{o}\}$ and $\alpha \sqcap \alpha_1 \sqcap \ldots \sqcap \alpha_k \in D$.
QP is defined as the set of $t \in \Lambda_D$ s.t. $|t| = 1$ i.e. $(\forall \alpha \in t)(|\alpha| = 1)$.
We have K, S, cc $\in$ QP ; $t, u \in$ QP $\Rightarrow$ $tu \in$ QP.
The model is *coherent* because $|t| = 1 \Rightarrow \mathrm{o} \notin t$ i.e. $t \star \{\mathrm{o}\} \notin \perp\!\!\!\perp$.
Lemma 1. $t \Vdash \top, \ldots, \top \to \bot$ iff $t = \{\mathrm{o}\}$.
Indeed, $t \star \emptyset . \ldots . \emptyset . \{\mathrm{o}\} \in \perp\!\!\!\perp \Rightarrow t = \{\mathrm{o}\}$ QED
Lemma 2. If $t \in |\top, \bot \to \bot| \cap |\bot, \top \to \bot|$ then $t = \{\mathrm{o}\}$.
We have $t \cap \emptyset . \{\mathrm{o}\} . \{\mathrm{o}\} \neq \emptyset$ and $t \cap \{\mathrm{o}\} . \emptyset . \{\mathrm{o}\} \neq \emptyset$; thus
$(\emptyset, a \to \mathrm{o}) \in t$ and $(b, \emptyset \to \mathrm{o}) \in t$ with $a, b \subset \{\mathrm{o}\}$.
These two formulas are compatible and therefore equal ; thus $a = b = \emptyset$. QED
It follows that $\mathsf{I} \Vdash |\top, \bot \to \bot| \cap |\bot, \top \to \bot| \to \bot$ i.e.
$\mathsf{I} \Vdash \forall x^{\gimel 2}(x \neq 0, x \neq 1 \to \bot) \to \bot$. Therefore :
The Boolean algebra $\gimel 2$ is non trivial.

Lemma 3. If $u \Vdash \bot, \bot \to \bot$ then u contains one of the formulas :

o ; {o} → o ; ∅, {o} → o ; {o}, {o} → o.

We have $u \cap \{o\} \bullet \{o\} \bullet \{o\} \neq \emptyset$, thus there exist $a, b \subset \{o\}$ s.t. $(a, b \to o) \in u$. QED

Lemma 4. Let $t \in \Lambda_D$ contain the 4 incompatible formulas :

{o} → o ; {{o} →o}, {o} → o ; {∅, {o} →o}, {o} → o ; {{o}, {o} →o}, {o} → o.

Then $t \Vdash |\top, \bot \to \bot| \cap |\bot, \top \to \bot|, \top \to \bot$ and $t \Vdash (\bot, \bot \to \bot), \bot \to \bot$.

By lemma 2, the first conclusion is $t \Vdash \bot \to \bot$; it is satisfied because $(\{o\} \to o) \in t$.

Now, let $u \Vdash \bot, \bot \to \bot$; we must show $t \cap u \bullet \{o\} \bullet \{o\} \neq \emptyset$

which follows immediately from lemma 3. QED

Theorem. The Boolean algebra ℷ2 is atomless.

We have $t \Vdash \forall x^{\daleth 2}\left(\forall y^{\daleth 2}(xy \neq 0, xy \neq x \to \bot), x \neq 0 \to \bot\right)$ iff

$t \Vdash |\top, \bot \to \bot| \cap |\bot, \top \to \bot|, \top \to \bot$ and $t \Vdash (\bot, \bot \to \bot), \bot \to \bot$.

Hence the result by lemma 4. QED

Integers

In the sequel, we use truth values defined by subsets $|U|$ of Λ.

They may be used in formulas only before a $\to$.

If $|U| \subset \Lambda, \|A\| \subset \Pi$, we define $\|U \to A\| = \{t \bullet \pi\ ;\ t \in |U|, \pi \in \|A\|\}$.

In particular $\|\neg U\| = \{t \bullet \pi\ ;\ t \in |U|, \pi \in \Pi\}$.

Lemma 5. If $(\forall t \in \Lambda)(t \in |U| \Rightarrow \theta t \in |U'|)$ then $\lambda x\, x \circ \theta \Vdash \neg U' \to \neg U$.

We shall sometimes write $\theta \Vdash U \to U'$ in such a case.

Now, define the formulas :

$\nu_0 = (\{\mathsf{o}\} \to \mathsf{o})\ ;\ \nu_1 = (\emptyset, \{\mathsf{o}\} \to \mathsf{o})\ ;\ \ldots\ ;\ \nu_n = (\emptyset, \ldots, \emptyset, \{\mathsf{o}\} \to \mathsf{o})\ ;\ \ldots\ ;$

and the terms $\overline{n} = \{\nu_n\}\ ;\ \mathsf{suc} = \{(\{\nu_0\} \to \nu_1), \ldots, (\{\nu_i\} \to \nu_{i+1}), \ldots\}$.

Define the unary predicate N by :

$|Nn| = \{\overline{n}\}$ if $n \in \mathbb{N}$; $|Nn| = \emptyset$ if $n \notin \mathbb{N}$.

Then we have easily $\lambda x(x)\overline{0} \Vdash \neg\neg N0$; $\mathsf{suc} \Vdash Nn \to N(n+1)$ for every n ;

i.e. $\lambda x\, x \circ \mathsf{suc} \Vdash \forall x(\neg N(x+1) \to \neg Nx)$.

We have shown : $\Vdash \forall x^{\mathrm{int}} \neg\neg Nx$.

Theorem 6. Let $u_n (n \in \mathbb{N})$ be any sequence of terms and define :
$\theta = \{(\{v_n\} \to \alpha) \; ; \; n \in \mathbb{N}, \alpha \in u_n\}$. Then $\theta \overline{n} = u_n$ for all $n \in \mathbb{N}$.
If every u_n is in QP, then $\theta \in$ QP.
We show that $\theta \in \Lambda_D$: if $(\{v_m\} \to \alpha) \asymp (\{v_n\} \to \beta)$ then $\{v_m, v_n\}$ is an antichain and therefore $m = n$; thus $\alpha, \beta \in u_m$; but $\alpha \asymp \beta$ and therefore $\alpha = \beta$.
$\theta\{v_n\} = u_n$ is obvious. QED
Define the unary predicate $\mathsf{ent}(x)$ by :
$|\mathsf{ent}(n)| = \{\underline{n}\}$ (Church integer) for $n \in \mathbb{N}$; $|\mathsf{ent}(n)| = \emptyset$ if $n \notin \mathbb{N}$.
We already know (general theory) that $\mathsf{ent}(x)$ is equivalent to $\mathsf{int}(x)$.
Apply lemma 5 and theorem 6 above with $u_n = \{\underline{n}\}$.
This gives $\theta \Vdash Nn \to \mathsf{ent}(n)$ and therefore $\lambda x\, x \circ \theta \Vdash \forall x(\neg \mathsf{ent}(x) \to \neg Nx)$.
Finally we have shown that the predicates Nx, $\mathsf{int}(x)$, $\mathsf{ent}(x)$ are equivalent.
In the following, we use Nx which is the simplest.

Corollary. If $\theta_n \Vdash F(n)$, with $\theta_n \in$ QP for all $n \in \mathbb{N}$, then there exists $\theta \in$ QP s.t. $\theta \Vdash \forall n^{\text{int}} F(n)$.

Applying theorem 6, we get $\theta \underline{n} \Vdash F(n)$ for all $n \in \mathbb{N}$, thus $\theta \Vdash \forall n^{\text{int}} F(n)$. QED

By the above corollary, the set of formulas which are realized by a proof-like term is closed by the ω-rule.

Thus there exists a realizability model *which is an ω-model*.

Let $\mathscr{B} = \mathscr{P}(\Pi)$ be the Boolean algebra of truth values.

The order is defined by $\|A\| \leq \|B\| \Leftrightarrow (\exists \theta \in \text{QP})(\theta \Vdash A \to B)$.

Theorem. $\mathscr{B}$ is a countably complete Boolean algebra :

If $\|B(n)\|_{n \in \mathbb{N}}$ is a sequence of truth values, then $\inf_{n \in \mathbb{N}} \|B(n)\| = \|\forall x^{\text{int}} B(x)\|$.

Let $\|A\| \leq \|B(n)\|$ for every $n \in \mathbb{N}$. Then $\theta_n \Vdash A \to B(n)$ for some sequence $\theta_n \in$ QP.

By the previous corollary, we get $\theta \Vdash \|A \to \forall x^{\text{int}} B(x)\|$ i.e. $\|A\| \leq \|\forall x^{\text{int}} B(x)\|$.

Conversely, $\|\forall x^{\text{int}} B(x)\| \leq \|B(n)\|$ because $\lambda x(x)\underline{n} \Vdash \forall x^{\text{int}} B(x) \to B(n)$. QED