

Using structures to fix (or improve) several systems of nets derived from Linear Logic.

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PPS - Université Paris Diderot

Collegium Logicum – 1 March 2011

Introduction

Interaction nets

Boxes

Problem

Systems to be fixed

Structures

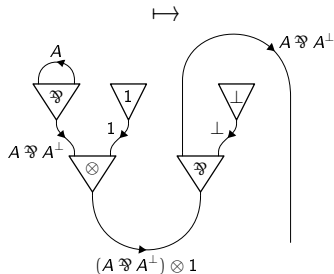
Localized Exponential

Conclusion

Introduction / Interaction nets

Born from multiplicative linear logic [Laf90]:

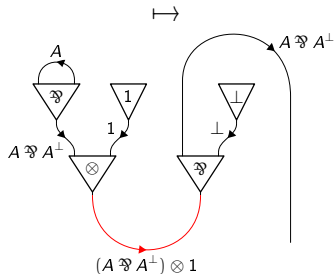
$$\begin{array}{c}
 \frac{}{\vdash A, A^\perp} \text{ ax} \quad \frac{}{\vdash 1} \text{ one} \quad \frac{}{\vdash (A^\perp \otimes A), A \wp A^\perp} \text{ ax} \\
 \frac{}{\vdash A \wp A^\perp} \text{ par} \quad \frac{}{\vdash 1} \text{ one} \quad \frac{}{\vdash (A^\perp \otimes A), \perp, A \wp A^\perp} \text{ bot} \\
 \frac{}{\vdash (A \wp A^\perp) \otimes 1} \text{ ten} \quad \frac{}{\vdash (A^\perp \otimes A) \wp \perp, A \wp A^\perp} \text{ par} \\
 \hline
 \vdash A \wp A^\perp \quad \text{cut}
 \end{array}$$



Introduction / Interaction nets

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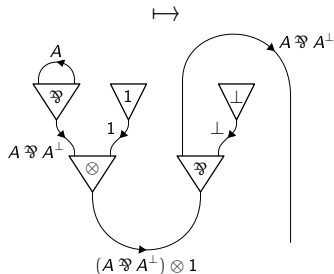
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 \quad
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 \quad
 \frac{
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 \frac{}{\vdash (A^\perp \otimes A), A \wp A^\perp} \text{ax}
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 }{\vdash A \wp A^\perp} \text{cut}$$



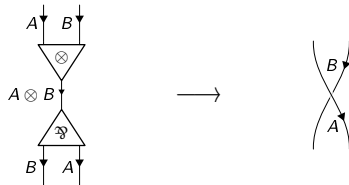
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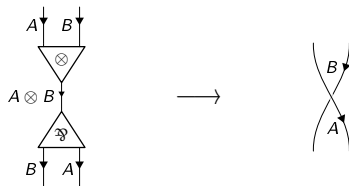


Introduction / Interaction nets

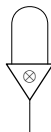


Flexible. . .

Introduction / Interaction nets

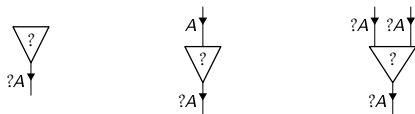


Flexible. . . too flexible.

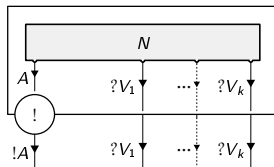


Introduction / Boxes

Exponentials:



Boxes are used to represent promotion: $\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} \text{ prom}$



Introduction / Problem

Nets are **too flexible**, but sequents are usually there to back them up.

New systems of nets appeared. What to do when sequents fail to back them up ? (Sequents are sometimes **too rigid**)

Introduction / Problem

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New systems of nets appeared. What to do when sequents fail to back them up ? (Sequents are sometimes **too rigid**)

- ▶ Differential Interaction Nets [ER06]
(sequent presentation is not fully satisfactory)
- ▶ Super-promotion
(actually broken without backup)
- ▶ Localized Boxes
(correctness criterion missing)

Introduction

Systems to be fixed

- Differential Interaction Nets

- Super-promotion

Structures

Localized Exponential

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Systems to be fixed / Differential Interaction Nets

DIN come from differential λ -calculus:

$$\frac{\partial}{\partial x} (\lambda z. (x, x)) \cdot u = \lambda z. (u, x) + \lambda z. (x, u)$$

$$\frac{\partial}{\partial x} (\lambda z. z) \cdot u = 0$$

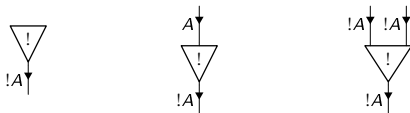
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Differential cells (co-weakening, co-dereliction, co-contraction):



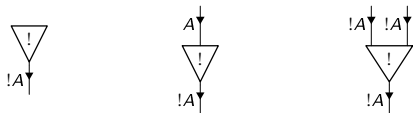
Systems to be fixed / Differential Interaction Nets

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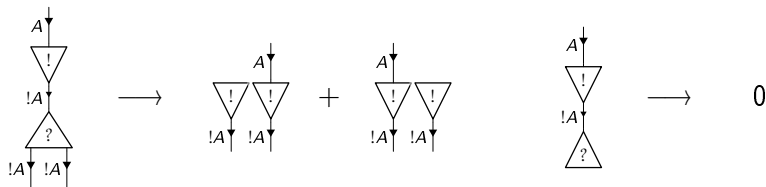
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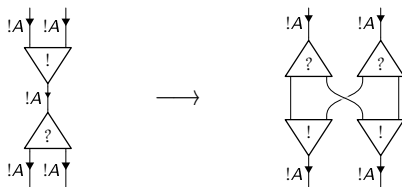


Routing of resources:



Systems to be fixed / Differential Interaction Nets

Having a look at this reduction:



Can it be written in sequent calculus ?

Systems to be fixed / Differential Interaction Nets

Multiplicative constructions:

$$\begin{array}{c}
 \frac{\vdash \Gamma_1, A \quad \vdash \Gamma_2, B}{\vdash \Gamma_1, \Gamma_2, A \otimes B} \text{ tensor} \\
 \frac{}{\vdash 1} \text{ unit} \\
 \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \text{ par} \\
 \frac{\vdash \Gamma}{\vdash \Gamma, \perp} \text{ bottom}
 \end{array}$$

Differential constructions can be introduced in the same way:

$$\begin{array}{c}
 \frac{\vdash \Gamma_1, !A \quad \vdash \Gamma_2, !A}{\vdash \Gamma_1, \Gamma_2, !A} \text{ co-contraction} \\
 \frac{}{\vdash !A} \text{ co-weakening} \\
 \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \text{ contraction} \\
 \frac{\vdash \Gamma}{\vdash \Gamma, ?A} \text{ weakening}
 \end{array}$$

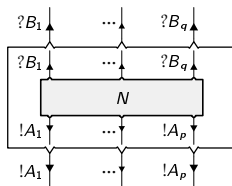
Systems to be fixed / Differential Interaction Nets

Cut elimination:

$$\begin{array}{c}
 \frac{\frac{\overline{\vdash \Gamma_1, !A}^{\pi_1} \quad \overline{\vdash \Gamma_2, !A}^{\pi_2}}{\vdash \Gamma_1, \Gamma_2, !A} \text{ co-contraction} \quad \frac{\overline{\vdash ?A^\perp, ?A^\perp, \Gamma_3}^{\pi_3}}{\vdash ?A^\perp, \Gamma_3} \text{ contraction}}{\vdash \Gamma_1, \Gamma_2, \Gamma_3} \text{ cut} \\
 \downarrow \\
 \frac{\frac{\overline{\vdash ?A^\perp, !A}^{\text{ax}} \quad \overline{\vdash ?A^\perp, !A}^{\text{ax}}}{\vdash ?A^\perp, ?A^\perp, !A} \text{ co-contraction} \quad \frac{\overline{\vdash ?A^\perp, !A}^{\text{ax}} \quad \overline{\vdash ?A^\perp, !A}^{\text{ax}}}{\vdash ?A^\perp, ?A^\perp, !A} \text{ co-contraction} \quad \overline{\vdash ?A^\perp, ?A^\perp, \Gamma_3}^{\pi_3}}{\vdash ?A^\perp, ?A^\perp, ?A^\perp, ?A^\perp, \Gamma_3} \text{ cut}^2 \\
 \frac{\overline{\vdash \Gamma_1, !A}^{\pi_1} \quad \overline{\vdash \Gamma_2, !A}^{\pi_2} \quad \frac{\frac{\vdash ?A^\perp, ?A^\perp, ?A^\perp, ?A^\perp, \Gamma_3}{\vdash ?A^\perp, ?A^\perp, ?A^\perp, \Gamma_3} \text{ commutation}}{\vdash ?A^\perp, ?A^\perp, \Gamma_3} \text{ contraction}^2}{\vdash \Gamma_1, \Gamma_2, \Gamma_3} \text{ cut}^2
 \end{array}$$

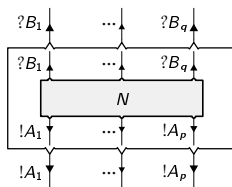
Systems to be fixed / Super-promotion

Super-promotion:

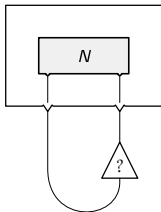


Systems to be fixed / Super-promotion

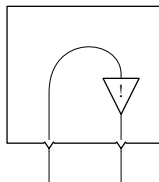
Super-promotion:



Correct:

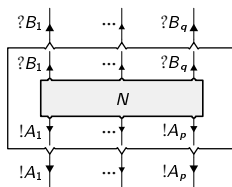


Hidden cycle:

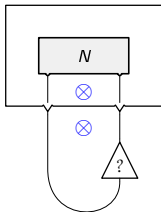


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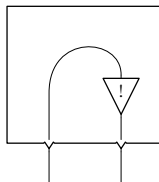
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Introduction

Systems to be fixed

Structures

- Typing nets

- Solving problems

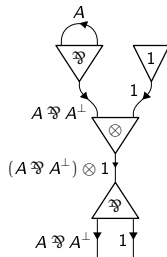
Localized Exponential

Conclusion

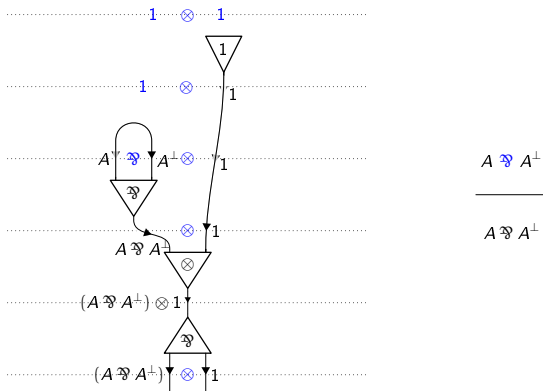
Structures / Typing nets

The problem: sequents can only type nets whose conclusions are in $\wp$ relation.

Solution: we should remember connectives between wires.



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We can use deep inference up to blue connectives.

Structures / Typing nets

Structures over linear logic formulaes:

$$\sigma ::= \cdot \mid \sigma ; \sigma \mid \circ \mid \sigma, \sigma \mid A$$

Derivations:

$$\frac{\sigma}{\vdots} \frac{\omega_i}{\omega_{i+1}} \frac{\vdots}{\tau}$$

Structures / Typing nets

Structures over linear logic formulae:

$$\sigma ::= \cdot \mid \sigma ; \sigma \mid \circ \mid \sigma, \sigma \mid A$$

Structural rules [Str03]:

$$\frac{(\sigma, \tau) ; \omega}{\sigma, (\tau ; \omega)} \text{ switch}$$

$$\frac{\cdot}{\sigma, \sigma^\perp} i\downarrow$$

$$\frac{\sigma^\perp ; \sigma}{\circ} i\uparrow$$

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Logical rules:

$$\frac{A; B}{A \otimes B} \text{ tensor}\downarrow$$

$$\frac{A, B}{A \wp B} \text{ par}\downarrow$$

$$\frac{A \wp B}{A, B} \text{ tensor}\uparrow$$

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Logical rules:

$$\frac{A ; B}{A \otimes B} \text{ tensor}\downarrow \qquad \frac{A, B}{A \wp B} \text{ par}\downarrow \qquad \frac{A \wp B}{A, B} \text{ tensor}\uparrow \qquad \frac{A \otimes B}{A ; B} \text{ par}\uparrow$$

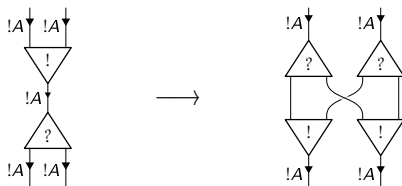
Symmetry, Deep inference!

Derivations:

$$\frac{\frac{\frac{\sigma}{\cdot}}{\omega_i}}{\omega_{i+1}} \dots \frac{\cdot}{\tau}$$

Structures / Solving problems

Differential interaction nets:



Using structures:

$$\frac{?A, ?A}{?A} \text{ contraction}\downarrow$$

$$\frac{!A; !A}{!A} \text{ co-contraction}\downarrow$$

$$\frac{\frac{!A; !A}{!A} \text{ co-contraction}\downarrow}{!A; !A} \text{ contraction}\uparrow \longrightarrow \frac{\frac{!A; !A}{!A; !A; !A; !A} \text{ contraction}\uparrow; \text{ contraction}\uparrow}{!A; !A; !A; !A} \diamond \frac{!A; !A; !A; !A}{!A; !A} \text{ co-contraction}\downarrow; \text{ co-contraction}\downarrow$$

Structures / Solving problems

Promotion: In order to remain as close as possible to boxes used in nets, we can use promotion rules that are parametrized by a derivation:

$$\frac{
 \begin{array}{c}
 !V_1; \cdots; !V_k \\
 \boxed{
 \begin{array}{c}
 !V_1; \cdots; !V_k \\
 \hline
 \vdots \\
 \hline
 A
 \end{array}
 } \\
 \hline
 !A
 \end{array}
 }{
 }
 \text{promotion}\downarrow$$

$$\frac{
 \begin{array}{c}
 ?A \\
 \boxed{
 \begin{array}{c}
 A \\
 \hline
 \vdots \\
 \hline
 ?V_1, \cdots, ?V_k
 \end{array}
 } \\
 ?V_1, \cdots, ?V_k
 \end{array}
 }{
 }
 \text{promotion}\uparrow$$

Structures / Solving problems

Super-promotion:

$$\begin{array}{c}
 !V_1 ; \cdots ; !V_q \\
 \hline
 \boxed{!V_1 ; \cdots ; !V_q} \\
 \hline
 \vdots \\
 \hline
 !A_1 ; \cdots ; !A_p \\
 \hline
 \boxed{!A_1 ; \cdots ; !A_p} \\
 \hline
 !A_1 ; \cdots ; !A_p
 \end{array}
 \text{super-promotion} \downarrow$$

$$\begin{array}{c}
 ?A_1 , \cdots , ?A_q \\
 \hline
 \boxed{?A_1 , \cdots , ?A_q} \\
 \hline
 \vdots \\
 \hline
 ?V_1 , \cdots , ?V_p \\
 \hline
 \boxed{?V_1 , \cdots , ?V_p} \\
 \hline
 ?V_1 , \cdots , ?V_p
 \end{array}
 \text{super-promotion} \uparrow$$

Not possible in sequent calculus.

Introduction

Systems to be fixed

Structures

Localized Exponential

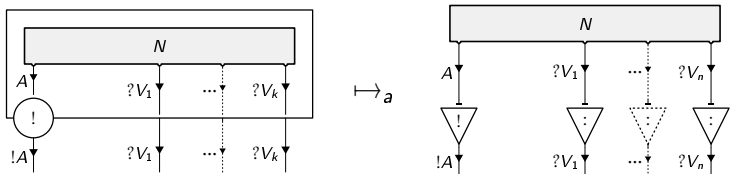
Localized Boxes

Benefits from/to Structures

Conclusion

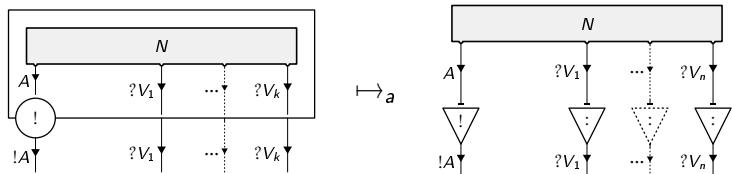
Localized Exponential / Localized Boxes

Idea: materialize only interfaces of boxes.

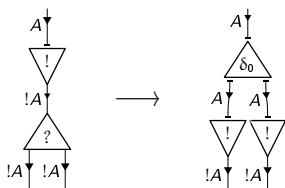


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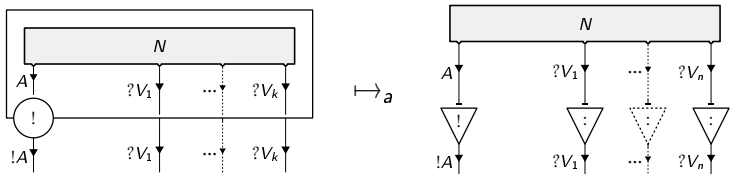


How it works:

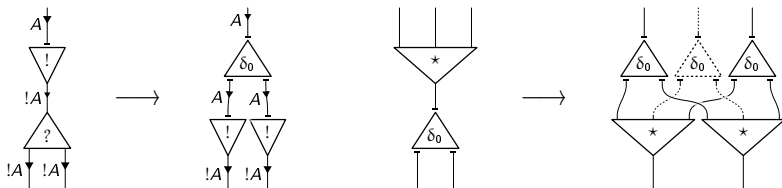


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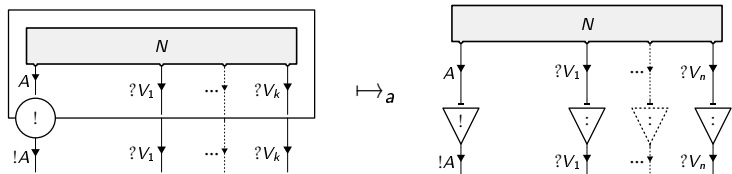


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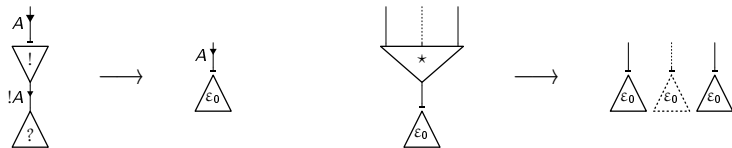


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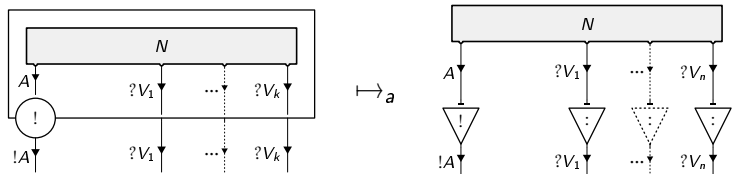


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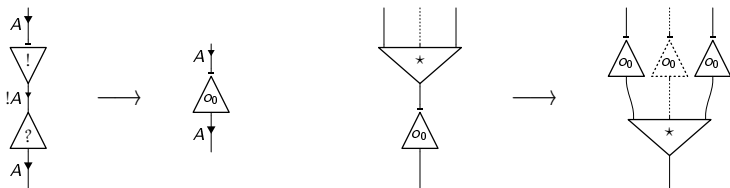


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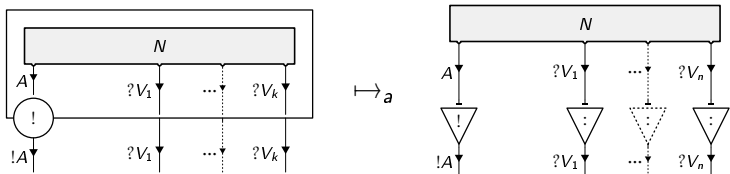


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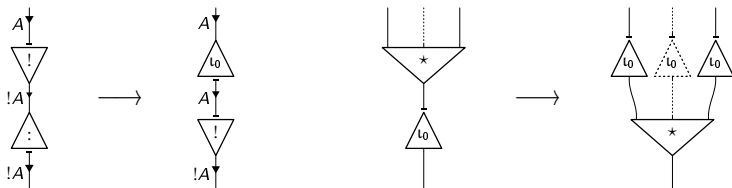


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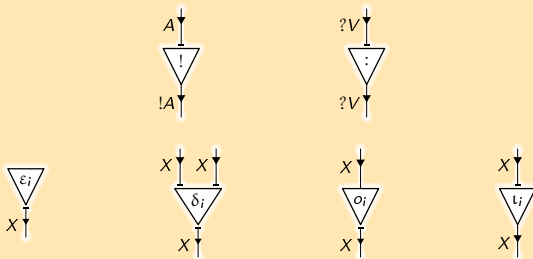
How it works:



Localized Exponential / Localized Boxes

Boxes can be replaced by simple cells.

Localized exponential system (**active** version)



- ▶ Similar to “sharing graphs” [GAL92].
- ▶ Reduction in **passive** and **controled** versions preserve box structure. (Correctness of intermediate reduction steps?)

Localized Exponential / Benefits from/to Structures

$$\sigma ::= \cdot \mid \sigma ; \sigma \mid \circ \mid \sigma , \sigma \mid [\sigma] \mid \{\sigma\} \mid A$$

Alternative structural rules for exponentials:

$$\begin{array}{cccc} \frac{[\omega]}{\cdot} \text{erase}\uparrow & \frac{[\omega]}{[\omega] ; [\omega]} \text{dupl.}\uparrow & \frac{[\omega]}{\omega} \text{open}\uparrow & \frac{[\omega]}{[[\omega]]} \text{in}\uparrow \\ & \frac{[\sigma] ; [\tau]}{[\sigma ; \tau]} & \frac{\cdot}{[\cdot]} & \end{array}$$

Logical rules:

$$\begin{array}{cccc} \frac{[A]}{!A} \text{prom-out}\downarrow & \frac{!A}{[!A]} \text{prom-in}\uparrow & & \\ \frac{\circ}{?A} \text{weak.}\downarrow & \frac{?A , ?A}{?A} \text{contr.}\downarrow & \frac{A}{?A} \text{derel.}\downarrow & \frac{??A}{?A} \text{digg.}\downarrow \end{array}$$

Localized Exponential / Benefits from/to Structures

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- Promotion is just inference under $[\]$ contexts.

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Work in progress

Conclusion / Work in progress

- ▶ Try structures when sequents are too rigid!
- ▶ Two levels of connectors fits nicely to Interaction Nets systems
- ▶ Boxes can be represented using:
 - ▶ a standard global construction
 - ▶ localized exponential structure
- ▶ A structured presentation of Nets might finally end up better than their graph presentation. (Simple induction principle)

Conclusion / Work in progress

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Thanks! Questions?



Thomas Ehrhard and Laurent Regnier.

Differential interaction nets.

Theoretical Computer Science, 364(2):166–195, 2006.



Georges Gonthier, Martín Abadi, and Jean-Jacques Lévy.

Linear logic without boxes.

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