Set-theoretic Foundation of Parametric Polymorphism and Subtyping

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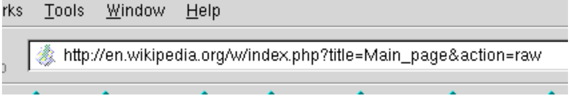
    - Intersection and negation permit XML typed programming with **overloading** and powerful **pattern matching**.
2. **Type variables:**
    - Parametric polymorphism already demonstrated its worth in practice.
    - Fulfills new needs specific to XML processing (*eg*, SOAP envelopes).
    - Sheds new light on the notion of **parametricity**.

Real case example: active pages



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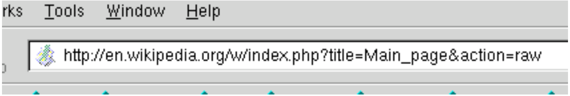

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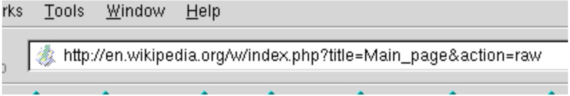

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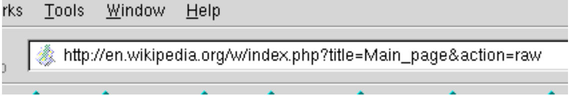

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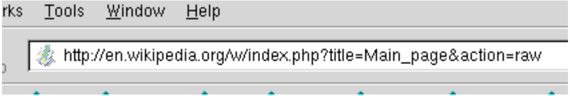

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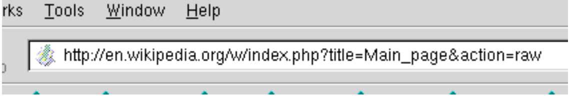

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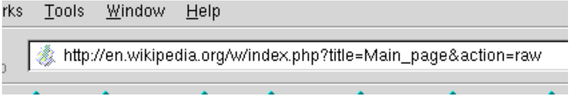

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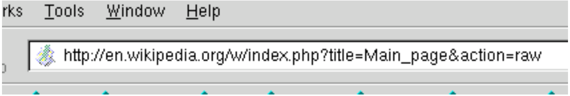

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Semantic Subtyping in a nutshell

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  constructors do not mix, *eg.*:

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Define subtyping semantically: [Hosoya, Pierce]

1. Interpret types as sets (of values)
2. *Define* subtyping as set containment.

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- **Then** *define* the **subtyping relation** as set-containment.

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Do not define what types are
define how they are related

Semantic subtyping: formalization

- **First**, define an interpretation of types into sets.

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Semantic subtyping [Benzaken, Castagna, Frisch]

1. Gives an interpretation satisfying the above constraints;
2. Gives an algorithm to decide the induced subtyping relation.

Polymorphic extension: adding type variables

Naive solution

$$t ::= B \mid t \times t \mid t \to t \mid t \vee t \mid t \wedge t \mid \neg t \mid \mathbb{0} \mid \mathbb{1}$$

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THIS IS A WRONG WAY: TOO MANY PROBLEMS

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- If $\alpha\leq\neg t$ then the left element of the union in (18) suffices;
- If $t\leq\alpha$, then $\alpha=(\alpha\backslash t)\vee t$. Thus $(t\times\alpha)=(t\times(\alpha\backslash t))\vee(t\times t)$. This union is contained component-wise in the one in (18).

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A SEMANTIC SOLUTION IS POSSIBLE

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The *leitmotif* of this work

A semantic characterization of models where *stuttering* is absent, should yield a subtyping relation that is:

1. Semantic
2. Intuitive for the programmer
3. Decidable

A semantic solution

Rough idea

Make indivisible types "splittable" so that type variables can range over strict subsets of every type, indivisible types included.
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and such that it satisfies:

$$[\![t_1{\to}s_1]\!]\eta \subseteq [\![t_2{\to}s_2]\!]\eta \iff \mathcal{P}(\overline{[\![t_1]\!]\eta \times \overline{[\![s_1]\!]\eta}}) \subseteq \mathcal{P}(\overline{[\![t_2]\!]\eta \times \overline{[\![s_2]\!]\eta}})$$

Subtyping relation

In this framework the natural definition of subtyping is

$$s \leq t \quad \overset{def}{\Longleftrightarrow} \quad \forall \eta \,.\, [\![s]\!]\eta \subseteq [\![t]\!]\eta$$

It "just" remains to find the uniformity condition to avoid stuttering and recover parametricity.

The magic property: **convexity**

Consider **only** models of semantic subtyping in which the following **convexity** property holds

$$\forall\eta.(\llbracket t_1\rrbracket\eta{=}\varnothing \text{ or } \llbracket t_2\rrbracket\eta{=}\varnothing) \iff (\forall\eta.\llbracket t_1\rrbracket\eta{=}\varnothing) \text{ or } (\forall\eta.\llbracket t_2\rrbracket\eta{=}\varnothing)$$

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Of course the problematic relation never holds, whatever the t:

$$(t \times \alpha) \not\leq (t \times \neg t) \vee (\alpha \times t)$$

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And we can prove far more complicated relations (see paper).

Subtyping algorithm

Subtyping Algorithm: $t_1 \leq t_2$

Step 1: Transform the subtyping problem into an emptiness decision problem:

$t_1 \leq t_2 \iff \forall\eta.[\![t_1]\!]\eta \subseteq [\![t_2]\!]\eta \iff \forall\eta.[\![t_1{\wedge}\neg t_2]\!]\eta{=}\varnothing \iff t_1{\wedge}\neg t_2 \leq \mathbb{0}$

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Step 2: **Put the type whose emptiness is to be decided in disjunctive normal form.**

$$\bigvee_{i\in I}\bigwedge_{j\in J}\ell_{ij}$$

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Step 3: Simplify mixed intersections:

Consider each summand of the union: cases such as $t_1{\times}t_2 \wedge t_1{\to}t_2$ or $t_1{\times}t_2 \wedge \neg(t_1{\to}t_2)$ are straightforward.

Solve:
$$\bigwedge_{i\in I} a_i \bigwedge_{j\in J} \neg a'_j \bigwedge_{h\in H} \alpha_h \bigwedge_{k\in K} \neg\beta_k$$

where all a are of the same kind.

Step 4: Eliminate toplevel negative variables.,

$$\forall\eta.[\![t]\!]\eta = \varnothing \iff \forall\eta.[\![t\{{}^{\neg\alpha}/_{\alpha}\}]\!]\eta = \varnothing$$

so replace $\neg\beta_k$ for β_k (forall $k \in K$)

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Step 5: Eliminate toplevel variables.

$$\bigwedge_{t_1\times t_2\in P} t_1\times t_2 \bigwedge_{h\in H} \alpha_h \quad\leq\quad \bigvee_{t'_1\times t'_2\in N} t'_1\times t'_2$$

holds if and only if

$$\bigwedge_{t_1\times t_2\in P} t_1\sigma \times t_2\sigma \bigwedge_{h\in H} \gamma^1_h \times \gamma^2_h \quad\leq\quad \bigvee_{t'_1\times t'_2\in N} t'_1\sigma \times t'_2\sigma$$

where $\sigma = \{{}^{(\gamma^1_h\times\gamma^2_h)\vee\alpha_h}/_{\alpha_h}\}_{h\in H}$ (similarly for arrows)

Step 6: Eliminate toplevel constructors, memoize, and recurse.
Thanks to *convexity* and (set-theoretic) product decomposition rules

$$\bigwedge_{t_1\times t_2\in P} t_1\times t_2 \quad\leq\quad \bigvee_{t_1'\times t_2'\in N} t_1'\times t_2' \qquad (2)$$

is equivalent to

$$\forall N'\subseteq N.\left(\bigwedge_{t_1\times t_2\in P} t_1 \leq \bigvee_{t_1'\times t_2'\in N'} t_1'\right) \text{ or } \left(\bigwedge_{t_1\times t_2\in P} t_2 \leq \bigvee_{t_1'\times t_2'\in N\setminus N'} t_2'\right)$$

(similarly for arrows)

Conclusion and New Directions

Conclusion

- We presented the first known solution to the problem of defining a semantic subtyping relation for a polymorphic regular tree types.
- A solution to this problem was considered unfeasible or even impossible.
- Our solution immediately applies to functional XML processing, but the potential fields of application seem much more numerous.
- Finally, our work opens both *practical* and *theoretical* new directions of research.

Practical problems

New typing possibilities:

```
fun even =
    | Int -> (x mod 2) == 0
    |  _  -> x
```

Intuitively we want to type it by

$$(\texttt{Int}{\longrightarrow}\texttt{Bool}) \wedge (\alpha\backslash\texttt{Int} \rightarrow \alpha\backslash\texttt{Int})$$

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Local type inference:

Let **map** : $(\alpha \to \beta) \to \alpha$ list $\to \beta$ list,

then for **map even** we wish to deduce the following type:

$(\ \mathtt{Int}\ \text{list} \to \mathtt{Bool}\ \text{list}\ ) \wedge$
$(\ (\alpha \backslash \mathtt{Int})\ \text{list} \to (\alpha \backslash \mathtt{Int})\ \text{list}\ ) \wedge$
$(\ \alpha\ \text{list} \to ((\alpha \backslash \mathtt{Int}) \vee \mathtt{Bool})\ \text{list}\ )$

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No principal typing (needs infinite connectives)

Practical problems

New typing possibilities: **new language design**

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Local type inference: **subtyping + instantiation**

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Convexity and parametricity?

In reality, the condition to be used is the generalization to n types:

$$\forall\eta.([\![t_1]\!]\eta{=}\varnothing \text{ or } \cdots \text{ or } [\![t_n]\!]\eta{=}\varnothing)$$
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Two examples of uniformity:

- $(t_1\times\ldots\times t_n)$ is empty if and only if exists at least one t_i empty
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... we have strong flavors of parametricity