A Gentle Introduction to Semantic Subtyping

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Outline

1. Motivations and goals.
2. Semantic subtyping.
3. Subtyping Algorithms.
4. Application to a language.
5. Extensions.

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The goal is to show how to take your favourite type constructors

$$\times,\ \rightarrow,\ \{\ldots\},\ \texttt{chan}(),\ \ldots$$

and add boolean combinators:

$$\vee,\quad \wedge,\quad \neg$$

so that they behave set-theoretically w.r.t. $\leq$

WHY?

Short answer: they are convenient and you need them to program XML in a typed language with **pattern matching**.

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Let:
- $t = \{v \mid v$ value of type $t\}$
- $\wr p \int = \{v \mid v$ matches pattern $p\}$

1. **Useful for typing:**

   `match` e `with` p_1 `->` e_1 `|` p_2 `->` e_2

   - To infer the type t_1 of e_1 we need $t \wedge \wr p_1 \int$ (where $e : t$);
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   - The type of the match is $t_1 \vee t_2$.

2. **Useful for programming:**

   `map` *catalog* `with`
   `x : (Car ∧ (¬Used ∨ Guarantee)) -> x`

   Select in *catalog* all the cars that if they are used then have a guarantee.

3. **Useful for other paradigms:**

   a general technique to add subtyping to different paradigms: functional (e.g. ML), concurrent (e.g. π-calculus), . . .

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- Handling subtyping without combinators is easy: constructors do not mix, e.g.:

$$\frac{s_2 \leq s_1 \qquad t_1 \leq t_2}{s_1 \rightarrow t_1 \leq s_2 \rightarrow t_2}$$

- With combinators is much harder: combinators distribute over constructors, e.g.

$$(s_1 \vee s_2) \rightarrow t \ \gtrless \ (s_1 \rightarrow t) \wedge (s_2 \rightarrow t)$$

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Instead of defining the subtyping relation so that it conforms to the semantic of types, define the semantics of types and derive the subtyping relation.

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$$[\![\ ]\!] : \textbf{Types} \longrightarrow \mathscr{P}(\mathscr{D})$$

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$$s \leq t \quad \stackrel{def}{\Longleftrightarrow} \quad [\![s]\!] \subseteq [\![t]\!]$$

KEY OBSERVATION 1:

The *model of types* may be independent from a *model of terms*

Hosoya and Pierce use the model of values:

$$[\![t]\!]_{\mathcal{V}} = \{v \mid \vdash v : t\}$$

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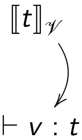

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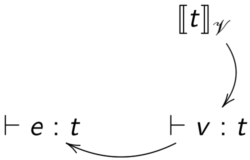

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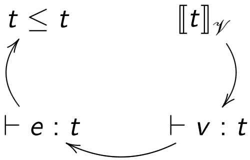

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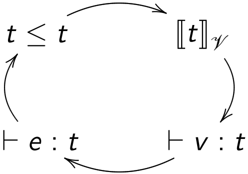

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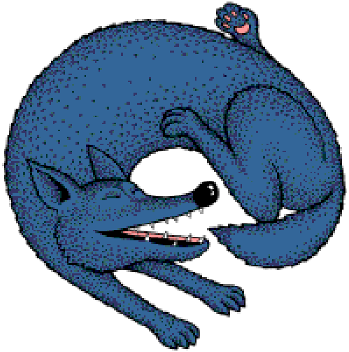

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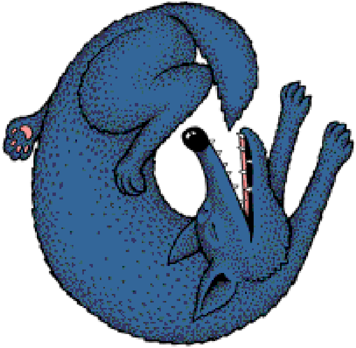

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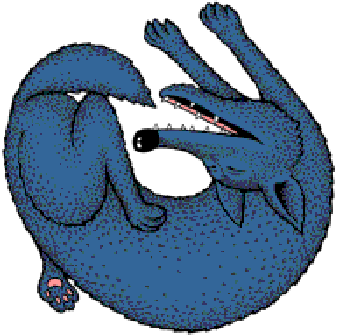

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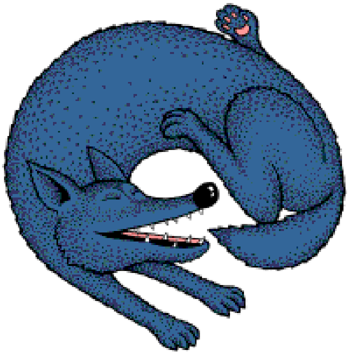

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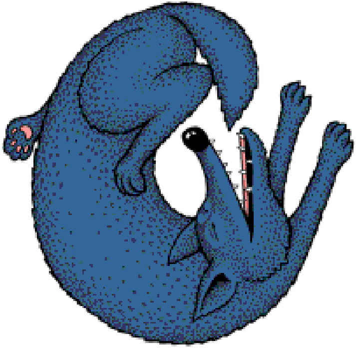

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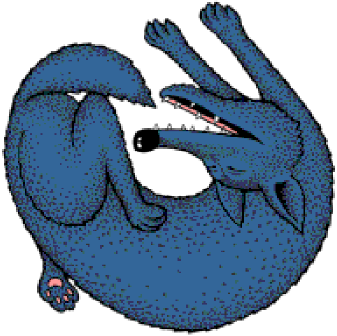

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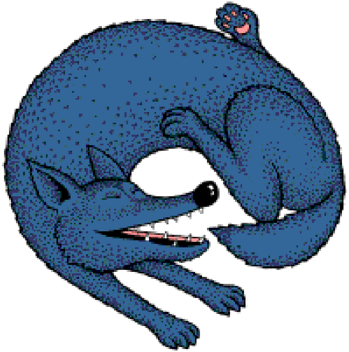

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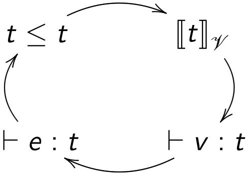

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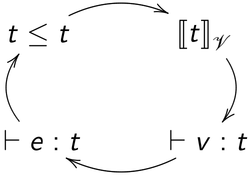

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$[\![t]\!]_{\mathscr{D}}$

$t \leq t$ $\quad\quad$ $[\![t]\!]_{\mathscr{V}}$

$\vdash e : t$ $\quad\quad$ $\vdash v : t$

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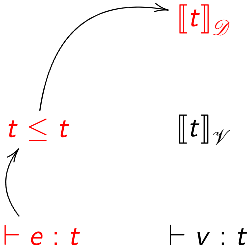

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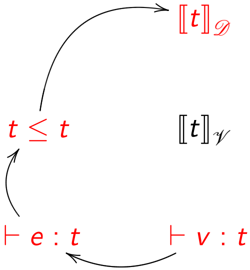

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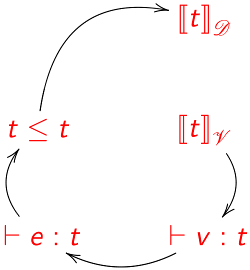

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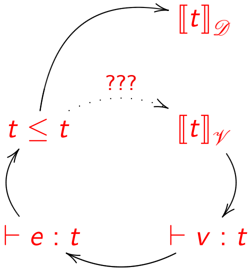

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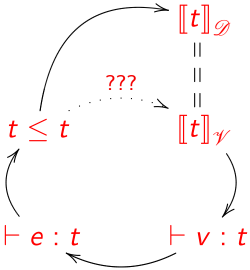

Step 1 : Model

Define when $[\![\ ]\!] : \mathbf{Types} \longrightarrow \mathscr{P}(\mathscr{D})$ yields a *set-theoretic* model.

- Easy for the combinators:

$$
\begin{aligned}
[\![t_1 \vee t_2]\!] &= [\![t_1]\!] \cup [\![t_2]\!] \\
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\end{aligned}
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- Hard for constructors:

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\begin{aligned}
[\![t_1 \times t_2]\!] &= [\![t_1]\!] \times [\![t_2]\!] \\
[\![t_1 \rightarrow t_2]\!] &= ???
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Think semantically!

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Intuition

$$[\![t \longrightarrow s]\!] = ???$$

Impossible since it requires $\mathscr{P}(\mathscr{D}^2) \subseteq \mathscr{D}$

KEY OBSERVATION 2:

We need the model to state how types are related rather than what the types are

Accept every $[\![\ ]\!]$ that behaves w.r.t. $\subseteq$ as if equation $(*)$ held, namely

$$[\![t_1 \longrightarrow s_1]\!] \subseteq [\![t_2 \longrightarrow s_2]\!] \iff \mathscr{P}(\overline{[\![t_1]\!] \times \overline{[\![s_1]\!]}}) \subseteq \mathscr{P}(\overline{[\![t_2]\!] \times \overline{[\![s_2]\!]}})$$

and similarly for any boolean combination of arrow types.

Intuition

$$[\![t \to s]\!] = \{\text{functions from } [\![t]\!] \text{ to } [\![s]\!]\}$$

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$$[\![t \to s]\!] = \{f \subseteq \mathscr{D}^2 \mid \forall (d_1, d_2) \in f.\ d_1 \in [\![t]\!] \Rightarrow d_2 \in [\![s]\!]\}$$

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We need the model to state **how types are related** rather than **what the types are**

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2. **Define** $\mathbb{E}[\![\_]\!]$: **Types** $\to \mathscr{P}(\mathscr{D}^2 + \mathscr{P}(\mathscr{D}^2))$ as follows

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To characterize $\leq$ all is needed is the test of emptyness

Indeed: $s \leq t \Leftrightarrow [\![s]\!] \subseteq [\![t]\!] \Leftrightarrow [\![s]\!] \cap \overline{[\![t]\!]} = \varnothing \Leftrightarrow [\![s \wedge \neg t]\!] = \varnothing$

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We relaxed our requirement but . . .

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$\mathbb{E}[\![\,]\!]$ characterizes the behavior of types (for what it concerns $\leq$ one can consider $[\![t]\!] = \mathbb{E}[\![t]\!]$): it depends on the language the types are intended for.

Variations are possible. Our choice

$$\mathbb{E}[\![t_1 \rightarrow t_2]\!] = \mathscr{P}(\overline{[\![t_1]\!] \times \overline{[\![t_2]\!]}})$$

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1. Take any model $(\mathscr{B}, [\![]\!]_{\mathscr{B}})$ to bootstrap the definition.
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$$s \leq_{\mathscr{B}} t \quad \iff \quad [\![s]\!]_{\mathscr{B}} \subseteq [\![t]\!]_{\mathscr{B}}$$
3. Take any "appropriate" language $\mathscr{L}$ and use $\leq_{\mathscr{B}}$ to type it
$$\Gamma \vdash_{\mathscr{B}} e : t$$
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$$s \leq_{\mathscr{V}} t \quad \iff \quad [\![s]\!]_{\mathscr{V}} \subseteq [\![t]\!]_{\mathscr{V}}$$

5. If $\mathscr{L}$ is "appropriate" ($\vdash_{\mathscr{B}} v : t \iff \not\vdash_{\mathscr{B}} v : \neg t$) then $[\![]\!]_{\mathscr{V}}$ is a model and

$$s \leq_{\mathscr{B}} t \quad \iff \quad s \leq_{\mathscr{V}} t$$

The circle is closed

Closing the circle

1. Take any model $(\mathscr{B}, [\![]\!]_{\mathscr{B}})$ to bootstrap the definition.
2. Define

$$s \leq_{\mathscr{B}} t \quad\iff\quad [\![s]\!]_{\mathscr{B}} \subseteq [\![t]\!]_{\mathscr{B}}$$

3. Take any "appropriate" language $\mathscr{L}$ and use $\leq_{\mathscr{B}}$ to type it

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Exhibit a model

Does a model exists? (i.e. a $[\![\ ]\!]$ such that $[\![t]\!] = \emptyset \iff \mathbb{E}[\![t]\!] = \emptyset$)

YES: take $(\mathscr{U}, [\![\ ]\!]_{\mathscr{U}})$ where

1. $\mathscr{U}$ least solution of $X = X^2 + \mathscr{P}_f(X^2)$
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Subtyping Algorithms.

Canonical forms

Every (recursive) type

$$t \ ::= \ B \ | \ t{\times}t \ | \ t{\rightarrow}t \ | \ t{\vee}t \ | \ t{\wedge}t \ | \ \neg t \ | \ \mathbf{0} \ | \ \mathbf{1}$$

is equivalent (semantically, that is w.r.t. $\leq$) to a type of the form:

$$\bigvee_{(P,N)\in\Pi}\Big(\big(\bigwedge_{s\times t\in P} s{\times}t\big)\wedge\big(\bigwedge_{s\times t\in N}\neg(s{\times}t)\big)\Big)\ \bigvee_{(P,N)\in\Sigma}\Big(\big(\bigwedge_{s\rightarrow t\in P} s{\rightarrow}t\big)\wedge\big(\bigwedge_{s\rightarrow t\in N}\neg(s{\rightarrow}t)\big)\Big)$$

1. Put it in disjunctive normal form, e.g.

   $$(a_1\wedge a_2\wedge\neg a_3)\vee(a_4\wedge\neg a_5)\vee(\neg a_6\wedge\neg a_7)\vee(a_8\wedge a_9)$$

2. Transform to have only homogeneous intersections, e.g.

   $$((s_1\times t_1)\wedge\neg(s_2\times t_2))\ \vee\ (\neg(s_3\rightarrow t_3)\wedge\neg(s_4\rightarrow t_4))\ \vee\ (s_5\times t_5)$$

3. Group negative and positive atoms in the intersections:

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Every (recursive) type

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Subtyping decomposition

Some ugly formulas:

$$\bigwedge_{i\in I} t_i{\times}s_i \leq \bigvee_{i\in J} t_i{\times}s_i$$

$$\iff \forall J' \subseteq J.\ \left(\bigwedge_{i\in I} t_i \leq \bigvee_{i\in J'} t_i\right) \text{ or } \left(\bigwedge_{i\in I} s_i \leq \bigvee_{i\in J\setminus J'} s_i\right)$$

$$\bigwedge_{i\in I} t_i{\rightarrow}s_i \leq \bigvee_{i\in J} t_i{\rightarrow}s_i$$

$$\iff \exists j \in J.\forall I' \subseteq I.\ \left(t_j \leq \bigvee_{i\in I'} t_i\right) \text{ or } \left(I' \neq I \text{ et } \bigwedge_{i\in I\setminus I'} s_i \leq s_j\right)$$

Decision procedure

$$s \leq t?$$

Recall that:

$$s \leq t \iff [\![s]\!] \cap \overline{[\![t]\!]} = \varnothing \iff [\![s \wedge \neg t]\!] = \varnothing \iff s \wedge \neg t = \mathbf{0}$$

1. Consider $s \wedge \neg t$
2. Put it in canonical form

$$\bigvee_{(P,N)\in\Pi} \Big(\big(\bigwedge_{s\times t\in P} s\times t\big) \wedge \big(\bigwedge_{s\times t\in N} \neg(s\times t)\big)\Big) \quad \bigvee_{(P,N)\in\Sigma} \Big(\big(\bigwedge_{s\to t\in P} s\to t\big) \wedge \big(\bigwedge_{s\to t\in N} \neg(s\to t)\big)\Big)$$

3. Decide (coinductively) whether the two summands are both empty by applying the ugly formulas of the previous slide.

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Application to a language.

Language

$e ::=$	x	variable
$\mid$	$\mu f^{(s_1 \to t_1; \ldots; s_n \to t_n)}(x).e$	abstraction, $n \geq 1$
$\mid$	$e_1 e_2$	application
$\mid$	(e_1, e_2)	pair
$\mid$	$\pi_i(e)$	projection, $i = 1, 2$
$\mid$	$(x = e \in t)?e_1 : e_2$	binding type case

Typing

$$\frac{\Gamma \vdash e : s \leq_{\mathscr{B}} t}{\Gamma \vdash e : t} \quad (subsumption)$$

$$\frac{(\forall i)\ \Gamma, (f : s_1{\to}t_1\wedge\ldots\wedge s_n{\to}t_n), (x : s_i) \vdash e : t_i}{\Gamma \vdash \mu f^{(s_1\to t_1;\ldots;s_n\to t_n)}(x).e : s_1{\to}t_1\wedge\ldots\wedge s_n{\to}t_n} \quad (abstr)$$

(for $s_1 \equiv s\wedge t$, $s_2 \equiv s\wedge\neg t$)

$$\frac{\Gamma \vdash e : s \qquad \Gamma, (x : s_1) \vdash e_1 : t_1 \qquad \Gamma, (x : s_2) \vdash e_2 : t_2}{\Gamma \vdash (x = e \in t)?e_1 : e_2 : \bigvee_{\{i \mid s_i \not\simeq \mathbf{0}\}} t_i} \quad (typecase)$$

Consider:

$$\mu f^{(\mathrm{Int}\to\mathrm{Int};\mathrm{Bool}\to\mathrm{Bool})}(x).(y = x \in \mathrm{Int})?(y+1):\mathrm{not}(y)$$

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Reduction

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\begin{array}{rcll}
(\mu f^{(\dots)}(x).e)v & \to & e[x/v, (\mu f^{(\dots)}(x).e)/f] & \\
(x = v \in t)?e_1 : e_2 & \to & e_1[x/v] & \text{if } v \in [\![t]\!] \\
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where

$$v \;::=\; \mu f^{(\dots)}(x).e \;\mid\; (v,v)$$

And we have

$$s \leq_{\mathscr{B}} t \qquad \Longleftrightarrow \qquad s \leq_{\mathscr{V}} t$$

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Why does it work?

$$s \leq_{\mathscr{B}} t \quad\Longleftrightarrow\quad s \leq_{\mathscr{V}} t \qquad (1)$$

Equation (1) (actually, $\Rightarrow$) states that the language is quite rich, since there always exists a value to separate two distinct types; i.e. its set of values is a model of types with "enough points"

For any model $\mathscr{B}$,

$$s \not\leq_{\mathscr{B}} t \Longrightarrow \text{there exists } v \text{ such that } \vdash v : s \text{ and } \not\vdash v : t$$

In particular, thanks to multiple arrows in λ-abstractions:

$$\bigwedge_{i=1..k} s_i \to t_i \not\leq t$$

then the two types are distinguished by $\mu f^{(s_1 \to t_1; \ldots; s_k \to t_k)}(x).e$

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Advantages for the programmer

The programmer does not need to know the gory details. All s/he needs to retain is

1. Types are the set of values of that type
2. Subtyping is set inclusion

Furthermore the property

$$s \not\leq t \Longrightarrow \text{there exists } v \text{ such that } \vdash v : s \text{ and } \not\vdash v : t$$

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Extensions.

ref types

$$[\![\texttt{ref}\ t]\!] = \{\texttt{ref}\ v \mid v \in [\![t]\!]\}$$

In practice equivalent to

$$[\![\texttt{ref}\ t]\!] = \begin{cases} \{[\![t]\!]\} & \text{if } [\![t]\!] \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases} \qquad (2)$$

Deduce the subtyping relation

$$\Big(\bigwedge_{\texttt{ref}\ s \in P} \texttt{ref}\ s\Big) \leq \Big(\bigvee_{\texttt{ref}\ t \in N} \texttt{ref}\ t\Big) \iff \begin{array}{l} \exists \texttt{ref}\ s \in P,\ \ s \simeq 0, \text{or} \\ \exists \texttt{ref}\ s_1 \in P,\ \exists \texttt{ref}\ s_2 \in P, s_1 \not\simeq s_2, \text{or} \\ \exists \texttt{ref}\ s \in P,\ \exists \texttt{ref}\ t \in N,\ \ s \simeq t \end{array}$$

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lazy types

If we define $t = \mathsf{Int}\times t$ then $t \simeq \mathbf{0}$.

Use $s = \mathsf{Int}\times \mathtt{lazy}\, s$.

$$(\mu f^{(\mathsf{Int}\to s)}(x).(x, \mathtt{lazy}\,(f(x+1))))0$$

$$[\![\mathtt{lazy}\, t]\!] = \{\mathtt{lazy}\, e \mid e \text{ is closed and } e : t\}$$

But each $\mathtt{lazy}\, e$ is identified by all the possible results it can return, namely $\{v \mid e \to^* v\}$, from which we deduce:

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Deduce the subtyping relation

$$(\textstyle\bigwedge_{\mathtt{lazy}\, s\in P} \mathtt{lazy}\, s) \leq (\bigvee_{\mathtt{lazy}\, t\in N} \mathtt{lazy}\, t) \iff \exists \mathtt{lazy}\, t \in N : \forall P' \subseteq P (\bigwedge_{\mathtt{lazy}\, s\in P'} s) \leq t$$

lazy types

If we define $t = \mathrm{Int}{\times}t$ then $t \simeq \mathbf{0}$.

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This theory applies to other paradigms, too.

For instance in a paper in LICS '05 it is applied to **π-calculus**. There you have nice things such as:

$$ch(t) \stackrel{\text{def}}{=} ch^{-}(t) \wedge ch^{+}(t)$$

[save a constructor]

$$ch^{+}(t_1) \vee ch^{+}(t_2) \lneqq ch^{+}(t_1 \vee t_2)$$

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Summarizing

$[\![\mathbf{0}]\!] = \emptyset$; $[\![\mathbf{1}]\!] = \mathscr{D}$;
$[\![t_1 \vee t_2]\!] = [\![t_1]\!] \cup [\![t_2]\!]$;
$[\![t_1 \wedge t_2]\!] = [\![t_1]\!] \cap [\![t_2]\!]$;
$[\![\neg t]\!] = \mathscr{D} \backslash [\![t]\!]$;
$[\![t]\!] = \emptyset \iff \mathbb{E}[\![t]\!] = \emptyset$

where the extensional interpretation associated to $[\![\;]\!]$ is defined as:

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\begin{array}{lcl}
\mathbb{E}[\![t \rightarrow s]\!] & = & \mathscr{P}(\overline{[\![t]\!] \times \overline{[\![s]\!]}}) \\
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Conclusion

La morale de l'histoire est . . .

If you have a strong semantic intuition of your favorite language and you want to add set-theoretic $\vee$, $\wedge$, $\neg$ types then:

1. Define $\mathbb{E}[\![\ ]\!]$ for your type constructors so that it matches your semantic intuition
2. Find a model (any model).
3. Use the subtyping relation induced by the model to type your language: if the intuition was right then the set of values is also a model, otherwise tweak it.
4. Use the set-theoretic properties of the model (actually of $\mathbb{E}[\![\ ]\!]$) to decompose the emptyness test for your type constructors, and hence derive a subtyping algorithm.
5. Enjoy.

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Addendum 1: a model may not exist

$$t = \texttt{int} \vee (\texttt{ref}(\texttt{int}) \wedge \texttt{ref}(t))$$

Is t equal to $\texttt{int}$?

$$t = \texttt{int} \iff (\texttt{ref}(\texttt{int}) \wedge \texttt{ref}(t)) = \varnothing \iff t \neq \texttt{int}$$

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Addendum 2: the real *abstr* typing rule

$$t \equiv (\bigwedge_{i=1..n} s_i \to t_i) \setminus (\bigvee_{j=1..m} s'_j \to t'_j) \not\leq \mathbf{0}$$

$$\frac{(\forall i)\ \Gamma, (f : t), (x : s_i) \vdash e : t_i}{\Gamma \vdash \mu f^{(s_1 \to t_1; \ldots; s_n \to t_n)}(x).e : t}\ (abstr)$$

Addendum 3: A different definition for $\mathbb{E}[\![\ ]\!]$

Note that according to the previous $\mathbb{E}[\![\ ]\!]$:

$$s \rightarrow t \ \leq \ \mathbf{1} \rightarrow \mathbf{1} \tag{3}$$

Every application is well typed. Add a distinguished Ω to denote a runtime type error, modify

$$\mathbb{E}[\![t{\longrightarrow}s]\!] = \{f \subseteq \mathscr{D}{\times}(\mathscr{D}{\cup}\{\Omega\}) \mid \forall(d_1, d_2) \in f.\, d_1 \in [\![t]\!] \Rightarrow d_2 \in [\![s]\!]\}$$

(3) no longer holds since the constant map $\neg s \mapsto \Omega$, is in the left hand type but not in the right one.

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