CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

PHILIPPE GAUCHER

About the globular homology of higher dimensional automata

Cahiers de topologie et géométrie différentielle catégoriques, tome 43, nº 2 (2002), p. 107-156.

<http://www.numdam.org/item?id=CTGDC_2002__43_2_107_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLIII-2 (2002)

ABOUT THE GLOBULAR HOMOLOGY OF HIGHER DIMENSIONAL AUTOMATA

by Philippe GAUCHER

RESUME. On introduit un nouveau nerf simplicial d'automate parallèle dont l'homologie simpliciale décalée de 1 foumit une nouvelle définition de l'homologie globulaire. Avec cette nouvelle définition, les inconvénients de la construction d'un article antérieur de l'auteur disparaissent. De plus les importants morphismes qui associent à tout globe les zones correspondantes de branchements et de confluences de chemins d'exécution deviennent ici des morphismes d'ensembles simpliciaux.

1 Introduction

One of the contributions of [11] is the introduction of two homology theories as a starting point for studying branchings and mergings in higher dimensional automata (HDA) from an homological point of view. However these homology theories had an important drawback : roughly speaking, they were not invariant by subdivisions of the observation. Later in [9], using a model of concurrency by strict globular ω-categories borrowed from [19], two new homology theories are introduced : the negative and positive corner homology theories H^- and H^+, also called the branching and the merging homologies. It is proved in [8] that they overcome the drawback of Goubault's homology theories.

Another idea of [9] is the construction of a diagram of abelian groups like in Figure 1, where H^{gl}_* is a new homology theory called the globular homology.

Geometrically, the non-trivial cycles of the globular homology must correspond to the oriented empty globes of $\mathcal{C}$, and the non-trivial cycles of the branching (resp. the merging) homology theory must correspond to the branching (resp. merging) areas of execution paths. And the morphisms h^- and h^+ must associate to any globe its corresponding branching area and merging area of execution paths. Many potential applications in computer science of these morphisms are put forward in [9].

Globular homology was therefore created in order to fulfill two conditions :

- Globular homology must take place in a diagram of abelian groups like in Figure 1. And the geometric meaning of h^- and h^+ must be exactly as above described.
- Globular homology must be an invariant of HDA with respect to reasonable deformations of HDA, that is of the corresponding ω-category.

What is a reasonable deformation of HDA was not yet very clear in [9]. This question is discussed with much more details in [10].

The *old globular homology* (i.e. the construction exposed in [9]) satisfied the first condition, and the second one was supposed to be satisfied by definition (cf. Definition 8.2 of two homotopic ω-categories in [9]), even if some problems were already mentioned, particularly the non-vanishing of the "old" globular homology of I^3, and more generally of I^n for any $n \geqslant 1$ in strictly positive dimension.

This latter problem is disturbing because the n-cube I^n (i.e. the corresponding automaton which consists of n 1-transitions carried out at the same time) can be deformed by crushing all the p-faces with $p > 1$ into an ω-category which has only 0-morphisms and 1-morphisms and because the globular homology is supposed to be an invariant by such deformations. The philosophy exposed in [10] tells us similar things : using S-deformations and T-deformations, the n-cube and the oriented line must be the same up to homotopy, and therefore must have the same globular homology.

The non-vanishing of the second globular homology group of I^3 (see Figure 2(c)) is due for instance to the 2-dimensional globular cycle

$$(R(-00) *_0 R(0++)) *_1 (R(-0-) *_0 R(0+0))$$
$$- (R(-00) *_0 R(0++)) - (R(-0-) *_0 R(0+0))$$

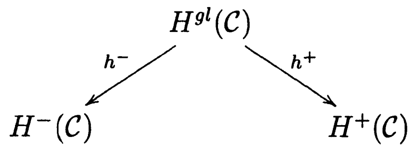


Figure 1: Associating to any globe its two corners

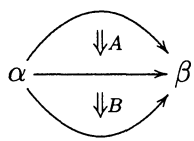
It is the reason why it was suggested in [9] to add the relation $A *_1 B = A + B$ at least to the 2-dimensional stage of the old globular complex.

But there is then no reason not to add the same relation in the rest of the definition of the old globular complex. For example, if we take the quotient of the old globular complex by the relation $A *_1 B = A + B$ for any pair (A, B) of 2-morphisms, then the ω-category defined as the free ω-category generated by the globular set generated by two 3-morphisms A and B such that $t_1 A = s_1 B$ gives rise to a 3-dimensional globular cycle $A *_1 B - A - B$ because $s_2(A *_1 B - A - B) = s_2 A *_1 s_2 B - s_2 A - s_2 B = 0$ and $t_2(A *_1 B - A - B) = t_2 A *_1 t_2 B - t_2 A - t_2 B = 0$. So putting the relation $A *_1 B - A - B = 0$ in the old globular complex for any pair of morphisms (A, B) of the same dimension sounds necessary. Similar considerations starting from the calculation of the $(n-1)$-th globular homology group of I^n entail the relations $A *_n B - A - B$ for any $n \geqslant 1$ and for any pair (A, B) of p-morphisms with $p \geqslant n + 1$ in the old globular chain complex.

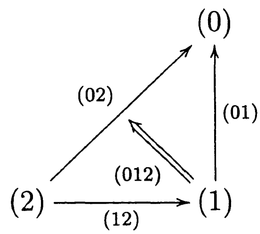
The *formal globular homology* of Definition 9.3 is exactly equal to the quotient of the old globular complex by these missing relations. It is conjectured (see conjecture 9.5) that this homology theory will coincide for free ω-categories generated by semi-cubical sets with the homology theory of Definition 5.2, this latter being the simplicial homology of the globular simplicial nerve $\mathcal{N}^{gl}$ shifted by one.

We claim that Definition 5.1 (and its simplicial homology shifted by one) cancels the drawback of the old globular homology at least for the following reasons :

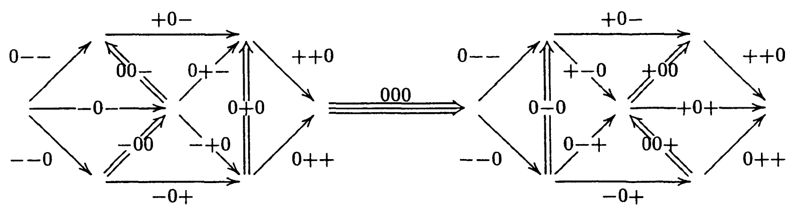
- It is noticed in [9] that both corner homologies come from the simplicial homology of two augmented simplicial nerves $\mathcal{N}^-$ and $\mathcal{N}^+$; there exists one and only natural transformation h^- (resp. h^+) from $\mathcal{N}^{gl}$ to $\mathcal{N}^-$ (resp. $\mathcal{N}^+$) preserving the interior labeling (Theorem 6.1).



(a) Composition of two 2-morphims



(b) The ω-category Δ^2



(c) The ω-category I^3

Figure 2: Some ω-categories (a k-fold arrow symbolizes a k-morphism)

- In homology, h^- and h^+ induce two natural linear maps from H_*^{gl} to resp. H_*^- and H_*^+ which do exactly what we want.

- The globular homology (formal or not) of I^n vanishes in strictly positive dimension for any $n \geqslant 0$. The globular homology of Δ^n (the n-simplex) and of 2_n (the free ω-category generated by one n-dimensional morphism) as well.

- Using Theorem 9.7 explaining the exact mathematical link between the old construction and the new one, one sees that one does not lose the possible applications in computer science pointed out in [9].

- The new globular homology, as well as the new globular cut are invariant by S-deformations, that is intuitively by contraction and dilatation of homotopies between execution paths. We will see however that it is not invariant by T-deformations, that is by subdivision of the time, as the old definition and this problem will be a little bit discussed.

This paper is two-fold. The first part introduces the new material. The second part justifies the new definition of the globular homology.

After Section 2 which recalls some conventions and some elementary facts about strict globular ω-categories (non-contracting or not) and about simplicial sets, the setting of *simplicial cuts* of non-contracting ω-categories and that of *regular cuts* are introduced. The first notion allows to enclose the new globular nerve of this paper and both corner nerves in one unique formalism. The notion of regular cuts gives an axiomatic framework for the generalization of the notion of negative and positive folding operators of [8]. Section 4 is an illustration of the previous new notions on the case of corner nerves. In the same section, some non-trivial facts about negative folding operators are recalled. Section 5 provides the definition of the globular nerve of a non-contracting ω-category.

The organization of the rest of the paper follows the preceding explanations. First in Section 6, the morphisms h^- and h^+ are constructed. Section 7 proves that the globular cut is regular. In particular, we get the globular folding operators. Section 8 proves the vanishing of the globular homology of the n-cube, the n-simplex and the free ω-category generated by one n-morphism. At last Section 9 makes explicit the exact relation between the new globular homology and the old one. Section 10 speculates about deformations of ω-categories considered as a model of HDA and the construction of the bisimplicial set of [10] is detailed.

2 Conventions and notations

2.1 Globular ω-category and cubical set

For us, an ω-category will be a strict globular ω-category with morphisms of finite dimension. More precisely (see [3] [23] [22] for more details) :

Definition 2.1. *A* 1-*category is a pair* $(A, (*, s, t))$ *satisfying the following axioms :*

1. *A is a set*
2. s *and* t *are set maps from* A *to* A *respectively called the source map and the target map*
3. *for* $x, y \in A$, $x * y$ *is defined as soon as* $tx = sy$
4. $x * (y * z) = (x * y) * z$ *as soon as both members of the equality exist*
5. $sx * x = x * tx = x$, $s(x * y) = sx$ *and* $t(x * y) = ty$ *(this implies* $ssx = sx$ *and* $ttx = tx$*).*

Definition 2.2. *A* 2-*category is a triple* $(A, (*_0, s_0, t_0), (*_1, s_1, t_1))$ *such that*

1. *both pairs* $(A, (*_0, s_0, t_0))$ *and* $(A, (*_1, s_1, t_1))$ *are* 1-*categories*
2. $s_0 s_1 = s_0 t_1 = s_0$, $t_0 s_1 = t_0 t_1 = t_0$, *and for* $i \geqslant j$, $s_i s_j = t_i s_j = s_j$ *and* $s_i t_j = t_i t_j = t_j$ *(Globular axioms)*
3. $(x *_0 y) *_1 (z *_0 t) = (x *_1 z) *_0 (y *_1 t)$ *(Godement axiom or interchange law)*
4. *if* $i \neq j$, *then* $s_i(x *_j y) = s_i x *_j s_i y$ *and* $t_i(x *_j y) = t_i x *_j t_i y$.

Definition 2.3. *A globular* ω-*category* $\mathcal{C}$ *is a set* A *together with a family* $(*_n, s_n, t_n)_{n \geqslant 0}$ *such that*

1. *for any* $n \geqslant 0$, $(A, (*_n, s_n, t_n))$ *is a* 1*-category*

2. *for any* $m, n \geqslant 0$ *with* $m < n$, $(A, (*_m, s_m, t_m), (*_n, s_n, t_n))$ *is a* 2*-category*

3. *for any* $x \in A$, *there exists* $n \geqslant 0$ *such that* $s_n x = t_n x = x$ *(the smallest of these* n *is called the dimension of* x*).*

A n-dimensional element of $\mathcal{C}$ is called a n-morphism. A 0-morphism is also called a *state* of $\mathcal{C}$, and a 1-morphism an *arrow*. If x is a morphism of an ω-category $\mathcal{C}$, we call $s_n(x) = d_n^-(x)$ the *n-source* of x and $t_n(x) = d_n^+(x)$ the *n-target* of x. The category of all ω-categories (with the obvious morphisms) is denoted by ωCat. The corresponding morphisms are called ω-functors. The set of morphisms of $\mathcal{C}$ of dimension at most n is denoted by $tr^n\mathcal{C}$; the set of morphisms of $\mathcal{C}$ of dimension exactly n is denoted by $\mathcal{C}_n$.

Sometime we will use the terminology *initial state* (resp. *final state*) for a state α which is not the 0-target (resp. the 0-source) of a 1-morphism.

Definition 2.4. *[4] [13] A cubical set consists of*

- *a family of sets* $(K_n)_{n\geqslant 0}$
- *a family of face maps* $K_n \xrightarrow{\partial_i^\alpha} K_{n-1}$ *for* $\alpha \in \{-,+\}$
- *a family of degeneracy maps* $K_{n-1} \xrightarrow{\epsilon_i} K_n$ *with* $1 \leqslant i \leqslant n$

which satisfy the following relations

1. $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha$ *for all* $i < j \leqslant n$ *and* $\alpha, \beta \in \{-,+\}$

2. $\epsilon_i \epsilon_j = \epsilon_{j+1} \epsilon_i$ *for all* $i \leqslant j \leqslant n$

3. $\partial_i^\alpha \epsilon_j = \epsilon_{j-1} \partial_i^\alpha$ *for* $i < j \leqslant n$ *and* $\alpha \in \{-,+\}$

4. $\partial_i^\alpha \epsilon_j = \epsilon_j \partial_{i-1}^\alpha$ *for* $i > j \leqslant n$ *and* $\alpha \in \{-,+\}$

5. $\partial_i^\alpha \epsilon_i = Id$

A family $(K_n)_{n\geqslant 0}$ *only equipped with a family of face maps* ∂_i^α *satisfying the same axiom as above is called a semi-cubical set.*

Definition 2.5. *The corresponding category of cubical sets, with an obvious definition of its morphisms, is isomorphic to the category of presheaves* $Sets^{\square^{op}}$ *over a small category* $\square$. *The corresponding category of semi-cubical sets , with an obvious definition of its morphisms, is isomorphic to the category of presheaves* $Sets^{\square^{semi^{op}}}$ *over a small category* $\square^{semi}$.

In a simplicial set, the face maps are always denoted by ∂_i, the degeneracy maps by ϵ_i. Here are the other conventions about simplicial sets (see for example [17] for further information) :

1. $Sets$: category of sets
2. $Sets^{\Delta^{op}}$: category of simplicial sets
3. $Comp(Ab)$: category of chain complexes of abelian groups
4. $C(A)$: unnormalized chain complex of the simplicial set A
5. $H_*(A)$: simplicial homology of a simplicial set A
6. Ab : category of abelian groups
7. Id : identity map
8. $\mathbb{Z}S$: free abelian group generated by the set S

HDA means *higher dimensional automaton.* In this paper, this is another term for *semi-cubical set*, or the corresponding free ω-category generated by it.

Various homology theories (see the diagram of Theorem 9.7) will appear in this paper. It is helpful for the reader to keep in mind that the total homology of a semi-cubical set is used nowhere in this work.

2.2 Non-contracting ω-category

Let $\mathcal{C}$ be an ω-category. We want to define an ω-category $\mathbb{P}\mathcal{C}$ ($\mathbb{P}$ for path) obtained from $\mathcal{C}$ by removing the 0-morphisms, by considering the 1-morphisms of $\mathcal{C}$ as the 0-morphisms of $\mathbb{P}\mathcal{C}$, the 2-morphisms of $\mathcal{C}$ as the 1-morphisms of $\mathbb{P}\mathcal{C}$ etc. with an obvious definition of the source and target maps and of the composition laws (this new ω-category is denoted by $\mathcal{C}[1]$

in [10]). The map $\mathbb{P} : \mathcal{C} \mapsto \mathbb{P}\mathcal{C}$ does not induce a functor from ωCat to itself because ω-functors can contract 1-morphisms and because with our conventions, a 1-source or a 1-target can be 0-dimensional. Hence the following definition

Proposition and definition 2.6. *For a globular ω-category $\mathcal{C}$, the following assertions are equivalent :*

(i) $\mathbb{P}\mathcal{C}$ is an ω-category ; in other terms, $*_i$, s_i and t_i for any $i \geqslant 1$ are internal to $\mathbb{P}\mathcal{C}$ and we can set $*_i^{\mathbb{P}\mathcal{C}} = *_{i+1}^{\mathcal{C}}$, $*_i^{\mathbb{P}\mathcal{C}} = *_{i+1}^{\mathcal{C}}$ and $*_i^{\mathbb{P}\mathcal{C}} = *_{i+1}^{\mathcal{C}}$ for any $i \geqslant 0$.

(ii) The maps s_1 and t_1 are non-contracting, that is if x is of strictly positive dimension, then s_1x and t_1x are 1-dimensional (a priori, one can only say that s_1x and t_1x are of dimension lower or equal than 1)

If Condition (ii) is satisfied, then one says that s_1 and t_1 are non-contracting and that $\mathcal{C}$ is non-contracting.

Proof. Suppose s_1 and t_1 non-contracting. Let x and y be two morphisms of strictly positive dimension and $p \geqslant 1$. Then $s_1 s_p x = s_1 x$ therefore $s_p x$ cannot be 0-dimensional. If $x *_p y$ then $s_1(x *_p y) = s_1 x$ if $p = 1$ and if $p > 1$ for two different reasons. Therefore $x *_p y$ cannot be 0-dimensional as soon as $p \geqslant 1$. $\square$

Definition 2.7. *Let f be an ω-functor from $\mathcal{C}$ to $\mathcal{D}$. The morphism f is non-contracting if for any 1-dimensional $x \in \mathcal{C}$, the morphism $f(x)$ is a 1-dimensional morphism of $\mathcal{D}$ (a priori, $f(x)$ could be either 0-dimensional or 1-dimensional).*

Definition 2.8. *The category of non-contracting ω-categories with the non-contracting ω-functors is denoted by ωCat_1.*

Notice that in [9], the word "non-1-contracting" is used instead of simply "non-contracting". Since [10], the philosophy behind the idea of deforming the ω-categories viewed as models of HDA is better understood. In particular, the idea of not contracting the morphisms is relevant only for 1-dimensional morphisms. So the "1" in "non-1-contracting" is not anymore necessary.

Definition 2.9. *Let C be a non-contracting ω-category. Then the ω-category $\mathbb{P}C$ above defined is called the path ω-category of C. The map $C \mapsto \mathbb{P}C$ induces a functor from ωCat_1 to ωCat.*

Here is a fundamental example of non-contracting ω-category. Consider a semi-cubical set K and consider the free ω-category $\Pi(K) := \int^{\underline{n}\in\square} K_n.I^n$ generated by it where

- I^n is the free ω-category generated by the faces of the n-cube, whose construction is recalled in Section 4.
- the integral sign denotes the coend construction and $K_n.I^n$ means the sum of "cardinal of K_n" copies of I^n (cf. [15] for instance).

Then one has

Proposition 2.10. *For any semi-cubical set K, $\Pi(K)$ is a non-contracting ω-category. The functor $\Pi : Sets^{\square^{semi^{op}}} \rightarrow \omega Cat$ from the category of semi-cubical sets to that of ω-categories yields a functor from $Sets^{\square^{semi^{op}}}$ to the category of non-contracting ω-categories ωCat_1.*

Proof. The characterization of Proposition 2.6 gives the solution. □

3 Cut of globular higher dimensional categories

Before introducing the globular nerve of an ω-category, let us introduce the formalism of *regular simplicial cuts* of ω-categories. The notion of *simplicial cuts* enables us to put together in the same framework both corner nerves constructed in [9, 8] and the new globular nerve of Section 5. The notion of *regular cuts* enables to generalize the notion of negative (resp. positive) folding operators associated to the branching (resp. merging) nerve (cf. [8]). It is also an attempt to finding a way of characterizing these three nerves. There are no much more things known about this problem.

Definition 3.1. *[5] An augmented simplicial set is a simplicial set*

$$((X_n)_{n\geqslant 0}, (\partial_i : X_{n+1} \longrightarrow X_n)_{0\leqslant i\leqslant n+1}, (\epsilon_i : X_n \longrightarrow X_{n+1})_{0\leqslant i\leqslant n})$$

together with an additional set X_{-1} and an additional map ∂_{-1} from X_0 to X_{-1} such that $\partial_{-1}\partial_0 = \partial_{-1}\partial_1$. A morphism of augmented simplicial set is a map of $\mathbb{N}$-graded sets which commutes with all face and degeneracy maps. We denote by $Sets_+^{\Delta^{op}}$ the category of augmented simplicial sets.

The "chain complex" functor of an augmented simplicial set X is defined by $C_n(X) = \mathbb{Z}X_n$ for $n \geqslant -1$ endowed with the simplicial differential map (denoted by ∂) in positive dimension and the map ∂_{-1} from $C_0(X)$ to $C_{-1}(X)$. The "simplicial homology" functor H_* from the category of augmented simplicial sets $Sets_+^{\Delta^{op}}$ to the category of abelian groups Ab is defined as the usual one for $* \geqslant 1$ and by setting $H_0(X) = Ker(\partial_{-1})/Im(\partial_0 - \partial_1)$ and $H_{-1}(X) = \mathbb{Z}X_{-1}/Im(\partial_{-1})$ whenever X is an augmented simplicial set.

Definition 3.2. *A (simplicial) cut is a functor $\mathcal{F} : \omega Cat_1 \to Sets_+^{\Delta^{op}}$ together with a family $ev = (ev_n)_{n\geqslant 0}$ of natural transformations $ev_n : F_n \longrightarrow tr^n\mathbb{P}$ where F_n is the set of n-simplexes of $\mathcal{F}$. A morphism of cuts from $(\mathcal{F}, ev)$ to $(\mathcal{G}, ev)$ is a natural transformation of functors ϕ from $\mathcal{F}$ to $\mathcal{G}$ which makes the following diagram commutative for any $n \geqslant 0$:*

$$\begin{array}{ccc} \mathcal{F}_n & \xrightarrow{ev_n} & tr^n\mathbb{P} \\ {\scriptstyle \phi_n}\downarrow & \nearrow{\scriptstyle ev_n} & \\ \mathcal{G}_n & & \end{array}$$

The terminology of "cuts" is borrowed from [21]. It will be explained later : cf. the explanations around Figure 3 and also Section 10.

There is no ambiguity to denote all ev_n by the same notation ev in the sequel. The map ev of $\mathbb{N}$-graded sets is called the *evaluation map* and a cut $(\mathcal{F}, ev)$ will be always denoted by $\mathcal{F}$.

If $\mathcal{F}$ is a functor from ωCat_1 to $Sets_+^{\Delta^{op}}$, let $C_{n+1}^{\mathcal{F}}(\mathcal{C}) := C_n(\mathcal{F}(\mathcal{C}))$ and let $H_{n+1}^{\mathcal{F}}$ be the corresponding homology theory for $n \geqslant -1$.

Let $M_n^{\mathcal{F}} : \omega Cat_1 \longrightarrow Ab$ be the functor defined as follows : the group $M_n^{\mathcal{F}}(\mathcal{C})$ is the subgroup generated by the elements $x \in \mathcal{F}_{n-1}(\mathcal{C})$ such that $ev(x) \in tr^{n-2}\mathbb{P}\mathcal{C}$ for $n \geqslant 2$ and with the convention $M_0^{\mathcal{F}}(\mathcal{C}) = M_1^{\mathcal{F}}(\mathcal{C}) = 0$ and the definition of $M_n^{\mathcal{F}}$ is obvious on non-contracting ω-functors. The elements of $M_*^{\mathcal{F}}(\mathcal{C})$ are called *thin*.

Let $CR_n^{\mathcal{F}} : \omega Cat_1 \longrightarrow Comp(Ab)$ be the functor defined by $CR_n^{\mathcal{F}} := C_n^{\mathcal{F}}/(M_n^{\mathcal{F}} + \partial M_{n+1}^{\mathcal{F}})$ and endowed with the differential map ∂. This chain complex is called the *reduced complex* associated to the cut $\mathcal{F}$ and the corresponding homology is denoted by $HR_*^{\mathcal{F}}$ and is called the *reduced homology* associated to $\mathcal{F}$. A morphism of cuts from $\mathcal{F}$ to $\mathcal{G}$ yields natural morphisms from $H_*^{\mathcal{F}}$ to $H_*^{\mathcal{G}}$ and from $HR_*^{\mathcal{F}}$ to $HR_*^{\mathcal{G}}$. There is also a canonical natural transformation $R^{\mathcal{F}}$ from $H_*^{\mathcal{F}}$ to $HR_*^{\mathcal{F}}$, functorial with respect to $\mathcal{F}$, that is making the following diagram commutative :

$$\begin{array}{ccc} H_*^{\mathcal{F}} & \xrightarrow{R^{\mathcal{F}}} & HR_*^{\mathcal{F}} \\ \downarrow & & \downarrow \\ H_*^{\mathcal{G}} & \xrightarrow{R^{\mathcal{G}}} & HR_*^{\mathcal{G}} \end{array}$$

Definition 3.3. *A cut $\mathcal{F}$ is regular if and only if it satisfies the following properties :*

1. *For any ω-category $\mathcal{C}$, the set $\mathcal{F}_{-1}(\mathcal{C})$ only depends on $tr^0\mathcal{C} = \mathcal{C}_0$: i.e. for any ω-categories $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C}_0 = \mathcal{D}_0$ implies $\mathcal{F}_{-1}(\mathcal{C}) = \mathcal{F}_{-1}(\mathcal{D})$.*
2. *$\mathcal{F}_0 := tr^0\mathbb{P}$.*
3. *$ev \circ \epsilon_i = ev$.*
4. *for any natural transformation of functors μ from $\mathcal{F}_{n-1}$ to $\mathcal{F}_n$ with $n \geqslant 1$, and for any natural map $\square$ from $tr^{n-1}\mathbb{P}$ to $\mathcal{F}_{n-1}$ such that $ev \circ \square = Id_{tr^{n-1}\mathbb{P}}$, there exists one and only one natural transformation $\mu.\square$ from $tr^n\mathbb{P}$ to $\mathcal{F}_n$ such that the following diagram commutes*

$$\begin{array}{ccccc} & & \overset{Id_{tr^n\mathbb{P}}}{\frown} & & \\ tr^n\mathbb{P} & \xrightarrow{\mu.\square} & \mathcal{F}_n & \xrightarrow{ev_n} & tr^n\mathbb{P} \\ {\scriptstyle i_n}\uparrow & & {\scriptstyle \mu}\uparrow & & {\scriptstyle i_n}\uparrow \\ tr^{n-1}\mathbb{P} & \xrightarrow{\square} & \mathcal{F}_{n-1} & \xrightarrow{ev_{n-1}} & tr^{n-1}\mathbb{P} \\ & & \underset{Id_{tr^{n-1}\mathbb{P}}}{\smile} & & \end{array}$$

where i_n is the canonical inclusion functor from $tr^{n-1}\mathbb{P}$ to $tr^n\mathbb{P}$.

5. *let* $\square_1^{\mathcal{F}} := Id_{\mathcal{F}_0}$ *and* $\square_n^{\mathcal{F}} := \epsilon_{n-2} \dots \epsilon_0 . \square_1^{\mathcal{F}}$ *a natural transformation from* $tr^{n-1}\mathbb{P}$ *to* $\mathcal{F}_{n-1}$ *for* $n \geqslant 2$ *; then the natural transformations* $\partial_i \square_n^{\mathcal{F}}$ *for* $0 \leqslant i \leqslant n-1$ *from* $tr^{n-1}\mathbb{P}$ *to* $\mathcal{F}_{n-2}$ *satisfy the following properties*
    (a) $\{ev\partial_{n-2}\square_n^{\mathcal{F}}, ev\partial_{n-1}\square_n^{\mathcal{F}}\} = \{s_{n-1}, t_{n-1}\}$.
    (b) *if for some* ω*-category* $\mathcal{C}$ *and some* $u \in \mathcal{C}_n$, $ev\partial_i\square_n^{\mathcal{F}}(u) = d_p^\alpha u$ *for some* $p \leqslant n$ *and for some* $\alpha \in \{-,+\}$, *then* $\partial_i\square_n^{\mathcal{F}} = \partial_i\square_n^{\mathcal{F}} d_p^\alpha$.
6. *Let* $\Phi_n^{\mathcal{F}} := \square_n^{\mathcal{F}} \circ ev$ *be a natural transformation from* $\mathcal{F}_{n-1}$ *to itself; then* $\Phi_n^{\mathcal{F}}$ *induces the identity natural transformation on* $CR_n^{\mathcal{F}}$.
7. *if* x, y *and* z *are three elements of* $\mathcal{F}_n(\mathcal{C})$, *and if* $ev(x) *_p ev(y) = ev(z)$ *for some* $1 \leqslant p \leqslant n$, *then* $x + y = z$ *in* $CR_{n+1}^{\mathcal{F}}(\mathcal{C})$ *and in a functorial way.*

If $\mathcal{F}$ *is a regular cut, then the natural transformation* $\Phi_n^{\mathcal{F}}$ *is called the* n*-dimensional folding operator of the cut* $\mathcal{F}$. *By convention, one sets* $\square_0^{\mathcal{F}} = Id_{\mathcal{F}_{-1}}$ *and* $\Phi_0^{\mathcal{F}} = Id_{\mathcal{F}_{-1}}$. *There is no ambiguity to set* $\Phi^{\mathcal{F}}(x) := \Phi_{n+1}^{\mathcal{F}}(x)$ *for* $x \in \mathcal{F}_n(\mathcal{C})$ *for some* ω*-category* $\mathcal{C}$. *So* $\Phi^{\mathcal{F}}$ *defines a natural transformation, and even a morphism of cuts, from* $\mathcal{F}$ *to itself. However beware of the fact that there is really an ambiguity in the notation* $\square^{\mathcal{F}}$ *: so this latter will not be used.*

Condition 3 tells us that the ϵ_i operations are really degeneracy maps. Condition 4 ensures the existence and the uniqueness of the folding operator associated to the cut.

Condition 5 tells us several things. A priori, a natural transformation like $ev\partial_i\square_n^{\mathcal{F}}$ from $tr^{n-1}\mathbb{P}$ to $tr^{n-2}\mathbb{P}$ is necessarily of the form d_p^α for some $p \leqslant n-1$ and for some $\alpha \in \{-,+\}$. Indeed consider the free ω-category $2_n(A)$ generated by some n-morphism A. Then $ev\partial_i\square_n^{\mathcal{F}}(A) \in 2_n(A)$ and therefore $ev\partial_i\square_n^{\mathcal{F}}(A) = d_p^\alpha(A)$ for some p and some α. By naturality, this implies that $ev\partial_i\square_n^{\mathcal{F}} = d_p^\alpha$. If $0 \leqslant i < n-2$, then

$$\begin{aligned}
ev\partial_i\square_n^{\mathcal{F}} &= ev\partial_i\square_n^{\mathcal{F}} d_{n-1}^\beta && \text{for some } \beta \in \{-,+\} \\
&= ev\partial_i\square_n^{\mathcal{F}} i_{n-1} d_{n-1}^\beta \\
&= ev\partial_i\epsilon_{n-2}\square_{n-1}^{\mathcal{F}} d_{n-1}^\beta && \text{by construction of } \square_n^{\mathcal{F}} \\
&= ev\epsilon_{n-3}\partial_i\square_{n-1}^{\mathcal{F}} d_{n-1}^\beta \\
&= ev\partial_i\square_{n-1}^{\mathcal{F}} d_{n-1}^\beta && \text{by rule 3} \\
&= d_p^\alpha d_{n-1}^\beta && \text{for some } p \leqslant n-2 \\
&= d_p^\alpha
\end{aligned}$$

Therefore $\partial_i\square_n^{\mathcal{F}}$ is thin. Now if $n-2 \leqslant i \leqslant n-1$, then

$$\begin{aligned}
ev\partial_i\square_n^{\mathcal{F}} &= ev\partial_i\square_n^{\mathcal{F}} d_{n-1}^\beta && \text{for some } \beta \in \{-,+\} \\
&= ev\partial_i\square_n^{\mathcal{F}} i_{n-1} d_{n-1}^\beta \\
&= ev\partial_i\epsilon_{n-2}\square_{n-1}^{\mathcal{F}} d_{n-1}^\beta && \text{by construction of } \square_n^{\mathcal{F}} \\
&= ev\square_{n-1}^{\mathcal{F}} d_{n-1}^\beta \\
&= d_{n-1}^\beta && \text{by construction of } \square_n^{\mathcal{F}}
\end{aligned}$$

Therefore $\left\{ev\partial_{n-2}\square_n^{\mathcal{F}}, ev\partial_{n-1}\square_n^{\mathcal{F}}\right\} \subset \{s_{n-1}, t_{n-1}\}$ always holds. Condition 5 states more precisely that these latter sets are actually equal. In other terms, the operator $\square_n^{\mathcal{F}}$ concentrates the "weight" on the faces $\partial_{n-2}\square_n^{\mathcal{F}}$ and $\partial_{n-1}\square_n^{\mathcal{F}}$.

Condition 6 explains the link between the thin elements of the cut and the folding operators. Intuitively, the folding operators move the labeling of the elements of the cuts in a *canonical position* without changing the total sum on the source and target sides. What is exactly this *canonical position* is precisely described by Proposition 3.5. Conditions 5 and 7 ensure that by moving the labeling of an element, we stay in the same equivalence class modulo thin elements.

Now here are some trivial remarks about regular cuts :

- Let f be a natural set map from $tr^0\mathbb{P}\mathcal{C} = \mathcal{C}_1$ to itself. Let 2_1 be the ω-category generated by one 1-morphism A. Then necessarily $f(A) = A$ and therefore $f = Id$. So the above axioms imply that $ev_0 = Id$.

- The map $\Phi_n^{\mathcal{F}}$ induces the identity natural transformation on $HR_n^{\mathcal{F}}$.
- For any $n \geqslant 1$, there exists non-thin elements x in $\mathcal{F}_{n-1}(\mathcal{C})$ as soon as $\mathcal{C}_n \neq \emptyset$. Indeed, if $u \in \mathcal{C}_n$, $ev\ \square_n^{\mathcal{F}}(u) = u$, therefore $\square_n^{\mathcal{F}}(u)$ is a non-thin element of $\mathcal{F}_{n-1}(\mathcal{C})$.

We end this section by some general facts about regular cuts.

Proposition 3.4. *Let f be a morphism of cuts from $\mathcal{F}$ to $\mathcal{G}$. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are regular. Then $\Phi^{\mathcal{G}} \circ f = f \circ \Phi^{\mathcal{F}}$ as natural transformation from $\mathcal{F}$ to $\mathcal{G}$. In other terms, the following diagram is commutative :*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{G} \\ {\scriptstyle \Phi^{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \Phi^{\mathcal{G}}} \\ \mathcal{F} & \xrightarrow{f} & \mathcal{G} \end{array}$$

Proof. Let $n \geqslant 0$ and let $P(n)$ be the property : "for any ω-category $\mathcal{C}$ and any $x \in tr^n\mathbb{P}\mathcal{C}$, then $f\square_{n+1}^{\mathcal{F}}(x) = \square_{n+1}^{\mathcal{G}}x$."

One has $\Phi_1^{\mathcal{F}} := Id_{\mathcal{F}_0}$, $\Phi_1^{\mathcal{G}} := Id_{G_0}$ and necessarily $f_0 = Id$ by definition of a morphism of cuts. Therefore $P(0)$ holds. Now suppose $P(n)$ proved for some $n \geqslant 0$. One has $ev f \square_{n+2}^{\mathcal{F}} = ev\square_{n+2}^{\mathcal{F}} = Id_{tr^{n+1}\mathbb{P}}$ since f is a morphism of cuts and

$$\begin{aligned} f\square_{n+2}^{\mathcal{F}} i_{n+1} &= f(\epsilon_n.\square_{n+1}^{\mathcal{F}})i_{n+1} \\ &= f\epsilon_n\square_{n+1}^{\mathcal{F}} && \text{by definition of } \epsilon_n.\square_{n+1}^{\mathcal{F}} \\ &= \epsilon_n f\square_{n+1}^{\mathcal{F}} && \text{since } f \text{ morphism of simplicial sets} \\ &= \epsilon_n\square_{n+1}^{\mathcal{G}} && \text{by induction hypothesis} \end{aligned}$$

Therefore the natural transformation $f\square_{n+2}^{\mathcal{F}}$ from $tr^{n+1}\mathbb{P}$ to $\mathcal{G}_{n+1}$ can be identified with $\epsilon_n.\square_{n+1}^{\mathcal{G}}$ which is precisely $\square_{n+2}^{\mathcal{G}}$. Therefore $P(n+1)$ is proved.

At last, if $x \in \mathcal{F}_n(\mathcal{C})$, then

$$\begin{aligned} \Phi^{\mathcal{G}} f(x) &= \square_{n+1}^{\mathcal{G}} ev f(x) && \text{by definition of folding operators} \\ &= \square_{n+1}^{\mathcal{G}} ev(x) && \text{since } f \text{ preserves the evaluation map} \\ &= f\square_{n+1}^{\mathcal{F}} ev(x) && \text{since } P(n) \text{ holds} \\ &= f\Phi^{\mathcal{F}}(x) && \text{by definition of folding operators} \end{aligned}$$

□

Proposition 3.5. *If u is a $(n+1)$-morphism of $\mathcal{C}$ with $n \geqslant 1$, then $\square^{\mathcal{F}}_{n+1}u$ is an homotopy within the simplicial set $\mathcal{F}(\mathcal{C})$ between $\square^{\mathcal{F}}_n s_n u$ and $\square^{\mathcal{F}}_n t_n u$.*

Proof. The natural map $ev\ \partial_i\square^{\mathcal{F}}_{n+1}$ for $0 \leqslant i \leqslant n$ from $tr^n\mathbb{P}$ to $tr^{n-1}\mathbb{P}$ is of the form $d^{\alpha_i}_{m_i}$ for $m_i \leqslant n$ with $m_i \leqslant n-1$ for $0 \leqslant i \leqslant n-2$ and $\{ev\partial_{n-1}\square^{\mathcal{F}}_{n+1}, ev\partial_n\square^{\mathcal{F}}_{n+1}\} = \{s_n, t_n\}$. Therefore for $0 \leqslant i \leqslant n-2$, $\partial_i\square^{\mathcal{F}}_{n+1} = \partial_i\square^{\mathcal{F}}_{n+1}s_n = \partial_i\square^{\mathcal{F}}_{n+1}t_n$ by rule 5b of Definition 3.3. And by construction of $\square^{\mathcal{F}}_{n+1}$, one obtains $\partial_i\square^{\mathcal{F}}_{n+1} = \epsilon_{n-2}\partial_i\square^{\mathcal{F}}_n s_n = \epsilon_{n-2}\partial_i\square^{\mathcal{F}}_n t_n$. □

Corollary 3.6. *If $x \in CR^{\mathcal{F}}_{n+1}(\mathcal{C})$, then $\partial x = \partial\square^{\mathcal{F}}_{n+1}x = \square^{\mathcal{F}}_n s_n x - \square^{\mathcal{F}}_n t_n x$ in $CR^{\mathcal{F}}_n(\mathcal{C})$. In other terms, the differential map from $CR^{\mathcal{F}}_{n+1}(\mathcal{C})$ to $CR^{\mathcal{F}}_n(\mathcal{C})$ with $n \geqslant 1$ is induced by the map $s_n - t_n$.*

4 The cuts of branching and merging nerves

We see now that the corner nerves $\mathcal{N}^\eta$ defined in [9] are two examples of regular cuts with the correspondence $\square^\eta_n := \square^{\mathcal{N}^\eta}_n$, $\Phi^\eta_n := \Phi^{\mathcal{N}^\eta}_n$, $H^\eta_n := H^{\mathcal{N}^\eta}_n$, $HR^\eta_n := HR^{\mathcal{N}^\eta}_n$ and $ev(x) = x(0_{dim(x)})$.

Let us first recall the construction of the free ω-category I^n generated by the faces of the n-cube. The faces of the n-cube are labeled by the words of length n in the alphabet $\{-, 0, +\}$, one word corresponding to the barycenter of one face. We take the convention that $00\dots0$ (n times) $=: 0_n$ corresponds to its interior and that $-_n$ (resp. $+_n$) corresponds to its initial state $--\cdots-$ (n times) (resp. to its final state $++\cdots+$ (n times)). If x is a face of the n-cube, let $R(x)$ be the set of faces of x. If X is a set of faces, then let $R(X) = \bigcup_{x\in X} R(x)$. Notice that $R(X \cup Y) = R(X) \cup R(Y)$ and that $R(\{x\}) = R(x)$. Then I^n is the free ω-category generated by the $R(x)$ with the rules

1. For x p-dimensional with $p \geqslant 1$,

$$s_{p-1}(R(x)) = R(s_x)$$

and

$$t_{p-1}(R(x)) = R(t_x)$$

where s_x and t_x are the sets of faces defined below.

2. If X and Y are two elements of I^n such that $t_p(X) = s_p(Y)$ for some p, then $X \cup Y$ belongs to I^n and $X \cup Y = X *_p Y$.

The set s_x is the set of subfaces of the faces obtained by replacing the i-th zero of x by $(-)^i$, and the set t_x is the set of subfaces of the faces obtained by replacing the i-th zero of x by $(-)^{i+1}$. For example, $s_{0+00} = \{\text{-+00}, \text{0++0}, \text{0+0-}\}$ and $t_{0+00} = \{\text{++00}, \text{0+-0}, \text{0+0+}\}$. Figure 2(c) represents the free ω-category generated by the 3-cube.

The branching and merging nerves are dual from each other. We set

$$\omega Cat(I^{n+1}, \mathcal{C})^\eta := \{x \in \omega Cat(I^{n+1}, \mathcal{C}), d_0^\eta(u) = \eta_{n+1} \text{ and } dim(u) = 1 \Longrightarrow dim(x(u)) = 1\}$$

where $\eta \in \{-, +\}$ and where η_{n+1} is the initial state (resp. final state) of I^{n+1} if $\eta = -$ (resp. $\eta = +$). For all (i, n) such that $0 \leqslant i \leqslant n$, the face maps ∂_i from $\omega Cat(I^{n+1}, \mathcal{C})^\eta$ to $\omega Cat(I^n, \mathcal{C})^\eta$ are the arrows ∂_{i+1}^η defined by

$$\partial_{i+1}^\eta(x)(k_1 \dots k_{n+1}) = x(k_1 \dots [\eta]_{i+1} \dots k_{n+1})$$

and the degeneracy maps ϵ_i from $\omega Cat(I^n, \mathcal{C})^\eta$ to $\omega Cat(I^{n+1}, \mathcal{C})^\eta$ are the arrows Γ_{i+1}^η defined by setting

$$\Gamma_i^-(x)(k_1 \dots k_n) := x(k_1 \dots \max(k_i, k_{i+1}) \dots k_n)$$
$$\Gamma_i^+(x)(k_1 \dots k_n) := x(k_1 \dots \min(k_i, k_{i+1}) \dots k_n)$$

with the order $- < 0 < +$.

Proposition and definition 4.1. *[9] Let $\mathcal{C}$ be an ω-category. The $\mathbb{N}$-graded set $\mathcal{N}^\eta(\mathcal{C})$ together with the convention $\mathcal{N}_{-1}^\eta(\mathcal{C}) = \mathcal{C}_0$, endowed with the maps ∂_i and ϵ_i above defined with moreover $\partial_{-1} = s_0$ (resp. $\partial_{-1} = t_0$) if $\eta = -$ (resp. $\eta = +$) and with $ev(x) = x(0_n)$ for $x \in \omega Cat(I^n, \mathcal{C})$ is a simplicial cut. It is called the η-corner simplicial nerve $\mathcal{N}^\eta$ of $\mathcal{C}$.*

Set $H_{n+1}^\eta(\mathcal{C}) := H_n(\mathcal{N}^\eta(\mathcal{C}))$ for $n \geqslant -1$. These homology theories are called *branching* and *merging homology* respectively and are exactly the same homology theories as that defined in [9] and studied in [8].

And we have

Theorem 4.2. *[8] The simplicial cut $\mathcal{N}^\eta$ is regular. The associated folding operator $\square_n^{\mathcal{N}^\eta}$ coincides with the operator $\square_n^\eta$ defined in [8]. And therefore the associated homology theory $HR_n^{\mathcal{N}^\eta}$ coincide with the reduced corner homology HR_n^η defined in [8].*

It is useful for the sequel to remind some important properties of the folding operators associated to corner nerves.

Theorem 4.3. *[8] Let $\mathcal{C}$ be an ω-category. Let x be an element of $\mathcal{N}_n^-(\mathcal{C})$. Then the following two conditions are equivalent :*

1. *the equality $x = \Phi_n^-(x)$ holds*
2. *for $1 \leqslant i \leqslant n$, one has $\omega\partial_i^+ x = \partial_i^+ x(0_n)$ is 0-dimensional and for $1 \leqslant i \leqslant n-2$, one has $\partial_i^- x \in Im(\Gamma_{n-2}^- \dots \Gamma_i^-)$.*

Another operator coming from [8] which matters for this paper is the operator θ_i^-.

Definition 4.4. *Let $x \in \mathcal{N}_n^-(\mathcal{C})$ for some $\mathcal{C}$ such that for any $1 \leqslant j \leqslant n+1$, $\partial_j^+ x$ is 0-dimensional. Then x is called a negative element of the branching nerve.*

Theorem 4.5. *Let $n \geqslant 2$. There exists natural transformations*

$$\theta_1^-, \dots, \theta_{n-1}^-$$

from $\mathcal{N}_n^-$ to itself satisfying the following properties :

1. *If x is a negative element of $\mathcal{N}_n^-(\mathcal{C})$, then for any $1 \leqslant i \leqslant n-1$, $\theta_i^- x$ is a negative element as well.*
2. *If x is a negative element of $\mathcal{N}_n^-(\mathcal{C})$, then for any $1 \leqslant i \leqslant n-1$, there exists a thin negative element y_i of $\mathcal{N}_{n+1}^-(\mathcal{C})$ such that $\partial^- y_i - x$ is a linear combination of thin negative elements.*
3. *There exists a composite of $\theta_1^-, \dots, \theta_{n-1}^-$ which coincides with the negative folding operators on negative elements of $\mathcal{N}_n^-$.*

Sketch of proof. Consider the $\theta_1^-, \dots, \theta_{n-1}^-$ of [8]. One has

$$\begin{aligned}
\partial_j^+\theta_i^- &= \begin{cases} \theta_{i-1}^-\partial_j^+ \text{ if } j < i \\ \theta_i^-\partial_j^+ \text{ if } j > i+2 \end{cases} \\
\partial_i^+\theta_i^- &= {}^v\psi_i^-\partial_i^+ \\
\partial_{i+1}^+\theta_i^- &= \epsilon_{i+1}\partial_{i+1}^+\partial_i^- +_i \epsilon_{i+1}\partial_{i+1}^+\partial_{i+1}^+ \\
\partial_{i+2}^+\theta_i^- &= {}^v\psi_i^+\partial_{i+2}^+
\end{aligned}$$

where, for the last formula, ${}^v\psi_i^\pm$ are other operators which is not important to explicitly define here : the only important thing is that $\partial_i^+\theta_i^-$ remains 0-dimensional if the argument is 0-dimensional. Hence property 1. As for property 2, it is enough to check it for $i = 1$. And in this case, y is a thin 4-cube satisfying

$$\begin{aligned}
\partial_1^+ y &= {}^v\psi_2^-\Gamma_1^-\partial_1^+ x \\
\partial_2^+ y &= \Gamma_2^-\partial_2^+ x \\
\partial_3^+ y &= \epsilon_3(\Gamma_1^-\partial_2^+\partial_1^- x +_1 \epsilon_2\partial_2^+\partial_2^+ x) \\
\partial_4^+ y &= {}^v\psi_2^+\Gamma_2^-\partial_3^+ x
\end{aligned}$$

Once again, we refer to [8] for the precise definition of the operators involved in the above formulas. The only thing that matters here is the dimension of $\partial_i^+ y$.

By [8], we know that $\Phi^- = \Theta \circ \Psi$ when Θ is a composite of θ_i^- and such that for x negative, $\Psi x = x$. Hence property 3. □

The graded set $(\omega Cat(I^n, \mathcal{C}))_{n\geqslant 0}$ endowed with the operations $\partial_i^\pm$ above defined and by the maps $\epsilon_i(x)(k_1 \dots k_{n+1}) = x(k_1 \dots \widehat{k_i} \dots k_{n+1})$ for $x \in \omega Cat(I^n, \mathcal{C})$ and $1 \leqslant i \leqslant n+1$ is a cubical set and is usually known as the *cubical singular nerve* of $\mathcal{C}$ [4]. The use of the same notation ϵ_i for the degeneracy maps of the cubical singular nerve and the degeneracy maps of the three simplicial nerves appearing in this paper is very confusing. Fortunately, we will not need the degeneracy maps of the cubical singular nerve in this work except for Theorem 4.5 right above.

5 The globular cut

The most direct way of constructing a cut of ω-categories consists of using the composite of both functors $\mathbb{P} : \mathcal{C} \mapsto \mathbb{P}\mathcal{C}$ and $\mathcal{N}$ where $\mathcal{N}$ is the simplicial nerve functor defined by Street [1].

Let us start this section by recalling the construction of the free ω-category Δ^n generated by the faces of the n-simplex. The faces of the n-simplex are labeled by the strictly increasing sequences of elements of $\{0, 1, \ldots, n\}$. The length of a sequence is equal to the dimension of the corresponding face plus one. If x is a face of the n-simplex, its subfaces are all increasing sequences of $\{0, 1, \ldots, n\}$ included in x. If x is a face of the n-simplex, let $R(x)$ be the set of faces of x. If X is a set of faces, then let $R(X) = \bigcup_{x \in X} R(x)$. Notice that $R(X \cup Y) = R(X) \cup R(Y)$ and that $R(\{x\}) = R(x)$. Then Δ^n is the free ω-category generated by the $R(x)$ with the rules

1. For x p-dimensional with $p \geqslant 1$,

$$s_{p-1}(R(x)) = R(s_x)$$

   and

$$t_{p-1}(R(x)) = R(t_x)$$

   where s_x and t_x are the sets of faces defined below.

2. If X and Y are two elements of Δ^n such that $t_p(X) = s_p(Y)$ for some p, then $X \cup Y$ belongs to Δ^n and $X \cup Y = X *_p Y$.

where s_x (resp. t_x) is the set of subfaces of x obtained by removing one element in odd position (resp. in even position). For instance, $s_{(04589)} = \{(4589), (0489), (0458)\}$ and $t_{(04589)} = \{(0589), (0459)\}$.

Sometimes we will write (for instance) $(0 < 4 < 5 < 8 < 9)$ instead of simply (04589). Figure 2(b) gives the example of the 2-simplex.

Let $x \in \omega Cat(\Delta^n, \mathcal{C})$. Then consider the labeling of the faces of respectively Δ^{n+1} and Δ^{n-1} defined by :

- $\epsilon_i(x)(\sigma_0 < \cdots < \sigma_r) = x(\sigma_0 < \cdots < \sigma_{k-1} < \sigma_k - 1 < \cdots < \sigma_r - 1)$ if $\sigma_{k-1} < i$ and $\sigma_k > i$.

[1]Of course, the functor $\mathcal{N}$ can be viewed as a functor from ωCat_1 to $Sets_+^{\Delta^{op}}$, but a "good" cut should not be extendable to a functor from ωCat to $Sets_+^{\Delta^{op}}$.

- $x(\sigma_0 < \cdots < \sigma_{k-1} < i < \sigma_{k+1} - 1 < \cdots < \sigma_r - 1)$ if $\sigma_{k-1} < i$, $\sigma_k = i$ and $\sigma_{k+1} > i + 1$.
- $x(\sigma_0 < \cdots < \sigma_{k-1} < i < \sigma_{k+2} - 1 < \cdots < \sigma_r - 1)$ if $\sigma_{k-1} < i$, $\sigma_k = i$ and $\sigma_{k+1} = i + 1$.

and

$$\partial_i(x)(\sigma_0 < \cdots < \sigma_s) = x(\sigma_0 < \cdots < \sigma_{k-1} < \sigma_k + 1 < \cdots < \sigma_s + 1)$$

where $\sigma_k, \ldots, \sigma_s \geqslant i$ and $\sigma_{k-1} < i$.

It can be checked that $\epsilon_i(x)$ (resp. $\partial_i(x)$) are ω-functors from Δ^{n+1} (resp. Δ^{n-1}) to $\mathcal{C}$ [23]. By construction, the map $[n] \mapsto \Delta^n$ induces then a functor from the well-known category Δ whose associated presheaves are the simplicial sets to ωCat. Therefore $\mathcal{N}(\mathcal{C}) = (\omega Cat(\Delta^*, \mathcal{C}), \partial_i, \epsilon_i)$ is a simplicial set which is called the *simplicial nerve* of $\mathcal{C}$.

Definition 5.1. *The globular cut $\mathcal{N}^{gl}$ (or the globular nerve) is the functor from ωCat_1 to $Sets_+^{\Delta^{op}}$ defined by $\mathcal{N}^{gl}_n(\mathcal{C}) = \omega Cat(\Delta^n, \mathbb{P}\mathcal{C})$ for $n \geqslant 0$ and with $\mathcal{N}^{gl}_{-1}(\mathcal{C}) = \mathcal{C}_0 \times \mathcal{C}_0$, and endowed with the augmentation map ∂_{-1} from $\mathcal{N}^{gl}_0(\mathcal{C}) = \mathcal{C}_1$ to $\mathcal{N}^{gl}_{-1}(\mathcal{C}) = \mathcal{C}_0 \times \mathcal{C}_0$ defined by $\partial_{-1}x = (s_0x, t_0x)$. The evaluation map ev is defined by $ev(x) = x((0\ldots n))$ for $x \in \omega Cat(\Delta^n, \mathbb{P}\mathcal{C})$. The homology theory $H^{gl}_n := H^{\mathcal{N}^{gl}}_n$ is called the globular homology and $HR^{gl}_n := HR^{\mathcal{N}^{gl}}_n$ the reduced globular homology.*

Geometrically, the elements of $\mathcal{N}^{gl}_n(\mathcal{C})$ are full $(n+1)$-globes. Figure 3 depicts a 2-simplex in the globular nerve. The simplexes seen by the globular cut are intuitively transverse to the execution paths, as well as those of corner nerves. Hence the terminology of cuts.

Here is now the new definition of the globular homology of a globular ω-category $\mathcal{C}$:

Definition 5.2. *Let $\mathcal{C}$ be a non-contracting ω-category. We set*

$$H^{gl}_{n+1}(\mathcal{C}) := H_n(\mathcal{N}^{gl}(\mathcal{C}))$$

for $n \geqslant -1$ and this homology theory is called the globular homology of $\mathcal{C}$.

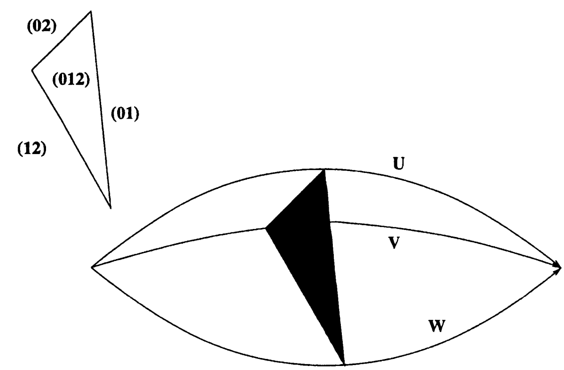


Figure 3: Globular 2-simplex

6 Associating to any globe its corners

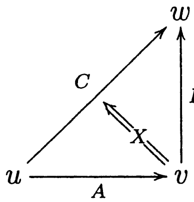
The purpose of the rest of the paper is to justify that Definition 5.2 is the right definition. This is not a mathematical statement of course ! We follow the order of the remarks at the very end of Section 1 which explain what kind of conditions the globular homology must fulfill. So we have first to construct h^- and h^+ and we must verify that geometrically, in homology, h^- and h^+ do what we expect to find. In fact, we refer to [10] for intuitive explanations of h^- and h^+. We only recall here Figure 4 as an illustration and care only about the construction of h^-.

Theorem 6.1. *Let $\alpha \in \{-,+\}$. There exists one and only one morphism of cuts h^α from $\mathcal{N}^{gl}$ to $\mathcal{N}^\alpha$. Moreover, for any non-contracting ω-category $\mathcal{C}$, both morphisms h^α from $\mathcal{N}^{gl}(\mathcal{C})$ to $\mathcal{N}^\alpha(\mathcal{C})$ are injective.*

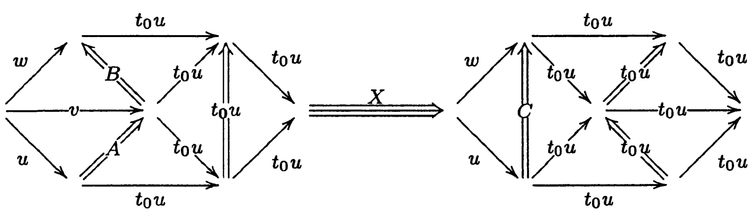
The rest of the section is devoted to the proof of Theorem 6.1. The following sequence of propositions establishes the existence of h^-. The term $\underline{cub}^n$ denotes the set of faces of the n-cube, as described in Section 4.

We briefly recall how filling shells in the cubical singular nerve. This technical tool already appears in [4] for ω-groupoids and in [1] for ω-categories. A particular case can be found in [9].

Definition 6.2. *A n-shell in the cubical singular nerve is a family of* $2(n+1)$



(a) A 2-globular simplex $\widetilde{X}$



(b) The 2-simplex $h^-(\widetilde{X})$

Figure 4: Illustration of h^-

elements $x_i^\pm$ *of* $\omega Cat(I^n, \mathcal{C})^-$ *such that* $\partial_i^\alpha x_j^\beta = \partial_{j-1}^\beta x_i^\alpha$ *for* $1 \leqslant i < j \leqslant n+1$ *and* $\alpha, \beta \in \{-,+\}$.

If $x_i^\pm$ is a n-shell, then it induces a labeling x on the set of faces of dimension at most n of the $(n+1)$-cube in the following manner : let $k_1 \dots k_{n+1}$ be a face of dimension at most n ; then there exists i such that $k_i \neq 0$; then let $x(k_1 \dots k_{n+1}) := x_i(k_1 \dots \widehat{k_i} \dots k_{n+1})$. The axiom satisfied by an n-shell ensures the coherence of the definition.

Proposition and definition 6.3. *Let* $x_i^\pm$ *be an* $(n-1)$*-shell with* $n \geqslant 1$.

- *The labeling of the faces of dimension at most* $(n-1)$ *of* I^n *defined by* $x_i^\pm$ *always induces an* ω*-functor and only one from* $I^n \backslash \{R(0_n)\}$ *to* $\mathcal{C}$. *Denote it by* x.
- *The* n*-shell* $(x_i^\pm)$ *is said fillable if there exists a morphism* u *of* $\mathcal{C}$ *such that* $s_{n-1}u = x\,(s_{n-1}R(0_n))$ *and* $t_{n-1}u = x\,(t_{n-1}R(0_n))$. *In this case, there exists a unique* ω*-functor* x *from* I^n *to* $\mathcal{C}$ *such that* $\partial_i^\pm x = x_i^\pm$ *for* $1 \leqslant i \leqslant n$ *and* $x(0_n) = u$.

Proof. Using the freeness of I^n, the construction in the proof of [9] Proposition 5.1 yields the ω-functor x from $I^n \backslash \{R(0_n)\}$ to $\mathcal{C}$. The hypotheses stated in [9] were too strong indeed. If moreover the shell is fillable in the above sense, one concludes still as in the proof of [9] Proposition 5.1. □

Now we can construct h^-.

Theorem 6.4. *Let* x *be an* n*-simplex of the globular simplicial nerve of* $\mathcal{C}$. *Then the map* $h_n^-(x)$ *from* $\underline{cub}^{n+1}$ *to* $\mathcal{C}$ *defined by*

1. $+ \in \{k_1 \dots k_{n+1}\}$ *implies* $h_n^-(x)(k_1 \dots k_{n+1}) = t_0 x((0))$ *(notice that* (0) *is the final state of* Δ^n*)*
2. $\{k_1, \dots, k_{n+1}\} \subset \{-, 0\}$ *and*
$$\{k_1, \dots, k_{n+1}\} \cap \{0\} = \{k_{\sigma_0+1}, \dots, k_{\sigma_r+1}\}$$
with $\sigma_0 < \dots < \sigma_r$ *implies* $h_n^-(x)(k_1 \dots k_{n+1}) = x((\sigma_0 \dots \sigma_r))$
3. $h_n^-(x)(-_{n+1}) = s_0 x((n))$ *(notice that* (n) *is the initial state of* Δ^n*)*

yields an ω-functor from I^{n+1} to $\mathcal{C}$. Moreover, h^- induces a morphism of simplicial sets from the globular nerve of $\mathcal{C}$ to its negative corner nerve. And the map from $\mathcal{N}^{gl}_{-1}(\mathcal{C})$ to $\mathcal{N}^{-}_{-1}(\mathcal{C})$ defined by $(x, y) \mapsto x$ extends the previous morphism to the corresponding augmented simplicial nerves. Moreover for $n \geqslant 0$, h^-_n is a one-to-one map and the image of h^-_n contains exactly all cubes x of the negative corner nerve such that as soon as $\partial^+_i x$ exists, then it is 0*-dimensional.*

There is no ambiguity to set $h^-(x) = h^-_n(x)$ if x is an n-simplex of the globular cut.

In the sequel, in order to make easier the reading of the calculations, we suppose that an expression like $(\sigma_0 < \sigma_j \leqslant \widehat{k} < \sigma_{j+1} < \ldots < \sigma_r)$ is the same thing as $(\sigma_0 < \sigma_j < \sigma_{j+1} < \ldots < \sigma_r)$ in Δ^* but with an additional information given within the calculation itself : here that $\sigma_j \leqslant \widehat{k} < \sigma_{j+1}$ holds.

Proof. One proves by induction on n the following property $P(n)$: " For any n-simplex x of the globular simplicial nerve of any ω-category $\mathcal{C}$, the map $h^-(x)$ from $\underline{cub}^{n+1}$ to $\mathcal{C}$ induces an ω-functor and moreover an element of $\omega Cat(I^{n+1}, \mathcal{C})^-$."

Let x be a 0-simplex of the globular nerve of $\mathcal{C}$. Then x is an ω-functor from Δ^0 to $\mathbb{P}\mathcal{C}$, and therefore it can be identified with the 1-morphism $x((0))$ of $\mathcal{C}$. Therefore

$$\begin{aligned} h^-(x)(0) &= x((0)) && \text{by rule 2} \\ h^-(x)(+) &= t_0 x((0)) && \text{by rule 1} \\ h^-(x)(-) &= s_0 x((0)) && \text{by rule 3} \end{aligned}$$

Therefore $P(0)$ is proved.

Now suppose that $P(n)$ is proved for $n \geqslant 0$. Let x be a $(n+1)$-simplex of the globular simplicial nerve of some ω-category $\mathcal{C}$. If $+ \in \{k_1, \ldots, k_{n+1}\}$, then

$$\begin{aligned} &\partial^-_i(h^-(x))(k_1 \ldots k_{n+1}) \\ &= h^-(x)(k_1 \ldots k_{i-1} - k_i \ldots k_{n+1}) && \text{by definition of } \partial^-_i \text{ for } 1 \leqslant i \leqslant n+2 \\ &= t_0 x((0)) && \text{by rule 1} \\ &= h^-(\partial_{i-1} x)(k_1 \ldots k_{n+1}) && \text{again by rule 1} \end{aligned}$$

If $+ \notin \{k_1, \ldots, k_{n+1}\}$, i.e. if $\{k_1, \ldots, k_{n+1}\} \subset \{-, 0\}$, set

$$\{k_1, \ldots, k_{n+1}\} \cap \{0\} = \{k_{\sigma_0+1}, \ldots, k_{\sigma_r+1}\}$$

with $\sigma_0 < \cdots < \sigma_r$. For a given i such that $1 \leqslant i \leqslant n+2$, set

$$w_1 \ldots w_{n+2} = k_1 \ldots k_{i-1} - k_i \ldots k_{n+1}$$

as word. Then let

$$\{w_1, \ldots, w_{n+2}\} \cap \{0\} = \{w_{\tau_0+1}, \ldots, w_{\tau_r+1}\}$$

with $\tau_0 < \cdots < \tau_r$. The relation between the sequence of σ_j and the sequence of τ_j is as follows :

$$\sigma_j + 1 \leqslant i - 1 \Longrightarrow \sigma_j = \tau_j$$
$$\sigma_j + 1 \geqslant i \Longrightarrow \sigma_j + 1 = \tau_j$$

And we have

$$\begin{aligned}
&\partial_i^-(h^-(x))(k_1 \ldots k_{n+1}) \\
&= h^-(x)(k_1 \ldots k_{i-1} - k_i \ldots k_{n+1}) \text{ by definition of } \partial_i^- \\
&= x((\tau_0 \ldots \tau_r)) \text{ by rule 2} \\
&= x((\sigma_0 < \cdots < \sigma_{j_0} \leqslant \widehat{i-2} < \widehat{i-1} < \sigma_{j_0+1} + 1 < \cdots < \sigma_r + 1)) \\
&= (\partial_{i-1}x)((\sigma_0 \ldots \sigma_r)) \text{ by definition of } \partial_{i-1} \\
&= h^-(\partial_{i-1}x)(k_1 \ldots k_{n+1}) \text{ by rule 2}
\end{aligned}$$

Therefore $\partial_i^-(h^-(x)) = h^-(\partial_{i-1}x)$. And by rule 1, $\partial_i^+(h^-(x))$ is the constant ω-functor from $\underline{cub}^{n+1}$ to $\mathcal{C}$ which sends any face of I^{n+1} on $t_0x((0))$. Therefore $(\partial_i^\pm(h^-(x)))_{1 \leqslant i \leqslant n+1}$ is a $(n+1)$-shell in the cubical nerve of $\mathcal{C}$ which is fillable. By Proposition 6.3, the labeling $h^-(x)$ of $\underline{cub}^{n+2}$ induces an ω-functor from I^{n+2} to $\mathcal{C}$ and $P(n+1)$ is proved.

By construction, the equality $\partial_i^-(h^-(x)) = h^-(\partial_{i-1}x)$ holds for any n-simplex x of the globular nerve and for $1 \leqslant i \leqslant n+1$. It remains to check that for such a simplex x, $\Gamma_i^-(h^-(x)) = h^-(\epsilon_{i-1}x)$ for $i \leqslant 1 \leqslant n+1$. Consider a face $k_1 \ldots k_{n+2}$ of the $(n+2)$-cube. If $+ \in \{k_1, \ldots, k_{n+2}\}$, then

$$\begin{aligned}
&\Gamma_i^-(h^-(x))(k_1 \ldots k_{n+2}) \\
&= h^-(x)(k_1 \ldots max(k_i, k_{i+1}) \ldots k_{n+2}) && \text{by definition of } \Gamma_i^- \\
&= t_0x((0)) && \text{by rule 1} \\
&= h^-(\epsilon_{i-1}x)(k_1 \ldots k_{n+2}) && \text{again by rule 1}
\end{aligned}$$

If $+ \notin \{k_1, \dots, k_{n+2}\}$, i.e. if $\{k_1, \dots, k_{n+2}\} \subset \{-, 0\}$, set

$$\{k_1, \dots, k_{n+2}\} \cap \{0\} = \{k_{\sigma_0+1}, \dots, k_{\sigma_r+1}\}$$

with $\sigma_0 < \dots < \sigma_r$. For a given i such that $1 \leqslant i \leqslant n+1$,

$$\{k_1, \dots, max(k_i, k_{i+1}), \dots, k_{n+2}\} \subset \{-, 0\}$$

and set $w_1 \dots w_{n+1} = k_1 \dots max(k_i, k_{i+1}) \dots k_{n+2}$ as word. Then let

$$\{w_1, \dots, w_{n+1}\} \cap \{0\} = \{w_{\tau_0+1}, \dots, w_{\tau_s+1}\}$$

with $\tau_0 < \dots < \tau_s$. One has to calculate

$$\begin{aligned} &\Gamma_i^-(h^-(x))(k_1 \dots k_{n+2}) \\ &= h^-(x)(k_1 \dots max(k_i, k_{i+1}) \dots k_{n+2}) \text{ by definition of } \Gamma_i^- \\ &= x((\tau_0 \dots \tau_s)) \qquad \text{by definition of } h^- \end{aligned}$$

for some $1 \leqslant i \leqslant n+2$.

The situation can be decomposed in three mutually exclusive cases :

1. $k_i = k_{i+1} = 0$. In this case, there exists a unique j_0 such that $\sigma_{j_0} + 1 = i$, $s = r - 1$ and

$$\begin{aligned} &\sigma_j + 1 \leqslant i - 1 \Longrightarrow \sigma_j = \tau_j \text{ (in this case, } j < j_0) \\ &\tau_{j_0} + 1 = i = \sigma_{j_0} + 1 \\ &\sigma_j + 1 \geqslant i + 2 \Longrightarrow \sigma_j - 1 = \tau_{j-1} \text{ (in this case, } j > j_0 + 1) \end{aligned}$$

Then $\sigma_{j_0+2} \geqslant i + 1$ and

$$\begin{aligned} &x((\tau_0 \dots \tau_s)) \\ &= x((\sigma_0 < \dots < \sigma_{j_0} = \widehat{i-1} < \sigma_{j_0+2} - 1 < \dots < \sigma_{s+1} - 1)) \\ &= (\epsilon_{i-1}x)(\sigma_0 \dots \sigma_{j_0}\sigma_{j_0+1}\sigma_{j_0+2} \dots \sigma_{s+1}) \text{ by definition of } \epsilon_i \\ &\quad \text{and since } \sigma_{j_0+1} = i \\ &= (h^-(\epsilon_{i-1}x))(k_1 \dots k_{n+2}) \text{ by definition of } h^- \end{aligned}$$

2. $k_i = k_{i+1} = -$. In this case, $s = r$ and
$$\sigma_j + 1 \leqslant i - 1 \Longrightarrow \sigma_j = \tau_j$$
$$\sigma_j + 1 \geqslant i + 2 \Longrightarrow \sigma_j - 1 = \tau_j$$
   Then for some k,
$$\begin{aligned} & x((\tau_0 \dots \tau_s)) \\ & = x((\sigma_0 < \dots < \sigma_k < \widehat{i-1} < \sigma_{k+1} - 1 < \dots < \sigma_r - 1)) \\ & = (\epsilon_{i-1}x)((\sigma_0 \dots \sigma_k \sigma_{k+1} \dots \sigma_r)) \text{ by definition of } \epsilon_i \\ & = (h^-(\epsilon_{i-1}x))(k_1 \dots k_{n+2}) \text{ by definition of } h^- \end{aligned}$$

3. $k_i \neq k_{i+1}$. Now $s = r$ and since $\{k_i, k_{i+1}\} \subset \{-, 0\}$, then there exists a unique j_0 such that $\sigma_{j_0} + 1 \in \{i, i+1\}$ and we have
$$\begin{aligned} & \sigma_j + 1 \leqslant i - 1 \Longrightarrow \sigma_j = \tau_j \text{ (in this case, } j < j_0) \\ & \tau_{j_0} + 1 = i \\ & \sigma_j + 1 \geqslant i + 2 \Longrightarrow \sigma_j - 1 = \tau_j \text{ (in this case, } j > j_0) \end{aligned}$$
   There are two subcases : $\sigma_{j_0} + 1 = i$ and $\sigma_{j_0} + 1 = i + 1$. In the first situation,
$$\begin{aligned} & x((\tau_0 \dots \tau_s)) \\ & = x((\sigma_0 < \dots < \sigma_{j_0-1} < \sigma_{j_0} = i - 1 < \sigma_{j_0+1} - 1 < \dots < \sigma_r - 1)) \\ & = x((\sigma_0 < \dots < \sigma_{j_0-1} < \sigma_{j_0} < \sigma_{j_0+1} - 1 < \dots < \sigma_r - 1)) \\ & = (\epsilon_{i-1}x)((\sigma_0 < \dots < \sigma_{j_0} < \sigma_{j_0+1} < \dots < \sigma_r)) \text{ by definition of } \epsilon_i \\ & = (h^-(\epsilon_{i-1}x))(k_1 \dots k_{n+2}) \text{ by definition of } h^- \end{aligned}$$
   In the second situation,
$$\begin{aligned} & x((\tau_0 \dots \tau_s)) \\ & = x((\sigma_0 < \dots < \sigma_{j_0-1} < \sigma_{j_0} - 1 = i - 1 < \sigma_{j_0+1} - 1 < \dots < \sigma_r - 1)) \\ & = x((\sigma_0 < \dots < \sigma_{j_0-1} < \sigma_{j_0} - 1 < \sigma_{j_0+1} - 1 < \dots < \sigma_r - 1)) \\ & = (\epsilon_{i-1}x)((\sigma_0 < \dots < \sigma_{j_0} < \sigma_{j_0+1} < \dots < \sigma_r)) \text{ by definition of } \epsilon_i \\ & = (h^-(\epsilon_{i-1}x))(k_1 \dots k_{n+2}) \text{ by definition of } h^- \end{aligned}$$

□

Notice that h^- induces a natural transformation from CR_*^{gl} to CR_*^- which is not injective. Consider for example the ω-category consisting of two composable 1-morphisms u and v with $t_0 u = s_0 v$. The 0-simplexes u and $u *_0 v$ of $\mathcal{N}_0^{gl}$ have indeed the same image by h^- in CR_1^-. To see that, consider the thin square c from I^2 to $\mathcal{C}$ defined by $c(-0) = u *_0 v$, $c(0+) = t_0 v$, $c(0-) = u$, $c(+0) = v$ and $c(00) = u *_0 v$.

Now we arrive at :

Theorem 6.5. *There exists one and only one morphism of cuts from $\mathcal{N}^{gl}$ to $\mathcal{N}^-$.*

The proof of this theorem uses Theorem 8.3 assertion 1 as shortcut. There is no vicious circle because the uniqueness of h^- and h^+ is used nowhere in this paper. The only fact which is used is that Theorem 6.4 provides a natural transformation from $\mathcal{N}^{gl}$ to $\mathcal{N}^-$ which is injective on the underlying sets.

Proof. Let h and h' be two morphisms of cuts from $\mathcal{N}^{gl}$ to $\mathcal{N}^-$. One proves by induction on n that h_n and h'_n from $\mathcal{N}_n^{gl}$ to $\mathcal{N}_n^-$ coincide. For $n = 0$, $\mathcal{N}_0^{gl} = \mathcal{N}_n^- = tr^0\mathbb{P}$. The only natural transformation from $tr^0\mathbb{P}$ to itself is $Id_{tr^0\mathbb{P}}$, therefore $h_0 = h'_0$.

Suppose $P(n)$ proved for some $n \geqslant 0$. Then for any $x \in \mathcal{N}_{n+1}^{gl}(\mathcal{C})$, and for any $0 \leqslant i \leqslant n+1$,

$$\begin{aligned}
\partial_{i+1}^- h_{n+1}(x) &= h_n(\partial_i x) && \text{since } h \text{ morphism of simplicial sets}\\
&= h'_n(\partial_i x) && \text{by induction hypothesis}\\
&= \partial_{i+1}^- h'_{n+1}(x) && \text{since } h' \text{ morphism of simplicial sets}
\end{aligned}$$

Now with $1 \leqslant j \leqslant n+2$,

$$\begin{aligned}
&(\partial_j^+ h_{n+1}(x))(-_{n+1})\\
&= h_{n+1}(x)(- \cdots - [+]_j - \cdots -)\\
&= h_{n+1}(x)\,(t_0 R(- \cdots - [0]_j - \cdots -))\\
&= t_0\,(h_{n+1}(x)(R(- \cdots - [0]_j - \cdots -))) \quad \text{since } h_{n+1}(x)\ \omega\text{-functor}\\
&= t_0\left((\partial_1^- \ldots \widehat{\partial_j^-} \ldots \partial_{n+2}^- h_{n+1}(x))(0)\right)\\
&= t_0\left(h_0(\partial_0 \ldots \widehat{\partial_{j-1}} \ldots \partial_{n+1} x)(0)\right) \quad \text{since } h \text{ morphism of simplicial sets}\\
&= t_0\left((\partial_0 \ldots \widehat{\partial_{j-1}} \ldots \partial_{n+1} x)((0))\right)
\end{aligned}$$

So the 0-morphism $\partial_j^+ h_{n+1}(x))(-_{n+1})$ is the value of the constant map $t_0 \circ x$ of Theorem 8.3 (denoted by $T(x)$ in Section 10).

Let $\mathcal{D}$ be the unique ω-category such that $\mathbb{P}\mathcal{D} = \Delta^{n+1}$ and with $\mathcal{D}_0 = \{\alpha, \beta\}$, $s_0(\mathbb{P}\mathcal{D}) = \{\alpha\}$, $t_0(\mathbb{P}\mathcal{D}) = \{\beta\}$ and $\alpha \neq \beta$. And consider $Id_{\Delta^{n+1}} \in \mathcal{N}^{gl}_{n+1}(\mathcal{D})$.

Suppose that $+ \in \{k_1, \ldots, k_{n+2}\} \subset \{-, +\}$ and suppose that at least two k_i are equal to $+$. Then there exists a 1-morphism u of I^{n+2} such that $s_0 u = \ell_1 \ldots \ell_{n+2}$ with exactly one ℓ_i equal to $+$ and such that $t_0 u = k_1 \ldots k_{n+2}$. Then

$$s_0\left(h_{n+1}(Id_{\Delta^{n+1}})(u)\right) = h_{n+1}(Id_{\Delta^{n+1}})(\ell_1 \ldots \ell_{n+2}) = \beta$$

by the previous calculation. Since β is the unique morphism of $\mathcal{D}$ with 0-source β, then $h_{n+1}(Id_{\Delta^{n+1}})(u) = \beta$ and therefore

$$h_{n+1}(Id_{\Delta^{n+1}})(k_1 \ldots k_{n+2}) = \beta.$$

Suppose now that $+ \in \{k_1, \ldots, k_{n+2}\}$ with perhaps some 0 in the set. Then

$$s_0\left(h_{n+1}(Id_{\Delta^{n+1}})(k_1 \ldots k_{n+2})\right) = \beta$$

and therefore

$$ev \circ h_{n+1}(Id_{\Delta^{n+1}})(k_1 \ldots k_{n+2}) = \beta = T\left(Id_{\Delta^{n+1}}\right).$$

The ω-functor x from Δ^{n+1} to $\mathbb{P}\mathcal{C}$ induces a non-contracting ω-functor $\overline{x}$ from $\mathcal{D}$ to $\mathcal{C}$ with $\overline{x}(\alpha) = S(x)$ ($S(x)$ being the value of the constant map

$s_0 \circ x$ by Theorem 8.3) and $\overline{x}(\beta) = T(x)$ which sends $Id_{\Delta^{n+1}} \in \mathcal{N}^{gl}_{n+1}(\mathcal{D})$ on $x \in \mathcal{N}^{gl}_{n+1}(\mathcal{C})$. So by naturality,

$$ev \circ h_{n+1}(x)(k_1 \dots k_{n+2}) = T(x).$$

Therefore for any $1 \leqslant j \leqslant n+2$, $\partial^+_j h_{n+1}(x) = \partial^+_j h'_{n+1}(x)$. By hypothesis, $ev(h_{n+1}(x)) = ev(x) = ev(h'_{n+1}(x))$. So $h_{n+1}(x)$ and $h'_{n+1}(x)$ induce the same labeling of the faces of I^{n+2} and $P(n+1)$ is proved. □

Without explanation, here is the construction of h^+ :

Proposition 6.6. *Let x be an n-simplex of the globular simplicial nerve of $\mathcal{C}$. Then the map $h^+_n(x)$ from $\underline{cub}^{n+1}$ to $\mathcal{C}$ defined by*

1. $- \in \{k_1 \dots k_{n+1}\}$ *implies* $h^+_n(x)(k_1 \dots k_{n+1}) = s_0 x((n))$ *(notice that (n) is the initial state of Δ^n)*
2. $\{k_1, \dots, k_{n+1}\} \subset \{+, 0\}$ *and*

$$\{k_1, \dots, k_{n+1}\} \cap \{0\} = \{k_{\sigma_0+1}, \dots, k_{\sigma_r+1}\}$$

with $\sigma_0 < \dots < \sigma_r$ *implies* $h^+_n(x)(k_1 \dots k_{n+1}) = x((\sigma_0 \dots \sigma_r))$

3. $h^+_n(x)(+_{n+1}) = t_0 x((0))$ *(notice that (0) is the final state of Δ^n)*

yields an ω-functor from I^{n+1} to $\mathcal{C}$. Moreover, h^+ induces a morphism of simplicial sets from the globular nerve of $\mathcal{C}$ to its positive corner nerve. And the map from $\mathcal{N}^{gl}_{-1}(\mathcal{C})$ to $\mathcal{N}^+_{-1}(\mathcal{C})$ defined by $(x, y) \mapsto y$ extends the previous morphism to the corresponding augmented simplicial nerves. Moreover for $n \geqslant 0$, h^+_n is a one-to-one map and the image of h^+_n contains exactly all cubes x of the positive corner nerve such that as soon as $\partial^-_i x$ exists, then it is 0-dimensional.

Question 6.7. *Is it possible to find an appropriate setting where the globular cut would be an initial object ? Is it possible to characterize the diagram of cuts of Figure 1 ?*

As immediate corollary of the construction of h^- and its injectivity, let us introduce the analogue of Proposition 6.3 in the globular nerve.

Definition 6.8. *In a simplicial set A, a n-shell is a family* $(x_i)_{i=0,\dots,n+1}$ *of* $(n+2)$ *n-simplexes of A such that for any* $0 \leqslant i < j \leqslant n+1$, $\partial_i x_j = \partial_{j-1} x_i$.

Proposition 6.9. *Let C be a non-contracting ω-category. Consider a n-shell* $(x_i)_{i=0,\dots,n+1}$ *of the globular simplicial nerve of C. Then*

1. *The labeling defined by* $(x_i)_{i=0,\dots,n+1}$ *yields an ω-functor x (and necessarily exactly one) from* $\Delta^{n+1} \backslash \{(01\dots n+1)\}$ *to* $\mathbb{P}C$.

2. *Let u be a morphism of C such that*

$$s_n u = x\left(s_n R((01\dots n+1))\right)$$

and

$$t_n u = x\left(t_n R((01\dots n+1))\right)$$

Then there exists one and only one ω-functor still denoted by x from Δ^{n+1} *to* $\mathbb{P}C$ *such that for any* $0 \leqslant i \leqslant n+1$, $\partial_i x = x_i$ *and*

$$x((01\dots n+1)) = u.$$

7 Regularity of the globular cut

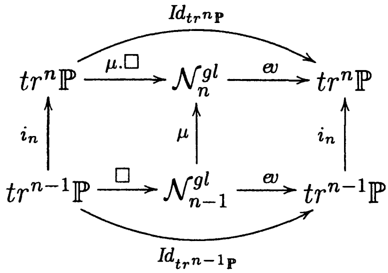
This section is devoted to the proof of the following theorem.

Theorem 7.1. *The globular cut is regular.*

The principle of this proof is to use the injectivity of the natural transformation h^- from $\mathcal{N}^{gl}$ to $\mathcal{N}^-$ and to use the regularity of $\mathcal{N}^-$.

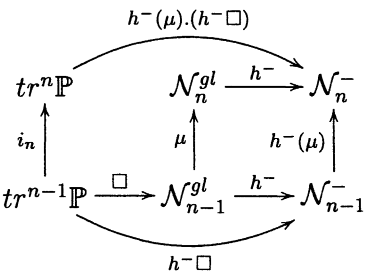
The folding operator $\Phi_n^{gl} := \Phi_n^{\mathcal{N}^{gl}}$ is called the n-dimensional globular folding operator and we set $\square_n^{gl} := \square_n^{\mathcal{N}^{gl}}$. It is clear that rule 1 and rule 2 of Definition 3.3 are satisfied. We have to check the rest of it.

Theorem 7.2. *For any natural transformation of functors μ from* $\mathcal{N}_{n-1}^{gl}$ *to* $\mathcal{N}_n^{gl}$ *with* $n \geqslant 1$, *and for any natural map* $\square$ *from* $tr^{n-1}\mathbb{P}$ *to* $\mathcal{N}_{n-1}^{gl}$ *such that* $ev \circ \square = Id_{tr^{n-1}\mathbb{P}}$, *there exists one and only one natural transformation* *denoted by* $\mu.\square$ *from* $tr^n\mathbb{P}$ *to* $\mathcal{N}_n^{gl}$ *such that the following diagram commutes*



where i_n *is the canonical inclusion functor from* $tr^{n-1}\mathbb{P}$ *to* $tr^n\mathbb{P}$.

Proof. The natural transformation $h^-\square$ from $tr^{n-1}\mathbb{P}$ to $\mathcal{N}_{n-1}^-$ can be lifted to a natural transformation $(h^-(\mu)).(h^-\square)$ from $tr^n\mathbb{P}$ to $\mathcal{N}_n^-$ since the cut $\mathcal{N}^-$ is regular. Since $h^-(\mu.\square) = (h^-(\mu)).(h^-\square)$ and since h^- is one-to-one in positive degree, there is at most one solution for this lifting problem.



Let $x \in \mathcal{C}_{n+1}$. For $0 \leqslant i \leqslant n$, the natural transformation

$$ev\ \partial_i\left(h^-(\mu).(h^-\square)\right) : tr^n\mathbb{P} \to tr^{n-1}\mathbb{P}$$

is of the form $d_{m_i}^{\alpha_i}$ for some $\alpha_i \in \{-,+\}$ and some $m_i \leqslant n$. Therefore

$$\begin{aligned}
&\partial_i\left(h^-(\mu).(h^-\square)\right) \\
&= \partial_i\left(h^-(\mu).(h^-\square)\right) i_n d_{m_i}^{\alpha_i} && \text{by Definition 3.3 rule 5b} \\
&= \partial_i h^-(\mu) h^-\square d_{m_i}^{\alpha_i} && \text{by hypothesis} \\
&= \partial_i h^-\mu\square d_{m_i}^{\alpha_i} \\
&= h^-\partial_i\mu\square d_{m_i}^{\alpha_i} && \text{since } h^- \text{ morphism of simplicial sets}
\end{aligned}$$

So $\partial_i\left(h^-(\mu).(h^-\square)\right)(x) \in h^-(\mathcal{N}^{gl}_{n-1}(\mathcal{C}))$ for any $0 \leqslant i \leqslant n$ and by Proposition 6.9, $\left(h^-(\mu).(h^-\square)\right)(x) \in h^-(\mathcal{N}^{gl}_n(\mathcal{C}))$. Let $\square'(x)$ be the unique element of $\mathcal{N}^{gl}_n(\mathcal{C})$ such that

$$h^-\square'(x) := \left(h^-(\mu).(h^-\square)\right)(x)$$

Then $\square'$ is a solution. □

Corollary 7.3. *The equalities $h^-\Phi^{gl} = \Phi^-h^-$ and $h^+\Phi^{gl} = \Phi^+h^+$ hold.*

Proof. It is a consequence of the naturality of h^- and h^+ and of Proposition 3.4. □

Now here is a characterization of globular folding operators :

Proposition 7.4. *Let x be a n-simplex of the globular nerve of $\mathcal{C}$. Then $x = \Phi^{gl}(x)$ if and only if for $0 \leqslant i \leqslant n-2$, $\partial_i x \in Im(\epsilon_{n-2}\dots\epsilon_i)$.*

Proof. The equality $x = \Phi^{gl}(x)$ implies $h^-(x) = \Phi^-(h^-(x))$, implies by Theorem 4.3 that for $1 \leqslant i \leqslant n-1$,

$$\begin{aligned} h^-(\partial_{i-1}x) &= \partial_i^-(h^-(x)) = \Gamma^-_{n-1}\dots\Gamma^-_i\square^-_i d_i^{(-)}h^-(x)(0_{n+1}) \\ &= h^-\left(\epsilon_{n-2}\dots\epsilon_{i-1}\square^{gl}_i s_i x((0\dots n))\right) \end{aligned}$$

therefore $\partial_{i-1}x \in Im(\epsilon_{n-2}\dots\epsilon_{i-1})$. Conversely, if for $0 \leqslant i \leqslant n-2$, $\partial_i x \in Im(\epsilon_{n-2}\dots\epsilon_i)$, then $h^-(x) = \Phi^-h^-(x) = h^-\Phi^{gl}(x)$ and therefore $x = \Phi^{gl}(x)$. □

Theorem 7.5. *The globular folding operator Φ^{gl} induces the identity map on the globular reduced chain complex CR^{gl}_*.*

Proof. Consider the θ_i^- operators of Theorem 4.5. If $x \in \mathcal{N}^{gl}_n$, then h^-x is negative. So $\theta_i^-h^-x$ is also negative by Theorem 4.5(1) and determines a unique element $\theta_i^{gl}x \in \mathcal{N}^{gl}_n$ such that $h^-\theta_i^{gl}x = \theta_i^-h^-x$. It is clear that these operators θ_i^{gl} induces the identity map on the reduced globular complex by Theorem 4.5(2). Since Φ^-h^-x is also negative, then by Theorem 4.5(3),

$$\Phi^-h^-x = \theta^-_{i_1}\dots\theta^-_{i_s}h^-x$$

for some sequence $i_1, \dots, i_s$. Therefore by the injectivity of h^-,

$$\Phi^{gl}x = \theta^{gl}_{i_1}\dots\theta^{gl}_{i_s}x$$

□

Theorem 7.6. *In the reduced globular complex, one has*

$$\square_n^{gl}(x *_p y) = \square_n^{gl}(x) + \square_n^{gl}(y)$$

for any morphisms x and y of $\mathcal{C}$ of dimension n and for $1 \leqslant p \leqslant n-1$.

Sketch of proof. One has

$$\begin{aligned} h^-(\square_n^{gl}(x *_p y)) &= \square_n^-(x *_p y) \\ &= \square_n^-(x) + \square_n^-(y) + t_1 + \partial^- t_2 \\ &= h^-(\square_n^{gl}(x)) + h^-(\square_n^{gl}(y)) + t_1 + \partial^- t_2 \end{aligned}$$

with t_1 a thin $(n+1)$-cube and t_2 a thin $(n+2)$-cube. The proof made in [8] shows that t_1 and t_2 are in the image of h^-. Indeed, the existence of t_1 and t_2 comes from the vanishing of some globular nerve. Therefore $t_1 = h^-(T_1)$ and $t_2 = h^-(T_2)$ where T_1 is a thin n-simplex and T_2 a thin $(n+1)$-simplex. This completes the proof. □

In fact one can explicitly verify that if x and y are two n-morphisms of $\mathcal{C}$, then $\square_n^{gl}(x *_{n-1} y) - \square_n^{gl}(x) - \square_n^{gl}(y)$ is a boundary in the normalized globular complex. It suffices to consider the thin $(n+1)$-cube $B_{n-1}^n(x,y)$ of [8] which turns to be in the image of h^- because it is negative. Therefore with $b(x,y) \in \omega Cat(\Delta^n, \mathcal{C})$ defined by $\partial_i b(x,y) = \epsilon_{n-2} \dots \epsilon_i \square_{i+1}^{gl} d_{i+1}^{(-)^{i+1}} x$ for $0 \leqslant i \leqslant n-3$ (observe that $d_{i+1}^{(-)^{i+1}} x = d_{i+1}^{(-)^{i+1}} y$), $\partial_{n-2} b(x,y) = \square_n^{gl} y$, $\partial_{n-1} b(x,y) = \square_n^{gl}(x *_{n-1} y)$, $\partial_n b(x,y) = \square_n^{gl} x$, one has

$$\partial b(x,y) = \pm \left(\square_n^{gl}(x *_{n-1} y) - \square_n^{gl}(x) - \square_n^{gl}(y)\right) + \text{ degenerate elements.}$$

8 Example of calculations of globular homology

The main goal of this section is to prove the vanishing of the globular homology of the n-cube in positive dimension for all $n \geqslant 0$. However we also study the case of the ω-category 2_n generated by one n-morphism and pose some questions about the globular homology of the ω-category generated by a composable pasting scheme in the sense of [12].

Theorem 8.1. *For any $p > 0$ and any $n \geqslant 0$, $H_p^{gl}(2_n) = 0$.*

Proof. For $p = 1$, it is obvious. For $p > 1$, one has

$$H_p^{gl}(2_n) \cong H_{p-1}(\mathbb{P}2_n) \cong H_{p-1}(2_{n-1}) = 0$$

where $H_*(\mathcal{D})$ means the simplicial homology of the simplicial nerve of the ω-category $\mathcal{D}$. □

Definition 8.2. *[9] Let $\mathcal{C}$ be an ω-category and let α and β be two 0-morphisms of $\mathcal{C}$. Then the bilocalization of $\mathcal{C}$ with respect to α and β is the ω-subcategory of $\mathcal{C}$ obtained by keeping in dimension 0 only α and β and by keeping in positive dimension all morphisms x such that $s_0x = \alpha$ and $t_0x = \beta$. It is denoted by $\mathcal{C}[\alpha, \beta]$.*

Theorem 8.3. *Let $\mathcal{C}$ be a non-contracting ω-category.*

1. *Let x be an ω-functor from Δ^n to $\mathbb{P}\mathcal{C}$ for some $n \geqslant 0$. Then the set maps*

$$(\sigma_0 \dots \sigma_r) \mapsto s_0x((\sigma_0 \dots \sigma_r))$$

and

$$(\sigma_0 \dots \sigma_r) \mapsto t_0x((\sigma_0 \dots \sigma_r))$$

from the underlying set of faces of Δ^n to $\mathcal{C}_0$ are constant. The unique value of $s_0 \circ x$ is denoted by $S(x)$ and the unique value of $t_0 \circ x$ is denoted by $T(x)$.

2. *For any pair (α, β) of 0-morphisms of $\mathcal{C}$, for any $n \geqslant 1$, and for any $0 \leqslant i \leqslant n$, then $\partial_i \left(\mathcal{N}_n^{gl}(\mathcal{C}[\alpha, \beta])\right) \subset \mathcal{N}_{n-1}^{gl}(\mathcal{C}[\alpha, \beta])$.*

3. *For any pair (α, β) of 0-morphisms of $\mathcal{C}$, for any $n \geqslant 0$, and for any $0 \leqslant i \leqslant n$, then $\epsilon_i \left(\mathcal{N}_n^{gl}(\mathcal{C}[\alpha, \beta])\right) \subset \mathcal{N}_{n+1}^{gl}(\mathcal{C}[\alpha, \beta])$.*

4. *By setting, $G^{\alpha,\beta}\mathcal{N}_n^{gl}(\mathcal{C}) := \mathcal{N}_n^{gl}(\mathcal{C}[\alpha, \beta])$ for $n \geqslant 0$ and $G^{\alpha,\beta}\mathcal{N}_{-1}^{gl}(\mathcal{C}) := \{(\alpha, \beta), (\beta, \alpha)\}$, one obtains a $(\mathcal{C}_0 \times \mathcal{C}_0)$-graduation on the globular nerve ; in particular, one has the direct sum of augmented simplicial sets*

$$\mathcal{N}_*^{gl}(\mathcal{C}) = \bigsqcup_{(\alpha,\beta) \in \mathcal{C}_0 \times \mathcal{C}_0} G^{\alpha,\beta}\mathcal{N}_*^{gl}(\mathcal{C})$$

and $G^{\alpha,\beta}\mathcal{N}_*^{gl}(\mathcal{C}) = \mathcal{N}_*^{gl}(\mathcal{C}[\alpha, \beta])$.

Proof. The only non-trivial part is the first assertion. Let $P(n)$ be the property : "for any non-contracting ω-category $\mathcal{C}$ and any ω-functor x from Δ^n to $\mathbb{P}\mathcal{C}$, the set map $(\sigma_0 \dots \sigma_r) \mapsto s_0 x((\sigma_0 \dots \sigma_r))$ from the set of faces of Δ^n to $\mathcal{C}_0$ is constant."

There is nothing to check for $P(0)$. For $P(1)$, if x is an ω-functor from Δ^1 to $\mathbb{P}\mathcal{C}$, then $s_1 x((01)) = x((1))$ and $t_1 x((01)) = x((0))$ in $\mathcal{C}$. Therefore

$$s_0 x((01)) = s_0 s_1 x((01)) = s_0 x((1))$$

and

$$s_0 x((0)) = s_0 t_1 x((01)) = s_0 x((01)).$$

Therefore $P(1)$ is true.

Suppose $P(n)$ proved for some $n \geqslant 1$ and let us prove $P(n+1)$. For any $1 \leqslant i \leqslant n$, the ω-functor $x : \Delta^{n+1} \to \mathbb{P}\mathcal{C}$ induces an ω-functor on the ω-category Δ_i^{n+1} generated by the face $(0 \dots \widehat{\imath} \dots n+1)$ and its subfaces. One has an isomorphism of ω-categories $\Delta^n \cong \Delta_i^{n+1}$. Therefore the restriction of $s_0 \circ x$ to the faces of Δ_i^{n+1} is constant by induction hypothesis. Now it is clear that $\Delta_i^{n+1} \cap \Delta_{i+1}^{n+1} \cong \Delta^{n-1} \neq \emptyset$ since $n \geqslant 1$. Therefore the set map $s_0 \circ x$ restricted to $\Delta_i^{n+1} \cup \Delta_{i+1}^{n+1}$ is constant. Therefore the restriction of the set map $s_0 \circ x$ to the faces of dimension at most n of Δ^{n+1} is constant. We know that

$$s_n R((01 \dots n+1)) = \Psi(X_0, X_1, \dots, X_s)$$

where $X_0, X_1, \dots, X_s$ are faces of Δ^{n+1} of dimension at most n. So

$$\begin{aligned} s_0 x((01 \dots n+1)) &= s_0 s_{n+1} x((01 \dots n+1)) \\ &= s_0 x\left(s_n R((01 \dots n+1))\right) \text{ since } x\ \omega\text{-functor} \\ &= s_0 x \Psi(X_0, X_1, \dots, X_s) \end{aligned}$$

where Ψ is a function using only the compositions of Δ^{n+1}. Then

$$x\Psi(X_0, X_1, \dots, X_s) = \Psi'(x(X_0), x(X_2), \dots, x(X_s))$$

where Ψ' is obtained from Ψ by replacing $*_i$ by $*_{i+1}$ since x is an ω-functor from Δ^{n+1} to $\mathbb{P}\mathcal{C}$. So

$$s_0 x((01 \dots n+1)) = \Psi'(s_0 x(X_0), s_0 x(X_2), \dots, s_0 x(X_s)) = s_0 x(X_0)$$

with the axioms of ω-categories. Therefore $P(n+1)$ is proved. □

Definition 8.4. *Let $\mathcal{C}$ be a non-contracting ω-category with exactly one initial state α and one final state β. Then the bilocalization $\mathcal{C}[\alpha, \beta]$ is also non-contracting and one can set $\Omega\mathcal{C} = \mathbb{P}(\mathcal{C}[\alpha, \beta])$.*

Theorem 8.5. *[18, 2, 16] Let $n \geqslant 1$. Then $\Omega\Delta^n = I^{n-1}$ and $\Omega I^{n-1} = P^{n-1}$ where P^{n-1} is the free ω-category generated by the composable pasting scheme of the faces of the $(n-1)$-dimensional permutohedron.*

Theorem 8.6. *For any $n \geqslant 0$, and any $p > 0$, $H_p^{gl}(I^n) = 0$.*

Proof. One has $H_p^{gl}(I^n) = \bigoplus_{(\alpha,\beta)\in\mathcal{C}_0\times\mathcal{C}_0} H_p^{gl}(I^n[\alpha,\beta])$ by Theorem 8.3. So it suffices to prove the vanishing of $H_p^{gl}(I^n[\alpha,\beta])$ as soon as $I^n[\alpha,\beta]$ contains morphisms in strictly positive dimension to prove the theorem.

Let α and β be two 0-morphisms of I^n such that $I^n[\alpha,\beta]$ contains other morphisms than α and β. Then in particular it contains some 1-morphisms from α to β which is a composite of 1-dimensional faces of I^n. Suppose that $\alpha = k_1 \dots k_n$. Then β is obtained from α by replacing some k_i equal to $-$ by $+$. Let $k_{\sigma_1}, \dots, k_{\sigma_r}$ be these k_i. Then

$$I^n[\alpha,\beta] \cong I^r[-_r, +_r]$$

as ω-category. Therefore it suffices to prove that $H_p^{gl}(I^n[-_n, +_n])$ vanishes.

The vanishing of $H_1^{gl}(I^n[-_n, +_n])$ is obvious. One has

$$H_p^{gl}(I^n[-_n, +_n]) = H_{p-1}(P^n)$$

for $p \geqslant 2$ by Theorem 8.5 and $H_{p-1}(P^n) = 0$ because the simplicial nerve of a composable pasting scheme is contractible [12]. □

Theorem 8.7. *For any $n \geqslant 0$, and any $p > 0$, $H_p^{gl}(\Delta^n) = 0$.*

Proof. By proceeding as in Theorem 8.6, we see that it suffices to prove that

$$H_p^{gl}(\Delta^n[(r),(s)]) = 0$$

for any pair $((r),(s))$ of 0-morphisms of Δ^n and for $n \geqslant 2$. However, $\Delta^n[(r),(s)]$ is non-empty if and only if $r > s$ with our conventions and in this case,

$$\Delta^n[(r),(s)] \cong \Delta^{r-s}[(r-s),(0)].$$

Therefore $H_p^{gl}(\Delta^n[(r),(s)]) \cong H_{p-1}(I^{r-s-1})$ by Theorem 8.5. □

More generally, as in [8], one sees that if $\mathcal{C}$ is a non-contracting ω-category such that $\mathbb{P}\mathcal{C}$ is the free ω-category generated by a composable pasting scheme in the sense of [12], then $H_p^{gl}(\mathcal{C}) = 0$ for $p \geqslant 1$. This is related to the problem of the existence of the derived pasting scheme of a given composable pasting scheme [14].

Conjecture 8.8. *Let $\mathcal{C}$ be an ω-category which is the free ω-category generated by a composable pasting scheme (therefore $\mathcal{C}$ is non-contracting). Then for any $p > 0$, $H_p^{gl}(\mathcal{C}) = 0$.*

9 Relation between the new globular homology and the old one

First of all, recall the definition of both formal corner homology theories from [8].

Definition 9.1. *Let $\mathcal{C}$ be a non-contracting ω-category. Set*

- $CF_0^-(\mathcal{C}) := \mathbb{Z}\mathcal{C}_0$
- $CF_1^-(\mathcal{C}) := \mathbb{Z}\mathcal{C}_1$
- $CF_n^-(\mathcal{C}) = \mathbb{Z}\mathcal{C}_n/\{x *_0 y = x, x *_1 y = x + y, \dots, x *_{n-1} y = x + y \mod \mathbb{Z}tr^{n-1}\mathcal{C}\}$ *for* $n \geqslant 2$

with the differential map $s_{n-1} - t_{n-1}$ from $CF_n^-(\mathcal{C})$ to $CF_{n-1}^-(\mathcal{C})$ for $n \geqslant 2$ and s_0 from $CF_1^-(\mathcal{C})$ to $CF_0^-(\mathcal{C})$. This chain complex is called the formal negative corner complex. The associated homology is denoted by $HF^-(\mathcal{C})$ and is called the formal negative corner homology of $\mathcal{C}$. The map CF_*^- (resp. HF_*^-) induces a functor from ωCat_1 to $Comp(Ab)$ (resp. Ab).

and symmetrically

Definition 9.2. *Let $\mathcal{C}$ be a non-contracting ω-category. Set*

- $CF_0^+(\mathcal{C}) := \mathbb{Z}\mathcal{C}_0$
- $CF_1^+(\mathcal{C}) := \mathbb{Z}\mathcal{C}_1$

- $CF_n^+(\mathcal{C}) = \mathbb{Z}\mathcal{C}_n/\{x *_0 y = y, x *_1 y = x + y, \dots, x *_{n-1} y = x + y \bmod \mathbb{Z}tr^{n-1}\mathcal{C}\}$ *for* $n \geqslant 2$

with the differential map $s_{n-1} - t_{n-1}$ *from* $CF_n^+(\mathcal{C})$ *to* $CF_{n-1}^+(\mathcal{C})$ *for* $n \geqslant 2$ *and* t_0 *from* $CF_1^+(\mathcal{C})$ *to* $CF_0^+(\mathcal{C})$. *This chain complex is called the formal positive corner complex. The associated homology is denoted by* $HF^+(\mathcal{C})$ *and is called the formal positive corner homology of* $\mathcal{C}$. *The map* CF_*^+ *(resp.* HF_*^+*) induces a functor from* ωCat_1 *to* $Comp(Ab)$ *(resp.* Ab*).*

The maps $\square_n^\pm$ from $\mathcal{C}_n$ to $C_n^\pm(\mathcal{C})$ induce a natural transformation from $CF_*^\pm$ to $CR_*^\pm$ and a natural transformation from $HF_*^\pm$ to $HR_*^\pm$.

Definition 9.3. *Let* $\mathcal{C}$ *be a non-contracting* ω*-category. Set*

- $CF_0^{gl}(\mathcal{C}) := \mathbb{Z}\mathcal{C}_0 \otimes \mathbb{Z}\mathcal{C}_0 \cong \mathbb{Z}(\mathcal{C}_0 \times \mathcal{C}_0)$
- $CF_1^{gl}(\mathcal{C}) := \mathbb{Z}\mathcal{C}_1$
- $CF_n^{gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_n/\{x *_1 y = x + y, \dots, x *_{n-1} y = x + y \bmod \mathbb{Z}tr^{n-1}\mathcal{C}\}$ *for* $n \geqslant 2$

with the differential map $s_{n-1} - t_{n-1}$ *from* $CF_n^{gl}(\mathcal{C})$ *to* $CF_{n-1}^{gl}(\mathcal{C})$ *for* $n \geqslant 2$ *and* $s_0 \otimes t_0$ *from* $CF_1^{gl}(\mathcal{C})$ *to* $CF_0^{gl}(\mathcal{C})$. *This chain complex is called the formal globular complex. The associated homology is denoted by* $HF^{gl}(\mathcal{C})$ *and is called the formal globular homology of* $\mathcal{C}$.

By Theorem 7.6 and Corollary 3.6, we see that the globular folding operators induce a natural morphism of chain complex from CF_*^{gl} to CR_*^{gl}, and therefore a natural transformation from HF_*^{gl} to HR_*^{gl}.

Question 9.4. *When is the natural morphism of chain complexes* R^{gl} *from* $CF_*^{gl}(\mathcal{C})$ *to* $CR_*^{gl}(\mathcal{C})$ *a quasi-isomorphism ?*

Conjecture 9.5. *(About the thin elements of the globular complex) Let* $\mathcal{C}$ *be a globular* ω*-category which is either the free globular* ω*-category generated by a semi-cubical set or the free globular* ω*-category generated by a globular set. Let* x_i *be elements of* $C_n^{gl}(\mathcal{C})$ *and let* λ_i *be natural numbers, where* i *runs over some set* I. *Suppose that for any* i, $ev(x_i)$ *is of dimension strictly lower than* n *(one calls it a thin element). Then* $\sum_i \lambda_i x_i$ *is a boundary if and only if it is a cycle.*

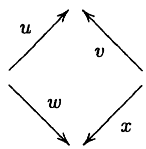


Figure 5: A false 1-globular cycle in the old globular homology

The above conjecture is clear for C_2^{gl} because all thin elements are degenerate. In higher dimension, there is enough room to have thin elements which are composition of degenerate elements, but which are not degenerate themselves.

The above conjecture is equivalent to claiming that the globular homology and the reduced one are equivalent for free globular ω-categories generated by either a semi-cubical set or a globular set.

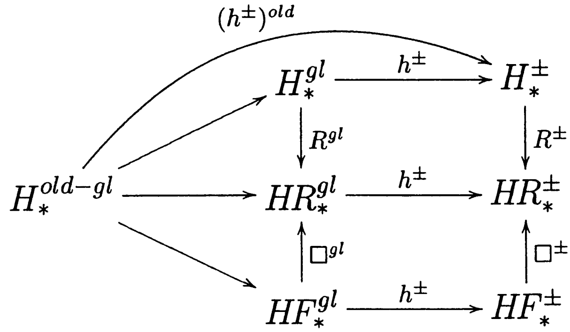
Now we are in position to give the exact statement relating the old globular homology of [9] and the new one.

Definition 9.6. *[9] Let $(C_*^{old-gl}(\mathcal{C}), \partial^{old-gl})$ be the chain complex defined as follows : $C_0^{old-gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_0 \oplus \mathbb{Z}\mathcal{C}_0$, and for $n \geqslant 1$, $C_n^{old-gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_n$, $\partial^{old-gl}(x) = (s_0 x, t_0 x)$ if $x \in \mathbb{Z}\mathcal{C}_1$ and for $n \geqslant 1$, $x \in \mathbb{Z}\mathcal{C}_{n+1}$ implies $\partial^{old-gl}(x) = s_n x - t_n x$. This complex is called the old globular complex of $\mathcal{C}$ and its corresponding homology the old globular homology.*

Instead of $C_0^{old-gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_0 \oplus \mathbb{Z}\mathcal{C}_0$, we set $C_0^{old-gl}(\mathcal{C}) = \mathbb{Z}(\mathcal{C}_0 \otimes \mathcal{C}_0)$ with the differential $\partial^{old-gl}(x) = s_0 x \otimes t_0 x$ for $x \in \mathcal{C}_1$. This makes H_1^{old-gl} slightly change. It does not matter because there is no influence on any potential applications. The difference appears in a situation like that of Figure 5. With $C_0^{old-gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_0 \oplus \mathbb{Z}\mathcal{C}_0$, $u + x - w - v$ is a old globular cycle. With $C_0^{old-gl}(\mathcal{C}) = \mathbb{Z}(\mathcal{C}_0 \otimes \mathcal{C}_0)$, this fake 1-globular cycle is killed.

Theorem 9.7. *We have the following commutative diagram of natural trans-*

formations for $* \geqslant 0$



where

- *the map* $H_*^{old-gl} \to H_*^{gl}$ *is the canonical map induced by* $x \mapsto \square_n^{gl}(x)$ *from* $\mathcal{C}_n$ *to* $\mathcal{N}_{n-1}^{gl}(\mathcal{C})$
- *the map* $H_*^{old-gl} \to HF_*^{gl}$ *is the canonical map making all identifications like* $A *_n B = A + B$ *for any* $n \geqslant 1$ *and any p-morphisms* A *and* B *with* $p \geqslant n+1$
- *the map* $HF_*^{gl} \to HF_*^{\pm}$ *is the canonical map making the supplemental identification* $x = x *_0 y$ *or* $y = x *_0 y$ *depending on the sign* $\pm$
- *the map* $HF_*^{\pm} \to HR_*^{\pm}$ *is the canonical map induced by the folding operators* $\square^{\pm}$ *of [8] (which is likely to be an isomorphism for any strict globular* ω*-category), and the map* $HF_*^{gl} \to HR_*^{gl}$ *is the canonical map induced by the folding operators* $\square^{gl}$ *(which is also likely to be an isomorphism for any strict globular* ω*-category)*
- *the maps* $R^{gl,\pm}$ *are the canonical maps from the globular or corner homology to the corresponding reduced homology (which are conjecturally an isomorphism for any free* ω*-category generated by a semi-cubical set or a globular set).*

Proof. This is due to the fact that for $n \geqslant 1$, the natural map $(h_n^{\pm})^{old}$ is induced by the set map $\square_n^-$ from $\mathcal{C}_n$ to $\omega Cat(I^n, \mathcal{C})^-$ ([9] Proposition 7.4). □

The difference between H_0^{old-gl} and H_0^{gl} is also not important. The group H_0^{old-gl} was indeed only introduced to define the morphisms h^- and h^+ in dimension 0. But H_0^{old-gl} does not have any computer-scientific meaning and is not involved in any potential applications.

10 Globular homology and deformation of HDA

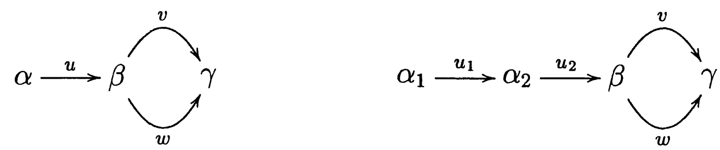
The following table summarizes how the globular nerve may be understood and compared with the two corner nerves of $\mathcal{C}$.

Geometric object	Formal theory	"True" theory	Simplicial cut
Branching	formal negative corner homology	negative corner homology	$\mathcal{N}^-(\mathcal{C})$
Merging	formal positive corner homology	positive corner homology	$\mathcal{N}^+(\mathcal{C})$
Globe	formal globular homology	globular homology	$\mathcal{N}^{gl}(\mathcal{C})$

Intuitively, the globular nerve of $\mathcal{C}$ contains all *achronal cuts* in the middle of all globes, whereas the negative and positive corner simplicial nerves contain all *achronal cuts* close to respectively the negative and the positive corners of the automaton. The expression "achronal" is borrowed from [6] and [7]. In these papers, HDA are modeled by local pospaces, and an achronal subspace Y of a local pospace is a topological subspace such that $x \leqslant y$ and $x, y \in Y$ imply $x = y$. The remarkable point is that the set of all achronal cuts of a given type can be enclosed into a simplicial set.

This could mean that the whole geometry of the free ω-category $\mathcal{C}$ generated by a semi-cubical set (i.e. a HDA) would be contained in the following diagram of augmented simplicial sets

$$\begin{array}{ccc} & \mathcal{N}^{gl}(\mathcal{C}) & \\ {}^{h^-}\swarrow & & \searrow^{h^+} \\ \mathcal{N}^-(\mathcal{C}) & & \mathcal{N}^+(\mathcal{C}) \end{array}$$



(a) $\mathcal{C}$

(b) Subdivision of u in $\mathcal{C}$

Figure 6: Subdivision of time

and in its temporal graph $tr^1\mathcal{C}$. This latter contains the information about the temporal structure of the HDA.

A problem, already mentioned in [10], is the question of the invariance of the globular homology of an ω-category up to a choice of a *cubification* [2] of the corresponding HDA. There are two types of deformations : the *spatial deformations* or *S-deformations* and the *temporal deformations* or *T-deformations*.

The globular cut is invariant by S-deformation, that is by deformations of p-morphisms with $p \geqslant 2$. This is simply due to the fact that such a deformation corresponds in the globular cut to a deformation of any simplex containing it as label. Therefore such a deformation corresponds to a deformation up to homotopy, in the usual sense, of the globular cut.

Unlike the corner homologies, the globular homology turns indeed to depend on the subdivision of time. The reason is contained in Figure 6. The obvious 1-functor from the left to the right such that $u \mapsto u_1 *_0 u_2$ should leave the globular homology invariant. This is not the case because the first globular homology is for the left member the free $\mathbb{Z}$-module generated by $v - w$ and $u *_0 v - u *_0 w$, and for the right member the free $\mathbb{Z}$-module generated by $v - w$ and $u_2 *_0 v - u_2 *_0 w$ and $u_1 *_0 u_2 *_0 v - u_1 *_0 u_2 *_0 w$. However in Figure 6, one can subdivide as many times as one wants for example v, and the globular homology will not change.

One way to overcome this problem is exposed in the last sections of [10], devoted to the description of a generic way to produce T-invariants

[2]Some authors [11] [21] use the term *cubicalation* : this means decomposing a HDA in cubes.

starting from the globular nerve. Let us prove [10] Claim 5.1 which enables to introduce the bisimplicial set mentioned in that paper.

Let $\mathcal{C}$ be a non-contracting ω-category. Using Theorem 8.3, recall that for some ω-functor x from Δ^n to $\mathbb{P}\mathcal{C}$, one calls $S(x)$ the unique element of the image of $s_0 \circ x$ and $T(x)$ the unique element of the image of $t_0 \circ x$. If (α, β) is a pair of $\mathcal{N}^{gl}_{-1}(\mathcal{C})$, set $S(\alpha, \beta) = \alpha$ and $T(\alpha, \beta) = \beta$.

Proposition 10.1. *Let $\mathcal{C}$ be a non-contracting ω-category. Let x and y be two ω-functors from Δ^n to $\mathbb{P}\mathcal{C}$ with $n \geqslant 0$. Suppose that $T(x) = S(y)$. Let $x * y$ be the map from the faces of Δ^n to $\mathcal{C}$ defined by*

$$(x * y)((\sigma_0 \ldots \sigma_r)) := x((\sigma_0 \ldots \sigma_r)) *_0 y((\sigma_0 \ldots \sigma_r)).$$

Then the following conditions are equivalent :

1. *The image of $x * y$ is a subset of $\mathbb{P}\mathcal{C}$.*
2. *The set map $x * y$ yields an ω-functor from Δ^n to $\mathbb{P}\mathcal{C}$ and $\partial_i(x * y) = \partial_i(x) * \partial_i(y)$ for any $0 \leqslant i \leqslant n$.*

On contrary, if for some $(\sigma_0 \ldots \sigma_r) \in \Delta^n$, $(x * y)((\sigma_0 \ldots \sigma_r))$ is 0-dimensional, then $x * y$ is the constant map $S(x) = T(y)$.

Proof. We have to prove that Condition 1 implies Condition 2. Let us consider $P(n)$: "for any non-contracting ω-category $\mathcal{C}$ and any ω-functor x and y from Δ^n to $\mathbb{P}\mathcal{C}$ such that $T(x) = S(y)$ and such that the image of $x * y$ is a subset of $\mathbb{P}\mathcal{C}$, then $x * y$ yields an ω-functor from Δ^n to $\mathbb{P}\mathcal{C}$ and $\partial_i(x * y) = \partial_i(x) * \partial_i(y)$ for any $0 \leqslant i \leqslant n$."

Property $P(0)$ is obvious. Suppose $P(n-1)$ proved for $n \geqslant 1$. For any $0 \leqslant i \leqslant n$, $\partial_i(x) * \partial_i(y)$ is a set map from Δ^{n-1} to $\mathbb{P}\mathcal{C}$ satisfying the hypothesis of the proposition, so by induction hypothesis, $\partial_i(x) * \partial_i(y)$ yields an ω-functor from Δ^{n-1} to $\mathbb{P}\mathcal{C}$. Let $z_i := \partial_i(x) * \partial_i(y)$. For $0 \leqslant j < i \leqslant n$,

$$\begin{aligned}
\partial_j(z_i) &= (\partial_j\partial_i(x)) * (\partial_j\partial_i(y)) && \text{by induction hypothesis}\\
&= (\partial_{i-1}\partial_j(x)) * (\partial_{i-1}\partial_j(y))\\
&= \partial_{i-1}(\partial_j(x) * \partial_j(y)) && \text{by induction hypothesis}\\
&= \partial_{i-1} z_j
\end{aligned}$$

Therefore $(z_i)_{0\leqslant i\leqslant n}$ is an $(n-1)$-shell. So it provides a unique ω-functor

$$z : \Delta^n \backslash \{(01\dots n)\} \to \mathbb{P}\mathcal{C}$$

by Proposition 6.9. It remains to check that

$$z\left(s_{n-1}R((01\dots n))\right) = s_n((x*y)((01\dots n)))$$

and

$$z\left(t_{n-1}R((01\dots n))\right) = t_n((x*y)((01\dots n)))$$

to complete the proof. Let us check the first equality. One has

$$s_{n-1}R((01\dots n)) = \Psi(X_1,\dots,X_s)$$

where Ψ uses only composition laws and where $X_1,\dots,X_s$ are faces of Δ^n of dimension at most $n-1$. Denote by Ψ' the same function as Ψ with $*_i$ replaced by $*_{i+1}$. Then

$z\left(s_{n-1}R((01\dots n))\right)$	
$= z\Psi(X_1,\dots,X_s)$	
$= \Psi'(z(X_1),\dots,z(X_s))$	since z ω-functor
$= \Psi'(x(X_1) *_0 y(X_1),\dots,x(X_s) *_0 y(X_s))$	by definition of z
$= \Psi'(x(X_1),\dots,x(X_s)) *_0 \Psi'(y(X_1),\dots,y(X_s))$	by interchange law
$= (x\Psi(X_1,\dots,X_s)) *_0 (y\Psi(X_1,\dots,X_s))$	since x and y ω-functors
$= (xs_{n-1}R((01\dots n))) *_0 (ys_{n-1}R((01\dots n)))$	
$= (s_n xR((01\dots n))) *_0 (s_n yR((01\dots n)))$	since x and y ω-functors
$= s_n\left(xR((01\dots n)) *_0 yR((01\dots n))\right)$	by interchange law
$= s_n((x*y)((01\dots n)))$	

Now let us suppose that $(x*y)((\sigma_0\dots\sigma_r))$ is 0-dimensional in $\mathcal{C}$ for some $(\sigma_0\dots\sigma_r)$. Then

$$s_1x((\sigma_0\dots\sigma_r)) *_0 s_1y((\sigma_0\dots\sigma_r))$$

is 0-dimensional. Either $s_0(\sigma_0 \dots \sigma_r) = (n)$ (the initial state of Δ^n) or there exists a 1-morphism U of Δ^n such that $s_0 U = (n)$ and $t_0 U = s_0(\sigma_0 \dots \sigma_r)$. In the first case, $x((n)) *_0 y((n))$ is 0-dimensional. In the second case,

$$x(t_0 U) *_0 y(t_0 U) = t_1 x(U) *_0 t_1 y(U) = t_1\left(x(U) *_0 y(U)\right)$$

is 0-dimensional. Then $x(U) *_0 y(U)$ is 0-dimensional as well as

$$x((n)) *_0 y((n)) = s_1\left(x(U) *_0 y(U)\right).$$

For any face $(\tau_0 \dots \tau_r)$ of $\Delta^n \backslash \{(n)\}$, there exists a 1-morphism V from $((n))$ to $s_0(\tau_0 \dots \tau_r)$ or $t_0(\tau_0 \dots \tau_r)$: let us say $s_0(\tau_0 \dots \tau_r)$. Since

$$s_1(x * y)(V) = (x * y)((n))$$

is 0-dimensional, then $(x * y)(V)$ is 0-dimensional, as well as

$$t_1(x * y)(V) = (x * y)(s_0(\tau_0 \dots \tau_r)) = s_1(x * y)((\tau_0 \dots \tau_r)).$$

Therefore $(x * y)((\tau_0 \dots \tau_r))$ is 0-dimensional. □

In the sequel, we set $(\alpha, \beta) * (\beta, \gamma) = (\alpha, \gamma)$, $S(\alpha, \beta) = \alpha$ and $T(\alpha, \beta) = \beta$. If x is an ω-functor from Δ^n to $\mathbb{P}\mathcal{C}$, and if y is the constant map $T(x)$ (resp. $S(x)$) from Δ^n to $\mathcal{C}_0$, then set $x * y := x$ (resp. $y * x := x$).

Theorem 10.2. *Suppose that $\mathcal{C}$ is an object of ωCat_1. Then for $n \geqslant 0$, the operations S, T and $*$ allow to define a small category $\underline{\mathcal{N}^{gl}_n(\mathcal{C})}$ whose morphisms are the elements of $\mathcal{N}^{gl}_n(\mathcal{C}) \cup \{$constant maps $\Delta^n \to \mathcal{C}_0\}$ and whose objects are the 0-morphisms of $\mathcal{C}$. If $\underline{\mathcal{N}^{gl}_{-1}(\mathcal{C})}$ is the small category whose morphisms are the elements of $\underline{\mathcal{C}_0 \times \mathcal{C}_0}$ and whose objects are the elements of $\mathcal{C}_0$ with the operations S, T and $*$ above defined, then one obtains (by defining the face maps ∂_i and degeneracy maps ϵ_i in an obvious way on $\{$constant maps $\Delta^n \to \mathcal{C}_0\}$) an augmented simplicial object $\underline{\mathcal{N}^{gl}_*}$ in the category of small categories.*

Proof. Equalities $S(x) = \partial_i S(x)$, $S(x) = \epsilon_i S(x)$, $T(x) = \partial_i T(x)$, $T(x) = \epsilon_i T(x)$ are consequences of Proposition 8.3. Equality $\partial_i(x * y) = \partial_i x * \partial_i y$ is proved right above. The verification of $\epsilon_i(x * y) = \epsilon_i x * \epsilon_i y$ is straightforward. □

By composing by the classifying space functor of small categories (cf. for example [20] for further details), one obtains a bisimplicial set which seems to be well-behaved with respect to subdivision of time. Indeed the first total homology groups associated to both ω-categories of Figure 6 are equal to $\mathbb{Z}$. Further explanations will be given in future papers.

To conclude, let us point out that in reasonable cases, i.e. when the p-morphisms (with $p \geqslant 2$) of a non-contracting ω-category $\mathcal{C}$ are invertible with respect to the composition laws $*_i$ of $\mathcal{C}$ for $i \geqslant 1$, then $\mathbb{P}\mathcal{C}$ becomes a globular ω-groupoid in the sense of Brown-Higgins. And therefore in such a case, it is well-known that the globular nerve of $\mathcal{C}$ satisfies the Kan property (see [23] or a generalization in [24]). However, this is not true in general for both corner nerves. To understand this fact, consider the 2-source of $R(000)$ in Figure 2(c) and remove $R(0+0)$. Consider both inclusion ω-functors from I^2 to respectively $R(-00)$ and $R(00-)$. Then the Kan condition fails because one cannot make the sum of $R(-00)$ and $R(00-)$ since $R(0+0)$ is removed.

References

[1] Fahd A. A. Al-Agl. *Aspects of multiple categories*. PhD thesis, University of Wales, Department of Pure Mathematics, University College of North Wales, Bangor, Gwynedd LL57 1UT, U.K., September 1989.

[2] H. J. Baues. Geometry of loop spaces and the cobar construction. *Mem. Amer. Math. Soc.*, 25(230):ix+171, 1980.

[3] R. Brown and P. J. Higgins. The equivalence of ∞-groupoids and crossed complexes. *Cahiers Topologie Géom. Différentielle*, 22(4):371–386, 1981.

[4] R. Brown and P. J. Higgins. On the algebra of cubes. *J. Pure Appl. Algebra*, 21(3):233–260, 1981.

[5] J. Duskin. Simplicial methods and the interpretation of "triple" cohomology. *Mem. Amer. Math. Soc.*, 3(issue 2, 163):v+135, 1975.

[6] L. Fajstrup, E. Goubault, and M. Raussen. Algebraic topology and concurrency. preprint, 1998.

[7] L. Fajstrup, E. Goubault, and M. Raußen. Detecting deadlocks in concurrent systems. In *CONCUR'98: concurrency theory (Nice)*, pages 332–347. Springer, Berlin, 1998.

[8] P. Gaucher. Combinatorics of branchings in higher dimensional automata. preprint math.CT/9912059, 2000.

[9] P. Gaucher. Homotopy invariants of higher dimensional categories and concurrency in computer science. *Math. Structures Comput. Sci.*, 10(4):481–524, 2000. Geometry and concurrency.

[10] P. Gaucher. From concurrency to algebraic topology. In Jeremy Gunawardena Maurice Herlihy Martin Raussen Patrick Cousot, Eric Goubault and Vladimiro Sassone, editors, *Electronic Notes in Theoretical Computer Science*, volume 39. Elsevier Science Publishers, 2001.

[11] E. Goubault. *The Geometry of Concurrency*. PhD thesis, Ecole Normale Supérieure, 1995.

[12] M. Johnson. The combinatorics of n-categorical pasting. *J. Pure Appl. Algebra*, 62(3):211–225, 1989.

[13] K.H. Kamps and T. Porter. *Abstract homotopy and simple homotopy theory*. World Scientific Publishing Co. Inc., River Edge, NJ, 1997.

[14] M. Kapranov and V. Voevodsky. Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders (list of results). *Cahiers Topologie Géom. Différentielle Catégoriques*, 32(1):11–27, 1991. International Category Theory Meeting (Bangor, 1989 and Cambridge, 1990).

[15] S. Mac Lane. *Categories for the working mathematician*. Springer-Verlag, New York, second edition, 1998.

[16] Yu. I. Manin and V. V. Schechtman. Arrangements of hyperplanes, higher braid groups and higher Bruhat orders. In *Algebraic number theory*, pages 289–308. Academic Press, Boston, MA, 1989.

[17] J.P. May. *Simplicial Objects in Algebraic Topology*. D. Van Nostrand Company, 1967.

[18] R. J. Milgram. Iterated loop spaces. *Ann. of Math. (2)*, 84:386–403, 1966.

[19] V. Pratt. Modeling concurrency with geometry. In ACM Press, editor, *Proc. of the 18th ACM Symposium on Principles of Programming Languages*, 1991.

[20] D. Quillen. Higher algebraic K-theory. I. pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[21] S. Sokolowski. Classifying holes of arbitrary dimensions in partially ordered cubes. Technical report, Kansas State University, 1999.

[22] R. Steiner. Tensor products of infinity-categories. University of Glasgow, 1991.

[23] R. Street. The algebra of oriented simplexes. *J. Pure Appl. Algebra*, 49(3):283–335, 1987.

[24] R. Street. Fillers for nerves. In *Categorical algebra and its applications (Louvain-La-Neuve, 1987)*, pages 337–341. Springer, Berlin, 1988.

Institut de Recherche Mathématique Avancée
ULP et CNRS
7 rue René Descartes
67084 Strasbourg Cedex
France
gaucher@irma.u-strasbg.fr