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From Concurrency to Algebraic Topology

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Abstract

This paper is a survey of the new notions and results scattered in [13,11,12]. However the speculations of Section 5 and Section 6 are new. Starting from a formalization of higher dimensional automata (HDA) by strict globular ω-categories, the construction of a diagram of simplicial sets over the three-object small category $- \leftarrow gl \rightarrow +$ is exposed. Some of the properties discovered so far on the corresponding simplicial homology theories are explained, in particular their links with geometric problems coming from concurrency theory in computer science.

1 Introduction

We have already argued in [13] for modeling higher dimensional automata (HDA) using strict globular ω-categories. To our knowledge, the link between globular ω-categories and concurrent automata was first noticed in [21]. Papers [13,11,12] show that this way of formalizing HDA is very well adapted to getting interesting new functors deeply related to the computer-scientific properties of the HDA. We would like to explain here the construction of these functors, some of their known properties and some perspectives. Many explanations are given in a very informal way. We refer to the bibliography for more details.

In [21,14], HDA are formalized using cubical sets in the sense of [6]. The link between cubical sets and ω-categories will be described at the end of Section 3. There are two equivalent approaches of the notion of ω-category: a cubical one and a globular one [3]. The globular approach will be used everywhere in this paper except in Figure 6 where cubical subdivisions are depicted. An example of globular subdivision is depicted in Figure 3(a). An informal topological description of HDA is also used in Section 2. The link between all these descriptions of HDA is summarized from the point of view of the homotopy of HDA in Section 6.

This survey is intended to be readable by non-specialists in algebraic topology. Only a small background is required: the definition of simplicial set (their face maps will be denoted by ∂_i and their degeneracy maps by ϵ_i) and of the as-

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sociated simplicial homology H_*, and therefore the definition of the homology of a chain complex of abelian groups [20,27,23].

Some geometric intuitions are introduced in Section 2. The definition of globular ω-category is recalled in Section 3. This section also provides a description of the ω-categories associated to the n-cubes and to the n-simplexes for all n. The three nerves and the two morphisms h^- and h^+ are described in Section 4. In Section 5, we speculate about what should be a good invariant of HDA.

I would like to thank Stefan Sokolowski for his useful comments about Section 2.

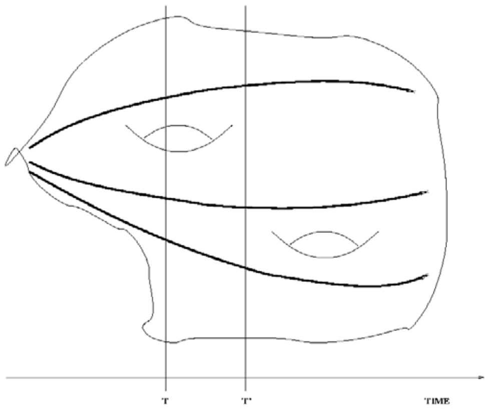
2 The topology of HDA in an informal way

To explain the geometric intuition which underlies the constructions of this paper, let us make a digression by considering a continuous model instead of a discrete one. The main ideas are indeed much simpler to understand in a continuous setting for a reader not familiar with the ω-categorical style. This section is the most informal part of the paper. Indeed the precise link between the topological and the ω-categorical approach is still an open question (see the end of Section 6). So we will use the continuous model of local pospaces as defined in [10]. Indeed a HDA can be seen as a flow of execution paths on a topological space.

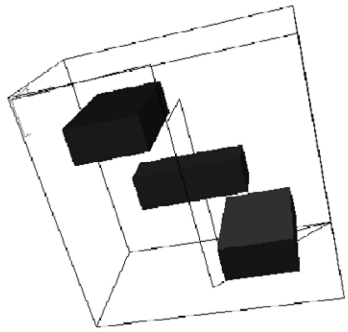
So let X be a local pospace. Recall that a subset U of a local pospace X is *achronal* if and only if for any $x, y \in U$, $x \leq y$ implies $x = y$. The easy case is when a clock is running concurrently to the HDA, that is when there exists a dimap D_X from X to $\mathbb{R}$ such that $X_t := D_X^{-1}(t)$ is achronal for every $t \in \mathbb{R}$ and such that for any $x, y \in X$, $x \leq y$ if and only if $D_X(x) \leq D_X(y)$. One says that X is an Euclidian local pospace [10]. This clock is supposed to represent an absolute time. This situation is drawn in Figure 1(a). The date map D_X, which associates to every point of the above topological space (that is to say a state of the corresponding HDA) the date when it occurs, is the projection map on the temporal line. Two achronal cuts $D_X^{-1}(T)$ and $D_X^{-1}(T')$ at the dates T and T' are depicted in Figure 1(a). In this situation, the execution paths ϕ are contained in the set of continuous map ϕ from $[0,1]$ to X such that $D_X \circ \phi$ is a non-decreasing map from $[0,1]$ to $\mathbb{R}$.

Deforming the HDA of the picture means deforming every cut of (X, D_X) in a continuous way. In other terms, in most cases, whenever f and g are two morphisms of topological spaces from X to Y (resp. from Y to X) such that for every $t \in \mathbb{R}$, the maps f and g induce reciprocal homotopy equivalences between $(D_X)^{-1}(t)$ and $(D_Y)^{-1}(t)$ and such that $D_X(X)$ and $D_Y(Y)$ are two homeomorphic subsets of $\mathbb{R}$, then one can say that f and g are reciprocal deformations between (X, D_X) and (Y, D_Y). In this case, both HDAs (X, D_X) and (Y, D_Y) are equal up to deformation.

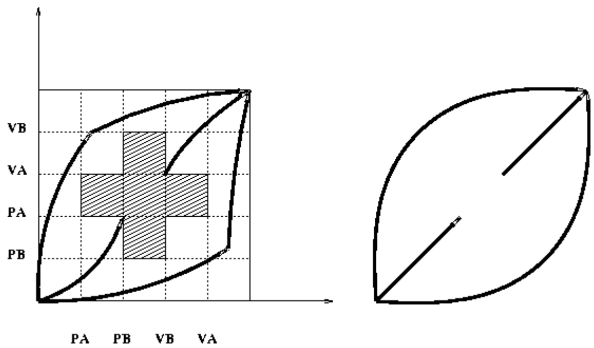
So a good way to start thinking of invariants of HDA as in Figure 1(a)



(a) A HDA X as a flow of execution paths



(b) The room with three barriers



(c) The Swiss Flag

Fig. 1. Examples of HDA

consists of thinking that an invariant of HDA is an invariant of cuts. And one can start constructing an invariant of HDA by choosing one real number t, one invariant coming from algebraic topology F, and by considering the map $(X, D_X) \mapsto F \circ (D_X)^{-1}(t)$.

Now let us remove the hypothesis of an absolute time. It becomes impossible to consider a map as above. The main idea of this paper is then as follows. There are three kind of geometric regions in a HDA : 1) the branching areas of execution paths, 2) the merging areas of execution paths, 3) and the oriented globes. And each of these regions gives rise to one simplicial set (the *branching nerve*, the *merging nerve* and the *globular nerve*) which represents respectively the topology of all cuts close to branching or merging areas of execution paths or in the middle of the globes. Moreover, as depicted in Figure 2, each topological cut in a middle of a globe can also be considered as a topological cut close to a branching and a merging area of execution paths (it suffices to hide –for example– the right part of Figure 2 with the hand to see a branching area appearing). Therefore there will be two morphisms of simplicial sets from the globular nerve to respectively the branching and the merging nerves.

There are some exceptions anyway to the above philosophy. Considering only achronal cuts is indeed not sufficient to characterize the homotopy type of an HDA. Loosely speacking, if it was sufficient, the simplicial set $N_*(X) = C^0_{achronal}(\Delta^*, X)$, where Δ^n would be here the usual topological n-simplex and where $C^0_{achronal}(\Delta^*, X)$ would be the continuous maps f from Δ^* to X such that $f(x) \leq f(y)$ if and only if $f(x) = f(y)$, would contain enough information to characterize the homotopy class of the HDA X. But the information that tells us how a given vertical simplex is related to another one along the flow of execution paths is missing. There are too many simplexes in this simplicial set and there is not enough information to make the required identifications.

For instance, in the room with three barriers of Figure 1(b) (borrowed from [10]), there are two non-dihomotopic execution paths which will give rise to a non-trivial cycle in the H_0 of the globular nerve althrough all achronal cuts in this example are path-connected. As other example, in a good theory it is indeed almost sure that an oriented line from a state α to another state β does not represent the same HDA as the HDA corresponding to one point. In the first case, there is a computation and in the second case there is not. Therefore these two HDAs (the oriented segment and the point) are not equal up to deformation. In other terms, one cannot contract a temporal line to one of these extremal points.

We will understand later that the three simplicial nerves constructed in this paper do see these situations. This means that the three nerves will also contain information not corresponding to any achronal cut.

The interest of considering such invariants is detailed in many papers [14,13]. Loosely speaking, such deformations leave the most interesting properties of a HDA unchanged. For example both HDAs of Figure 1(c) are essentially the same (beware of the fact that there is no absolute time in this latter example). As explained in [9], the state γ represents a deadlock and the state δ an unreachable state. Compare the possible execution paths on the left and the four execution paths on the right. These are essentially the same! This means that an algorithm reducing an HDA by a deformation before doing calculations would be more efficient than any other algorithm.

3 Modeling HDA by means of globular ω-categories

Roughly speaking, in a given globular ω-category $\mathcal{C}$, 0-morphisms represent the states of the corresponding automaton, 1-morphisms represent all possible execution paths in the HDA and higher dimensional morphisms rep-

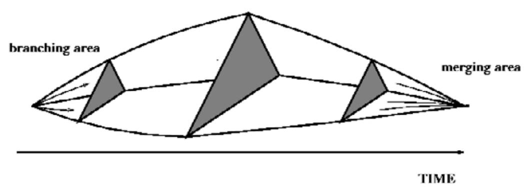


Fig. 2. The fundamental structure

resent homotopies between morphisms of lower dimension. They represent the execution of several tasks carried out at the same time. In particular 2-morphisms represent homotopies between execution paths. The composition of execution paths matches exactly the composition of 1-morphisms. Higher dimensional composition laws formalize the composition of higher dimensional homotopies (cf. Figure 3(a)). As already noticed in [21], the axioms of globular ω-categories encode the geometric properties of compositions of execution paths and homotopies between them.

Let us recall the definition of ω-category in three steps (see [5,26,24] for more details):

Definition 3.1 A 1-*category* is a pair $(A, (*, s, t))$ satisfying the following axioms:

(i) A is a set

(ii) s and t are set maps from A to A respectively called the source map and the target map

(iii) for $x, y \in A$, $x * y$ is defined as soon as $tx = sy$

(iv) $x * (y * z) = (x * y) * z$ as soon as both members of the equality exist

(v) $sx * x = x * tx = x$, $s(x * y) = sx$ and $t(x * y) = ty$ (this implies $ssx = sx$ and $ttx = tx$).

Definition 3.2 A 2-*category* is a triple $(A, (*_0, s_0, t_0), (*_1, s_1, t_1))$ such that

(i) both pairs $(A, (*_0, s_0, t_0))$ and $(A, (*_1, s_1, t_1))$ are 1-categories

(ii) $s_0 s_1 = s_0 t_1 = s_0$, $t_0 s_1 = t_0 t_1 = t_0$, and for $i \geq j$, $s_i s_j = t_i s_j = s_j$ and $s_i t_j = t_i t_j = t_j$ (Globular axioms)

(iii) $(x *_0 y) *_1 (z *_0 t) = (x *_1 z) *_0 (y *_1 t)$ (Godement axiom)

(iv) if $i \neq j$, then $s_i(x *_j y) = s_i x *_j s_i y$ and $t_i(x *_j y) = t_i x *_j t_i y$.

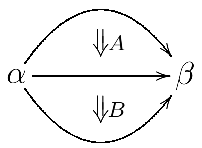
Definition 3.3 A *globular ω-category* $\mathcal{C}$ is a set A together with a family $(*_n, s_n, t_n)_{n \geq 0}$ such that

(i) for any $n \geq 0$, $(A, (*_n, s_n, t_n))$ is a 1-category

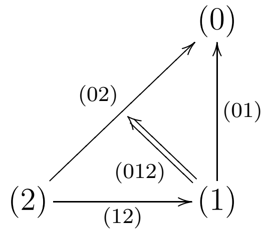
(ii) for any $m, n \geq 0$ with $m < n$, $(A, (*_m, s_m, t_m), (*_n, s_n, t_n))$ is a 2-category

(iii) for any $x \in A$, there exists $n \geq 0$ such that $s_n x = t_n x = x$ (the smallest of these n is called the dimension of x).

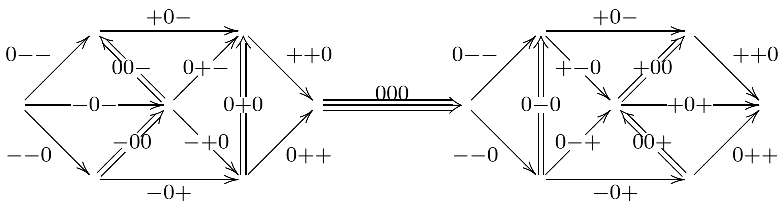
A n-dimensional element of $\mathcal{C}$ is called a n-morphism. A 0-morphism is



(a) Composition of two 2-morphims



(b) The ω-category Δ^2



(c) The ω-category I^3

Fig. 3. Some ω-categories (a k-fold arrow symbolizes a k-morphism)

also called a state of $\mathcal{C}$, and a 1-morphism an arrow. If x is a morphism of an ω-category $\mathcal{C}$, we call $s_n(x)$ the n-source of x and $t_n(x)$ the n-target of x. The category of all ω-categories (with the obvious morphisms) is denoted by ωCat. The corresponding morphisms are called ω-functors.

As fundamental examples of ω-categories, there are the ω-category I^n associated to the n-dimensional cube and that of the n-dimensional simplex (this latter is denoted by Δ^n). For the cube, the older attempt of constructing a structure of ω-category on the set of faces of the n-cube is maybe in [1]. As for the n-simplex, the seminal work is [26]. Since then, many constructions have been proposed.

Both families of ω-categories can be characterized in the same way. The first step consists of labelling all faces of the n-cube and of the n-simplex. For the n-cube, this consists of considering all words of length n in the alphabet $\{-, 0, +\}$, one word corresponding to the barycenter of a face (with $00\ldots 0$ (n times) $=: 0_n$ corresponding to its interior). As for the n-simplex, its faces are in bijection with strictly increasing sequences of elements of $\{0, 1, \ldots, n\}$. A sequence of length $p+1$ will be of dimension p. If x is a face, let $R(x)$ be the set of faces of x seen respectively as a sub-cube or a sub-simplex. If X is a set of faces, then let $R(X) = \bigcup_{x \in X} R(x)$. Notice that $R(X \cup Y) = R(X) \cup R(Y)$ and that $R(\{x\}) = R(x)$. Then I^n and Δ^n are the free ω-categories generated by the $R(x)$ with the rules

(i) For x p-dimensional with $p \geq 1$, $s_{p-1}(R(x)) = R(s_x)$ and $t_{p-1}(R(x)) =$

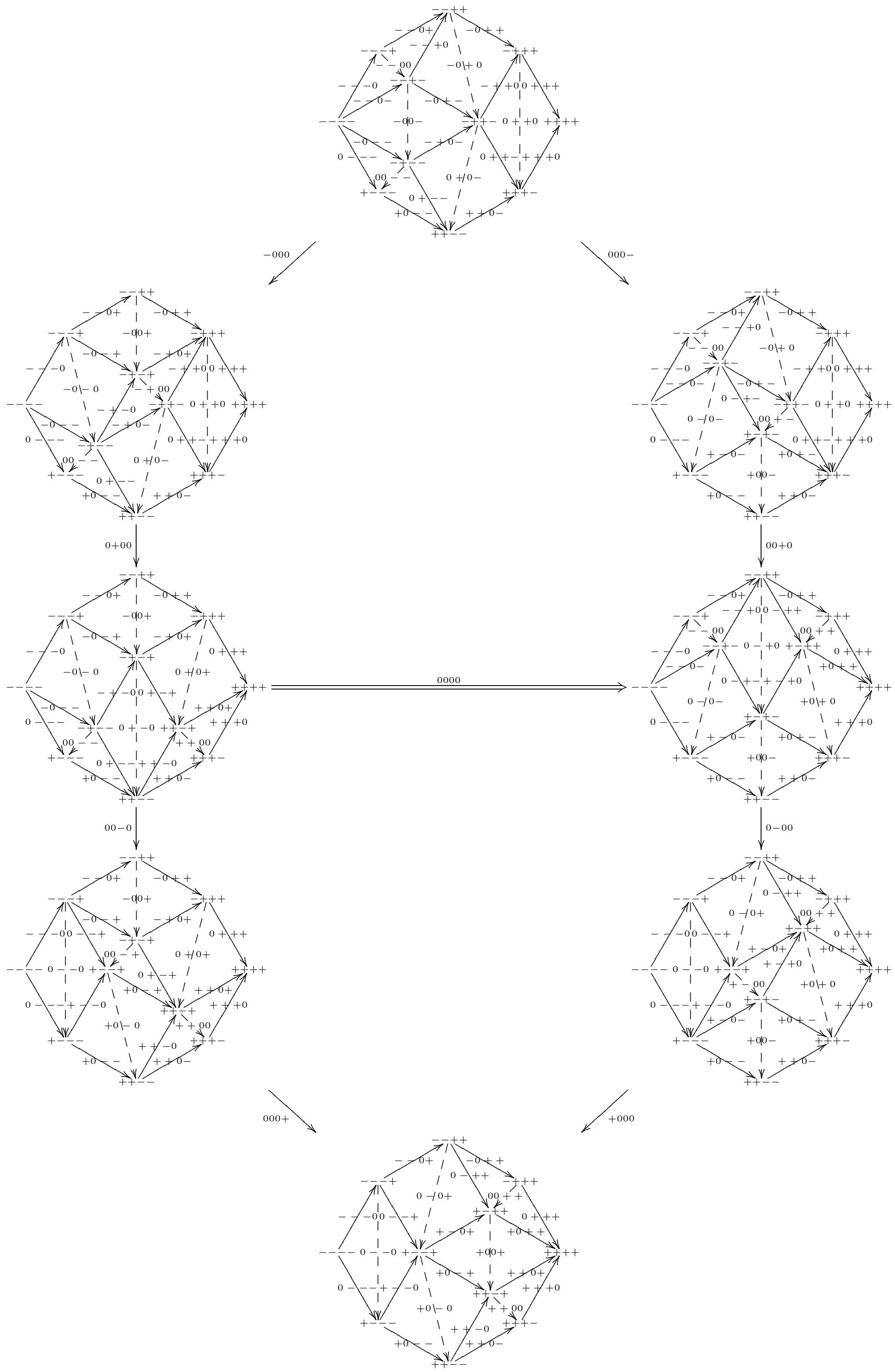


Fig. 4. The ω-category I^4

$R(t_x)$ where s_x and t_x are the sets of faces defined below.

(ii) If X and Y are two elements of I^n (resp. Δ^n) such that $t_p(X) = s_p(Y)$ for some p, then $X \cup Y$ belongs to I^n (resp. Δ^n) and $X \cup Y = X *_p Y$. The slogan is : "Composition means union".

Only the definition of s_x and t_x differs from one case to the other one. Let us give the computation rule in some examples. For the cube, the i-th zero is replaced by $(-)^i$ (resp. $(-)^{i+1}$) for s_x (resp. t_x). For example, one has $s_{0+00} = \{$-+00, 0++0, 0+0-$\}$ and $t_{0+00} = \{$++00, 0+-0, 0+0+$\}$. Figure 4 represents the 4-cube (the TeX program is due to Emmanuel Peyre).

For the simplex, $s_{(04589)} = \{(4589), (0489), (0458)\}$ (the elements in odd position are removed) and $t_{(04589)} = \{(0589), (0459)\}$ (the elements in even position are removed).

The above constructions are examples of free ω-categories generated by some data (see for example [15,17,25] for possible descriptions of these data). Using the construction of I^n, one can construct the free ω-category generated by a cubical set. A cubical set K is indeed a set-valued presheaf over some small category $\square$ whose objects are natural numbers and whose morphisms encode the axioms of cubical sets (exactly in the same way that the small category Δ does for simplicial sets) [7,16]. It is a general fact that a cubical set K is in a canonical way the direct limit of its cubes : $K = \int^{\underline{n}\in\square} K_n.\square(-,\underline{n})$ where the integral sign is the coend construction [19] and $K_n.\square(-,\underline{n})$ means the sum of "cardinal of K_n" copies of $\square(-,\underline{n})$. It then suffices to paste the ω-categories associated to every n-cube of K in the same way that they are pasted in K, that is to consider $\int^{\underline{n}\in\square} K_n.I^n$, to obtain the *$\omega$-categorical realization* of the cubical set K. For instance the ω-categorical realization of a 1-dimensional cubical set is an ω-category whose 1-morphisms are exactly the arrows of the cubical set and all possible compositions of these arrows (further details in the informal part of [13]). Exactly in the same way, the topological space $\int^{\underline{n}\in\square} K_n.[0,1]^n$, where $[0,1]^n$ is the topological n-cube, is nothing else but the usual geometric realization of the cubical set K [19].

4 Fundamental constructions

First of all here are some important definitions. The $\mathbb{N}$-graded set $\mathcal{C}[1]$ is obtained from $\mathcal{C}$ by removing the 0-morphisms, by considering the 1-morphisms of $\mathcal{C}$ as the 0-morphisms of $\mathcal{C}[1]$, the 2-morphisms of $\mathcal{C}$ as the 1-morphisms of $\mathcal{C}[1]$, etc. with an obvious definition of the source and target maps and of the composition laws. The map $T : \mathcal{C} \mapsto \mathcal{C}[1]$ does not induce a functor from ωCat to itself because ω-functors can contract 1-morphisms and because with our conventions, a 1-source or a 1-target can be 0-dimensional. Hence the following definition :

Definition 4.1 [13] An ω-category $\mathcal{C}$ is *non-1-contracting* if $\mathcal{C}[1]$ is an ω-category (or equivalently if $s_1 x$ and $t_1 x$ are 1-dimensional as soon as x is not 0-dimensional). Let f be an ω-functor from $\mathcal{C}$ to $\mathcal{D}$. The morphism f is *non-1-contracting* if for any 1-dimensional $x \in \mathcal{C}$, the morphism $f(x)$ is a 1-dimensional morphism of $\mathcal{D}$.

Definition 4.2 The category of non-1-contracting ω-categories with the non-1-contracting ω-functors is denoted by ωCat_1.

Notice that the image of the realization functor from cubical sets to globular ω-categories is included in ωCat_1. So from a computer-scientific point of view, this restriction does not matter.

Following [8], an augmented simplicial set is a simplicial set $(X_n)_{n\geq 0}$ endowed with an additional set X_{-1} and an additional set map ∂_{-1} from X_0 to X_{-1} such that $\partial_{-1}\partial_0 = \partial_{-1}\partial_1$ where ∂_0 and ∂_1 are the two face maps from X_1 to X_0. The "simplicial homology" functor H_* from the category of augmented simplicial sets $Sets_+^{\Delta^{op}}$ to the category of abelian groups Ab is defined as the usual one for $* \geq 1$ and by setting $H_0(X) = Ker(\partial_{-1})/Im(\partial_0 - \partial_1)$ and $H_{-1}(X) = \mathbb{Z}X_{-1}/Im(\partial_{-1})$ whenever X is an augmented simplicial set.

4.1 The branching and merging nerves

The branching and merging nerves are dual from each other. We set

$$\omega Cat(I^{n+1}, \mathcal{C})^\eta := \{x \in \omega Cat(I^{n+1}, \mathcal{C}), x(\eta \dots [0]_i \dots \eta) \text{ 1-dimensional}\}$$

where $\eta \in \{-,+\}$ and where the expression $\eta \dots [0]_i \dots \eta$ denotes the word on $\{\eta, 0\}$ with exactly one zero in the i-th position and for all (i,n) such that $0 \leq i \leq n$, the face maps ∂_i from $\omega Cat(I^{n+1}, \mathcal{C})^\eta$ to $\omega Cat(I^n, \mathcal{C})^\eta$ are the arrows ∂_{i+1}^η defined by

$$\partial_{i+1}^\eta(x)(k_1 \dots k_{n+1}) = x(k_1 \dots [\eta]_{i+1} \dots k_{n+1})$$

and the degeneracy maps ϵ_i from $\omega Cat(I^n, \mathcal{C})^\eta$ to $\omega Cat(I^{n+1}, \mathcal{C})^\eta$ are the arrows Γ_{i+1}^η defined by setting

$$\Gamma_i^-(x)(k_1 \dots k_n) := x(k_1 \dots \max(k_i, k_{i+1}) \dots k_n)$$
$$\Gamma_i^+(x)(k_1 \dots k_n) := x(k_1 \dots \min(k_i, k_{i+1}) \dots k_n)$$

with the order $- < 0 < +$.

Definition 4.3 [13] The *η-corner simplicial nerve* $\mathcal{N}^\eta$ is the functor from ωCat_1 to $Sets_+^{\Delta^{op}}$ defined by $\mathcal{N}_n^\eta(\mathcal{C}) := \omega Cat(I^{n+1}, \mathcal{C})^\eta$ for $n \geq 0$ and with $\mathcal{N}_{-1}^\eta(\mathcal{C}) = \mathcal{C}_0$ and endowed with the augmentation map ∂_{-1} from $\mathcal{N}_0^\eta(\mathcal{C}) = \mathcal{C}_1$ to $\mathcal{N}_{-1}^\eta(\mathcal{C}) = \mathcal{C}_0$ defined by $\partial_{-1} = s_0$ (resp. $\partial_{-1} = t_0$) if $\eta = -$ (resp. $\eta = +$).

In the sequel, "$-$-corner" means *branching* and "$+$-corner" means *merging*. Set

$$H_{n+1}^\eta(\mathcal{C}) := H_n(\mathcal{N}^\eta(\mathcal{C}))$$

for $n \geq -1$. These homology theories are called branching and merging homology respectively. The abelian group $H_0^-(\mathcal{C})$ (resp. $H_0^+(\mathcal{C})$) is the free abelian group generated by the final (resp. initial) states of $\mathcal{C}$.

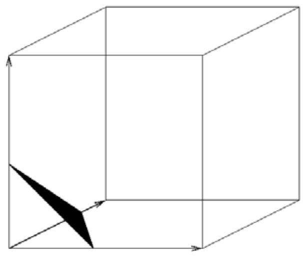


Fig. 5. model of 2-simplex in the branching nerve

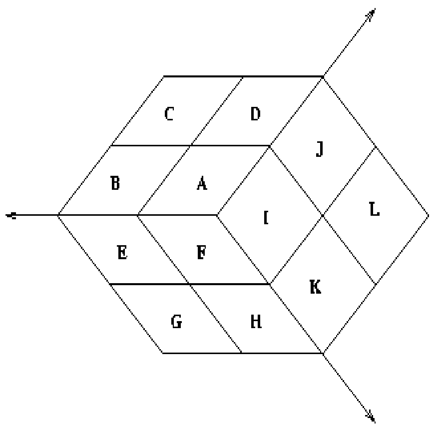
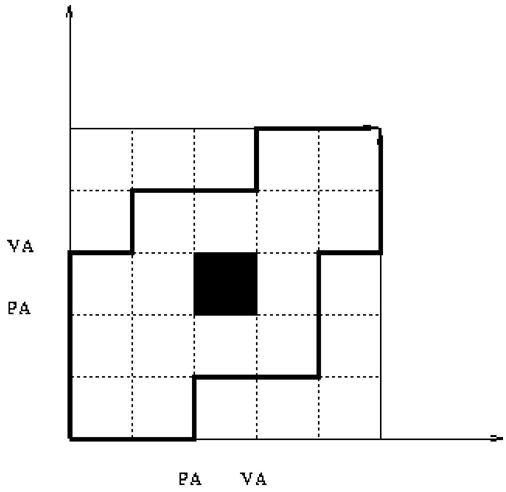


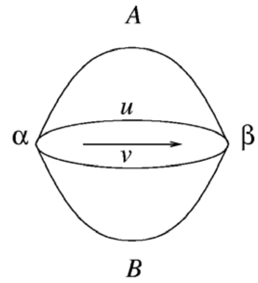
Fig. 6. A 2-dimensional branching area

The evaluation map ev defined by $ev(x) = x(0_{n+1})$ for $x \in \omega Cat(I^{n+1}, \mathcal{C})^\eta$ associates to any such x the label of the interior of x. A 2-simplex of the branching nerve (that is an ω-functor from I^3 to $\mathcal{C}$) is depicted in Figure 5. The "2-simplex part" is described by the dark triangle. The simplicial structure of these two nerves comes from the fact that close to a corner, the intersection of an n-cube by an hyperplane is an $(n-1)$-simplex. Considering the maps Γ^-_{i+1} and Γ^+_{i+1} is not new (cf. the operations Γ_{i+1} in [6] and $\Gamma_{i+1}, \Gamma'_{i+1}$ in [2]). Our notations are adapted to the simplicial structure of Definition 4.3 noticed for the first time in [13].

Figure 6 represents a 2-dimensional branching area. It corresponds to the homology class of the cycle $(A) - (F) + (I)$. One can prove that the cycles $(A, B, C, D) - (E, F, G, H) + (I, J, K, L)$, $(A) - (F, H) + (I, K)$ correspond to the same homology class as $(A) - (F) + (I)$. The exact statement can be found in [11] : it uses the cubical analogue of the globular composition laws $*_n$ of Definition 3.2 and Definition 3.3. It means that negative (resp. positive) corner homology theories describe the branching (resp. merging) areas of execution paths in a HDA. In other terms, the homology class does not depend on a cubification of the HDA. Therefore they correct the main drawback of the homology theories of [14].



(a) Example from distributed database theory



(b) A 3-dimensional hole

Fig. 7. Examples of cycles

4.2 The globular nerve

If $\mathcal{C}$ is an ω-category and if $x \in \omega Cat(\Delta^n, \mathcal{C}[1])$, one can set

$$\begin{aligned} &\epsilon_i(x)(\sigma_0 < \ldots < \sigma_{n+1}) = x(\sigma_0 < \ldots < \widehat{i} < \sigma_k - 1 < \ldots < \sigma_{n+1} - 1) \\ &\partial_i(x)(\sigma'_0 < \ldots < \sigma'_{n-1}) = \\ &\qquad x(\sigma'_0 < \ldots < \sigma'_{k-1} < \sigma'_k + 1 < \ldots < \sigma'_{n-1} + 1) \end{aligned}$$

where $\sigma'_k, \ldots, \sigma'_{n-1} \geq i$ and where $\widehat{i}$ means that i is removed if it appears and that bigger σ_ℓ are replaced by $\sigma_\ell - 1$.

It can be checked that $\epsilon_i(x)$ (resp. $\partial_i(x)$) are ω-functors from Δ^{n+1} (resp. Δ^{n-1}) to $\mathcal{C}[1]$ and $(\omega Cat(\Delta^*, \mathcal{C}[1]), \partial_i, \epsilon_i)$ is a simplicial set which will be called the simplicial nerve of $\mathcal{C}[1]$.

Definition 4.4 [12] The *globular simplicial nerve* $\mathcal{N}^{gl}$ is the functor from ωCat_1 to $Sets_+^{\Delta^{op}}$ defined by $\mathcal{N}^{gl}_n(\mathcal{C}) = \omega Cat(\Delta^n, \mathcal{C}[1])$ for $n \geq 0$ and with $\mathcal{N}^{gl}_{-1}(\mathcal{C}) = \mathcal{C}_0 \times \mathcal{C}_0$, and endowed with the augmentation map ∂_{-1} from $\mathcal{N}^{gl}_0(\mathcal{C}) = \mathcal{C}_1$ to $\mathcal{N}^{gl}_{-1}(\mathcal{C}) = \mathcal{C}_0 \times \mathcal{C}_0$ defined by $\partial_{-1} x = (s_0 x, t_0 x)$.

The evaluation map ev defined for $x \in \omega Cat(\Delta^n, \mathcal{C}[1])$ by

$$ev(x) = x((0 \ldots n))$$

associates to any such x the label of the interior of x. Intuitively, elements of $\mathcal{N}^{gl}_n(\mathcal{C})$ are full $(n+1)$-globes. Figure 8 depicts a 2-simplex in the globular nerve. Indeed, if x is an ω-functor from Δ^n to $\mathcal{C}[1]$ for some n, then the set map from Δ^n to $\mathcal{C}$ given by $(\sigma_0 \ldots \sigma_r) \mapsto s_0 x((\sigma_0 \ldots \sigma_r))$ (resp. $(\sigma_0 \ldots \sigma_r) \mapsto t_0 x((\sigma_0 \ldots \sigma_r)))$ is the constant map because of the globular equations in Defi-

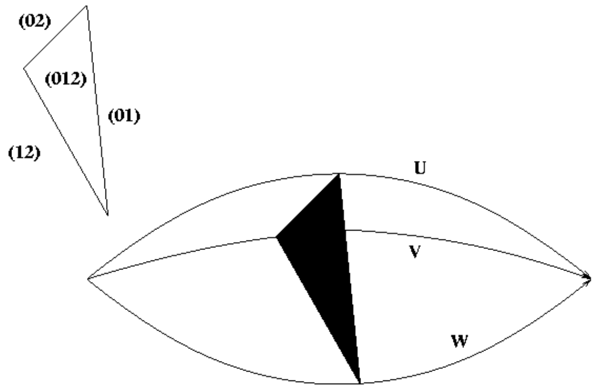


Fig. 8. Globular 2-simplex

nition 3.2. In other terms, the globular nerve contains some information about the simplexes included in the HDA and which are in vertical position.

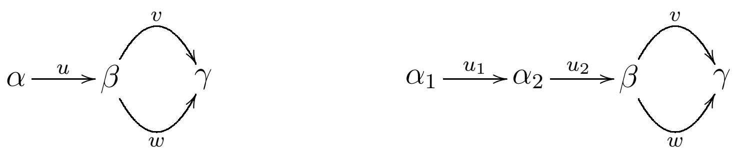
Set

$$H^{gl}_{n+1}(\mathcal{C}) := H_n(\mathcal{N}^{gl}(\mathcal{C}))$$

for $n \geq -1$. This homology theory is called the globular homology. An example of globular cycle of dimension 1 is $\gamma_1 - \gamma_2$ of Figure 7(a) where γ_1 and γ_2 are the execution paths drawn from the point of coordinates $(0,0)$ to the point of coordinates $(5,5)$. We call it an oriented 1-dimensional loop. An example of a globular cycle of dimension 2 is $A - B$ of Figure 7(b) (more precisely, A means here the ω-functor from $\Delta^1 = I^1$ to $\mathcal{C}[1]$ such that the interior is labelled by A and idem for B). Looking back to Figure 1(b), we see that the corresponding first globular group does not vanish : this means in this case that the globular nerve contains information not related to any achronal cuts of the HDA. Of course this property of the globular nerve comes from the fact that the simplicial nerve is augmented.

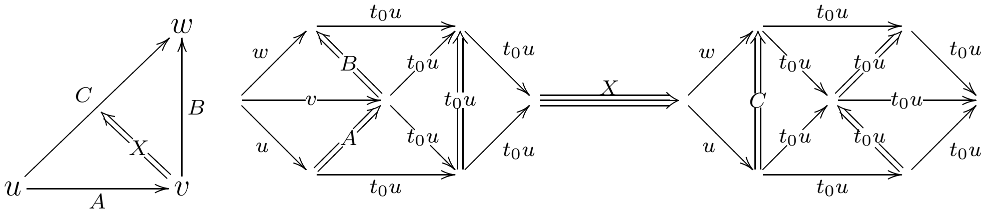
Like for the corner homologies, we can arrive at similar conclusions with the globular homology. Subdividing p-morphisms with $p \geq 2$ (Figure 3(a) can be seen as the subdivision of a 2-morphism in two parts A and B) in a HDA leaves the globular homology unchanged.

As for the subdivision of 1-morphisms, one can prove that subdivisions of indecomposable 1-morphisms leave both corner homology theories unchanged. This is not the case for the globular homology. Indeed if both corner homologies of HDAs of Figure 9(a) and Figure 9(b) are equal (to $\mathbb{Z}$), this property fails for the globular homology : the first globular homology group of Figure 9(a) is equal to $\mathbb{Z}^{\oplus 2}$ (the free abelian group generated by $v - w$, $u *_0 v - u *_0 w$) and the first globular homology group of Figure 9(b) is equal to $\mathbb{Z}^{\oplus 3}$ (the free abelian group generated by $v - w$, $u_2 *_0 v - u_2 *_0 w$, $u_1 *_0 u_2 *_0 v - u_2 *_0 w$).



(a) $\mathcal{C}$ (b) Subdivision of u in $\mathcal{C}$

Fig. 9. Example of T-deformation



(a) A 2-globular simplex $\widetilde{X}$

(b) The 2-simplex $h^-(\widetilde{X})$

Fig. 10. Illustration of h^-

4.3 The morphisms from the globular to the corner nerves

Both morphisms of simplicial sets h^η from the globular nerve to the corner nerves arise from the canonical inclusion map from $\mathcal{N}^{gl}(\mathcal{C})$ to $\mathcal{N}^\eta(\mathcal{C})$. They can be characterized by the following statement :

Theorem 4.5 [12] *Let $\eta \in \{-,+\}$. There exists one and only one natural transformation h^η from $\mathcal{N}^{gl}$ to $\mathcal{N}^\eta$ such that $ev \circ h^\eta = ev$.*

Take a globular 2-simplex as in Figure 8. It can be mapped to an ω-functor from I^3 to $\mathcal{C}$ as in Figure 5 by labelling the faces of I^3 with only 0 and $-$ in their description with the corresponding label of the original figure and by labelling all other faces of I^3 by the common 0-target of each label of the globular 2-simplex. For example Figure 10(b) represents the image of the globular 2-simplex of Figure 10(a) by h^- (see the convention of labelling in Figure 3(b) and Figure 3(c)) : notice that $t_0u = t_0v = t_0w = t_0A = t_0B = t_0C = t_0X$.

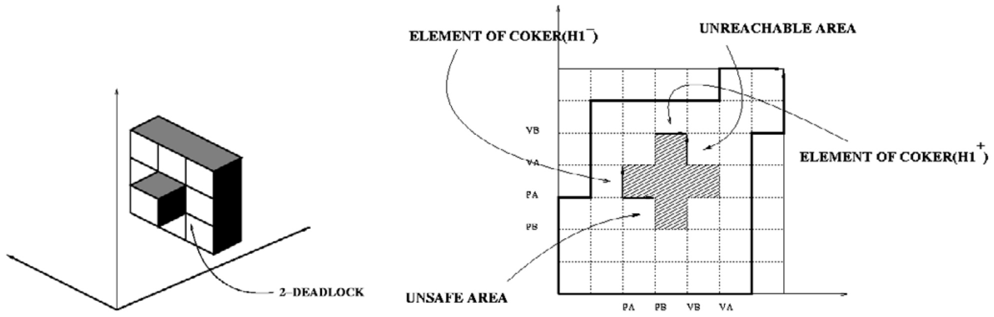
In homology , the morphism h^- (resp. h^+) associates to any n-globe its corresponding n-dimensional branching (resp. merging) area of execution paths.

We already explained in [13] the link between the simplicial homology of the cone of h^- (resp. h^+) with the n-dimensional deadlocks (resp. the unreachable n-morphisms) for some classes of HDAs.

Indeed the morphisms h^- and h^+ associate in homology to any oriented loop of any dimension its corresponding negative or positive corners. We can

$$\begin{tikzcd} & \mathcal{N}^{gl}(\mathcal{C}) \arrow[dl, "h^-"'] \arrow[dr, "h^+"] & \\ \mathcal{N}^-(\mathcal{C}) & & \mathcal{N}^+(\mathcal{C}) \end{tikzcd}$$

Fig. 11. The fundamental diagram



(a) Example of 2-deadlock

(b) Unsafe area and unreachable area in a concurrent machine with semaphores

Fig. 12. Applications in computer science

immediately see an application of these maps. Looking back to the Swiss Flag example of Figure 1(c), it is clear that the cokernel of h_1^- does not vanish, because of the deadlock and the unsafe area. A negative corner which yields a non trivial element in this cokernel is drawn in Figure 12(b). In the same way, the cokernel of h_1^+ in the Swiss Flag example still does not vanish, because of the unreachable state and the unreachable area. A positive corner which yields a non zero element of this cokernel is represented in Figure 12(b).

In Figure 12(a), execution paths are supposed to be the continuous maps γ from $[0,1]$ to the complement of the depicted obstacle such that the composite $\pi_x \circ \gamma$, $\pi_y \circ \gamma$ and $\pi_z \circ \gamma$ are non-decreasing maps from $[0,1]$ to $\mathbb{R}$, where π_x, π_y and π_z are the projections on the axes. One sees a non-trivial element of the cokernel of h_2^- which detects the presence of the 2-deadlock.

5 Deforming HDA : some speculations and perspectives

Now we would like to speculate about what should be a "good" invariant of HDA. An ω-category seen as a HDA can be deformed in different ways : by deforming p-morphisms with $p \geq 2$ (spatial deformations or S-deformations), by concatening or subdividing 1-morphisms (temporal deformations or T-deformations), i.e. execution paths in the corresponding automaton. Table 1 explains the behavior of the objects introduced in this paper with respect to these deformations. About the "almost" : the deformation of a p-morphism u for $p \geq 2$ corresponds in the globular nerve to a simplicial deformation of

Deformation type	S-invariant	T-invariant
Globular nerve	yes	no
Corner nerves	almost	no
Globular homology	yes	no
Corner homologies	yes	yes

Table 1
Behavior of the constructions w.r.t. deformations of HDA

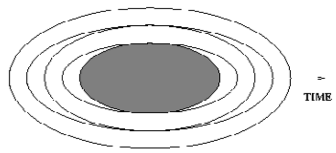


Fig. 13. Thick 1-dimensional oriented globe

any simplex x such that the image of x contains u ; the same deformation corresponds to a simplicial deformation of an n-simplex y of, for example, the negative corner nerve only if the label u belongs to the negative part of y. In the branching nerve, the positive part of an ω-functor from I^{n+1} to $\mathcal{C}$ is a "dead part".

The non-invariance with respect to T-deformations is also an obstacle to find appropriate invariants for local pospaces [10]. Consider for instance the HDA of Figure 9(a). By thickening the 1-morphisms as in Figure 13, one obtains a HDA such that its first globular homology group is an infinite sum of copies of $\mathbb{Z}$. To understand this fact, let us come back to Figures 9. The more subdivisions in u there are , the bigger the rank of the first globular homology group is. At the limit with the real line, one obtains an infinite number of globular 1-cycles. It is therefore necessary to make supplemental identifications within the nerves in order to get "good" invariants. Here is now a fundamental fact :

Claim 5.1 *Suppose that $\mathcal{C}$ is an object of ωCat_1 such that its 1-morphisms are never invertible (or equivalently if x and y are two 1-morphisms, $x *_0 y$ is 1-dimensional if the expression makes sense). For $n \geq 0$ and $x \in \mathcal{N}^{gl}_n(\mathcal{C})$, let $S(x) := s_0 ev(x)$ and $T(x) := t_0 ev(x)$. Suppose that for another $y \in \mathcal{N}^{gl}_n(\mathcal{C})$, $T(x) = S(y)$. Let $x * y$ be the map from Δ^n to $\mathcal{C}[1]$ defined by*

$$(x * y)((\sigma_0 \dots \sigma_r)) := x((\sigma_0 \dots \sigma_r)) *_0 y((\sigma_0 \dots \sigma_r)).$$

Then :

(i) *For $n \geq 0$, $(\mathcal{N}^{gl}_n(\mathcal{C}) \cup \mathcal{C}_0, S, T, *)$ is a 1-category by considering elements of $\mathcal{C}_0$ as 0-morphisms. In particular $x * y$ is an ω-functor for $n \geq 0$. In this 1-category, 1-morphisms are still never invertible and therefore it can be seen as a small category by adding 1-dimensional identities.*

(ii) *With the obvious structure of a small category on* $(\mathcal{C}_0 \times \mathcal{C}_0, \mathcal{C}_0, S, T, *)$ *defined by setting* $S(u,v) := u$, $T(u,v) := v$ *and* $(u,v) * (v,w) := (u,w)$, *then the globular nerve becomes an augmented simplicial object in the category of small categories.*

Intuitively, for $n \geq 0$, the 1-category $(\mathcal{N}^{gl}_n(\mathcal{C}) \cup \mathcal{C}_0, S, T, *)$ represents the temporal structure of the $(n+1)$-dimensional paths. Using the canonical inclusion of the globular nerve in both corner nerves, both corner nerves can be endowed with a structure of augmented simplicial object in the category of 1-category with source and target maps only partially defined.

Let $\overrightarrow{N}$ be a functor from the category of small categories to any good category whose simplicial objects have satisfactory properties with respect to the usual structures in algebraic topology (for example the usual category of simplicial sets up to homotopy). We have a wide degree of freedom for the choice of this good category. Indeed Baues's book [4] explains that the main theorems of algebraic topology (as the Hurewicz theorem or some Whitehead theorems) can be recovered from the axiomatic theory of cofibration categories. Suppose that any subdivision of a morphism in two morphisms (as for example the canonical functor from the HDA of Figure 9(a) to that of Figure 9(b) such that $u \mapsto u_1 *_0 u_2$ and being the identity map elsewhere) induces an homotopy equivalence by $\overrightarrow{N}$. Then the composite functor $\overrightarrow{N} \circ \mathcal{N}^{gl}$ would be S-invariant because of $\mathcal{N}^{gl}$ and T-invariant because of $\overrightarrow{N}$. So far only corner homologies are invariant by all possible types of deformations of HDA. Street's simplicial nerve [26] of 1-categories (our 1-categories have no invertible 1-morphisms and therefore they can be viewed as small category by adding identity 1-dimensional elements) and the classifying space of small categories (see for example [22] for further details) satisfy this property. Unfortunately, the total homology of the bisimplicial set associated to the ω-category of Figure 9(a) is equal to $\mathbb{Z}$ in both cases, and not to $\mathbb{Z} \oplus \mathbb{Z}$ as required by the fact that one does not want the 1-morphisms of a given ω-category to be contracted in the same homotopy class. So either the construction of another nerve of small categories will be required, or the distinction between the case of Figure 9(a) and the same case with no calculation before the branching (i.e. the 1-morphism u contracted) will be made by the homotopical side of the theory.

6 Concluding remarks

Throughout this paper, we have presented a new approach of the topology of HDA based upon the introduction of three new simplicial nerves and two natural morphisms of simplicial sets for any HDA. It opens the perspective of deeply relating the geometry of HDA with homological algebra and other usual tools developed for algebraic topology. The construction has the following properties:

(i) Every orthogonal (or anti-diagonal) cut of a HDA behaves like a true topological space.

(ii) There are four fundamental types of cuts : close to branching or merging areas of execution paths, in the middle of the globes, and all cuts.

(iii) The three first types of cuts give rise to three new simplicial sets, and two morphisms of simplicial sets ; the last one is the direct limit of the diagram depicted in Figure 11.

There are two types of deformations of HDA (T and S). As shown in the above table, most of the functors introduced here are S-invariant but not T-invariant. The T-invariance could be related to finding a new nerve of small categories (with the source and target maps not necessarily defined everywhere).

Getting three T-invariant and S-invariant nerves would enable us to define the notion of homotopy equivalent HDAs as follows : the maps f from a HDA X to a HDA Y and g from Y to X would be reciprocal homotopy equivalences of HDA if and only if $\mathcal{N}^{gl}(f)$ and $\mathcal{N}^{gl}(g)$ (resp . $\mathcal{N}^{-}(f)$ and $\mathcal{N}^{-}(g)$, $\mathcal{N}^{+}(f)$ and $\mathcal{N}^{+}(g)$) were reciprocal homotopy equivalences of simplicial sets. Similar constructions for local pospaces would enable us to make precise the following idea : up to homotopy of HDA, the category of local pospaces (at least a sub-category which would be the analogue of CW-complexes in this theory), the category of cubical sets satisfying some Kan conditions, and the category of non-1-contracting ω-categories with some additional technical conditions (for the three simplicial nerves to be Kan) should be equivalent. A similar equivalence in the framework of usual algebraic topology is proved in [18] between CW-complexes and weak ω-groupoids modulo weak equivalences.

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