CLOSED SYMMETRIC MONOIDAL STRUCTURE AND FLOW

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ABSTRACT. The category of flows is not cartesian closed. We construct a closed symmetric monoidal structure which has moreover a satisfactory behavior from the computer scientific viewpoint.

CONTENTS

1. INTRODUCTION

The category of flows was introduced in [Gau03a] as a convenient framework for the study of higher dimensional automata up to homotopy. However the category of flows is not cartesian closed (cf. Proposition 5.1). Moreover, the categorical product of flows is badly behaved from a computer scientific viewpoint for very simple reasons. Indeed, if X and Y are two flows with non empty path spaces $\mathbb{P}X$ and $\mathbb{P}Y$, then the path space $\mathbb{P}(X \times Y)$ is isomorphic to $\mathbb{P}X \times \mathbb{P}Y$. In other terms, if γ is a non-constant execution path of X and if $\alpha \in Y^0$ is a state of Y, then (γ, α) does not correspond to any non-constant execution path of $X \times Y$. This problem disappears by considering the tensor product $X \otimes Y$ (cf. Proposition 5.2) since the path space $\mathbb{P}(X \otimes Y)$ is exactly $(\mathbb{P}X \times Y^0) \sqcup (X^0 \times \mathbb{P}Y) \sqcup (\mathbb{P}X \times \mathbb{P}Y)$ with the composition law that one expects to find. And it turns out that this new symmetric monoidal structure is closed.

Section 3 is a short reminder about flows. Section 4 recalls the definition of a *non-contracting topological 1-category* and the construction of the closed monoidal structure $\otimes$ made in [Gau03a]. Section 5 shows that the tensor product of flows is actually a closed monoidal structure, using Section 4. Section 6 provides a more explicit way of establishing the closedness of the tensor product of flows. This new method does not seem to be applicable to the case of non-contracting topological 1-categories. Section 7 proves a negative

result: this new closed monoidal structure together with the model structure constructed in [Gau03a] does not provide the category of flows a structure of monoidal model category.

2. Warning

This paper belongs to a set of papers corresponding to the content of "A Convenient Category for the Homotopy Theory of Concurrency" [Gau]. Indeed, the latter paper does not seem to be publishable because of its length. A detailed abstract (in French) of [Gau] can be found in [Gau03b] and [Gau03c].

3. Reminder about the category of flows

3.1. **Compactly generated topological space.** The category **Top** of compactly generated topological spaces is complete, cocomplete and cartesian closed (more details for this kind of topological spaces in [Bro88, May99], the appendix of [Lew78] and also the preliminaries of [Gau03a]). Let us denote by $\mathbf{TOP}(X, -)$ the right adjoint of the functor $- \times X : \mathbf{Top} \longrightarrow \mathbf{Top}$. For the sequel, any topological space will be supposed to be compactly generated.

Let $n \geqslant 1$. Let $\mathbf{D}^n$ be the closed n-dimensional disk. Let $\mathbf{S}^{n-1} = \partial \mathbf{D}^n$ be the boundary of $\mathbf{D}^n$ for $n \geqslant 1$. Notice that $\mathbf{S}^0$ is the discrete two-point topological space $\{-1, +1\}$. Let $\mathbf{D}^0$ be the one-point topological space. Let $\mathbf{S}^{-1} = \varnothing$ be the empty set.

3.2. **Definition of a flow.**

Definition 3.1. *A* flow *X consists of a topological space $\mathbb{P}X$, a discrete space X^0, two continuous maps s and t from $\mathbb{P}X$ to X^0 and a continuous and associative map $* : \{(x, y) \in \mathbb{P}X \times \mathbb{P}X ; t(x) = s(y)\} \longrightarrow \mathbb{P}X$ such that $s(x * y) = s(x)$ and $t(x * y) = t(y)$. A morphism of flows $f : X \longrightarrow Y$ consists of a set map $f^0 : X^0 \longrightarrow Y^0$ together with a continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ such that $f(s(x)) = s(f(x))$, $f(t(x)) = t(f(x))$ and $f(x * y) = f(x) * f(y)$. The corresponding category will be denoted by* **Flow**.

The continuous map $s : \mathbb{P}X \to X^0$ is called the *source map*. The continuous map $t : \mathbb{P}X \to X^0$ is called the *target map*. One can canonically extend these two maps to the whole underlying topological space $X^0 \sqcup \mathbb{P}X$ of X by setting $s(x) = x$ and $t(x) = x$ for $x \in X^0$.

The topological space X^0 is called the *0-skeleton* of X. The 0-dimensional elements of X are also called *states* or *constant execution paths*.

The elements of $\mathbb{P}X$ are called *non constant execution paths*. If γ_1 and γ_2 are two non-constant execution paths, then $\gamma_1 * \gamma_2$ is called the concatenation or the composition of γ_1 and γ_2. For $\gamma \in \mathbb{P}X$, $s(\gamma)$ is called the *beginning* of γ and $t(\gamma)$ the *ending* of γ.

Notation 3.2. *For $\alpha, \beta \in X^0$, let $\mathbb{P}_{\alpha,\beta}X$ be the subspace of $\mathbb{P}X$ equipped the Kelleyfication of the relative topology consisting of the non-execution paths of X with beginning α and with ending β.*

Notation 3.3. *An element $x \in X^0$ is achronal if $x \notin s(\mathbb{P}X) \cup t(\mathbb{P}X)$. The subspace of achronal points of X^0 is denoted by $Ach(X)$. The image of s is denoted by $\mathbb{P}_0 X$ and the image of t by $\mathbb{P}_1 X$. Therefore $Ach(X) = X^0 \backslash (\mathbb{P}_0 X \cup \mathbb{P}_1 X)$.*

Definition 3.4. *For X a flow, a point α of X^0 such that there are no non-constant execution paths γ such that $t(\gamma) = \alpha$ (resp. $s(\gamma) = \alpha$) is called* initial state *(resp.* final state*).*

Definition 3.5. [Gau03a] *Let Z be a topological space. Then the* globe *of Z is the flow* $\mathrm{Glob}(Z)$ *defined as follows:* $\mathrm{Glob}(Z)^0 = \{0, 1\}$, $\mathbb{P}\mathrm{Glob}(Z) = Z$, $s = 0$, $t = 1$ *and the composition law is trivial.*

Definition 3.6. [Gau03a] *The* directed segment $\overrightarrow{I}$ *is the flow defined as follows:* $\overrightarrow{I}^0 = \{0, 1\}$, $\mathbb{P}\overrightarrow{I} = \{[0, 1]\}$, $s = 0$ *and* $t = 1$.

Notation 3.7. *The space* $\mathbf{FLOW}(X, Y)$ *is the set* $\mathbf{Flow}(X, Y)$ *equipped with the Kelleyfication of the compact-open topology.*

Theorem 3.8. [Gau03a] *The category* **Flow** *is complete and cocomplete. In particular, a terminal object is the flow* $\mathbf{1}$ *having the discrete set* $\{0, u\}$ *as underlying topological space with 0-skeleton* $\{0\}$ *and path space* $\{u\}$. *And a initial object is the unique flow* $\varnothing$ *having the empty set as underlying topological space.*

Definition 3.9. [Gau03a] *A morphism of flows* $f : X \to Y$ *is said* synchronized *if and only if it induces a bijection of sets between the 0-skeleton of X and the 0-skeleton of Y.*

4. Reminder about non-contracting topological 1-categories

Definition 4.1. [Gau03a] *A non-contracting topological 1-category X is a pair of compactly generated topological spaces $(X^0, \mathbb{P}X)$ together with continuous maps s, t and $*$ satisfying the same properties as in the definition of flow except that X^0 is not necessarily discrete. The corresponding category is denoted by* $\mathbf{1Cat}_1^{\mathbf{top}}$.

Theorem 4.2. [Gau03a] *The category* $\mathbf{1Cat}_1^{\mathbf{top}}$ *is complete and cocomplete. The inclusion functor* $\widetilde{\omega} : \mathbf{Flow} \to \mathbf{1Cat}_1^{\mathbf{top}}$ *preserves finite limits.*

Theorem 4.3. [Gau03a] *The inclusion functor* $\widetilde{\omega} : \mathbf{Flow} \to \mathbf{1Cat}_1^{\mathbf{top}}$ *has a right adjoint that will be denoted by* $\widetilde{D}$. *In particular, this implies that the canonical inclusion functor* $\mathbf{Flow} \to \mathbf{1Cat}_1^{\mathbf{top}}$ *preserves colimits. Moreover, one has* $\widetilde{D} \circ \widetilde{\omega} = Id_{\mathbf{Flow}}$ *and*

$$\varprojlim_i X_i \cong \varprojlim_i \widetilde{D} \circ \widetilde{\omega}(X_i) \cong \widetilde{D}\left(\varprojlim_i \widetilde{\omega}(X_i)\right).$$

Proposition 4.4. [Gau03a] *Let X and Y be two objects of* $\mathbf{1Cat}_1^{\mathbf{top}}$. *There exists a unique structure of topological 1-category $X \otimes Y$ on the topological space $X \times Y$ such that*

(1) $(X \otimes Y)^0 = X^0 \times Y^0$.
(2) $\mathbb{P}(X \otimes Y) = (\mathbb{P}X \times \mathbb{P}X) \sqcup (X^0 \times \mathbb{P}Y) \sqcup (\mathbb{P}X \times Y^0)$.
(3) $s(x, y) = (sx, sy)$, $t(x, y) = (tx, ty)$, $(x, y) * (z, t) = (x * z, y * t)$.

Theorem 4.5. [Gau03a] *The tensor product of* $\mathbf{1Cat}_1^{\mathbf{top}}$ *is a closed symmetric monoidal structure, that is there exists a bifunctor*

$$[\mathbf{1Cat}_1^{\mathbf{top}}] : \mathbf{1Cat}_1^{\mathbf{top}} \times \mathbf{1Cat}_1^{\mathbf{top}} \longrightarrow \mathbf{1Cat}_1^{\mathbf{top}}$$

contravariant with respect to the first argument and covariant with respect to the second argument such that one has the natural isomorphism of sets

$$\mathbf{1Cat}_1^{\mathbf{top}}(X \otimes Y, Z) \cong \mathbf{1Cat}_1^{\mathbf{top}}\left(X, [\mathbf{1Cat}_1^{\mathbf{top}}](Y, Z)\right)$$

for any topological 1-categories X, Y and Z. Moreover, one has the natural homeomorphism

$$\left([\mathbf{1Cat}_1^{\mathbf{top}}](Y, Z)\right)^0 \cong \mathbf{FLOW}(Y, Z).$$

5. Tensor product of flows

Proposition 5.1. [Gau03a] *The category of flows* **Flow** *(as well as the category of non-contracting topological 1-categories* $\mathbf{1Cat_1^{top}}$*) is not cartesian closed.*

Proof. We recall here the proof given in [Gau03a]. If **Flow** was cartesian closed, then its product would commute with colimit. This latter property fails as we can see with the following example. Let 2_0 be the flow consisting of one achronal point $*$. Consider the flows $\overrightarrow{I}_v$ and $\overrightarrow{I}_w$ such that $\overrightarrow{I}_v \cong \overrightarrow{I}$, $\mathbb{P}\overrightarrow{I}_v = \{v\}$, $\overrightarrow{I}_w \cong \overrightarrow{I}$, $\mathbb{P}\overrightarrow{I}_v = \{w\}$. Consider the diagram of **Flow**

$$\overrightarrow{I}_v \xleftarrow{\ i_1\ } 2_0 \xrightarrow{\ i_0\ } \overrightarrow{I}_w$$

where $i_0(*) = 0$ and $i_1(*) = 1$. Then the colimit $\overrightarrow{I}_v.\overrightarrow{I}_w$ of this diagram is the flow representing the concatenation of v and w. For any flow X, $\mathbb{P}\left(\overrightarrow{I} \times X\right) \cong \mathbb{P}X$ as sets and the paths of $\overrightarrow{I} \times X$ are never composable because $s\left(\mathbb{P}\left(\overrightarrow{I} \times X\right)\right) \subset \{0\} \times X$ and $t\left(\mathbb{P}\left(\overrightarrow{I} \times X\right)\right) \subset \{1\} \times X$. Moreover, the flow $\overrightarrow{I} \times 2_0$ is the achronal flow $\{(0,*),(1,*)\}$. Therefore the path space of the colimit of

$$\overrightarrow{I} \times \overrightarrow{I}_v \xleftarrow{\ i_1\ } \overrightarrow{I} \times 2_0 \xrightarrow{\ i_0\ } \overrightarrow{I} \times \overrightarrow{I}_w$$

consists exactly of the two non-composable non-constant execution paths $([0,1],v)$ and $([0,1],w)$. On the contrary, the path space of $\overrightarrow{I} \times \left(\overrightarrow{I}_v.\overrightarrow{I}_w\right)$ consists exactly of the three non-composable non-constant execution paths $([0,1],v)$, $([0,1],w)$ and $([0,1],v*w)$. $\qquad\square$

Proposition 5.2. *Let X and Y be two flows. There exists a unique structure of flows $X \otimes Y$ on the set $X \times Y$ such that*

(1) $(X \otimes Y)^0 = X^0 \times Y^0$
(2) $\mathbb{P}(X \otimes Y) = (\mathbb{P}X \times \mathbb{P}Y) \cup \left(X^0 \times \mathbb{P}Y\right) \cup \left(\mathbb{P}X \times Y^0\right)$
(3) $s(x,y) = (sx,sy)$, $t(x,y) = (tx,ty)$, $(x,y)*(z,t) = (x*z,y*t)$.

Moreover one has $Ach(X \otimes Y) = Ach(X) \times Ach(Y)$ and $\widetilde{\omega}(X \otimes Y) = \widetilde{\omega}(X) \otimes \widetilde{\omega}(Y)$.

Proof. The first part of the statement is clear and is analogue to Proposition 4.4. By definition, the following equality holds:

$$\mathbb{P}_0(X \otimes Y) = \left(X^0 \times \mathbb{P}_0 Y\right) \cup \left(\mathbb{P}_0 X \times Y^0\right) \cup (\mathbb{P}_0 X \times \mathbb{P}_0 Y)$$
$$\mathbb{P}_1(X \otimes Y) = \left(X^0 \times \mathbb{P}_1 Y\right) \cup \left(\mathbb{P}_1 X \times Y^0\right) \cup (\mathbb{P}_1 X \times \mathbb{P}_1 Y)$$

Therefore

$$
\begin{aligned}
\left(X^0 \times Y^0\right) \backslash \mathbb{P}_0(X \otimes Y) &= \left(X^0 \times \left(Y^0 \backslash \mathbb{P}_0 Y\right)\right) \cap \left(\left(X^0 \backslash \mathbb{P}_0 X\right) \times Y^0\right) \\
&\quad \cap \left[\left(X^0 \times \left(Y^0 \backslash \mathbb{P}_0 Y\right)\right) \cup \left(\left(X^0 \backslash \mathbb{P}_0 X\right) \times Y^0\right)\right] \\
&= \left(\left(X^0 \backslash \mathbb{P}_0 X\right) \times \left(Y^0 \backslash \mathbb{P}_0 Y\right)\right) \\
&\quad \cap \left[\left(X^0 \times \left(Y^0 \backslash \mathbb{P}_0 Y\right)\right) \cup \left(\left(X^0 \backslash \mathbb{P}_0 X\right) \times Y^0\right)\right] \\
&= \left(X^0 \backslash \mathbb{P}_0 X\right) \times \left(Y^0 \backslash \mathbb{P}_0 Y\right)
\end{aligned}
$$

So

$$
\begin{aligned}
Ach\left(X \otimes Y\right) &= \left(X^0 \times Y^0\right) \backslash \mathbb{P}_0\left(X \otimes Y\right) \cap \left(X^0 \times Y^0\right) \backslash \mathbb{P}_1\left(X \otimes Y\right) \\
&= \left(X^0 \backslash \mathbb{P}_0 X\right) \times \left(Y^0 \backslash \mathbb{P}_0 Y\right) \cap \left(X^0 \backslash \mathbb{P}_1 X\right) \times \left(Y^0 \backslash \mathbb{P}_1 Y\right) \\
&= Ach\left(X\right) \times Ach\left(Y\right)
\end{aligned}
$$

The equality $\widetilde{\omega}\left(X \otimes Y\right) = \widetilde{\omega}\left(X\right) \otimes \widetilde{\omega}\left(Y\right)$ comes from Proposition 4.4. $\square$

Theorem 5.3. *The tensor product of* **Flow** *is a closed symmetric monoidal structure, that is there exists a bifunctor* [**Flow**] : **Flow** $\times$ **Flow** $\to$ **Flow** *contravariant with respect to the first argument and covariant with respect to the second argument such that one has the natural bijection of sets*

$$
\mathbf{Flow}\left(X \otimes Y, Z\right) \cong \mathbf{Flow}\left(X, [\mathbf{Flow}]\left(Y, Z\right)\right)
$$

for any flows X, Y and Z.

Proof. One has with Theorem 4.3 and Theorem 4.5:

$$
\begin{aligned}
\mathbf{Flow}\left(X \otimes Y, Z\right) &\cong \mathbf{1Cat_1^{top}}\left(\widetilde{\omega}\left(X \otimes Y\right), \widetilde{\omega}\left(Z\right)\right) \\
&\cong \mathbf{1Cat_1^{top}}\left(\widetilde{\omega}\left(X\right) \otimes \widetilde{\omega}\left(Y\right), \widetilde{\omega}\left(Z\right)\right) \\
&\cong \mathbf{1Cat_1^{top}}\left(\widetilde{\omega}\left(X\right), [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}\left(Y\right), \widetilde{\omega}\left(Z\right)\right)\right) \\
&\cong \mathbf{Flow}\left(X, \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}\left(Y\right), \widetilde{\omega}\left(Z\right)\right)\right)
\end{aligned}
$$

so it suffices to set $[\mathbf{Flow}]\left(Y, Z\right) := \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}\left(Y\right), \widetilde{\omega}\left(Z\right)\right)$. $\square$

In particular one has therefore

$$
[\mathbf{Flow}]\left(\varinjlim_i Y_i, Z\right) \cong \varprojlim_i[\mathbf{Flow}]\left(Y_i, Z\right)
$$

and

$$
[\mathbf{Flow}]\left(Y, \varprojlim_i Z_i\right) \cong \varprojlim_i[\mathbf{Flow}]\left(Y, Z_i\right).
$$

The first isomorphism is not surprising because the functor $\widetilde{\omega}$ commutes with colimits by Theorem 4.3. So one has

$$
\begin{aligned}
[\mathbf{Flow}]\left(\varinjlim_i Y_i, Z\right) &\cong \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}\left(\varinjlim_i Y_i\right), \widetilde{\omega}\left(Z\right)\right) \\
&\cong \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\varinjlim_i \widetilde{\omega}\left(Y_i\right), \widetilde{\omega}\left(Z\right)\right) \\
&\cong \widetilde{D} \circ \varprojlim_i[\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}\left(Y_i\right), \widetilde{\omega}\left(Z\right)\right) \\
&\cong \varprojlim_i \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}\left(Y_i\right), \widetilde{\omega}\left(Z\right)\right) \\
&\cong \varprojlim_i[\mathbf{Flow}]\left(Y_i, Z\right)
\end{aligned}
$$

A similar verification does not seem to be possible for the second isomorphism because the functor $\widetilde{\omega}$ does not commute in general with limits ! In fact something slightly more complicated happens.

Proposition 5.4. *One has the natural isomorphism of flows*

$$\widetilde{D} \circ [\mathbf{1Cat_1^{top}}](\widetilde{\omega}(Y), Z) \cong \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}(Y), \widetilde{\omega} \circ \widetilde{D}(Z)\right)$$

Moreover one cannot remove the $\widetilde{D}$ on the left from this isomorphism.

Proof. Indeed

$$\mathbf{Flow}\left(X, \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}(Y), \widetilde{\omega} \circ \widetilde{D}(Z)\right)\right)$$
$$\cong \quad \mathbf{1Cat_1^{top}}\left(\widetilde{\omega}(X), [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}(Y), \widetilde{\omega} \circ \widetilde{D}(Z)\right)\right)$$
$$\cong \quad \mathbf{1Cat_1^{top}}\left(\widetilde{\omega}(X) \otimes \widetilde{\omega}(Y), \widetilde{\omega} \circ \widetilde{D}(Z)\right)$$
$$\cong \quad \mathbf{1Cat_1^{top}}\left(\widetilde{\omega}(X \otimes Y), \widetilde{\omega} \circ \widetilde{D}(Z)\right)$$
$$\cong \quad \mathbf{Flow}\left(X \otimes Y, \widetilde{D}(Z)\right)$$
$$\cong \quad \mathbf{1Cat_1^{top}}(\widetilde{\omega}(X \otimes Y), Z)$$
$$\cong \quad \mathbf{1Cat_1^{top}}(\widetilde{\omega}(X) \otimes \widetilde{\omega}(Y), Z)$$
$$\cong \quad \mathbf{1Cat_1^{top}}\left(\widetilde{\omega}(X), [\mathbf{1Cat_1^{top}}](\widetilde{\omega}(Y), Z)\right)$$
$$\cong \quad \mathbf{Flow}\left(X, \widetilde{D} \circ [\mathbf{1Cat_1^{top}}](\widetilde{\omega}(Y), Z)\right)$$

The conclusion follows by Yoneda's lemma. Now suppose that the isomorphism

$$[\mathbf{1Cat_1^{top}}](\widetilde{\omega}(Y), Z) \cong [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}(Y), \widetilde{\omega} \circ \widetilde{D}(Z)\right)$$

was true. Then one would get, by considering the 0-skeletons of the two members,

$$\mathbf{1CAT_1^{top}}(\widetilde{\omega}(Y), Z) \cong \mathbf{1CAT_1^{top}}\left(\widetilde{\omega}(Y), \widetilde{\omega} \circ \widetilde{D}(Z)\right)$$

for any object Z of $\mathbf{1Cat_1^{top}}$. So one would have the isomorphisms of topological spaces

$$\mathbf{FLOW}\left(Y, \varprojlim_i Z_i\right)$$

$$\cong \mathbf{1CAT_1^{top}}\left(\widetilde{\omega}(Y), \widetilde{\omega}\left(\varprojlim_i Z_i\right)\right)$$

$$\cong \mathbf{1CAT_1^{top}}\left(\widetilde{\omega}(Y), \widetilde{\omega} \circ \widetilde{D} \circ \varprojlim_i \circ \widetilde{\omega}(Z_i)\right) \text{ using Theorem 4.2}$$

$$\cong \mathbf{1CAT_1^{top}}\left(\widetilde{\omega}(Y), \varprojlim_i \circ \widetilde{\omega}(Z_i)\right)$$

$$\cong \left([\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}(Y), \varprojlim_i \circ \widetilde{\omega}(Z_i)\right)\right)^0$$

$$\cong \varprojlim_i \left([\mathbf{1Cat_1^{top}}](\widetilde{\omega}(Y), \widetilde{\omega}(Z_i))\right)^0$$

$$\cong \varprojlim_i \mathbf{1CAT_1^{top}}(\widetilde{\omega}(Y), \widetilde{\omega}(Z_i))$$

$$\cong \varprojlim_i \mathbf{FLOW}(Y, Z_i)$$

which is known to be false in general because the functor $\mathbf{FLOW}(Y, -)$ does not commute with all limits[Gau03a]. □

Now we can make the verification of the second isomorphism. Indeed

$$[\mathbf{Flow}]\left(Y, \varprojlim_i Z_i\right)$$

$$\cong \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}(Y), \widetilde{\omega}\left(\varprojlim_i Z_i\right)\right)$$

$$\cong \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}(Y), \widetilde{\omega} \circ \widetilde{D} \circ \varprojlim_i \circ \widetilde{\omega}(Z_i)\right) \text{ using Theorem 4.2}$$

$$\cong \widetilde{D} \circ [\mathbf{1Cat_1^{top}}]\left(\widetilde{\omega}(Y), \varprojlim_i \widetilde{\omega}(Z_i)\right) \qquad \text{by Proposition 5.4}$$

$$\cong \widetilde{D} \circ \varprojlim_i [\mathbf{1Cat_1^{top}}](\widetilde{\omega}(Y), \widetilde{\omega}(Z_i))$$

$$\cong \varprojlim_i \widetilde{D} \circ [\mathbf{1Cat_1^{top}}](\widetilde{\omega}(Y), \widetilde{\omega}(Z_i))$$

$$\cong \varprojlim_i [\mathbf{Flow}](Y, Z_i).$$

6. Explicit construction of the right adjoint

This section is devoted to proving the above fact in a more explicit way.

If A is a topological space and if Z is a flow, let $[\mathbf{Flow}](\mathrm{Glob}(A), Z)$ be the flow defined as follows:

- The 0-skeleton $[\mathbf{Flow}]\left(\mathrm{Glob}\left(A\right),Z\right)^{0}$ is the set

$$\mathbf{Flow}\left(\mathrm{Glob}\left(A\right),Z\right) \cong \bigsqcup_{(\alpha,\beta)\in Z^0\times Z^0} \mathbf{Top}\left(A,\mathbb{P}_{\alpha,\beta}Z\right)$$

  equipped with the discrete topology.
- The path space $\mathbb{P}[\mathbf{Flow}]\left(\mathrm{Glob}\left(A\right),Z\right)$ is the disjoint sum (in $\mathbf{Top}$)

$$\bigsqcup_{(\alpha,\beta,\gamma,\delta)\in Z^0\times Z^0\times Z^0\times Z^0} (A,Z)_{\alpha,\beta}^{\gamma,\delta}$$

  where $(A,Z)_{\alpha,\beta}^{\gamma,\delta}$ is the pullback

$$
\begin{array}{ccc}
(A,Z)_{\alpha,\beta}^{\gamma,\delta} & \longrightarrow & \mathbf{Top}\left(A,\mathbb{P}_{\alpha,\beta}Z\right)\times\mathbb{P}_{\beta,\delta}Z \\
\downarrow & & \downarrow \\
\mathbb{P}_{\alpha,\gamma}Z\times\mathbf{Top}\left(A,\mathbb{P}_{\gamma,\delta}Z\right) & \longrightarrow & \mathbf{TOP}\left(A,\mathbb{P}_{\alpha,\delta}Z\right)
\end{array}
$$

  where $\mathbf{Top}\left(-,-\right)$ means the discrete topology and $\mathbf{TOP}\left(-,-\right)$ the Kelleyfication of the compact-open topology.
- The source map $(A,Z)_{\alpha,\beta}^{\gamma,\delta} \to [\mathbf{Flow}]\left(\mathrm{Glob}\left(A\right),Z\right)^{0}$ sends an element of $(A,Z)_{\alpha,\beta}^{\gamma,\delta}$ to its projection on $\mathbf{Top}\left(A,\mathbb{P}_{\alpha,\beta}Z\right)$ equipped with the discrete topology.
- The target map $(A,Z)_{\alpha,\beta}^{\gamma,\delta} \to [\mathbf{Flow}]\left(\mathrm{Glob}\left(A\right),Z\right)^{0}$ sends an element of $(A,Z)_{\alpha,\beta}^{\gamma,\delta}$ to its projection on $\mathbf{Top}\left(A,\mathbb{P}_{\gamma,\delta}Z\right)$ equipped with the discrete topology.
- The composition of an element of $(A,Z)_{\alpha,\beta}^{\gamma,\delta}$ with an element of the pullback

$$
\begin{array}{ccc}
(A,Z)_{\gamma,\delta}^{\zeta,\epsilon} & \longrightarrow & \mathbf{Top}\left(A,\mathbb{P}_{\gamma,\delta}Z\right)\times\mathbb{P}_{\delta,\epsilon}Z \\
\downarrow & & \downarrow \\
\mathbb{P}_{\gamma,\zeta}Z\times\mathbf{Top}\left(A,\mathbb{P}_{\zeta,\epsilon}Z\right) & \longrightarrow & \mathbf{TOP}\left(A,\mathbb{P}_{\gamma,\epsilon}Z\right)
\end{array}
$$

  is defined as follows. Consider the pullback P of the diagram

$$
\begin{array}{ccc}
P & \longrightarrow & (A,Z)_{\alpha,\beta}^{\gamma,\delta} \\
\downarrow & & \downarrow \\
(A,Z)_{\gamma,\delta}^{\zeta,\epsilon} & \longrightarrow & \mathbf{Top}\left(A,\mathbb{P}_{\gamma,\delta}Z\right)
\end{array}
$$

  There are canonical continuous maps

$$P \to \mathbf{Top}\left(A,\mathbb{P}_{\alpha,\beta}Z\right)\times\mathbb{P}_{\beta,\delta}Z\times\mathbb{P}_{\delta,\epsilon}Z$$

  and

$$P \to \mathbb{P}_{\alpha,\gamma}Z\times\mathbb{P}_{\gamma,\zeta}Z\times\mathbf{Top}\left(A,\mathbb{P}_{\zeta,\epsilon}Z\right)$$

  giving rise to the commutative diagram

$$
\begin{array}{ccc}
P & \longrightarrow & \mathbf{Top}\left(A,\mathbb{P}_{\alpha,\beta}Z\right)\times\mathbb{P}_{\beta,\epsilon}Z \\
\downarrow & & \downarrow \\
\mathbb{P}_{\alpha,\gamma}Z\times\mathbf{Top}\left(A,\mathbb{P}_{\zeta,\epsilon}Z\right) & \longrightarrow & \mathbf{TOP}\left(A,\mathbb{P}_{\alpha,\epsilon}Z\right)
\end{array}
$$

and therefore to a natural continuous map $P \to (A, Z)_{\alpha,\beta}^{\zeta,\epsilon}$. The latter map yields a natural associative composition law.

Proposition 6.1. *For any topological space A and B, one has the natural bijection of sets*

$$\mathbf{Flow}\left(\operatorname{Glob}(B) \otimes \operatorname{Glob}(A), Z\right) \cong \mathbf{Flow}\left(\operatorname{Glob}(B), [\mathbf{Flow}]\left(\operatorname{Glob}(A), Z\right)\right).$$

Proof. First of all, one has to calculate the tensor product $\operatorname{Glob}(B) \otimes \operatorname{Glob}(A)$. The latter flow can be conventionally represented as follows:

where the vertices are the elements of the 0-skeleton of $\operatorname{Glob}(B) \otimes \operatorname{Glob}(A)$ and where the labels of the arrows between them are the components of the path space of $\operatorname{Glob}(B) \otimes \operatorname{Glob}(A)$. Let $(B, A, Z)_{\alpha,\beta}^{\gamma,\delta}$ be the elements f of

$$\mathbf{Flow}\left(\operatorname{Glob}(B) \otimes \operatorname{Glob}(A), Z\right)$$

such that $f(0,0) = \alpha$, $f(0,1) = \beta$, $f(1,0) = \gamma$ and $f(1,1) = \delta$. It is helpful to notice that the locations of α, β, γ and δ in the expression $(B, A, Z)_{\alpha,\beta}^{\gamma,\delta}$ correspond to the locations of $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ in the above diagram. Then by definition of a morphism of flows, one has the pullback

so one has the pullback

In the other hand, the set $\mathbf{Flow}\left(\operatorname{Glob}(B), [\mathbf{Flow}]\left(\operatorname{Glob}(A), Z\right)\right)$ is in natural bijection with

$$\bigsqcup_{u,v \in \mathbf{Flow}(\operatorname{Glob}(A), Z)} \mathbf{Top}\left(B, \mathbb{P}_{u,v}[\mathbf{Flow}]\left(\operatorname{Glob}(A), Z\right)\right)$$

A pair $(u, v) \in \mathbf{Flow}\left(\operatorname{Glob}(A), Z\right) \times \mathbf{Flow}\left(\operatorname{Glob}(A), Z\right)$ is determined by $u(0) = \alpha$, $u(1) = \beta$, $v(0) = \gamma$, $v(1) = \delta$ and by the continuous map $\mathbb{P}u \in \mathbf{Top}\left(A, \mathbb{P}_{\alpha,\beta}Z\right)$ and $\mathbb{P}v \in \mathbf{Top}\left(A, \mathbb{P}_{\gamma,\delta}Z\right)$. Then one has a natural bijection of sets between

$$\mathbf{Flow}\left(\operatorname{Glob}(B), [\mathbf{Flow}]\left(\operatorname{Glob}(A), Z\right)\right)$$

and the disjoint sum over

$$(\alpha, \beta, \gamma, \delta, m, n) \in Z^0 \times Z^0 \times Z^0 \times Z^0 \times \mathbf{Top}\left(A, \mathbb{P}_{\alpha,\beta}Z\right) \times \mathbf{Top}\left(A, \mathbb{P}_{\gamma,\delta}Z\right)$$

of elements $f \in \mathbf{Top}\left(B, (A, Z)_{\alpha,\beta}^{\gamma,\delta}\right)$ such that such that the composite of f with the canonical projection map $(A, Z)_{\alpha,\beta}^{\gamma,\delta} \to \mathbf{Top}\left(A, \mathbb{P}_{\alpha,\beta}Z\right)$ is the constant map m and such that the composite of f with the canonical projection map $(A, Z)_{\alpha,\beta}^{\gamma,\delta} \to \mathbf{Top}\left(A, \mathbb{P}_{\gamma,\delta}Z\right)$ is the constant map n hence the result. $\square$

We need to recall the following theorems for the sequel:

Theorem 6.2. [Gau03a] *Any flow is the colimit in* **Flow** *of points and globes in a canonical way, i.e. there exists for any flow X a diagram $\mathbb{D}(X)$ of flows containing only points, globes and concatenations of globes such that the mapping $X \mapsto \mathbb{D}(X)$ is functorial and such that $X \cong \varinjlim \mathbb{D}(X)$ in a canonical way.*

Corollary 6.3. [Gau03a] *Let $P(X)$ be a statement depending on a flow X and satisfying the following property: if $D : \mathcal{I} \to$ **Flow** is a diagram of flows such that for any object i of $\mathcal{I}$, $P(D(i))$ holds, then $P\left(\varinjlim D\right)$ holds. Then the following assertions are equivalent:*

(i) *The statement $P(X)$ holds for any flow X of* **Flow**.
(ii) *The statements $P(\{*\})$ and $P(\mathrm{Glob}(Z))$ hold for any object Z of* **Top**.

Theorem 6.4. *Let Y be a flow. Then the functor $- \otimes Y :$ **Flow** $\to$ **Flow** has a right adjoint denoted by $[\mathbf{Flow}](Y, -)$. So the tensor product of flows is a closed symmetric monoidal structure. In particular one has the natural isomorphisms of flows*

$$[\mathbf{Flow}]\left(\varinjlim_i Y_i, Z\right) \cong \varprojlim_i [\mathbf{Flow}](Y_i, Z)$$

and

$$[\mathbf{Flow}]\left(Y, \varprojlim_i Z\right) \cong [\mathbf{Flow}](Y_i, Z_i).$$

Proof. Let $[\mathbf{Flow}](\{*\}, Z) := Z$ so that for any flow X one has

$$\mathbf{Flow}(X \otimes \{*\}, Z) \cong \mathbf{Flow}(X, Z)$$

and let $[\mathbf{Flow}](\mathrm{Glob}(A), Z)$ be defined as above for any flow Z. Using Theorem 6.2, let $Y = \varinjlim \mathbb{D}(Y)$. Then let

$$[\mathbf{Flow}](Y, Z) := \varprojlim_i [\mathbf{Flow}](\mathbb{D}(Y)(i), Z)$$

Then $[\mathbf{Flow}](Y, Z)^0 \cong \varprojlim_i [\mathbf{Flow}](\mathbb{D}(Y)(i), Z)^0$ as set therefore the following natural bijections of sets hold

$$[\mathbf{Flow}](Y, Z)^0 \cong \varprojlim_i \mathbf{Flow}(\mathbb{D}(Y)(i), Z) \cong \mathbf{Flow}(Y, Z).$$

Then for any topological space B, one has

$$
\begin{aligned}
\mathbf{Flow}(\mathrm{Glob}(B) \otimes Y, Z) \quad &\cong \quad \mathbf{Flow}\left(\mathrm{Glob}(B) \otimes \left(\varinjlim \mathbb{D}(Y)\right), Z\right) \\
&\cong \quad \varprojlim \mathbf{Flow}(\mathrm{Glob}(B) \otimes \mathbb{D}(Y), Z) \\
&\cong \quad \varprojlim \mathbf{Flow}(\mathrm{Glob}(B), [\mathbf{Flow}](\mathbb{D}(Y), Z)) \\
&\cong \quad \mathbf{Flow}\left(\mathrm{Glob}(B), \varprojlim[\mathbf{Flow}](\mathbb{D}(Y), Z)\right) \\
&\cong \quad \mathbf{Flow}(\mathrm{Glob}(B), [\mathbf{Flow}](Y, Z))
\end{aligned}
$$

Moreover one has

$$\mathbf{Flow}\left(\{*\}\otimes Y,Z\right)\cong\mathbf{Flow}\left(Y,Z\right)\cong[\mathbf{Flow}]\left(Y,Z\right)^{0}\cong\mathbf{Flow}\left(\{*\},[\mathbf{Flow}]\left(Y,Z\right)\right)$$

so the natural isomorphism $\mathbf{Flow}\left(X\otimes Y,Z\right)\cong\mathbf{Flow}\left(X,[\mathbf{Flow}]\left(Y,Z\right)\right)$ holds if X is a point or a globe. We could be tempted to concluding that the proof is complete using Corollary 6.3. However this would not be correct because we do not know yet that $\varprojlim_{i}[\mathbf{Flow}]\left(Y_{i},Z\right)\cong[\mathbf{Flow}]\left(\varinjlim Y_{i},Z\right)$ for any diagram of flows $i\mapsto Y_{i}$! Let B_{1} and B_{2} be two topological spaces. Then the set $\mathbf{Flow}\left(\left(\mathrm{Glob}\left(B_{1}\right)*\mathrm{Glob}\left(B_{2}\right)\right)\otimes Y,Z\right)$ is the pullback

$$
\begin{array}{ccc}
\mathbf{Flow}\left(\left(\mathrm{Glob}\left(B_{1}\right)*\mathrm{Glob}\left(B_{2}\right)\right)\otimes Y,Z\right) & \longrightarrow & \mathbf{Flow}\left(\mathrm{Glob}\left(B_{2}\right)\otimes Y,Z\right) \\
\downarrow & & \downarrow \\
\mathbf{Flow}\left(\mathrm{Glob}\left(B_{1}\right)\otimes Y,Z\right) & \longrightarrow & \mathbf{Flow}\left(\{*\}\otimes Y,Z\right)
\end{array}
$$

since the tensor product of flows commutes with colimits. So the set

$$\mathbf{Flow}\left(\left(\mathrm{Glob}\left(B_{1}\right)*\mathrm{Glob}\left(B_{2}\right)\right)\otimes Y,Z\right)$$

is the pullback

$$
\begin{array}{ccc}
\mathbf{Flow}\left(\left(\mathrm{Glob}\left(B_{1}\right)*\mathrm{Glob}\left(B_{2}\right)\right)\otimes Y,Z\right) & \longrightarrow & \mathbf{Flow}\left(\mathrm{Glob}\left(B_{2}\right),[\mathbf{Flow}]\left(Y,Z\right)\right) \\
\downarrow & & \downarrow \\
\mathbf{Flow}\left(\mathrm{Glob}\left(B_{1}\right),[\mathbf{Flow}]\left(Y,Z\right)\right) & \longrightarrow & \mathbf{Flow}\left(\{*\},[\mathbf{Flow}]\left(Y,Z\right)\right)
\end{array}
$$

by the preceding calculations of this proof. So the natural bijection of sets

$$\mathbf{Flow}\left(\left(\mathrm{Glob}\left(B_{1}\right)*\mathrm{Glob}\left(B_{2}\right)\right)\otimes Y,Z\right)\cong\mathbf{Flow}\left(\left(\mathrm{Glob}\left(B_{1}\right)*\mathrm{Glob}\left(B_{2}\right)\right),[\mathbf{Flow}]\left(Y,Z\right)\right)$$

holds. Now we can conclude using Theorem 6.2 by

$$
\begin{aligned}
\mathbf{Flow}\left(X\otimes Y,Z\right) & \cong \mathbf{Flow}\left(\left(\varinjlim\mathbb{D}\left(X\right)\right)\otimes Y,Z\right) \\
& \cong \varprojlim\mathbf{Flow}\left(\mathbb{D}\left(X\right)\otimes Y,Z\right) \\
& \cong \varprojlim\mathbf{Flow}\left(\mathbb{D}\left(X\right),[\mathbf{Flow}]\left(Y,Z\right)\right) \\
& \cong \mathbf{Flow}\left(\varinjlim\mathbb{D}\left(X\right),[\mathbf{Flow}]\left(Y,Z\right)\right) \\
& \cong \mathbf{Flow}\left(X,[\mathbf{Flow}]\left(Y,Z\right)\right)
\end{aligned}
$$

$$\square$$

Notice that in general, the topological spaces

$$\mathbf{FLOW}\left(X\otimes Y,Z\right)$$

and

$$\mathbf{FLOW}(X,[\mathbf{Flow}](Y,Z))$$

are not homeomorphic! Indeed for $X=\{*\}$, then

$$\mathbf{FLOW}\left(X\otimes Y,Z\right)\cong\mathbf{FLOW}\left(Y,Z\right)$$

and

$$\mathbf{FLOW}\left(X,[\mathbf{Flow}]\left(Y,Z\right)\right)\cong[\mathbf{Flow}]\left(Y,Z\right)^{0}\cong\mathbf{Flow}\left(Y,Z\right)$$

which is always discrete. However one has

Proposition 6.5. *Let B be a topological space. Then for any flow Y and Z, there is a canonical homeomorphism*

$$\mathbf{FLOW}\left(\mathrm{Glob}\left(B\right)\otimes Y,Z\right)\cong\mathbf{FLOW}\left(\mathrm{Glob}\left(B\right),[\mathbf{Flow}]\left(Y,Z\right)\right).$$

Proof. If $Y = \{*\}$, then

$$\mathbf{FLOW}\left(\mathrm{Glob}\left(B\right) \otimes Y, Z\right) \cong \mathbf{FLOW}\left(\mathrm{Glob}\left(B\right), \left[\mathbf{Flow}\right]\left(Y, Z\right)\right)$$

It then suffices to show the homeomorphism

$$\mathbf{FLOW}\left(\mathrm{Glob}\left(B\right) \otimes \mathrm{Glob}\left(A\right), Z\right) \cong \mathbf{FLOW}\left(\mathrm{Glob}\left(B\right), \left[\mathbf{Flow}\right]\left(\mathrm{Glob}\left(A\right), Z\right)\right).$$

to be able to conclude with Corollary 6.3 and Theorem 6.4.

There is an inclusion of topological spaces

$$\mathbf{FLOW}\left(\mathrm{Glob}\left(B\right) \otimes \mathrm{Glob}\left(A\right), Z\right) \to \mathbf{TOP}\left(\mathrm{Glob}\left(B\right) \otimes \mathrm{Glob}\left(A\right), Z\right)$$

so if $(B, A, Z)_{\alpha,\beta}^{\gamma,\delta}$ is equipped with the Kelleyfication of the relative topology induced by

$$\mathbf{FLOW}\left(\mathrm{Glob}\left(B\right) \otimes \mathrm{Glob}\left(A\right), Z\right)$$

one has the pullback in **Top**:

In the other hand, the topological space

$$\mathbf{FLOW}\left(\mathrm{Glob}\left(B\right), \left[\mathbf{Flow}\right]\left(\mathrm{Glob}\left(A\right), Z\right)\right)$$

is isomorphic to

$$\bigsqcup_{u,v \in \mathbf{Flow}(\mathrm{Glob}(A), Z)} \mathbf{TOP}\left(B, \mathbb{P}_{u,v}[\mathbf{Flow}]\left(\mathrm{Glob}\left(A\right), Z\right)\right)$$

As above, the pairs $(u, v) \in \mathbf{Flow}\left(\mathrm{Glob}\left(A\right), Z\right) \times \mathbf{Flow}\left(\mathrm{Glob}\left(A\right), Z\right)$ are determined by $u(0) = \alpha$, $u(1) = \beta$, $v(0) = \gamma$, $v(1) = \delta$ and by the continuous map $\mathbb{P}u \in \mathbf{Top}\left(A, \mathbb{P}_{\alpha,\beta}Z\right)$ and $\mathbb{P}v \in \mathbf{Top}\left(A, \mathbb{P}_{\gamma,\delta}Z\right)$. Then one has a natural isomorphism of topological spaces between

$$\mathbf{FLOW}\left(\mathrm{Glob}\left(B\right), \left[\mathbf{Flow}\right]\left(\mathrm{Glob}\left(A\right), Z\right)\right)$$

and the disjoint sum over

$$(\alpha, \beta, \gamma, \delta, m, n) \in Z^0 \times Z^0 \times Z^0 \times Z^0 \times \mathbf{Top}\left(A, \mathbb{P}_{\alpha,\beta}Z\right) \times \mathbf{Top}\left(A, \mathbb{P}_{\gamma,\delta}Z\right)$$

of elements $f \in \mathbf{TOP}\left(B, (A, Z)_{\alpha,\beta}^{\gamma,\delta}\right)$ such that such that the composite of f with the canonical projection map $(A, Z)_{\alpha,\beta}^{\gamma,\delta} \to \mathbf{TOP}\left(A, \mathbb{P}_{\alpha,\beta}Z\right)$ is the constant map m and such that the composite of f with the canonical projection map $(A, Z)_{\alpha,\beta}^{\gamma,\delta} \to \mathbf{TOP}\left(A, \mathbb{P}_{\gamma,\delta}Z\right)$ is the constant map n hence the result. $\square$

7. Model structure of **Flow** and tensor product of flows

Some useful references for the notion of model category are [Hov99, GJ99]. See also [DHK97, Hir01].

Theorem 7.1. [Gau03a] *The category of flows can be given a model structure such that:*

(1) *The weak equivalences are the weak S-homotopy equivalences.*

(2) *The fibrations are the continuous maps satisfying the RLP (right lifting property) with respect to the morphisms* $\mathrm{Glob}(\mathbf{D}^n) \longrightarrow \mathrm{Glob}([0,1] \times \mathbf{D}^n)$ *for* $n \geqslant 0$. *The fibration are exactly the morphisms of flows* $f : X \longrightarrow Y$ *such that* $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ *is a Serre fibration of* **Top**.

(3) *The cofibrations are the morphisms satisfying the LLP (left lifting property) with respect to any map satisfying the RLP with respect to the morphisms* $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$ *with* $\mathbf{S}^{-1} = \varnothing$ *and for* $n \geqslant 0$ *and with respect to the morphisms* $\varnothing \longrightarrow \{0\}$ *and* $\{0,1\} \longrightarrow \{0\}$.

(4) *Any flow is fibrant.*

Proposition 7.2. [Gau03a] *Let U and V be two topological spaces. Let $\phi : U \to V$ be a continuous map. Let $f : X \to Y$ be a morphism of flows and let $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ the corresponding continuous maps between the two path spaces. Assume that at least one of the following conditions holds:*

(1) *both topological spaces U and V are connected*

(2) *the topological space V is connected and the morphism of flows f is synchronized.*

Then the following conditions are equivalent:

(1) *the morphism f satisfies the RLP with respect to the morphism of flows* $\mathrm{Glob}(\phi) : \mathrm{Glob}(U) \to \mathrm{Glob}(V)$

(2) *the continuous map $\mathbb{P}f$ satisfies the RLP with respect to the continuous map* $\phi : U \to V$.

The following definition is an adaptation of the more general notion of monoidal model category (cf. [Hov99]).

If $(\mathcal{C}, \otimes)$ is a monoidal category, if $f : U \to V$ and $g : W \to X$ are two morphisms of $\mathcal{C}$, then let

$$f \square g : P(f, g) = (V \otimes W) \sqcup_{U \otimes W} (U \otimes X) \to V \otimes X$$

Definition 7.3. *Let $\mathcal{C}$ be a cofibrantly generated model category equipped with a closed symmetric monoidal structure $\otimes$. Let I be the set of generating cofibrations and let J be the set of generating acyclic cofibrations. Then $\mathcal{C}$ together with $\otimes$ is a monoidal model category if the following conditions hold:*

(1) *The monoidal structure $\otimes$ is a Quillen bifunctor, which means here that any morphism of $I \square I$ is a cofibration and that any morphism of $I \square J$ and $J \square I$ is an acyclic cofibration.*

(2) *Let $q : QS \to S$ be the cofibrant replacement for the unit S of $\otimes$. Then the natural morphism $q \otimes Id_X : QS \otimes X \to S \otimes X$ is a weak equivalence for any X.*

Unfortunately, one has:

Proposition 7.4. *The model category* **Flow** *together with the closed monoidal structure $\otimes$ does not satisfy the axioms of monoidal model category.*

Proof. Consider the two cofibrations of flows $f : \{0,1\} \to \{0\}$ and $g : \mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{Glob}(\mathbf{D}^n)$ for $n \geqslant 1$. Then

$P(f,g)$

$= \{0\} \otimes \mathrm{Glob}(\mathbf{S}^{n-1}) \sqcup_{\{0,1\} \otimes \mathrm{Glob}(\mathbf{S}^{n-1})} \{0,1\} \otimes \mathrm{Glob}(\mathbf{D}^n)$

$\cong \{0\} \otimes \mathrm{Glob}(\mathbf{S}^{n-1}) \sqcup_{\{0\} \otimes \mathrm{Glob}(\mathbf{S}^{n-1}) \sqcup \{1\} \otimes \mathrm{Glob}(\mathbf{S}^{n-1})} (\{0\} \otimes \mathrm{Glob}(\mathbf{D}^n) \sqcup \{1\} \otimes \mathrm{Glob}(\mathbf{D}^n))$

$\cong \{0\} \otimes \mathrm{Glob}(\mathbf{D}^n) \sqcup_{\mathrm{Glob}(\mathbf{S}^{n-1})} \{1\} \otimes \mathrm{Glob}(\mathbf{D}^n)$

$\cong \mathrm{Glob}(\mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n)$

$\cong \mathrm{Glob}(\mathbf{S}^n)$

If $\mathrm{Glob}(\mathbf{S}^n) \to \mathrm{Glob}(\mathbf{D}^n)$ was a cofibration for the model structure of **Flow**, then by Proposition 7.2, and since $\mathbf{S}^1$ and $\mathbf{D}^1$ are connected, then $\mathbf{S}^1 \to \mathbf{D}^1$ would be a cofibration for the model structure of **Top**. Contradiction. $\qquad\square$

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