

THE CHAIN REPLACEMENT OF A POSET FLOW

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ABSTRACT. We introduce the chain replacement of a poset flow: it is obtained by considering the simplicial nerves of the posets of strictly increasing chains in the given poset, ordered by refinement. It maps posets to q -cofibrant flows and inclusions of posets to q -cofibrations. Using the combinatorial properties of the chain replacement, we prove that pushouts along the chain replacement of an order-reflecting inclusion of posets induce trivial h -cofibrations on spaces of execution paths between states of the original flow. By introducing the Hurewicz model structure on flows (or H -model structure), we deduce the same property for any q -cofibrant replacement of an order-reflecting inclusion of posets.

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1. INTRODUCTION

Presentation. Flows provide a convenient topological model for directed phenomena arising in concurrency [4]: they are small semicategories enriched over a convenient category of topological spaces, whose morphism spaces are spaces of execution paths. In this framework, posets are regarded as flows, with one execution path from x to y precisely when $x < y$. Such poset flows occur naturally in the homotopy theory of flows, for instance in the definition of dihomotopy [8] and in the study of globular subdivisions [13]. Poset flows are not q -cofibrant in general. We introduce an explicit q -cofibrant replacement, the *chain replacement*. For a poset P , the flow $\text{Ch}(P)$ has the same states as P , and its space of execution paths from x to y is the geometric realization of the nerve of the poset of strict chains from x to y , ordered by refinement. This construction is functorial in strictly increasing maps, takes posets to q -cofibrant flows, and sends inclusions of posets to q -cofibrations.

2020 *Mathematics Subject Classification.* 55U35, 05E45, 06A07, 18N40.

Key words and phrases. directed homotopy, flow, poset, cofibrant replacement, model category, Hurewicz model structure, chain replacement, enriched semicategory.

The main result concerns the effect of attaching such a replacement to an arbitrary flow. We prove that:

Theorem (Theorem 8.4). *Let $i : P \subset Q$ be an order-reflecting inclusion of posets, and let $i^{\text{cof}} : P^{\text{cof}} \rightarrow Q^{\text{cof}}$ be any q -cofibrant replacement of i . In every pushout diagram*

$$\begin{array}{ccc} P^{\text{cof}} & \longrightarrow & X \\ i^{\text{cof}} \downarrow & & \downarrow f \\ Q^{\text{cof}} & \longrightarrow & Y, \end{array}$$

the induced map

$$\mathbb{P}_{u,v}X \longrightarrow \mathbb{P}_{f(u),f(v)}Y$$

is a trivial h -cofibration of spaces for all states $u, v \in X^0$.

We recover [13, Proposition 8.8] as a particular case. This result is used in [13] to prove that globular subdivisions are dihomotopy equivalences in the sense of [8]. Thus the homotopy type of the space of execution paths between any two states of X is preserved. The order-reflecting hypothesis is essential: if $x \not\leq y$ in P but $x < y$ in Q , then $\mathbb{P}_{x,y} \text{Ch}(P)$ is empty and $\mathbb{P}_{x,y} \text{Ch}(Q)$ is nonempty (Figure 2).

We first prove the result for the chain replacement (Theorem 5.4). The simplices in the nerves of the posets of chains give a factorization

$$\text{Ch}(P) \hookrightarrow \text{Ch}(Q)|_P \hookrightarrow \text{Ch}(Q).$$

The first map is a trivial q -cofibration when $P \subset Q$ is order-reflecting (Theorem 3.5). A pushout along the second map preserves the spaces of execution paths between the states of the original flow (Proposition 4.4). To prove the result for an arbitrary q -cofibrant replacement, we use the Hurewicz model structure defined by the topological enrichment. All flows are H -cofibrant, every q -cofibration is an H -cofibration, and weak equivalences of the q -model structure between q -cofibrant flows are weak equivalences of the H -model structure. The gluing lemma then gives the comparison with any q -cofibrant replacement of $P \subset Q$.

We deduce by a standard transfinite induction:

Corollary (Corollary 8.5). *Let $f : X \rightarrow Y$ be a retract of a transfinite composition of pushouts of q -cofibrant replacements of order-reflecting inclusions of posets. Then, for any pair of states $u, v \in X^0$, f induces a trivial h -cofibration of spaces*

$$\mathbb{P}_{u,v}X \xrightarrow{\cong} \mathbb{P}_{f(u),f(v)}Y.$$

In particular, it is a homotopy equivalence.

Future work. The combinatorial constructions developed in this paper provide the main technical tool for a subsequent paper, where we prove the cubical analogue of [13], namely that barycentric subdivisions are dihomotopy equivalences in the sense of [8].

Outline of the paper. Section 2 fixes the conventions on flows, posets, and order-reflecting inclusions. Section 3 defines the chain replacement and proves its basic cellular properties. Section 4 proves the full subflow pushout lemma. Section 5 proves the main result for the chain replacement. Section 6 constructs the Hurewicz model structure on

flows. Section 7 compares this model structure with the q-model structure. Finally, Section 8 combines these ingredients to prove the theorem for arbitrary q-cofibrant replacements.

2. NOTATION AND ORDER-REFLECTING INCLUSIONS

Top denotes either the category of Δ -generated spaces or the category of Δ -Hausdorff Δ -generated spaces (cf. [11, Section 2 and Appendix B]). It can be equipped with its q-model structure or its h-model structure. **Top** is locally presentable [5, Corollary 3.7] and cartesian closed [9, Proposition 2.5]; its internal hom is denoted by $\mathbf{TOP}(-, -)$. By compact, we mean quasi-compact Hausdorff.

A *flow* is a small semicategory enriched over **Top**. The set of objects of a flow X , called its *states*, is denoted by X^0 . The space of morphisms, called *execution paths*, from u to v is denoted by $\mathbb{P}_{u,v}X$. The strictly associative composition law is denoted by $*$. The category of flows is denoted by **Flow**. Every set can be viewed as a flow without execution paths. The flow $\mathbf{Glob}(Z)$ is defined by

$$\mathbf{Glob}(Z)^0 = \{0, 1\}, \quad \mathbb{P} \mathbf{Glob}(Z) = \mathbb{P}_{0,1} \mathbf{Glob}(Z) = Z.$$

Flow can be equipped with its q-model structure (e.g. [12, Theorem 7.4]). The q-model structure is cofibrantly generated. The set of generating q-cofibrations is $I^{\mathbf{gl}} \cup \{C : \emptyset \rightarrow \{0\}, R : \{0, 1\} \rightarrow \{0\}\}$ with

$$I^{\mathbf{gl}} = \{\mathbf{Glob}(\mathbf{S}^{n-1}) \subset \mathbf{Glob}(\mathbf{D}^n) \mid n \geq 0\}$$

where $\mathbf{S}^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 = 1\}$ is the $(n-1)$ -dimensional sphere, $\mathbf{S}^{-1} = \emptyset$ and $\mathbf{D}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 \leq 1\}$ is the n -dimensional disk. The set of generating trivial q-cofibrations is

$$J^{\mathbf{gl}} = \{\mathbf{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathbf{Glob}(\mathbf{D}^n \times [0, 1]) \mid n \geq 0\}.$$

A *relative cell complex* is a transfinite composition of pushouts of maps in $I^{\mathbf{gl}} \cup \{C, R\}$. A *cell complex* X is a flow X such that the canonical map $\emptyset \rightarrow X$ is a relative cell complex.

If B is a flow whose states are identified with Q , we write $B|_P$ for the *full* subflow on the states of P :

$$(B|_P)^0 = P, \quad \mathbb{P}_{x,y}(B|_P) = \mathbb{P}_{x,y}B \quad (x, y \in P).$$

A (small) poset R is viewed as the flow with set of states R , with one execution path from x to y if $x < y$, and no execution path otherwise.

Let $\mathbf{PoSet}^+$ denote the category of posets and strictly increasing maps. An inclusion of posets $P \subset Q$ is called *order-reflecting* if for all $x, y \in P$,

$$x \leq_Q y \implies x \leq_P y.$$

Equivalently, the order of P is the order induced from Q . Every order-reflecting inclusion of posets is a morphism of $\mathbf{PoSet}^+$.

For a small category $\mathcal{C}$, $N(\mathcal{C})$ denotes the simplicial nerve of $\mathcal{C}$ and $|N(\mathcal{C})|$ denotes its geometric realization. The geometric realization of a simplicial set is obtained by attaching an n -dimensional cell for each nondegenerate n -simplex of the given simplicial set. By [11, Proposition B.15 and B.16], the topology of the geometric realization is therefore the final topology for the cells. This fact is used in the proofs of Proposition 3.3

and Proposition 3.4 to prove that some continuous bijections are homeomorphisms. By [16, Lemma 3.1.8], the comparison map

$$|\Delta^m \times \Delta^n| \longrightarrow |\Delta^m| \times |\Delta^n|$$

is a homeomorphism which is natural in m and n , where Δ^k denotes the representable k -dimensional simplex. Thus the realization functor in any cartesian closed category of topological spaces preserves finite products. There are natural homeomorphisms

$$|N(\mathcal{C} \times \mathcal{D})| \cong |N(\mathcal{C}) \times N(\mathcal{D})| \cong |N(\mathcal{C})| \times |N(\mathcal{D})|$$

since the simplicial nerve functor is a right adjoint [15, Theorem 14.1.5].

3. THE CHAIN REPLACEMENT OF A POSET

Let P be a poset. For $x < y$ in P , let $\mathcal{C}_P(x, y)$ be the poset of strict chains

$$\gamma = (x = x_0 < x_1 < \cdots < x_k = y)$$

ordered by refinement: $\gamma \leq \gamma'$ if γ is obtained from γ' by removing intermediate vertices. Thus the chain $(x < y)$ is an initial object of $\mathcal{C}_P(x, y)$. If $x \not< y$, put $\mathcal{C}_P(x, y) = \emptyset$. If $x < y$ in P , and if

$$\sigma = (\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_m)$$

is an m -simplex of $N\mathcal{C}_P(x, y)$, write $V(\gamma_i)$ for the set of vertices of the strict chain γ_i , and put

$$V_P(\sigma) = \bigcap_{i=0}^m V(\gamma_i) = V(\gamma_0).$$

Thus $V_P(\sigma)$ is the set of vertices of P through which every chain γ_i passes. It contains x and y . We say that σ is indecomposable if

$$V_P(\sigma) = \{x, y\}.$$

Equivalently, there is no intermediate element $z \in P$, $x < z < y$, through which all the chains γ_i factor. Note that this does not mean that the open interval $]x, y[$ is empty. We call σ *decomposable* if it is not indecomposable.

For each m -simplex σ of $N\mathcal{C}_P(x, y)$, write τ_σ for its unique nondegenerate reduction given by the Eilenberg–Zilber lemma:

$$\sigma = (s_\sigma)^*(\tau_\sigma), \quad \tau_\sigma \text{ nondegenerate}, \quad s_\sigma : \Delta^m \rightarrow \Delta^{n_\sigma}, \quad m \geq n_\sigma.$$

The pair (s_σ, τ_σ) is unique by [6, Chapter II, §3, Proposition 1, pp. 26–27].

Define a flow $\text{Ch}(P)$ as follows. Its set of states is P , and

$$\mathbb{P}_{x,y} \text{Ch}(P) = \begin{cases} |N\mathcal{C}_P(x, y)|, & x < y, \\ \emptyset, & x \not< y. \end{cases}$$

Composition is induced by concatenation of strict chains:

$$\mathcal{C}_P(x, y) \times \mathcal{C}_P(y, z) \longrightarrow \mathcal{C}_P(x, z), \quad (\gamma, \delta) \longmapsto \gamma * \delta.$$

It is strictly associative. There is an augmentation

$$\varepsilon_P : \text{Ch}(P) \longrightarrow P$$

which is the identity on states and collapses each nonempty path space to the unique path of the poset flow P .

Since a map of PoSet^+ sends a strict chain to a strict chain, we obtain the following definition:

3.1. Definition. The assignment

$$P \mapsto \text{Ch}(P)$$

defines a functor $\text{Ch} : \text{PoSet}^+ \rightarrow \text{Flow}$ called the *chain replacement functor*.

3.2. Lemma (Normal form). *Let P be a poset and let*

$$\sigma \in \text{NC}_P(x, y)_m.$$

There exist unique elements

$$x = z_0 < z_1 < \cdots < z_r = y$$

and a unique decomposition

$$\sigma = \sigma_1 * \cdots * \sigma_r$$

such that

$$\sigma_j \in \text{NC}_P(z_{j-1}, z_j)_m$$

is indecomposable for every $1 \leq j \leq r$. Each σ_j has a unique Eilenberg–Zilber decomposition

$$\sigma_j = (s_j)^*(\tau_j),$$

where τ_j is nondegenerate and indecomposable. Consequently, every simplex σ has a unique normal form

$$\sigma = (s_1)^*(\tau_1) * \cdots * (s_r)^*(\tau_r)$$

as a composite of degeneracies of nondegenerate indecomposable simplices.

Proof. The elements $z_0, \dots, z_r$ are precisely the elements of

$$V_P(\sigma) = \bigcap_{i=0}^m V(\gamma_i) = V(\gamma_0),$$

written in increasing order. Every chain in σ contains each z_j , so restriction to the intervals $[z_{j-1}, z_j]$ yields the decomposition

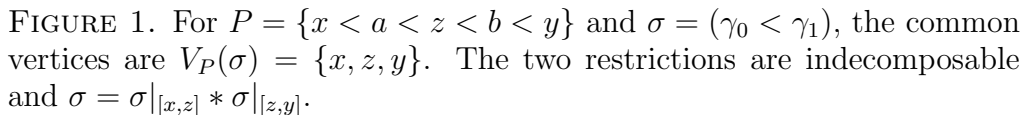
$$\sigma = \sigma|_{[z_0, z_1]} * \cdots * \sigma|_{[z_{r-1}, z_r]}.$$

These restrictions are indecomposable, and any decomposition into indecomposable simplices is determined by the same common vertices. The decomposition is therefore unique; see Figure 1.

The Eilenberg–Zilber lemma gives, for each indecomposable simplex σ_j , a unique pair (s_j, τ_j) such that $\sigma_j = (s_j)^*(\tau_j)$ and τ_j is nondegenerate. Moreover, τ_j is indecomposable. This proves the uniqueness of the normal form. $\square$

3.3. Proposition. *For every poset P , the flow $\text{Ch}(P)$ is q -cofibrant and $\varepsilon_P : \text{Ch}(P) \rightarrow P$ is a trivial q -fibration of flows. Moreover, if $P \subset Q$ is an inclusion of posets, the induced map $\text{Ch}(P) \rightarrow \text{Ch}(Q)$ is a q -cofibration. More precisely, it is a relative cell complex.*

Proof. We construct $\text{Ch}(P)$ as a cell complex.


$$\text{Glob}(|\partial\Delta^m|) \longrightarrow \text{Glob}(|\Delta^m|),$$
$$\begin{array}{ccccc}
\coprod_{\sigma} \mathrm{Glob}(|\partial\Delta^m|) & \longrightarrow & G^{m-1}(P) & \longrightarrow & \mathrm{Ch}(P) \\
\downarrow & & \downarrow & & \parallel \\
\coprod_{\sigma} \mathrm{Glob}(|\Delta^m|) & \xrightarrow{\coprod_{\sigma} \chi_{\sigma}} & G^m(P) & \dashrightarrow & \mathrm{Ch}(P), \\
& \searrow \scriptstyle \coprod_{\sigma} \Psi_{\sigma} & & \nearrow & \\
& & & &
\end{array}$$

We check that the attaching maps are well defined for $m \geq 0$. We first introduce some notations. For a general indecomposable m -simplex σ , write

$$\chi_\sigma : \text{Glob}(|\Delta^m|) \xrightarrow{\text{Glob}(|s_\sigma|)} \text{Glob}(|\Delta^{n_\sigma}|) \xrightarrow{\chi_{\tau_\sigma}} G^{n_\sigma}(P)$$

where (s_σ, τ_σ) is as above. For a general m -simplex σ , if $V_P(\sigma) = \{x = z_0 < z_1 < \cdots < z_r = y\}$ for $r \geq 1$, then every chain in σ contains every z_j . Each restriction $\sigma|_{[z_{j-1}, z_j]}$ is indecomposable and σ is therefore the unique composite of indecomposable simplices: $\sigma = \sigma|_{[z_0, z_1]} * \cdots * \sigma|_{[z_{r-1}, z_r]}$. Write

$$\chi_\sigma = \chi_{\sigma|_{[z_0, z_1]}} * \cdots * \chi_{\sigma|_{[z_{r-1}, z_r]}}.$$

Let $\sigma = (\gamma_0, \dots, \gamma_m)$ be an indecomposable nondegenerate m -simplex of $N\mathcal{C}_P(x, y)$, and let $\partial_i\sigma$ be one of its faces of codimension one. If $i > 0$, then $\partial_i\sigma$ is indecomposable, since γ_0 remains one of its vertices. This implies that the characteristic map $\chi_{\partial_i\sigma}$ is already defined. Otherwise, write

$$V_P(\partial_0\sigma) = \{x = z_0 < z_1 < \cdots < z_r = y\}.$$

Every chain in $\partial_0\sigma$ contains every z_j . Therefore $\partial_0\sigma$ is the unique composite of its restrictions

$$(\partial_0\sigma)|_{[z_{j-1}, z_j]}, \quad (1 \leq j \leq r).$$

These restrictions are indecomposable simplices of the nerves

$$N\mathcal{C}_P(z_{j-1}, z_j),$$

possibly degenerate; their nondegenerate reductions are indecomposable by construction. Thus each such nondegenerate reduction has smaller dimension and has already been attached. This implies that the characteristic map $\chi_{\partial_0\sigma}$ is already defined. It follows that every face of σ has already been constructed by the time the cell corresponding to σ is attached.

Let $G^\omega(P) = \varinjlim G^m(P)$ be the flow obtained by the above cellular construction. We claim that

$$G^\omega(P) = \text{Ch}(P).$$

By Lemma 3.2, the map $G^\omega(P) \rightarrow \text{Ch}(P)$ induces injective maps on each path space. Indeed, all cells used in the construction are indecomposable nondegenerate simplices of the nerves $N\mathcal{C}_P(x, y)$, and the composition is induced by concatenation of chains. This follows from the uniqueness of the normal form.

Conversely, let τ be any nondegenerate simplex of $N\mathcal{C}_P(x, y)$. If τ is indecomposable, then it is one of the attached cells. If it is not indecomposable, write

$$V_P(\tau) = \{x = z_0 < z_1 < \cdots < z_r = y\}.$$

Then every chain in τ contains each z_j , and τ is the composite of its indecomposable restrictions to the intervals $[z_{j-1}, z_j]$. The nondegenerate reductions of these restrictions are indecomposable. Hence τ is obtained by composition from the attached cells. Therefore the map of flows $G^\omega(P) \rightarrow \text{Ch}(P)$ induces a bijection on the set of states and a continuous bijection

$$f_{x,y} : \mathbb{P}_{x,y}G^\omega(P) \longrightarrow \mathbb{P}_{x,y}\text{Ch}(P)$$

for each pair of states (x, y) .

Each $\mathbb{P}_{x,y}\text{Ch}(P)$ is the geometric realization of a simplicial nerve, namely $N\mathcal{C}_P(x, y)$. Each m -simplex σ is indecomposable if and only if its nondegenerate reduction τ_σ is indecomposable. When σ is indecomposable, including the degenerate case, we obtain the commutative diagram of sets

$$\begin{array}{ccccc} |\Delta^m| & \xrightarrow{|\sigma|} & \mathbb{P}_{x,y}\text{Ch}(P) & \xrightarrow{f_{x,y}^{-1}} & \mathbb{P}_{x,y}G^\omega(P) \\ |s_\sigma| \downarrow & \nearrow & & \nearrow & \\ |\Delta^{n_\sigma}| & & & \nearrow & \\ & & & \text{P}_{0,1}\chi_{\tau_\sigma} & \end{array}$$

where (s_σ, τ_σ) is as above. We have proved that for any indecomposable simplex σ , including the degenerate case, the composite map

$$p_\sigma : |\Delta^m| \xrightarrow{|\sigma|} \mathbb{P}_{x,y}\text{Ch}(P) \xrightarrow{f_{x,y}^{-1}} \mathbb{P}_{x,y}G^\omega(P)$$

is equal to $\mathbb{P}_{0,1}\chi_\sigma$, and therefore is continuous. When σ is decomposable, write

$$\sigma = \sigma_1 * \cdots * \sigma_p, \quad p > 1, \quad \sigma_1, \dots, \sigma_p \text{ indecomposable.}$$

Let $x = z_0 < \cdots < z_p = y$ be the common vertices of the chains in σ . The formula $p_\sigma = p_{\sigma_1} * \cdots * p_{\sigma_p}$ gives the commutative diagram

$$\begin{array}{ccc} |\Delta^m| & \xrightarrow{\delta} & \prod_{j=1}^p |\Delta^m| \\ p_\sigma \downarrow & & \downarrow \prod_j p_{\sigma_j} \\ \mathbb{P}_{x,y} G^\omega(P) & \xleftarrow{*} & \prod_{j=1}^p \mathbb{P}_{z_{j-1}, z_j} G^\omega(P). \end{array}$$

Here δ is the diagonal and $*$ is composition. All three maps in this factorization are continuous, so p_σ is continuous.

The topology of $\mathbb{P}_{x,y} \text{Ch}(P)$ is the final topology with respect to each $|\sigma| : |\Delta^m| \rightarrow \mathbb{P}_{x,y} \text{Ch}(P)$ for σ running over all nondegenerate simplices of $N\mathcal{C}_P(x, y)$. Since each composite

$$|\Delta^m| \xrightarrow{|\sigma|} \mathbb{P}_{x,y} \text{Ch}(P) \xrightarrow{f_{x,y}^{-1}} \mathbb{P}_{x,y} G^\omega(P)$$

is continuous, in both cases, we deduce that each $f_{x,y}$ is a homeomorphism. We have proved that $G^\omega(P) = \text{Ch}(P)$. Hence $\text{Ch}(P)$ is a cell complex, and in particular it is q-cofibrant.

For $x < y$, the poset $\mathcal{C}_P(x, y)$ has the initial object $(x < y)$. Consequently

$$|N\mathcal{C}_P(x, y)|$$

is contractible. The map

$$\varepsilon_P : \text{Ch}(P) \longrightarrow P$$

is the identity on states and sends each nonempty contractible path space to the singleton path space of the poset flow P . If $x \not< y$, both path spaces are empty. Thus every map on path spaces induced by ε_P is a trivial q-fibration of spaces. Since the map on states is a bijection, ε_P is a trivial q-fibration of flows.

It remains to prove the last assertion. Let $P \subset Q$ be an inclusion of posets. The cellular construction just described for P is a subcomplex of the corresponding cellular construction for Q . Indeed, a simplex of $N\mathcal{C}_P(x, y)$ has the same set of vertices whether it is regarded in P or in Q . Therefore $\text{Ch}(Q)$ is obtained from $\text{Ch}(P)$ by first adding the new states of $Q \setminus P$, and then attaching the remaining globular cells of the cellular decomposition of $\text{Ch}(Q)$. Hence

$$\text{Ch}(P) \longrightarrow \text{Ch}(Q)$$

is a q-cofibration of flows. □

Every chain in $\partial_0\sigma$ contains each z_j . Hence $\partial_0\sigma$ is the composite of its restrictions to the intervals

$$[z_{j-1}, z_j] \quad (1 \leq j \leq r).$$

Each restriction is a possibly degenerate simplex of

$$NC_Q(z_{j-1}, z_j).$$

Its nondegenerate reduction is P -indecomposable. Indeed, a common intermediate vertex of P in such a restriction would be an element of $V_P(\partial_0\sigma)$ strictly between two consecutive elements z_{j-1} and z_j , which is impossible. Thus each nondegenerate reduction either lies in $NC_P(z_{j-1}, z_j)$, and has already been constructed in $\text{Ch}(P)$, or is one of the P -indecomposable cells already attached in lower dimension. Therefore every face of σ has already been constructed, and the attaching map is well defined.

Let $G^\omega(P, Q) = \varinjlim G^m(P, Q)$ be the flow obtained after all these cells have been attached. We claim that

$$G^\omega(P, Q) = \text{Ch}(Q)|_P.$$

The P -relative analogue of Lemma 3.2, obtained by replacing indecomposable simplices with P -indecomposable simplices, shows that the canonical morphism

$$G^\omega(P, Q) \longrightarrow \text{Ch}(Q)|_P$$

induces injective maps on all path spaces. Indeed, the attached cells correspond precisely to P -indecomposable nondegenerate simplices of $NC_Q(x, y)$, with $x, y \in P$, and composition corresponds to concatenation of chains in Q . This follows from the uniqueness of the normal form.

Conversely, let τ be a nondegenerate simplex of $NC_Q(x, y)$ with $x, y \in P$. If τ belongs to $NC_P(x, y)$, then it has already been constructed in $\text{Ch}(P)$. If τ is P -indecomposable and does not belong to $NC_P(x, y)$, then it has been attached. Otherwise, τ is P -decomposable. Writing

$$V_P(\tau) = \{x = z_0 < z_1 < \cdots < z_r = y\},$$

the simplex τ is the composite of its restrictions to the intervals $[z_{j-1}, z_j]$. The nondegenerate reductions of these restrictions are P -indecomposable by the same argument as above; hence they either lie in $\text{Ch}(P)$ or are among the attached cells. Thus τ belongs to the path space of $G^\omega(P, Q)$. Therefore every simplex of every $NC_Q(x, y)$, with $x, y \in P$, belongs to $G^\omega(P, Q)$, and so, using the final topology as in Proposition 3.3, we deduce the equality

$$G^\omega(P, Q) = \text{Ch}(Q)|_P.$$

Thus

$$\text{Ch}(P) \longrightarrow \text{Ch}(Q)|_P$$

is a relative cell complex, hence a q -cofibration of flows. Since $\text{Ch}(P)$ is a cell complex by Proposition 3.3 and $\text{Ch}(Q)|_P$ has just been obtained from $\text{Ch}(P)$ by attaching cells, both flows are q -cofibrant. $\square$

3.5. Theorem. *Let $P \subset Q$ be an inclusion of posets. Then the following conditions are equivalent:*

1. *The inclusion $P \subset Q$ is order-reflecting.*
2. *The canonical map which is bijective on states*

$$\mathrm{Ch}(P) \longrightarrow \mathrm{Ch}(Q)|_P$$

is a trivial q -cofibration between q -cofibrant flows.

Proof. Assume that the inclusion $P \subset Q$ is order-reflecting. By Proposition 3.4, the map $\mathrm{Ch}(P) \rightarrow \mathrm{Ch}(Q)|_P$ is a q -cofibration. It remains only to verify that this q -cofibration is a weak equivalence. If $x \not\leq y$ in P , then $x \not\leq y$ in Q , because $P \subset Q$ is order-reflecting. Both path spaces are therefore empty. If $x < y$ in P , then also $x < y$ in Q . The initial objects in the two posets of chains give the commutative triangle

$$\begin{array}{ccc} |\mathrm{NC}_P(x, y)| & \longrightarrow & |\mathrm{NC}_Q(x, y)| \\ & \searrow \simeq & \downarrow \simeq \\ & & \{*\}. \end{array}$$

The maps to a point are homotopy equivalences. Hence the horizontal map is a weak homotopy equivalence, and $\mathrm{Ch}(P) \rightarrow \mathrm{Ch}(Q)|_P$ is a weak equivalence of flows.

Assume now that the inclusion $P \subset Q$ is not order-reflecting. Then there exist $x, y \in P$ such that $x \not\leq y$ in P and $x < y$ in Q . Then $\mathbb{P}_{x,y} \mathrm{Ch}(P) = \emptyset$ and $\mathbb{P}_{x,y} \mathrm{Ch}(Q)|_P$ is nonempty. Thus the map $\mathrm{Ch}(P) \rightarrow \mathrm{Ch}(Q)|_P$ cannot be a weak equivalence of the q -model structure of Flow. $\square$

$$\begin{array}{ccccc} P & & & & Q \\ & \bullet & \bullet & \xrightarrow{i} & \bullet \longrightarrow \bullet \longrightarrow \bullet \\ & x & y & & x \quad z \quad y \\ \mathbb{P}_{x,y} \mathrm{Ch}(P) = \emptyset & & & & \mathbb{P}_{x,y} \mathrm{Ch}(Q) \cong [0, 1] \end{array}$$

FIGURE 2. An inclusion which is not order-reflecting. In P , the two elements are incomparable. In Q , the two chains $(x < y)$ and $(x < z < y)$ form the poset $\mathcal{C}_Q(x, y)$. Its nerve has one nondegenerate 1-simplex.

4. FULL SUBFLOWS AND PUSHOUTS

Write $\mathbf{Top}_\Delta$ for the category of Δ -generated spaces, $\mathbf{HTop}_\Delta$ for its full subcategory of Δ -Hausdorff spaces, and $\mathbf{TOP}_\Delta(-, -)$ for the internal hom of $\mathbf{Top}_\Delta$. For either category $\mathcal{T}$, let $\mathbf{Flow}(\mathcal{T})$ denote the category of small semicategories enriched over $\mathcal{T}$.

4.1. Proposition. *The reflector $w_\Delta : \mathbf{Top}_\Delta \rightarrow \mathbf{HTop}_\Delta$ preserves finite products. Applying it to each space of execution paths, without changing the states, defines a left adjoint*

$$\widehat{w}_\Delta : \mathbf{Flow}(\mathbf{Top}_\Delta) \longrightarrow \mathbf{Flow}(\mathbf{HTop}_\Delta)$$

to the inclusion. For every $F \in \mathbf{Flow}(\mathbf{Top}_\Delta)$ and every subset $A \subset F^0$,

$$\widehat{w}_\Delta(F|_A) \cong (\widehat{w}_\Delta F)|_A.$$

Proof. Let $\eta_K : K \rightarrow w_\Delta K$ be the reflection unit [11, Proposition B.7]. Cartesian closedness, reflection, and [11, Proposition B.6] give, for $K, L \in \mathbf{Top}_\Delta$ and $Z \in \mathbf{HTop}_\Delta$,

$$\begin{aligned} \mathbf{Top}_\Delta(K \times L, Z) &\cong \mathbf{Top}_\Delta(w_\Delta L, \mathbf{TOP}_\Delta(K, Z)) \\ &\cong \mathbf{Top}_\Delta(w_\Delta K, \mathbf{TOP}_\Delta(w_\Delta L, Z)) \\ &\cong \mathbf{Top}_\Delta(w_\Delta K \times w_\Delta L, Z). \end{aligned}$$

Products are formed in $\mathbf{Top}_\Delta$; those of Δ -Hausdorff spaces remain Δ -Hausdorff by [11, Proposition B.4] and Δ -kelleyfication. Thus the canonical comparison is an isomorphism:

$$\begin{aligned} \kappa_{K,L} : w_\Delta(K \times L) &\xrightarrow{\cong} w_\Delta K \times w_\Delta L, \\ \kappa_{K,L} \eta_{K \times L} &= \eta_K \times \eta_L, \quad w_\Delta\{*\} \cong \{*\}. \end{aligned}$$

Write $F_{x,y} = \mathbb{P}_{x,y}F$ and $\mu_{x,y,z} : F_{x,y} \times F_{y,z} \rightarrow F_{x,z}$ for composition. On $\widehat{w}_\Delta F$, put

$$\begin{aligned} \widehat{\mu}_{x,y,z} : w_\Delta F_{x,y} \times w_\Delta F_{y,z} &\longrightarrow w_\Delta F_{x,z}, \\ \widehat{\mu}_{x,y,z} &= w_\Delta(\mu_{x,y,z}) \circ \kappa_{F_{x,y}, F_{y,z}}^{-1}, \\ \widehat{\mu}_{x,y,z}(\eta_{F_{x,y}} \times \eta_{F_{y,z}}) &= \eta_{F_{x,z}} \mu_{x,y,z}. \end{aligned}$$

Precomposition with finite products of units is injective for Δ -Hausdorff targets. Hence the reflected composition is associative and factorizations of path maps preserve composition, giving, for $G \in \mathbf{Flow}(\mathbf{HTop}_\Delta)$,

$$\mathbf{Flow}(\mathbf{HTop}_\Delta)(\widehat{w}_\Delta F, G) \cong \mathbf{Flow}(\mathbf{Top}_\Delta)(F, G).$$

Restriction commutes by definition. □

Let $\mathbf{Graph}(\mathcal{T})$ consist of directed graphs with vertices G^0 and edge spaces $G_{x,y} \in \mathcal{T}$. A morphism $f : G \rightarrow H$ is a function $f^0 : G^0 \rightarrow H^0$ together with continuous maps $f_{x,y} : G_{x,y} \rightarrow H_{f^0(x), f^0(y)}$. The edge space $G^1 = \coprod_{x,y} G_{x,y}$ has source and target in the discrete space G^0 . Write $\mathcal{U} : \mathbf{Flow}(\mathcal{T}) \rightarrow \mathbf{Graph}(\mathcal{T})$ for the functor forgetting composition.

4.2. Proposition. *For either choice of $\mathcal{T}$, the functor $\mathcal{U}$ has a left adjoint $\mathcal{F}$, the free flow functor, with $(\mathcal{F}G)^0 = G^0$ and*

$$\mathbb{P}_{x,y} \mathcal{F}G = \coprod_{n \geq 1} \coprod_{x_1, \dots, x_{n-1} \in G^0} \prod_{i=1}^n G_{x_{i-1}, x_i}, \quad x_0 = x, \quad x_n = y.$$

Composition is induced by concatenation.

Proof. By cartesian closedness, products in $\mathcal{T}$ distribute over coproducts, so concatenation is continuous and associative. The summands with $n = 1$ define $\iota_G : G \rightarrow \mathcal{U}\mathcal{F}G$, with identity map on vertices.

A graph morphism $f : G \rightarrow \mathcal{U}X$ extends with $\bar{f}^0 = f^0$ and

$$\mathbb{P}_{x,y}\bar{f}|_{\prod_{i=1}^n G_{x_{i-1},x_i}} = *^{(n)} \circ \prod_{i=1}^n f_{x_{i-1},x_i},$$

where $*^{(n)}$ is iterated composition in X , with $*^{(1)} = \text{Id}$. These continuous maps preserve composition by associativity. Since words are composites of letters, the extension is unique:

$$\begin{array}{ccc} G & \xrightarrow{\iota_G} & \mathcal{U}\mathcal{F}G \\ & \searrow f & \downarrow \mathcal{U}\bar{f} \\ & & \mathcal{U}X. \end{array}$$

This proves $\mathcal{F} \dashv \mathcal{U}$. □

4.3. Proposition. *Every pushout of flows on the left induces a pushout on the right, where $U' = U|_{\varphi(V^0)}$ and $Y' = Y|_{\psi(X^0)}$:*

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \varphi \downarrow & & \downarrow \psi \\ U & \xrightarrow{\theta} & Y \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} V & \xrightarrow{\alpha} & X \\ \varphi \downarrow & & \downarrow \\ U' & \xrightarrow{\quad} & Y'. \end{array}$$

Proof. First suppose that $\mathbf{Top} = \mathbf{Top}_\Delta$. Put

$$S = \varphi(V^0), \quad T = U^0 \amalg_{V^0} X^0 = Y^0, \quad B = \psi(X^0).$$

The relations $\varphi(v) \sim \alpha(v)$, $v \in V^0$, leave $U^0 \setminus S$ untouched. Hence

$$T \cong B \amalg (U^0 \setminus S), \quad B \cong X^0 \amalg_{V^0} S, \quad (\theta^0)^{-1}(B) = S.$$

Let G have vertices T , edges $\mathbb{P}X \amalg \mathbb{P}U$, and endpoint maps induced by ψ^0, θ^0 . By Proposition 4.2, distributing the edge coproducts expresses $W_{p,q} = \mathbb{P}_{p,q}\mathcal{F}G$ as a coproduct of products of original path spaces, with sources and targets fixed. Let $\sim$ be the congruence generated by composing paths composable in X or U , and by $\alpha(\nu) \sim \varphi(\nu)$, $\nu \in \mathbb{P}V$.

Products and quotients are formed in $\mathbf{Top}_\Delta$. Its inclusion into ordinary topological spaces is left adjoint to Δ -kelleyfication k_Δ [11, Appendix B, p. 603], so coequalizers have the ordinary final topology. A quotient $q : K \rightarrow L$ is the coequalizer of $k_\Delta(K \times_L^{\text{TOP}} K) \rightrightarrows K$: kelleyfication leaves the underlying relation unchanged, and finality gives continuous factorization. The left adjoint $(-) \times K$ preserves coequalizers, so products of quotient maps are quotient maps. Thus concatenation descends continuously:

$$\begin{array}{ccc} W_{p,q} \times W_{q,r} & \xrightarrow{\text{concatenation}} & W_{p,r} \\ \pi_{p,q} \times \pi_{q,r} \downarrow & & \downarrow \pi_{p,r} \\ (W_{p,q}/\sim) \times (W_{q,r}/\sim) & \xrightarrow{*_\sim} & W_{p,r}/\sim, \end{array}$$

Here $\pi_{p,q}$ is the quotient map. The adjunction $\mathcal{F} \dashv \mathcal{U}$ identifies the quotient flow with Y , so

$$\mathbb{P}_{p,q}Y \cong W_{p,q}/\sim.$$

Let $a : X \rightarrow Z$ and $b : U' \rightarrow Z$ satisfy $a\alpha = b\varphi$, inducing $\ell^0 : B \rightarrow Z^0$. Cut words with endpoints in B at every intermediate state in B . Each piece is one path of X or consists of paths of U with internal states in $T \setminus B$. By the state identifications above, the latter paths are composable in U with endpoints in S . Define E by applying a to paths of X , b to the composites in U' , and composing in Z . Since a^0, b^0 factor through ℓ^0 , adjacent images compose in Z even when distinct original endpoints have the same image in B .

Composition in X is respected by a . Composing in U preserves a piece's composite when the junction is outside S , or joins two pieces with matching original endpoints when it is in S ; b respects the latter composition. For $\nu \in \mathbb{P}V$, both images are single pieces because their endpoints map into B , so

$$\begin{aligned} E(w_1 * \varphi(\nu) * w_2) &= E(w_1) * b(\varphi(\nu)) * E(w_2) \\ &= E(w_1) * a(\alpha(\nu)) * E(w_2) \\ &= E(w_1 * \alpha(\nu) * w_2). \end{aligned}$$

Factors with empty w_i are omitted. The composition equalities hold on words before descending E to the quotient. By symmetry, reversing composition relations along actual factorizations in X or U exposes factors $\alpha(\nu)$ or $\varphi(\nu)$, respectively, without changing E . Thus E respects $\sim$.

On each product summand, the original sources, targets and cuts are fixed, so E is continuous. Since joins lie in B , $E(w * w') = E(w) * E(w')$. Every word is a composite of its pieces, so the induced morphism $\ell : Y' \rightarrow Z$ is the unique extension of a, b .

For $\mathbf{Top} = \mathbf{HTop}_\Delta$, form the pushout $\tilde{Y}$ in $\mathbf{Flow}(\mathbf{Top}_\Delta)$, identify $\tilde{Y}^0 = Y^0 = U^0 \amalg_{V^0} X^0$, and put $B = \psi(X^0)$. Proposition 4.1 and the first case give

$$Y' \cong \widehat{w}_\Delta(\tilde{Y}|_B) \cong X \amalg_V U',$$

where the last pushout is formed in $\mathbf{Flow}(\mathbf{HTop}_\Delta)$. □

4.4. Proposition (Full subflow pushout lemma). *Let U be a flow and $S \subset U^0$. For a morphism $g : U|_S \rightarrow X$, form the pushout*

$$\begin{array}{ccc} U|_S & \xrightarrow{g} & X \\ \downarrow & & \downarrow \iota \\ U & \longrightarrow & Y. \end{array}$$

The canonical maps are isomorphisms:

$$\begin{aligned} X &\xrightarrow{\cong} Y|_{\iota(X^0)}, \\ \mathbb{P}_{\alpha, \beta} X &\xrightarrow{\cong} \mathbb{P}_{\iota(\alpha), \iota(\beta)} Y \quad (\alpha, \beta \in X^0). \end{aligned}$$

Proof. Proposition 4.3 gives $Y|_{\iota(X^0)} \cong X \amalg_{U|_S} U|_S \cong X$. □

4.5. Corollary. *Let $P \subset Q$ be an order-reflecting inclusion of posets. Consider a pushout of flows*

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \iota \\ Q & \longrightarrow & Y. \end{array}$$

For all states $\alpha, \beta \in X^0$, the induced map

$$\mathbb{P}_{\alpha, \beta} X \longrightarrow \mathbb{P}_{\iota(\alpha), \iota(\beta)} Y$$

is a homeomorphism.

Proof. Order reflection gives $P = Q|_P$; apply Proposition 4.4. □

5. PUSHOUT ALONG A CHAIN REPLACEMENT

5.1. Definition ([3, p. 686]). An injective continuous map $i : A \rightarrow X$ of topological spaces is *relative- T_1* if for any open subset U of A and any point $z \in X \setminus i(U)$, there is an open set W of X with $i(U) \subset W$ and $z \notin W$.

Every closed- T_1 inclusion is relative- T_1 by [11, Proposition 2.3]: we shall use the following result.

5.2. Theorem. *Let $X : \lambda \rightarrow \mathbf{Flow}$ be a transfinite tower of flows such that for all $\mu < \lambda$, the map $\mathbb{P}X_\mu \rightarrow \mathbb{P}X_{\mu+1}$ is a relative- T_1 inclusion. Then the canonical map*

$$\varinjlim (\mathbb{P} \cdot X) \longrightarrow \mathbb{P} \varinjlim X$$

is a homeomorphism. Moreover the topology of $\mathbb{P} \varinjlim X$ is the final topology.

Proof. It is [11, Theorem 5.5]. □

Theorem 5.2 applies in particular when each map $\mathbb{P}X_\mu \rightarrow \mathbb{P}X_{\mu+1}$ is an h-cofibration since every h-cofibration of $\mathbf{Top}$ is relative- T_1 by [11, Proposition 2.6].

5.3. Proposition. *Let $f : X \rightarrow Y$ be a trivial q-cofibration of flows. Then for every $u, v \in X^0$, the induced map*

$$\mathbb{P}_{u, v} X \xrightarrow{f} \mathbb{P}_{f(u), f(v)} Y$$

is a trivial h-cofibration of spaces.

When X and Y are q-cofibrant, the result follows from [11, Theorem 5.7]. The following proof does not require this assumption.

Proof. Every trivial q-cofibration is a retract of a relative $J^{\mathbf{gl}}$ -cell complex. Since trivial h-cofibrations are closed under retracts, it suffices to consider a transfinite composition of pushouts of maps in $J^{\mathbf{gl}}$:

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow \varinjlim X_\lambda = Y.$$

By [11, Theorem 5.4(2)], each pushout of a generating trivial q-cofibration of flows induces a trivial h-cofibration on the path spaces (i.e. a trivial cofibration of the h-model structure of $\mathbf{Top}$). Thus we obtain a transfinite tower of trivial h-cofibrations of spaces

$$\mathbb{P}_{u, v} X = \mathbb{P}_{u, v} X_0 \longrightarrow \mathbb{P}_{u, v} X_1 \longrightarrow \cdots \longrightarrow \varinjlim \mathbb{P}_{u, v} X_\lambda.$$

A transfinite composite of trivial h-cofibrations is again a trivial h-cofibration since they are trivial cofibrations of the h-model structure of $\mathbf{Top}$. Since every h-cofibration of $\mathbf{Top}$ is relative- T_1 by [11, Proposition 2.6], the homeomorphism of Theorem 5.2 induces the map

$$\mathbb{P}_{u,v}X \longrightarrow \mathbb{P}_{f(u),f(v)}Y$$

which is the composite of a trivial h-cofibration and a homeomorphism. $\square$

5.4. Theorem. *Let $i : P \subset Q$ be an order-reflecting inclusion of posets, and let*

$$\mathrm{Ch}(i) : \mathrm{Ch}(P) \longrightarrow \mathrm{Ch}(Q)$$

be the chain replacement of the inclusion $P \subset Q$. Let

$$\begin{array}{ccc} \mathrm{Ch}(P) & \longrightarrow & X \\ \mathrm{Ch}(i) \downarrow & & \downarrow f \\ \mathrm{Ch}(Q) & \longrightarrow & Y \end{array}$$

be a pushout of flows. Then for all states $u, v \in X^0$, the induced map

$$\mathbb{P}_{u,v}X \longrightarrow \mathbb{P}_{f(u),f(v)}Y$$

is a trivial h-cofibration of topological spaces.

Proof. Factor $\mathrm{Ch}(i)$ through $\mathrm{Ch}(Q)|_P$, and form the two pushout squares

$$\begin{array}{ccc} \mathrm{Ch}(P) & \longrightarrow & X \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{Ch}(Q)|_P & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ \mathrm{Ch}(Q) & \longrightarrow & Y. \end{array}$$

Thus $X_1 = X \amalg_{\mathrm{Ch}(P)} \mathrm{Ch}(Q)|_P$ and $Y \cong X_1 \amalg_{\mathrm{Ch}(Q)|_P} \mathrm{Ch}(Q)$. By Theorem 3.5, the induced map

$$X \longrightarrow X_1$$

is a trivial q-cofibration of flows. By Proposition 5.3, for all $u, v \in X^0$, the map

$$\mathbb{P}_{u,v}X \longrightarrow \mathbb{P}_{u,v}X_1$$

is a trivial h-cofibration. Apply Proposition 4.4 with

$$U = \mathrm{Ch}(Q), \quad S = P, \quad U|_S = \mathrm{Ch}(Q)|_P, \quad X = X_1.$$

It gives the isomorphism of flows

$$X_1 \cong Y|_{f(X^0)}.$$

For $u, v \in X^0 = X_1^0$, the induced maps form the commutative triangle

$$\begin{array}{ccc} \mathbb{P}_{u,v}X & \xrightarrow{\cong} & \mathbb{P}_{u,v}X_1 \\ & \searrow & \downarrow \cong \\ & & \mathbb{P}_{f(u),f(v)}Y. \end{array}$$

The horizontal map is a trivial h-cofibration and the vertical map is a homeomorphism. Their composite is therefore a trivial h-cofibration. $\square$

6. THE H-MODEL STRUCTURE OF FLOWS

6.1. Lemma (well-known). *Every object U of $\mathbf{Top}$ is canonically homeomorphic, in $\mathbf{Top}$, to the coproduct of its path-connected components:*

$$\coprod_{c \in \pi_0(U)} U_c \xrightarrow{\cong} U.$$

Proof. See [9, Proposition 2.8]. $\square$

The following consequence will be used repeatedly. For every $A \in \mathbf{Top}$, Lemma 6.1 gives a natural homeomorphism

$$(1) \quad \mathbf{TOP}(U, A) \cong \prod_{c \in \pi_0(U)} \mathbf{TOP}(U_c, A).$$

This follows from the universal property of the coproduct $U \cong \coprod_c U_c$.

We denote by $\mathbf{FLOW}(X, Y)$ the usual enriched hom space of morphisms of flows from X to Y . Thus the category of flows is regarded with its standard $\mathbf{Top}$ -enrichment. By [7, Theorem 5.10 (1)], the comparison map

$$\mathbf{FLOW}(\varinjlim X_i, Y) \xrightarrow{\cong} \varinjlim \mathbf{FLOW}(X_i, Y)$$

is a homeomorphism. In the Δ -generated setting, and unlike the k -space setting, the functor $\tilde{\omega}$ of [7, Theorem 5.4] does commute with all limits, and not only with finite limits, because a limit of discrete spaces remains discrete in $\mathbf{Top}$. Thus the same calculation as in the proof of [7, Theorem 5.10 (2)] proves that the comparison map

$$\mathbf{FLOW}(X, \varprojlim Y_i) \xrightarrow{\cong} \varprojlim \mathbf{FLOW}(X, Y_i)$$

is a homeomorphism for any limit.

For $U \in \mathbf{Top}$ and a flow Y , recall the flow $\{U, Y\}_S$:

$$\{U, Y\}_S^0 = Y^0, \quad \mathbb{P}_{\alpha, \beta}\{U, Y\}_S = \mathbf{TOP}(U, \mathbb{P}_{\alpha, \beta}Y),$$

with pointwise composition. Theorem 7.8 of [7] says that the functor $\{U, -\}_S : \mathbf{Flow} \rightarrow \mathbf{Flow}$ has a left adjoint, denoted by

$$U \boxtimes - : \mathbf{Flow} \longrightarrow \mathbf{Flow}.$$

In particular, there are natural bijections

$$\mathbf{Flow}(U \boxtimes X, Y) \cong \mathbf{Flow}(X, \{U, Y\}_S).$$

When C is non-empty and path-connected, there are natural bijections of sets

$$\text{Top}(C, \text{FLOW}(X, Y)) \cong \text{Flow}(C \boxtimes X, Y) \cong \text{Flow}(X, \{C, Y\}_S)$$

by [7, Theorem 7.9]. We shall use the standard strengthened form for a non-empty path-connected parameter space C :

$$(2) \quad \text{TOP}(C, \text{FLOW}(X, Y)) \cong \text{FLOW}(C \boxtimes X, Y) \cong \text{FLOW}(X, \{C, Y\}_S).$$

The left-hand homeomorphism is precisely [7, Theorem 9.6]. The right-hand homeomorphism states that the adjunction is compatible with the Top -enrichment. This statement does not appear explicitly in [7]. It is derived from the Yoneda lemma as follows. Let $Z \in \text{Top}$. First we observe that

$$\begin{aligned} \text{Top}(Z, \text{FLOW}(C \boxtimes X, Y)) &\cong \prod_{c \in \pi_0(Z)} \text{Top}(Z_c, \text{FLOW}(C \boxtimes X, Y)), \\ \text{Top}(Z, \text{FLOW}(X, \{C, Y\}_S)) &\cong \prod_{c \in \pi_0(Z)} \text{Top}(Z_c, \text{FLOW}(X, \{C, Y\}_S)). \end{aligned}$$

Thus we can assume that Z is non-empty and path-connected. Then, using [7, Theorems 7.8 and 7.9], we write the sequence of natural bijections

$$\begin{aligned} \text{Top}(Z, \text{FLOW}(C \boxtimes X, Y)) &\cong \text{Flow}(C \boxtimes (Z \boxtimes X), Y) \\ &\cong \text{Flow}((C \times Z) \boxtimes X, Y) \\ &\cong \text{Flow}(Z \boxtimes X, \{C, Y\}_S) \\ &\cong \text{Top}(Z, \text{FLOW}(X, \{C, Y\}_S)). \end{aligned}$$

By the Yoneda lemma, the natural map

$$\text{FLOW}(C \boxtimes X, Y) \cong \text{FLOW}(X, \{C, Y\}_S)$$

is therefore a homeomorphism. We shall also use the globe formula from [7, Theorem 7.8]: for C path-connected and non-empty and for every $Z \in \text{Top}$,

$$(3) \quad C \boxtimes \text{Glob}(Z) \cong \text{Glob}(C \times Z).$$

Let $U \in \text{Top}$ and let X be a flow. Define

$$(4) \quad U \otimes X := \coprod_{c \in \pi_0(U)} (U_c \boxtimes X),$$

where the coproduct is taken in Flow . If $U = \emptyset$, this is the initial flow. Observe that

$$(U \otimes X)^0 \cong \coprod_{c \in \pi_0(U)} X^0,$$

as expected: a family of maps $U \rightarrow \text{FLOW}(X, Y)$ may choose a different map on states $X^0 \rightarrow Y^0$ on each path-connected component of U .

6.2. Proposition. *For any two flows X, Y and any space U , there is a natural homeomorphism*

$$\text{FLOW}(U \otimes X, Y) \cong \text{TOP}(U, \text{FLOW}(X, Y)).$$

Proof. Using the definition (4), the universal property of coproducts in $\mathbf{Flow}$, the connected case (2), and finally (1), we obtain natural homeomorphisms

$$\begin{aligned} \mathbf{FLOW}(U \otimes X, Y) &\cong \prod_{c \in \pi_0(U)} \mathbf{FLOW}(U_c \boxtimes X, Y) \\ &\cong \prod_{c \in \pi_0(U)} \mathbf{TOP}(U_c, \mathbf{FLOW}(X, Y)) \\ &\cong \mathbf{TOP}\left(\prod_{c \in \pi_0(U)} U_c, \mathbf{FLOW}(X, Y)\right) \\ &\cong \mathbf{TOP}(U, \mathbf{FLOW}(X, Y)). \end{aligned}$$

All homeomorphisms are natural in U , X and Y . □

Let $U \in \mathbf{Top}$ and let Y be a flow. Define

$$(5) \quad Y^U := \prod_{c \in \pi_0(U)} \{U_c, Y\}_S,$$

where the product is taken in $\mathbf{Flow}$. If $U = \emptyset$, this is the terminal flow.

6.3. Proposition. *For any two flows X, Y and any space U , there is a natural homeomorphism*

$$\mathbf{FLOW}(X, Y^U) \cong \mathbf{TOP}(U, \mathbf{FLOW}(X, Y)).$$

Proof. Using the definition (5), the universal property of products in $\mathbf{Flow}$, the connected case (2), and (1), we obtain natural homeomorphisms

$$\begin{aligned} \mathbf{FLOW}(X, Y^U) &\cong \prod_{c \in \pi_0(U)} \mathbf{FLOW}(X, \{U_c, Y\}_S) \\ &\cong \prod_{c \in \pi_0(U)} \mathbf{TOP}(U_c, \mathbf{FLOW}(X, Y)) \\ &\cong \mathbf{TOP}\left(\prod_{c \in \pi_0(U)} U_c, \mathbf{FLOW}(X, Y)\right) \\ &\cong \mathbf{TOP}(U, \mathbf{FLOW}(X, Y)). \end{aligned}$$

Again, the construction is natural in U , X and Y . □

Combining Propositions 6.2 and 6.3, the standard $\mathbf{Top}$ -enriched category of flows is tensored and cotensored over $\mathbf{Top}$ in the sense of [2, Section 6.5]. Explicitly, for $U \in \mathbf{Top}$ and flows X, Y , the tensor and cotensor are

$$U \otimes X = \prod_{c \in \pi_0(U)} (U_c \boxtimes X), \quad Y^U = \prod_{c \in \pi_0(U)} \{U_c, Y\}_S,$$

and they satisfy the enriched adjunction homeomorphisms

$$\mathbf{FLOW}(U \otimes X, Y) \cong \mathbf{TOP}(U, \mathbf{FLOW}(X, Y)) \cong \mathbf{FLOW}(X, Y^U).$$

When U is path-connected and non-empty, the formulas reduce to the classical $U \boxtimes X$ and $\{U, Y\}_S$.

Since **Flow** is complete and cocomplete, it is therefore topologically bicomplete in the sense of Barthel–Riehl [1]. Since **Flow** is locally presentable by [10, Theorem 6.11] and topologically bicomplete, it satisfies the monomorphism hypothesis of [1, Definition 5.16] by [1, Remark 5.20].

We now write **H** with a capital letter for the Hurewicz model structure on **Flow** obtained from [1, Corollary 5.23]. Thus the weak equivalences are the homotopy equivalences defined by the topological enrichment, the fibrations are the maps that have the right lifting property against all cylinder inclusions of the form $i_0 : Z \rightarrow I \otimes Z$, and the cofibrations are the maps that have the left lifting property against all **H**-fibrations that are also homotopy equivalences. The **H**-cofibrations have the left lifting property against all maps of the form $p_0 : Z^I \rightarrow Z$. This terminology avoids any conflict with the **h**-model structure of flows of [12]: in both model structures, all objects are fibrant; however, there exist flows which are not **h**-cofibrant by [12, Proposition 7.9], whereas all flows are **H**-cofibrant.

7. COMPARISON BETWEEN THE **Q**- AND THE **H**-MODEL STRUCTURES

We first compare the homotopies in the two model structures.

7.1. Lemma. *Let $f, g : X \rightarrow Y$ be two maps of flows. If f and g are homotopic in the **H**-model structure, then $f^0 = g^0 : X^0 \rightarrow Y^0$, and the maps*

$$\mathbb{P}f, \mathbb{P}g : \mathbb{P}X \longrightarrow \mathbb{P}Y$$

are homotopic maps of spaces. More precisely, for every $\alpha, \beta \in X^0$, the restrictions

$$\mathbb{P}_{\alpha, \beta} f, \mathbb{P}_{\alpha, \beta} g : \mathbb{P}_{\alpha, \beta} X \longrightarrow \mathbb{P}_{f(\alpha), f(\beta)} Y$$

are homotopic.

Proof. A homotopy from f to g is a map

$$H : I \otimes X \longrightarrow Y, \quad I = [0, 1],$$

whose restrictions along the two endpoint maps $X \cong \{0\} \otimes X \rightarrow I \otimes X$ and $X \cong \{1\} \otimes X \rightarrow I \otimes X$ are f and g . Since I is path-connected, our formula for the tensor gives

$$I \otimes X = I \boxtimes X \quad \text{and} \quad (I \otimes X)^0 = X^0.$$

The two endpoint maps are the identity on the set of states. Hence the two restrictions of H have the same map on states, i.e. $f^0 = g^0$. By the tensor adjunction, H is equivalently a continuous map

$$\widehat{H} : I \longrightarrow \mathbf{FLOW}(X, Y)$$

from f to g . The composite

$$I \times \mathbb{P}X \xrightarrow{\widehat{H} \times \text{Id}} \mathbf{FLOW}(X, Y) \times \mathbb{P}X \xrightarrow{\text{ev}} \mathbb{P}Y$$

is a homotopy from $\mathbb{P}f$ to $\mathbb{P}g$. Since $f^0 = g^0$, it restricts to a homotopy $H_{\alpha,\beta}$ for every $\alpha, \beta \in X^0$:

$$\begin{array}{ccc}
 \mathbb{P}_{\alpha,\beta}X & & \\
 \downarrow i_0 & \searrow \mathbb{P}_{\alpha,\beta}f & \\
 I \times \mathbb{P}_{\alpha,\beta}X & \xrightarrow{H_{\alpha,\beta}} & \mathbb{P}_{f(\alpha),f(\beta)}Y \\
 \uparrow i_1 & \nearrow \mathbb{P}_{\alpha,\beta}g & \\
 \mathbb{P}_{\alpha,\beta}X & &
 \end{array}$$

□

7.2. Proposition. *Let $f : X \rightarrow Y$ be a weak equivalence of the H-model structure of flows. Then $f^0 : X^0 \rightarrow Y^0$ is a bijection, and*

$$\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$$

is a homotopy equivalence of spaces. In fact, for every $\alpha, \beta \in X^0$, the induced map

$$\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$$

is a homotopy equivalence.

In the language of [12], every weak equivalence of the H-model structure of flows is a weak equivalence of the h-model structure. The converse is false, since a weak equivalence of the h-model structure is not necessarily invertible up to homotopy.

Proof. By definition of the weak equivalences in the H-model structure, f is a homotopy equivalence in the topologically enriched category **Flow**. Choose a homotopy inverse $g : Y \rightarrow X$ and homotopies

$$gf \simeq \text{Id}_X, \quad fg \simeq \text{Id}_Y.$$

Applying Lemma 7.1 to these homotopies gives

$$g^0 f^0 = \text{Id}_{X^0}, \quad f^0 g^0 = \text{Id}_{Y^0},$$

so f^0 is a bijection. The same lemma gives homotopies of spaces

$$\mathbb{P}g \mathbb{P}f \simeq \text{Id}_{\mathbb{P}X}, \quad \mathbb{P}f \mathbb{P}g \simeq \text{Id}_{\mathbb{P}Y}.$$

Thus $\mathbb{P}f$ is a homotopy equivalence of spaces, with homotopy inverse $\mathbb{P}g$. Finally, because the maps on states of gf and Id_X are equal, and likewise for fg and Id_Y , the same argument restricts to each summand $\mathbb{P}_{\alpha,\beta}X$. □

7.3. Lemma. *Let $p : E \rightarrow B$ be a fibration of the H-model structure of flows. Then for every $\alpha, \beta \in E^0$, the induced map*

$$\mathbb{P}_{\alpha,\beta}p : \mathbb{P}_{\alpha,\beta}E \longrightarrow \mathbb{P}_{p(\alpha),p(\beta)}B$$

is a Hurewicz fibration of spaces.

Proof. Let $Z \in \mathbf{Top}$. A lifting problem in spaces

$$\begin{array}{ccc} Z \times \{0\} & \longrightarrow & \mathbb{P}_{\alpha,\beta} E \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ Z \times I & \longrightarrow & \mathbb{P}_{p(\alpha),p(\beta)} B \end{array}$$

is equivalently a lifting problem of flows

$$\begin{array}{ccc} \text{Glob}(Z) & \longrightarrow & E \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ \text{Glob}(Z \times I) & \longrightarrow & B \end{array}$$

Here the left vertical map is induced by $Z \times \{0\} \subset Z \times I$, and the maps on states are determined by α, β and $p(\alpha), p(\beta)$. By the globe formula (3), this left vertical map is the cylinder inclusion

$$\text{Glob}(Z) \longrightarrow I \otimes \text{Glob}(Z).$$

Since p is an H-fibration, it has the right lifting property with respect to all such cylinder inclusions. The resulting lift is precisely a lift in the original topological lifting problem. Hence $\mathbb{P}_{\alpha,\beta} p$ is a Hurewicz fibration. $\square$

7.4. Theorem. *Every q -cofibration of flows is an H-cofibration of flows.*

Proof. Since the q -cofibrations are the retracts of relative $(I^{\text{gl}} \cup \{C, R\})$ -cell complexes and since the class of H-cofibrations is closed under pushouts, transfinite compositions and retracts, it suffices to prove that every element of $I^{\text{gl}} \cup \{C, R\}$ is an H-cofibration.

Let $p : E \rightarrow B$ be a trivial H-fibration. By Proposition 7.2, the map $p^0 : E^0 \rightarrow B^0$ is a bijection. Therefore C has the left lifting property with respect to p : a map $\{0\} \rightarrow B$ is just a state of B , and it has a unique preimage in E^0 . The fact that this map is a bijection also implies that R has the left lifting property with respect to p : if two states of E have the same image in B , then they are equal. It remains to treat the globular generators of I^{gl} . Let $j : S^{n-1} \rightarrow D^n$ be the standard inclusion. A lifting problem

$$\begin{array}{ccc} \text{Glob}(S^{n-1}) & \longrightarrow & E \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ \text{Glob}(D^n) & \longrightarrow & B \end{array}$$

fixes two states $\alpha, \beta \in E^0$ and is exactly the same thing as a lifting problem in $\mathbf{Top}$

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathbb{P}_{\alpha,\beta} E \\ j \downarrow & \nearrow \text{dashed} & \downarrow \mathbb{P}_{\alpha,\beta} p \\ D^n & \longrightarrow & \mathbb{P}_{p(\alpha),p(\beta)} B \end{array}$$

By Lemma 7.3, the right vertical map is a Hurewicz fibration of spaces. By Proposition 7.2, it is also a homotopy equivalence. Thus it is a trivial fibration in the Hurewicz model structure of $\mathbf{Top}$. The inclusion $S^{n-1} \subset D^n$ is a closed NDR pair, hence a cofibration in that Hurewicz model structure. Therefore the displayed lifting problem has a solution. This proves that every globular generator is an H-cofibration, and the theorem follows. $\square$

7.5. Proposition. *Let $f : X \rightarrow Y$ be a weak equivalence of the q -model structure of flows. If X and Y are q -cofibrant, then f is a weak equivalence of the H -model structure. Equivalently, f is a homotopy equivalence for the topological enrichment of $\mathbf{Flow}$.*

Proof. All flows are fibrant in the q -model structure. Hence X and Y are cofibrant-fibrant objects of the q -model structure. By the usual Whitehead theorem for model categories, a weak equivalence between cofibrant-fibrant objects admits a homotopy inverse, with homotopies computed using any cylinder object of the q -model structure. By [7, Corollary 7.11], and since $I = [0, 1]$ is path-connected, two maps of flows $f, g : X \rightarrow Y$ are homotopic for the q -model structure if there exists a map

$$H : I \boxtimes X = I \otimes X \longrightarrow Y$$

such that the composite

$$X \amalg X \longrightarrow I \otimes X \xrightarrow{H} Y, \quad I = [0, 1],$$

is equal to (f, g) . Therefore the homotopy inverse supplied by the Whitehead criterion is also an inverse up to homotopy in the H -model structure. This is precisely to say that f is a weak equivalence for the H -model structure. $\square$

8. PUSHOUT ALONG A Q -COFIBRANT REPLACEMENT

We now generalize Theorem 5.4 to any q -cofibrant replacement of any order-reflecting inclusion of posets $P \subset Q$.

8.1. Proposition. *Let $\mathcal{M}$ be a model category. Consider a commutative diagram of solid arrows*

$$\begin{array}{ccccc} A & \overset{\ell}{\dashrightarrow} & B & \xrightarrow{\simeq} & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \overset{\ell'}{\dashrightarrow} & B' & \xrightarrow{\simeq} & C' \end{array}$$

such that the maps $B \rightarrow C$ and $B' \rightarrow C'$ are trivial fibrations and such that the map $A \rightarrow A'$ is a cofibration between cofibrant objects. Then there exist maps $\ell : A \rightarrow B$ and $\ell' : A' \rightarrow B'$ making the diagram commutative.

Proof. It is [13, Proposition 2.5]. $\square$

8.2. Proposition. *Let $f : X \rightarrow Y$ be a q -cofibration of flows. Then for every $u, v \in X^0$, the induced map*

$$\mathbb{P}_{u,v}X \xrightarrow{f} \mathbb{P}_{f(u),f(v)}Y$$

is an h -cofibration of spaces.

When X and Y are q -cofibrant, the result follows from [11, Theorem 5.7]. The following proof does not require this assumption.

Proof. Every q -cofibration is a retract of a relative $(I^{\mathbb{N}} \cup \{C, R\})$ -cell complex. After refining the cellular presentation, we may suppose that one generating q -cofibration is attached at each successor stage.

For a pushout of a map of I^{gl} , the induced maps on spaces of execution paths are h-cofibrations by [11, Theorem 5.4(1)]. A pushout of C induces a homeomorphism on spaces of execution paths. Finally, consider a pushout of R , and denote its map on states by q . If the two chosen states are already equal, the pushout is an isomorphism. Otherwise, the description by reduced words of this pushout gives, for every pair of states a, b , a homeomorphism

$$\mathbb{P}_{q(a),q(b)}Y \cong \mathbb{P}_{a,b}X \amalg Z_{a,b}$$

for some space $Z_{a,b}$, under which

$$\mathbb{P}_{a,b}X \longrightarrow \mathbb{P}_{q(a),q(b)}Y$$

is the inclusion of the first coproduct summand. Since every topological space is h-cofibrant, this map is an h-cofibration.

Consider now a cellular tower

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\lambda = Y,$$

and let u_α, v_α be the images in X_α^0 of fixed states $u, v \in X^0$. The preceding discussion shows that every successor map

$$\mathbb{P}_{u_\alpha, v_\alpha}X_\alpha \longrightarrow \mathbb{P}_{u_{\alpha+1}, v_{\alpha+1}}X_{\alpha+1}$$

is an h-cofibration. It also shows that every map

$$\mathbb{P}X_\alpha \longrightarrow \mathbb{P}X_{\alpha+1}$$

is an h-cofibration, and hence a relative- T_1 inclusion. At every nonzero limit ordinal $\beta \leq \lambda$, the homeomorphism of Theorem 5.2 restricts to a homeomorphism

$$\varinjlim_{\alpha < \beta} \mathbb{P}_{u_\alpha, v_\alpha}X_\alpha \cong \mathbb{P}_{u_\beta, v_\beta}X_\beta.$$

Consequently,

$$\mathbb{P}_{u,v}X \longrightarrow \mathbb{P}_{f(u),f(v)}Y$$

is a transfinite composite of h-cofibrations and is therefore an h-cofibration. The result follows since h-cofibrations are closed under retracts. $\square$

8.3. Definition. A *q-cofibrant replacement* of a map of flows $f : X \rightarrow Y$ is a commutative square

$$\begin{array}{ccc} X^{\text{cof}} & \xrightarrow{\sim} & X \\ f^{\text{cof}} \downarrow & & \downarrow f \\ Y^{\text{cof}} & \xrightarrow{\sim} & Y \end{array}$$

in which X^{cof} and Y^{cof} are q-cofibrant, the horizontal maps are trivial q-fibrations, and f^{cof} is a q-cofibration.

8.4. **Theorem.** Let $P \subset Q$ be an order-reflecting inclusion of posets, and let

$$i^{\text{cof}} : P^{\text{cof}} \longrightarrow Q^{\text{cof}}$$

be a q -cofibrant replacement. Let

$$\begin{array}{ccc} P^{\text{cof}} & \xrightarrow{\varphi} & X \\ \downarrow i^{\text{cof}} & & \downarrow f \\ Q^{\text{cof}} & \xrightarrow{\varphi'} & Y \end{array} \quad \lrcorner$$

be a pushout of flows. Then for all states $u, v \in X^0$, the induced map

$$\mathbb{P}_{u,v}X \longrightarrow \mathbb{P}_{f(u),f(v)}Y$$

is a trivial h -cofibration of topological spaces.

We apply the gluing lemma [14, Lemma 8.12] in the H-model structure of Section 6. All flows are H-cofibrant and H-fibrant, so this model structure is proper.

Proof. Consider the commutative diagram of flows

$$\begin{array}{ccccc} & & \text{Ch}(P) & \xrightarrow{\ell} & P^{\text{cof}} & \xrightarrow{\simeq} & P \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{Ch}(Q) & \xrightarrow{\ell'} & Q^{\text{cof}} & \xrightarrow{\simeq} & Q \end{array}$$

The existence of ℓ and ℓ' is a consequence of Proposition 8.1: they are weak equivalences of flows by the two-out-of-three property. Consider the flow Y^{ch} defined by the pushout diagram

$$\begin{array}{ccc} \text{Ch}(P) & \xrightarrow{\varphi^\ell} & X \\ \text{Ch}(i) \downarrow & & \downarrow g \\ \text{Ch}(Q) & \xrightarrow{j} & Y^{\text{ch}} \end{array} \quad \lrcorner$$

Let $k : Y^{\text{ch}} \rightarrow Y$ be the canonical map, so that $kg = f$ and $kj = \varphi'\ell'$. We obtain the commutative diagram

$$\begin{array}{ccccc} \text{Ch}(P) & \xrightarrow{\quad} & X & & \\ \downarrow & \searrow \simeq & \downarrow & \searrow & \\ & & P^{\text{cof}} & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow & & \downarrow \\ \text{Ch}(Q) & \xrightarrow{\quad} & Y^{\text{ch}} & \xrightarrow{k} & Y \\ & \searrow \simeq & \downarrow & \searrow & \\ & & Q^{\text{cof}} & \xrightarrow{\quad} & Y \end{array}$$

The flows $\text{Ch}(P)$ and $\text{Ch}(Q)$ are q -cofibrant by Proposition 3.3, and P^{cof} and Q^{cof} are q -cofibrant by hypothesis. Thus the maps $\text{Ch}(P) \rightarrow P^{\text{cof}}$ and $\text{Ch}(Q) \rightarrow Q^{\text{cof}}$ are weak equivalences of the H-model structure of flows by Proposition 7.5. The vertical maps $\text{Ch}(i) : \text{Ch}(P) \rightarrow \text{Ch}(Q)$ and $i^{\text{cof}} : P^{\text{cof}} \rightarrow Q^{\text{cof}}$ are H-cofibrations of flows by Theorem 7.4. By the gluing lemma [14, Lemma 8.12] applied in the H-model structure of flows, the map $Y^{\text{ch}} \rightarrow Y$ is a weak equivalence of the H-model structure of flows. For $u, v \in X^0$, consider the commutative triangle

$$\begin{array}{ccc} \mathbb{P}_{u,v}X & \xrightarrow{\simeq} & \mathbb{P}_{g(u),g(v)}Y^{\text{ch}} \\ & \searrow & \downarrow \simeq \\ & & \mathbb{P}_{f(u),f(v)}Y. \end{array}$$

The horizontal map is a trivial h-cofibration by Theorem 5.4. The vertical map is a homotopy equivalence by Proposition 7.2. Thus their composite is a homotopy equivalence. Moreover, f is a q -cofibration, as a pushout of a q -cofibration. Proposition 8.2 shows that the same composite is an h-cofibration. It is therefore a trivial h-cofibration. $\square$

8.5. Corollary. *Let $f : X \rightarrow Y$ be a retract of a transfinite composition of pushouts of q -cofibrant replacements of order-reflecting inclusions of posets. Then, for any pair of states $u, v \in X^0$, f induces a trivial h-cofibration of spaces*

$$\mathbb{P}_{u,v}X \xrightarrow{\simeq} \mathbb{P}_{f(u),f(v)}Y.$$

In particular, it is a homotopy equivalence.

Proof. We first consider a transfinite composition

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\lambda$$

in which, for every $\alpha < \lambda$, the map

$$j_\alpha : X_\alpha \longrightarrow X_{\alpha+1}$$

is a pushout of a q -cofibrant replacement of an order-reflecting inclusion of posets.

The map on states of every such q -cofibrant replacement is injective. Indeed, in the commutative square of Definition 8.3, the two trivial q -fibrations induce bijections on the sets of states, and hence the map on states of the q -cofibrant replacement is conjugate, by these bijections, to the given inclusion of posets. Since pushouts and transfinite compositions of injections are injections in **Set**, all the maps

$$j_\alpha^0 : X_\alpha^0 \longrightarrow X_{\alpha+1}^0$$

are injective.

Fix $u, v \in X_0^0$, and denote by u_α, v_α their images in X_α^0 . By Theorem 8.4, for every $\alpha < \lambda$, the map

$$\mathbb{P}_{u_\alpha, v_\alpha}X_\alpha \longrightarrow \mathbb{P}_{u_{\alpha+1}, v_{\alpha+1}}X_{\alpha+1}$$

is a trivial h-cofibration.

We now verify the behavior at limit ordinals. For every $\alpha < \lambda$, the map

$$\mathbb{P}X_\alpha \longrightarrow \mathbb{P}X_{\alpha+1}$$

factors as

$$\coprod_{(a,b) \in X_\alpha^0 \times X_\alpha^0} \mathbb{P}_{a,b} X_\alpha \longrightarrow \coprod_{(a,b) \in X_\alpha^0 \times X_\alpha^0} \mathbb{P}_{j_\alpha(a), j_\alpha(b)} X_{\alpha+1} \longrightarrow \mathbb{P} X_{\alpha+1}.$$

The first map is a coproduct of trivial h-cofibrations by Theorem 8.4. The second map is the inclusion of a coproduct summand, because j_α^0 is injective. Since every topological space is h-cofibrant, this second map is an h-cofibration. Consequently,

$$\mathbb{P} X_\alpha \longrightarrow \mathbb{P} X_{\alpha+1}$$

is an h-cofibration and therefore a relative- T_1 inclusion.

Let $\beta \leq \lambda$ be a nonzero limit ordinal. Theorem 5.2 gives a homeomorphism

$$\varinjlim_{\alpha < \beta} \mathbb{P} X_\alpha \cong \mathbb{P} X_\beta.$$

Since all maps on states in the tower are injective, this homeomorphism restricts to a homeomorphism

$$\varinjlim_{\alpha < \beta} \mathbb{P}_{u_\alpha, v_\alpha} X_\alpha \cong \mathbb{P}_{u_\beta, v_\beta} X_\beta.$$

Thus the tower of path spaces

$$(\mathbb{P}_{u_\alpha, v_\alpha} X_\alpha)_{\alpha \leq \lambda}$$

is continuous at every limit ordinal, and all its successor maps are trivial h-cofibrations. Since trivial h-cofibrations are closed under transfinite composition, the induced map

$$\mathbb{P}_{u,v} X_0 \longrightarrow \mathbb{P}_{u_\lambda, v_\lambda} X_\lambda$$

is a trivial h-cofibration.

Finally, let $f : X \rightarrow Y$ be a retract of such a transfinite composition $g : A \rightarrow B$. Write the retraction as the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{r_0} & A & \xrightarrow{s_0} & X \\ f \downarrow & & \downarrow g & & \downarrow f \\ Y & \xrightarrow{r_1} & B & \xrightarrow{s_1} & Y, \end{array} \quad s_0 r_0 = \text{Id}_X, \quad s_1 r_1 = \text{Id}_Y.$$

For every $u, v \in X^0$, the map

$$\mathbb{P}_{u,v} X \longrightarrow \mathbb{P}_{f(u), f(v)} Y$$

is then a retract, in the arrow category of topological spaces, of

$$\mathbb{P}_{r_0(u), r_0(v)} A \longrightarrow \mathbb{P}_{g(r_0(u)), g(r_0(v))} B.$$

The latter is a trivial h-cofibration by the preceding argument. Since trivial h-cofibrations are closed under retracts, the map induced by f is a trivial h-cofibration. $\square$

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