GLOBULAR SUBDIVISIONS ARE DIHOMOTOPY EQUIVALENCES

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ABSTRACT. We prove that any globular subdivision of multipointed d-spaces gives rise to a dihomotopy equivalence between the associated flows. The proof involves a new Reedy category which is a tweak of the one used to prove the left properness of the q-model category of flows. As a straightforward application, the flows associated to two multipointed d-spaces related by a finite zigzag of globular subdivisions have isomorphic branching and merging homology theories.

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1. INTRODUCTION

Presentation. Directed Algebraic Topology (DAT) studies mathematical objects arising from the geometric study of concurrent systems up to homotopy [6]. There are two main classes of geometric models: the *continuous* models like Grandis' directed spaces (Definition 4.3) and Krishnan's streams [24], and the *multipointed* models like multipointed d-spaces (Definition 3.1) and flows (Definition 3.4) which have a distinguished set of states. The main multipointed *combinatorial* model of concurrency is the category of precubical sets, a *n*-cube representing the concurrent execution of *n* actions [6]. A precubical set can be realized in any of the geometric models of concurrency above.

There is no known convenient model category structure on continuous DAT models, as all attempts to date cannot prevent the directed segment from being contracted by weak equivalences, which implies that the weak equivalences destroy the causal structure. On the other hand, combinatorial model category structures have been introduced for

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multipointed d-spaces and flows. They are related by a zigzag of Quillen equivalences by [14, Theorem 10.9] and [18, Theorem 14]. There is also a functor cat : $\mathcal{M}dTop \rightarrow Flow$ from multipointed d-spaces to flows which is neither a left adjoint nor a right adjoint and such that the total left derived functor in the sense of [5] induces an equivalence of categories between the homotopy categories of the model structures [18, Theorem 15]. Unfortunately, there are two issues: 1) the weak equivalences are extremely rigid, the weak equivalences inducing bijections between the distinguished sets of states; 2) the weak equivalences do not preserve the causal structure *between* the distinguished set of states. The latter problem is explained in detail in [19, Section 10 and Section 11].

To overcome the first issue, the paper [10] introduces a notion of flow up to dihomotopy: two flows are dihomotopy equivalent if they can be related by a finite zigzag of weak equivalences and of retracts of transfinite compositions of pushouts of generating subdivisions in the sense of Definition 8.3. A Whitehead theorem is even proved [10, Theorem 4.6 and 4.7], namely that any dihomotopy equivalence between two q-cofibrant homotopy continuous flows is invertible up to homotopy, the notion of homotopy continuous object playing the role of fibrant object in this theory. Unfortunately, it can also be proved that the categorical localization of flows up to dihomotopy is not the homotopy category of a model structure on flows [9, Theorem 5.7]. Not much more is known about this categorical localization. However, it is conjecturally the homotopy category of a cofibration category. Concerning the second issue, and after [19] proving the globular analogue of Dubut's results from [4], the category of cellular multipointed d-spaces (i.e. the cellular objects of their q-model structure) up to globular subdivisions (Definition 6.1) seems to be a framework which deserves to be studied more carefully. Indeed, not only it contains all examples coming from computer science, but also, unlike dihomotopy equivalences of flows, globular subdivisions preserve the causal structure between the distinguished sets of states in the following sense. The natural systems in Dubut's sense of the associated directed spaces of two cellular multipointed d-spaces related by a finite zigzag of globular subdivisions are bisimilar up to homotopy, whether the discrete or the continuous versions of natural system is used [19, Theorem 8.16 and Theorem 9.4].

The purpose of this paper is to link the approaches of [10] and [19] by the theorem stated now:

Theorem. (Corollary 10.4) Let X and Y be two cellular multipointed d-spaces related by a finite zigzag sequence of globular subdivisions. Then the associated flows cat(X) and cat(Y) are related by a finite zigzag of maps of $cell(\mathcal{T}^{cof})$ (i.e. transfinite compositions of pushouts of generating subdivisions) and of weak equivalences of flows. Using the language of [10], the two flows cat(X) and cat(Y) are dihomotopy equivalent.

The induced functor from the category of multipointed *d*-spaces up to globular subdivisions to the category of flows up to dihomotopy equivalences cannot be an equivalence of categories. Indeed, [19, Section 10] provides in [19, Proposition 10.1] an example of a trivial q-fibration between two cellular multipointed *d*-spaces $f : A \to B$ such that the natural systems in Dubut's sense of the associated directed spaces $\overrightarrow{Sp}(A)$ and $\overrightarrow{Sp}(B)$ (see Notation 4.4) cannot be bisimilar up to homotopy. By [19, Theorem 8.16 and Theorem 9.4], this implies that the cellular multipointed *d*-spaces A and B cannot be isomorphic in the categorical localization of the cellular multipointed *d*-spaces by the globular subdivisions. It would be interesting to prove the same results for the cubical subdivisions, since cubical subdivisions of realizations of precubical sets as directed spaces preserve the bisimilarity type [4]. This problem is left for a future work.

As a straightforward application of Corollary 10.4 and [11, Corollary 11.3], if $f: X \to Y$ is a globular subdivision, then the associated map of flows $\operatorname{cat}(f) : \operatorname{cat}(X) \to \operatorname{cat}(Y)$ induces an isomorphism on the branching and merging homology theories. These homology theories detect the non-deterministic branching and merging areas of execution paths. There are methods to define the branching and merging homology theories of a multipointed *d*-space without using the functor $\operatorname{cat} : \mathcal{M} \operatorname{dTop} \to \operatorname{Flow}$ and the constructions of [11]. These methods will make the preservation of the branching and merging homology theories by the globular subdivisions much easier to prove. This will be the subject of a subsequent paper.

Outline of the paper. Section 2 recalls some basic facts in category theory and in model category theory.

Section 3 recalls some basic facts about multipointed *d*-spaces and flows and everything the reader needs to know about the functor cat : $\mathcal{M}dTop \rightarrow Flow$ from multipointed *d*-spaces to flows which is gathered in Theorem 3.6. Note that it is not necessary to recall the definition of a Moore flow. It is only required to know that there exists some model category between the model categories of multipointed *d*-spaces and flows relating the latter by a zigzag of Quillen equivalences satisfying some very specific properties. The reader is invited to read the statement of Theorem 3.6 carefully because some properties are unusual.

Section 4 recalls the link between the formalism of multipointed d-spaces and flows and the formalism of directed spaces and enriched small categories. This enables us to obtain simpler statements for Theorem 6.8 and Theorem 9.5 and to use some results from [19] in this paper.

Section 5 introduces the cellular multipointed *d*-spaces and expounds some preparatory lemmas to study globular subdivisions in Section 6. We introduce in Proposition 5.3 a cellular multipointed *d*-space $\operatorname{Glob}^{top}(\mathbf{D}^n)_F$ which is the (n + 1)-dimensional topological globe $\operatorname{Glob}^{top}(\mathbf{D}^n)$ with a finite set *F* of additional distinguished states belonging to the interior of the underlying topological space. The main result is Proposition 5.5 which states that the the map of multipointed *d*-spaces $\operatorname{Glob}^{top}(\mathbf{S}^{n-1}) \subset \operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is cellular.

Section 6 studies the notion of globular subdivision (Definition 6.1) which was introduced in [20, Definition 4.10] for a very specific kind of cellular multipointed d-spaces and in [19, Definition 9.1] for general cellular multipointed d-spaces. The important notion of connection map of a globular subdivision is introduced in Definition 6.6. Theorem 6.8 is the first key fact of the paper: it is a consequence of the definition of the connection maps and of the specific geometric properties of the topological globes, which are already used in [19]. Finally, Theorem 6.9 proves that the source and target of any globular subdivision have compatible cellular decompositions. It is the second key fact of the paper.

Section 7 introduces a new Reedy category $\mathcal{P}^{C}(S)$ where $C: P \to S$ is a set map from the underlying set of a nonempty poset (P, \leq) to a nonempty set. It is a tweak of the one introduced in [16, Section 3]. We recover the Reedy category of [16] with $(P, \leq) = \{0 < 1\}$. It is necessary for the proof of Theorem 8.5 which leads to Corollary 8.6. Section 8 introduces the notion of generating subdivision (see Definition 8.3). An arbitrary choice of q-cofibrant replacements is made to introduce an essentially small class \mathcal{T}^{cof} of generating subdivisions. We also prove that the spaces of execution paths are preserved by a pushout along a generating subdivision in Corollary 8.6. It is the third key fact of the paper.

Section 9 introduces a specific set \mathcal{T}^{gl} of generating subdivisions (see Notation 9.4) which is specially designed for the calculations of Theorem 9.5. We then prove Theorem 9.5 which is the analogue for the maps of \mathcal{T}^{gl} of Theorem 6.8 for the globular subdivisions. This is the fourth key fact of the paper.

Section 10 uses all preceding results to prove that the image by the functor cat : $\mathcal{M}dTop \rightarrow Flow$ of every globular subdivision factors as a map of $cell(\mathcal{T}^{cof})$ followed by a weak equivalence of flows. Finally, we obtain Corollary 10.4 which is the goal of the paper. Some comments follow about the notion of underlying homotopy type of a flow.

2. Prerequisites and categorical preliminaries

The knowledge of [10], in particular the exact notion of a dihomotopy of flows, is not required to understand this paper. On the other hand, the results expounded in this work rely on [14–16, 18, 19]. Reminders are given throughout the paper.

All facts of this section are well-known or are variants of well-known facts. The arguments are sketched for the ease of the reader (this also enables us to fix the notations).

2.1. **Proposition.** Let \mathcal{K} be a cocomplete category. Consider the commutative diagram of objects of \mathcal{K}



If \underline{C} and \underline{D} are pushout squares, then the composite square $\underline{C} + \underline{D}$ is a pushout square. If \underline{C} and \underline{E} are pushout squares, then the composite square $\underline{C} + \underline{E}$ is a pushout square.

Proof. The proof is straightforward.

2.2. Corollary. Let \mathcal{K} be a cocomplete category. Consider the commutative diagram of objects of \mathcal{K}



If <u>C</u> and <u>C</u> + <u>D</u> are pushout squares, then the commutative square <u>D</u> is a pushout square. If <u>C</u> and <u>C</u> + <u>E</u> are pushout squares, then the commutative square <u>E</u> is a pushout square.

Proof. Assume that \underline{C} and $\underline{C} + \underline{D}$ are pushout squares. Replace the commutative square \underline{D} by a pushout square $\underline{D'}$. Then by Proposition 2.1, $\underline{C} + \underline{D'}$ is a pushout square. Hence $\underline{C} + \underline{D} \cong \underline{C} + \underline{D'}$, $\underline{C} + \underline{D}$ being a pushout square. We deduce that the commutative square \underline{D} is a pushout square. The other case is similar.

Let \mathcal{K} be a cocomplete category. A transfinite tower in \mathcal{K} consists of a colimitpreserving functor $X : \lambda \to \mathcal{K}$ from a transfinite ordinal λ viewed as a small category to \mathcal{K} . Let $X_{\lambda} = \varinjlim X$: it is called the transfinite composition of the transfinite tower. The notation $f \boxtimes g$ means that f satisfies the left lifting property (LLP) with respect to g, or equivalently that g satisfies the right lifting property (RLP) with respect to f. For a class of maps \mathcal{C} , let $\operatorname{inj}(\mathcal{C}) = \{g \in \mathcal{K}, \forall f \in \mathcal{C}, f \boxtimes g\}$, $\operatorname{cof}(\mathcal{C}) = \{f \mid \forall g \in \operatorname{inj}(\mathcal{C}), f \boxtimes g\}$ and $\operatorname{cell}(\mathcal{C})$ denotes the class of transfinite compositions of pushouts of elements of \mathcal{C} . In a locally presentable category, and if \mathcal{C} is a set of maps, $\operatorname{cof}(\mathcal{C})$ is the class of maps which are retracts of maps of $\operatorname{cell}(\mathcal{C})$ by [23, Corollary 2.1.15].

Let \mathcal{K} be a bicomplete category. Let $\mathcal{D}: I \to \mathcal{K}$ be a diagram over a Reedy category (I, I_+, I_-) . The latching category at $i \in I$ is denoted by $\partial(I_+\downarrow i)$, the latching object at $i \in I$ by $L_i\mathcal{D} := \varinjlim_{\partial(I_+\downarrow i)} \mathcal{D}$, the matching category at $i \in I$ by $\partial(i\downarrow I_-)$ and the matching object at $i \in I$ by $M_i\mathcal{D} = \varprojlim_{\partial(i\downarrow I_-)} \mathcal{D}$. See e.g. [22, Definition 15.2.3 and Definition 15.2.5] or [23, Definition 5.1.2] for the general definitions of a latching/matching category/object.

A model category is a bicomplete category \mathcal{K} equipped with a class of cofibrations \mathcal{C} , a class of fibrations \mathcal{F} and a class of weak equivalences \mathcal{W} such that: 1) \mathcal{W} is closed under retract and satisfies two-out-of-three property, 2) the pairs $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are functorial weak factorization systems. We refer to [23, Chapter 1] and to [22, Chapter 7] for the basic notions about general model categories. A cofibration is denoted by $\bullet \longrightarrow \bullet$, a fibration by $\bullet \longrightarrow \bullet$ and a weak equivalence by $\bullet \stackrel{\simeq}{\longrightarrow} \bullet$.

A cellular object X of a cofibrantly generated model category is an object such that the canonical map $\emptyset \to X$ belongs to **cell**(I) where I is the set of generating cofibrations. The transfinite sequence of pushouts is called a *cellular decomposition* of X and each pushout is called a *cell*. In a combinatorial model category (i.e. a model category such that the underlying category is locally presentable [25]), every map $X \to Y$ factors as a composite $X \to Z \to Y$ such that the left-hand map belongs to **cell**(I) and the right-hand map to **inj**(I) by [2, Proposition 1.3].

2.3. **Proposition.** Let \mathcal{K} be a model category. Let $F, G : \lambda \to \mathcal{K}$ be two transfinite towers of cofibrant objects such that all maps of the towers are cofibrations. Let $\mu : F \Rightarrow G$ be an objectwise weak equivalence. Then $\lim_{n \to \infty} \mu : \lim_{n \to \infty} F \to \lim_{n \to \infty} G$ is a weak equivalence.

Proof. By [23, Theorem 5.1.3], the colimit functor is a left Quillen functor if the category of towers \mathcal{K}^{λ} is equipped with its Reedy model structure and the two towers are Reedy cofibrant. Hence $\varinjlim \mu : \varinjlim F \to \varinjlim G$ is a weak equivalence.

By replacing in the statement of Proposition 2.4 cocomplete category by model category and cell(I) by cofibration, the same argument as the one of Proposition 2.3 completes the proof. Since Proposition 2.4 is slightly more general, we borrow the argument from [26, Lemma 9.3.4] which is initially written in the setting of cofibration categories. 2.4. **Proposition.** (variant of [23, Corollary 5.1.5] and [26, Lemma 9.3.4]) Let \mathcal{K} be a cocomplete category. Let I be a set of maps of \mathcal{K} . Let λ be a transfinite ordinal. Consider two transfinite towers $A : \lambda \to \mathcal{C}$ and $B : \lambda \to \mathcal{C}$ and a natural transformation $f : A \Rightarrow B$. Assume that for each ordinal $\nu < \lambda$, the map $B_{\nu} \sqcup_{A_{\nu}} A_{\nu+1} \to B_{\nu+1}$ belongs to cell(I). Then the map

$$B_0 \sqcup_{A_0} \varinjlim A_\lambda \longrightarrow \varinjlim B_\lambda$$

belongs to $\operatorname{cell}(I)$ as well.

Proof. Denote $A = \lim_{n \to \infty} A_n$ and $B = \lim_{n \to \infty} B_n$. We consider the commutative diagram of \mathcal{K}



The map $B_0 \sqcup_{A_0} A_\lambda \to B_\lambda$ is the transfinite composition of the maps $B_\nu \sqcup_{A_\nu} A_\lambda \to B_{\nu+1} \sqcup_{A_{\nu+1}} A_\lambda$ with $\nu < \lambda$, and each map of the transfinite sequence is a pushout of $B_\nu \sqcup_{A_\nu} A_{\nu+1} \to B_{\nu+1}$ along the map $B_\nu \sqcup_{A_\nu} A_{\nu+1} \to B_\nu \sqcup_{A_\nu} A_\lambda$. The proof is complete, each map $B_\nu \sqcup_{A_\nu} A_{\nu+1} \to B_{\nu+1}$ belonging to **cell**(*I*) by hypothesis. \Box

Finally, Proposition 2.5 is used in Proposition 10.1.

2.5. **Proposition.** Let \mathcal{K} be a model category. Consider a commutative diagram of solid arrows



such that the maps $B \to C$ and $B' \to C'$ are trivial fibrations and such that the map $A \to A'$ is a cofibration between cofibrant objects. Then there exist two maps $\ell : A \to B$ and $\ell' : A' \to B'$ making the diagram commutative.

Proof. Consider the Reedy model category $\mathcal{K}^{1\to 2}$ of functors from the direct Reedy category $1 \to 2$ to \mathcal{K} . It coincides with the projective model structures since the Reedy category $1 \to 2$ is direct. In this model category of maps, the map $A \to A'$ is a cofibration, and the commutative square



is a trivial fibration from the map $B \to B'$ to the map $C \to C'$. The commutative diagram



is a map in $\mathcal{K}^{1\to 2}$ from the cofibrant object $A \to A'$ to the object $C \to C'$. Hence the existence of a lift (ℓ, ℓ') from $A \to A'$ to $B \to B'$ making the whole diagram in the statement of the proposition commutative.

3. Multipointed d-spaces and flows

The category **Top** denotes either the category of Δ -generated spaces or of Δ -Hausdorff Δ -generated spaces (cf. [16, Section 2 and Appendix B]). It is Cartesian closed by a result due to Dugger and Vogt recalled in [13, Proposition 2.5] and locally presentable by [7, Corollary 3.7]. The internal hom is denoted by **TOP**(-, -). The right adjoint of the inclusion from Δ -generated spaces to general topological spaces is called the Δ -kelleyfication. We will make use of the well-known three model structures of **Top**, namely the q-model structure, the h-model structure and the m-model structure: see the end of [16, Appendix B] for an overview and the bibliography of the latter paper for many other references (e.g. [3]).

Let γ_1 and γ_2 be two continuous maps from [0, 1] to some topological space such that $\gamma_1(1) = \gamma_2(0)$. The composite defined by

$$(\gamma_1 *_N \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

is called the normalized composition.

Let \mathcal{M} be the set of non decreasing surjective maps from [0,1] to [0,1] equipped with the Δ -kelleyfication of the relative topology induced by the set inclusion $\mathcal{M} \subset$ **TOP**([0,1], [0,1]).

3.1. **Definition.** [18, Definition 3.4] A multipointed d-space X is a triple $(|X|, X^0, \mathbb{P}^{top}X)$ such that

• The pair $(|X|, X^0)$ is a multipointed space. The space |X| is called the underlying space of X and the set X^0 the set of states of X.

- The set $\mathbb{P}^{top}X$ is a set of continuous maps from [0, 1] to |X| called the *execution paths*, satisfying the following axioms:
 - For any execution path γ , one has $\gamma(0), \gamma(1) \in X^0$.
 - Let γ be an execution path of X. Then any composite $\gamma \phi$ with $\phi \in \mathcal{M}$ is an execution path of X.
 - Let γ_1 and γ_2 be two composable execution paths of X; then the normalized composition $\gamma_1 *_N \gamma_2$ is an execution path of X.

A map $f : X \to Y$ of multipointed *d*-spaces is a map of multipointed spaces from $(|X|, X^0)$ to $(|Y|, Y^0)$ such that for any execution path γ of X, the map $\mathbb{P}^{top}f : \gamma \mapsto f.\gamma$ is an execution path of Y. The category of multipointed *d*-spaces is denoted by \mathcal{M} dTop. Let $\mathbb{P}^{top}_{\alpha,\beta}X = \{\gamma \in \mathbb{P}^{top}X \mid \gamma(0) = \alpha, \gamma(1) = \beta\}$. The set $\mathbb{P}^{top}_{\alpha,\beta}X$ is equipped with the Δ -kelleyfication of the relative topology with respect to the inclusion $\mathbb{P}^{top}_{\alpha,\beta}X \subset \mathbf{TOP}([0,1],|X|)$. Thus a set map $Z \to \mathbb{P}^{top}_{\alpha,\beta}X$ where Z is Δ -generated, is continuous if and only if the associated map $Z \times [0,1] \to |X|$ is continuous.

The category $\mathcal{M}d\mathbf{Top}$ is locally presentable, and in particular bicomplete, by [18, Proposition 3.6]. The functor which forgets the execution paths $\Omega: X \mapsto (|X|, X^0)$ from $\mathcal{M}d\mathbf{Top}$ to the category \mathbf{mTop} of multipointed topological spaces is topological in the sense of [1, Section 21] by [18, Theorem 3.17]. In particular, it creates limits and colimits thanks to the initial and final structures respectively.

Every set S can be viewed a multipointed d-spaces (S, S, \emptyset) . Let X be a multipointed d-space. Let $S \subset X^0$. Denote by $X \upharpoonright_S$ the triple $(|X|, S, \mathbb{P}^{top}X)$; in particular $X \upharpoonright_{X^0} = X$. The topological globe of a topological space Z, which is denoted by $\operatorname{Glob}^{top}(Z)$, is the multipointed d-space defined as follows

• the underlying topological space is the quotient space

$$\frac{\{0,1\} \sqcup (Z \times [0,1])}{(z,0) = (z',0) = 0, (z,1) = (z',1) = 1}$$

- the set of states is $\{0, 1\}$
- the set of execution paths is the set of continuous maps

$$\{\delta_z \phi \mid \phi \in \mathcal{M}, z \in Z\}$$

with $\delta_z(t) = (z, t)$. It is equal to the underlying set of the space $Z \times \mathcal{M}$.

In particular, $\operatorname{Glob}^{top}(\emptyset)$ is the multipointed d-space $\{0, 1\} = (\{0, 1\}, \{0, 1\}, \emptyset)$.

3.2. Notation. Let $n \ge 1$. Denote by $\mathbf{D}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, x_1^2 + \cdots + x_n^2 \le 1\}$ the *n*-dimensional disk, and by $\mathbf{S}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, x_1^2 + \cdots + x_n^2 = 1\}$ the (n-1)dimensional sphere. By convention, let $\mathbf{D}^0 = \{0\}$ and $\mathbf{S}^{-1} = \emptyset$.

3.3. Notation. Denote by $\overrightarrow{I}^{top} = \text{Glob}^{top}(\mathbf{D}^0)$ the directed segment multipointed with the extremities.

The *q*-model structure of multipointed d-spaces [18, Section 4] is the unique combinatorial model structure such that

$$I^{gl,top} \cup \{C : \emptyset \to \{0\}, R : \{0,1\} \to \{0\}\}$$

with $I^{gl,top} = { \text{Glob}^{top}(\mathbf{S}^{n-1}) \subset \text{Glob}^{top}(\mathbf{D}^n) \mid n \ge 0 }$ is the set of generating cofibrations, the maps between globes being induced by the closed inclusions $\mathbf{S}^{n-1} \subset \mathbf{D}^n$, and such

that

$$J^{gl,top} = \{ \operatorname{Glob}^{top}(\mathbf{D}^n) \subset \operatorname{Glob}^{top}(\mathbf{D}^{n+1}) \mid n \ge 0 \}$$

is the set of generating trivial cofibrations, the maps between globes being induced by the closed inclusions $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. The weak equivalences are the maps of multipointed *d*-spaces $f : X \to Y$ inducing a bijection $f^0 : X^0 \cong Y^0$ and a weak homotopy equivalence $\mathbb{P}^{top}f : \mathbb{P}^{top}_{\alpha,\beta}X \to \mathbb{P}^{top}_{f(\alpha),f(\beta)}Y$ for all $(\alpha, \beta) \in X^0 \times X^0$ and the fibrations are the maps of multipointed *d*-spaces $f : X \to Y$ inducing a q-fibration $\mathbb{P}^{top}_{\alpha,\beta}f : \mathbb{P}^{top}_{\alpha,\beta}X \to \mathbb{P}^{top}_{f(\alpha),(\beta)}Y$ of topological spaces for all $(\alpha, \beta) \in X^0 \times X^0$.

3.4. **Definition.** [8, Definition 4.11] A *flow* X is a small enriched semicategory. Its set of objects (preferably called *states*) is denoted by X^0 and the space of morphisms (preferably called *execution paths*) from α to β is denoted by $\mathbb{P}_{\alpha,\beta}Y$ (e.g. [14, Definition 10.1]). The category is denoted by **Flow**.

The category **Flow** is locally presentable. Every set can be viewed as a flow with an empty path space. This give rise to a functor from sets to flows which is limit-preserving and colimit-preserving. More generally, any poset can be viewed as a flow, with a unique execution path from u to v if and only if u < v. This gives rise to a functor from posets to flows which is still limit-preserving but not colimit-preserving: loops are crushed because of the antisymmetry axiom in the category of posets (which is locally presentable, hence bicomplete) whereas they are not crushed in the category of flows.

3.5. Notation. For any topological space Z, the flow $\operatorname{Glob}(Z)$ is the flow having two states 0 and 1 and such that the only nonempty space of execution paths, when Z is nonempty, is $\mathbb{P}_{0,1}\operatorname{Glob}(Z) = Z$. It is called *the globe of* Z.

The q-model structure of flows [17, Theorem 7.6] is the unique combinatorial model structure such that

$$I^{gl} \cup \{C : \emptyset \to \{0\}, R : \{0, 1\} \to \{0\}\}$$

with $I^{gl} = {\text{Glob}(\mathbf{S}^{n-1}) \subset \text{Glob}(\mathbf{D}^n) \mid n \ge 0}$ is the set of generating cofibrations, the maps between globes being induced by the closed inclusions $\mathbf{S}^{n-1} \subset \mathbf{D}^n$, and such that

$$J^{gl} = \{ \operatorname{Glob}(\mathbf{D}^n) \subset \operatorname{Glob}(\mathbf{D}^{n+1}) \mid n \ge 0 \}$$

is the set of generating trivial cofibrations, the maps between globes being induced by the closed inclusions $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. The weak equivalences are the maps of flows $f: X \to Y$ inducing a bijection $f^0: X^0 \cong Y^0$ and a weak homotopy equivalence $\mathbb{P}f: \mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),f(\beta)}Y$ for all $(\alpha,\beta) \in X^0 \times X^0$ and the fibrations are the maps of flows $f: X \to Y$ inducing a q-fibration $\mathbb{P}_{\alpha,\beta}f: \mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),(\beta)}Y$ of topological spaces for all $(\alpha,\beta) \in X^0 \times X^0$.

There is a unique functor

$\operatorname{cat}: \mathcal{M}\mathbf{dTop} \longrightarrow \mathbf{Flow}$

from the category of multipointed *d*-spaces to the category of flows taking a multipointed *d*-space X to the unique flow $\operatorname{cat}(X)$ such that $\operatorname{cat}(X)^0 = X^0$ and such that $\mathbb{P}_{\alpha,\beta}\operatorname{cat}(X)$ is the quotient of the space of execution paths $\mathbb{P}_{\alpha,\beta}^{top}X$ by the equivalence relation generated by the reparametrization by \mathcal{M} , the composition of $\operatorname{cat}(X)$ being induced by the normalized composition. We gather now all what the reader needs to know about multipointed d-spaces and flows for this paper. Theorem 3.6 relies mostly on the results of [14, 15, 18]. Note that it is not necessary at all to know what a Moore flow is. It is only necessary to know that it exists.

3.6. Theorem. There exists a combinatorial q-model category \mathcal{M} Flow (whose objects are called Moore flows) satisfying the following properties:

(1) There is a Quillen equivalence

 $\mathbb{M}_{1}^{top}:\mathcal{M}\mathbf{Flow}\leftrightarrows\mathcal{M}\mathbf{dTop}:\mathbb{M}^{top}$

between the q-model structures such that the unit and the counit maps induce isomorphisms on q-cofibrant objects. Moreover, the right Quillen adjoint \mathbb{M}^{top} : $\mathcal{M}dTop \rightarrow \mathcal{M}Flow$ takes q-cofibrant (cellular resp.) multipointed d-spaces to q-cofibrant (cellular resp.) Moore flows.

(2) There is a Quillen equivalence

$\mathbb{M}_{!}:\mathcal{M}\mathbf{Flow}\leftrightarrows\mathbf{Flow}:\mathbb{M}$

between the q-model structures.

- (3) The functor cat : $\mathcal{M}d\mathbf{Top} \to \mathbf{Flow}$ satisfies cat $\cong \mathbb{M}_!\mathbb{M}^{top}$; as a corollary, for any topological space Z, there is the natural isomorphism of flows cat(Glob^{top}(Z)) \cong Glob(Z).
- (4) The functor cat : MdTop → Flow is neither a left adjoint nor a right adjoint; however its left derived functor in the sense of [5] induces an equivalence of categories between the homotopy categories of the q-model structures; in particular, the functor cat : MdTop → Flow takes weak equivalences between q-cofibrant multipointed d-spaces to weak equivalences between q-cofibrant flows.

Proof. (1) is [18, Theorem 14] and [18, Corollary 9]. (2) is [14, Theorem 10.9]. (3) and (4) are [18, Theorem 15]. \Box

The following additional facts play an important role in the sequel.

3.7. Proposition. Consider a pushout diagram of multipointed d-spaces of the form



with A cellular and $n \ge 0$. Then there is a pushout diagram of flows



Proof. The proof is sketched in [19, Proposition 7.1]. A proof is given for the convenience of the reader. The pushout diagram of multipointed d-spaces gives rise using [18,

Corollary 8] to a pushout diagram of Moore flows



From the isomorphism of functors cat $\cong \mathbb{M}_! \mathbb{M}^{top}$, we obtain the pushout diagram of flows



The proof is complete since $\operatorname{cat}(\operatorname{Glob}^{top}(Z)) = \operatorname{Glob}(Z)$ for any topological space Z. \Box

3.8. Theorem. Consider a pushout diagram of cellular multipointed d-spaces



such that the vertical maps belong to $\operatorname{cell}(I^{gl,top})$. Then there is the pushout diagram of cellular flows



and the vertical maps belong to $\operatorname{cell}(I^{gl})$.

Proof. Assume first that there is a pushout diagram of cellular multipointed d-spaces



for some $n \ge 0$. By Proposition 2.1, there is the pushout square of cellular multipointed *d*-spaces



Using Proposition 3.7, we obtain the pushout squares



Using Corollary 2.2, we deduce the pushout diagram of cellular flows



We then deduce the theorem when the vertical maps of \underline{C} are a pushout of a finite composition of maps of $I^{gl,top}$. There is the isomorphism of functors cat $\cong \mathbb{M}_{!}\mathbb{M}^{top}$. Since $\mathbb{M}_{!}$ is a left adjoint, we obtain the pushout diagram of flows



thanks to [18, Theorem 5] in the non-finite case. All involved flows are cellular by Proposition 3.7 and [18, Theorem 5] again.

4. ENRICHED SMALL CATEGORIES AND DIRECTED SPACES

To have simpler statements for Theorem 6.8 and Theorem 9.5, we introduced the category of topologically enriched small categories.

4.1. Notation. The category of (topologically) enriched small categories is denoted by Cat_{Top} . The forgetful functor $Cat_{Top} \subset Flow$ has a left adjoint denoted by $I^+ : Flow \to Cat_{Top}$.

The execution path of a flow X from α to β is still denoted by $\mathbb{P}_{\alpha,\beta}X$, whereas the space of morphisms from α to β in $I^+(X)$ is denoted by $I^+(X)(\alpha,\beta)$. One has the homeomorphisms

$$\mathbf{I}^{+}(X)(\alpha,\beta) = \begin{cases} \mathbb{P}_{\alpha,\beta}X & \text{if } \alpha \neq \beta\\ \{\mathrm{Id}_{\alpha}\} \sqcup \mathbb{P}_{\alpha,\beta}X & \text{if } \alpha = \beta \end{cases}$$

4.2. Notation. Denote by \mathcal{I} the set of non-decreasing continuous maps from [0,1] to [0,1]. Note that an element of \mathcal{I} can be a constant map.

We also need to introduced Grandis' notion of directed space to be able to use some results from [19].

4.3. **Definition.** [21, Definition 1.1] [6, Definition 4.1] A directed space is a pair X = (|X|, d(X)) consisting of a topological space |X| and a set d(X) of continuous paths from [0, 1] to |X| satisfying the following axioms:

- d(X) contains all constant paths;
- d(X) is closed under normalized composition;
- d(X) is closed under reparametrization by an element of \mathcal{I} .

The space |X| is called the *underlying topological space* or the *state space*. The elements of d(X) are called *directed paths*. A morphism of directed spaces is a continuous map between the underlying topological spaces which takes a directed path of the source to a directed path of the target. The category of directed spaces is denoted by **dTop**. Write $\overrightarrow{P}(X)(u, v)$ for the space of directed paths of X from u to v equipped with the Δ -kelleyfication of the compact-open topology.

The category of traces of a directed space X, denoted by $\overrightarrow{T}(X)$, has for objects the points of X and the set of maps $\overrightarrow{T}(X)(a,b)$ from $a \in X$ to $b \in X$ is the set of traces $\langle \gamma \rangle$ of directed paths γ going from a to b, i.e. the set of directed paths from a to b up to reparametrization by a map of \mathcal{M} . The composition of traces, denoted by *, is induced by the normalized composition of directed paths, i.e. $\langle \gamma \rangle * \langle \gamma' \rangle = \langle \gamma *_N \gamma' \rangle$. It is strictly associative.

By [19, Proposition 3.7 and Theorem 3.8], the mapping $\overrightarrow{\Omega} : Y = (|Y|, d(Y)) \mapsto (|Y|, |Y|, d(Y))$ induces a full and faithful functor $\overrightarrow{\Omega} : \mathbf{dTop} \to \mathcal{M}\mathbf{dTop}$ which is a right adjoint. Moreover there is the equality $\overrightarrow{\mathrm{Sp}}(\overrightarrow{\Omega}(X)) = X$ for all directed spaces X.

4.4. Notation. Denote by $\overrightarrow{Sp} : \mathcal{M}d\mathbf{Top} \to d\mathbf{Top}$ the left adjoint.

By [19, Proposition 3.6], the left adjoint $\overrightarrow{Sp} : \mathcal{M}d\mathbf{Top} \to d\mathbf{Top}$ is defined as follows. The underlying space of $\overrightarrow{Sp}(X)$ is |X| and the set of directed spaces d(X) consists of all constant paths and all Moore compositions of the form $[(\gamma_1\phi_1\mu_{\ell_1}) * \cdots * (\gamma_n\phi_n\mu_{\ell_n})$ such that $\ell_1 + \cdots + \ell_n = 1$ where $\gamma_1, \ldots, \gamma_n$ are execution paths of X and $\phi_i \in \mathcal{I}$ for $i = 1, \ldots, n$, and where $\mu_{\ell} : [0, \ell] \to [0, 1]$ is defined by $\mu_{\ell}(t) = t/\ell$ with $\ell > 0$.

4.5. Notation. For a multipointed *d*-space *X*, let $\overrightarrow{T}(X) = \overrightarrow{T}(\overrightarrow{Sp}(X))$. For all $\alpha, \beta \in |X|$, let $\overrightarrow{P}(X)(\alpha, \beta) = \overrightarrow{P}(\overrightarrow{Sp}(X))(\alpha, \beta)$.

4.6. **Proposition.** Let X be a q-cofibrant multipointed d-space X. Let $\alpha, \beta \in X^0$. Then there are the homeomorphisms

$$\overrightarrow{P}(X)(\alpha,\beta) \cong \begin{cases} \{\alpha\} \sqcup \mathbb{P}^{top}_{\alpha,\beta}X & \text{if } \alpha = \beta \\ \mathbb{P}^{top}_{\alpha,\beta}X & \text{if } \alpha \neq \beta \end{cases}$$

where α denotes the constant path α . There is also the homeomorphism

$$\vec{T}(X)(\alpha,\beta) \cong \mathrm{I}^+(\mathrm{cat}(X))(\alpha,\beta)$$

for all $\alpha, \beta \in X^0$. Moreover the spaces $\overrightarrow{P}(X)(\alpha, \beta)$ are m-cofibrant and the spaces $\overrightarrow{T}(X)(\alpha, \beta)$ are q-cofibrant.

Proof. Every q-cofibrant multipointed d-space is a retract of a cellular one. Thus one can suppose that X is cellular. The first part is then a consequence of [19, Theorem 4.9]. The second part is the consequence of the definitions of a trace and of the functor cat. The spaces $\mathbb{P}^{top}_{\alpha,\beta}X$ are m-cofibrant by [18, Theorem 16] and the spaces $\mathbb{P}^{\alpha,\beta}_{\alpha,\beta}\operatorname{cat}(X)$ are q-cofibrant by [14, Theorem 9.11], $\operatorname{cat}(X)$ being a q-cofibrant flow by Theorem 3.6. Hence the proof is complete.

Consider the example of the non q-cofibrant multipointed d-space X



It is defined as follows. The underlying space |X| is the topological space $|X| = \{(u, u) \mid u \in [0,1]\} \cup \{(u, 1 - u) \mid u \in [0,1]\}$. Let $X^0 = \{(0,0), (0,1), (1,0), (1,1)\}$. Let $\mathbb{P}_{(0,0),(1,1)}^{top}X = \{t \mapsto (\phi(t), \phi(t)) \mid \phi \in \mathcal{M}\}, \mathbb{P}_{(0,1),(1,0)}^{top}X = \{t \mapsto (\phi(t), 1 - \phi(t)) \mid \phi \in \mathcal{M}\}$ and $\mathbb{P}_{\alpha,\beta}^{top}X = \emptyset$ otherwise. Then $\overrightarrow{P}(X)((0,0), (1,0)) \cong \mathcal{M}$ and $\mathbb{P}_{(0,0),(1,0)}^{top}X = \emptyset$. The multipointed *d*-space *X* is not q-cofibrant because a q-cofibrant replacement of *X* is the disjoint sum $\overrightarrow{I}^{top} \sqcup \overrightarrow{I}^{top}$ of two directed segments multipointed with the extremities.

For every precubical set K, consider its realization |K| as a multipointed d-space: the set of states is K_0 , the underlying space is the geometric realization of K and the set of execution path from α to β is the set of all nonconstant directed paths in the geometric realization of K from α to β ([19, Definition 3.12]). This yields a functor from precubical sets to multipointed d-spaces. It satisfies trivially the homeomorphisms

$$\overrightarrow{P}(|K|)(\alpha,\beta) \cong \begin{cases} \{\alpha\} \sqcup \mathbb{P}^{top}_{\alpha,\beta}|K| & \text{if } \alpha = \beta \\ \mathbb{P}^{top}_{\alpha,\beta}|K| & \text{if } \alpha \neq \beta \end{cases}$$

and

$$\overrightarrow{T}(|K|)(\alpha,\beta) \cong \mathrm{I}^+(\mathrm{cat}(|K|))(\alpha,\beta)$$

for all $\alpha, \beta \in K_0$.

This implies that the q-cofibrancy hypothesis is a sufficient but not necessary condition to obtain the conclusion of Proposition 4.6.

4.7. **Theorem.** Consider a map of multipointed d-spaces $f : X \to Y$ between two q-cofibrant multipointed d-spaces. The following statements are equivalent:

- (1) f is a weak equivalence of the q-model structure.
- (2) f induces a bijection between the states and for all $(\alpha, \beta) \in X^0 \times X^0$ a weak homotopy equivalence $\overrightarrow{P}(X)(\alpha, \beta) \simeq \overrightarrow{P}(Y)(f(\alpha), f(\beta))$.
- (3) f induces a bijection between the states and for all (α, β) ∈ X⁰ × X⁰ a weak homotopy equivalence T (X)(α, β) ≃ T (Y)(f(α), f(β)).
 (4) f induces a bijection between the states and for all (α, β) ∈ X⁰ × X⁰ a homotopy
- (4) f induces a bijection between the states and for all $(\alpha, \beta) \in X^0 \times X^0$ a homotopy equivalence $\overrightarrow{P}(X)(\alpha, \beta) \simeq \overrightarrow{P}(Y)(f(\alpha), f(\beta))$.
- (5) f induces a bijection between the states and for all $(\alpha, \beta) \in X^0 \times X^0$ a homotopy equivalence $\overrightarrow{T}(X)(\alpha, \beta) \simeq \overrightarrow{T}(Y)(f(\alpha), f(\beta))$.

Proof. The equivalence $(1) \Leftrightarrow (2)$ is a consequence of the definition of a weak equivalence and of Proposition 4.6. By Proposition 4.6 and [18, Theorem 16], the quotient maps $\overrightarrow{P}(X)(\alpha,\beta) \to \overrightarrow{T}(X)(\alpha,\beta)$ and $\overrightarrow{P}(Y)(f(\alpha),f(\beta)) \to \overrightarrow{T}(Y)(f(\alpha),f(\beta))$ are homotopy equivalences. For all $\alpha, \beta \in X^0$, there is the commutative diagram of spaces



The equivalence $(2) \Leftrightarrow (3)$ is then a consequence of the two-out-of-three property. Since X and Y are q-cofibrant, the topological spaces $\overrightarrow{P}(X)(\alpha,\beta)$ and $\overrightarrow{P}(Y)(f(\alpha), f(\beta))$ are m-cofibrant by Proposition 4.6. Moreover, the topological spaces $\overrightarrow{T}(X)(\alpha,\beta)$ and $\overrightarrow{T}(Y)(f(\alpha), f(\beta))$ are q-cofibrant by Proposition 4.6 as well. We obtain the equivalences $(2) \Leftrightarrow (4)$ and $(3) \Leftrightarrow (5)$ thanks to [3, Corollary 3.4].

5. Cellular multipointed d-spaces

All cellular multipointed d-spaces for the q-model structure of $\mathcal{M}d\mathbf{Top}$ can be reached from \varnothing without using the cofibration $R : \{0, 1\} \to \{0\}$ and by regrouping the pushouts of $C : \varnothing \to \{0\}$ at the very beginning. Thus, for the sequel, a cellular decomposition of a cellular multipointed d-space of the q-model category $\mathcal{M}d\mathbf{Top}$ consists of a colimitpreserving functor $X : \lambda \longrightarrow \mathcal{M}d\mathbf{Top}$ from a transfinite ordinal λ to the category of multipointed d-spaces such that

- The multipointed d-space X_0 is a set, in other terms $X_0 = (X^0, X^0, \emptyset)$ for some set X^0 .
- For all $\nu < \lambda$, there is a pushout diagram of multipointed *d*-spaces



with $n_{\nu} \ge 0$.

The underlying topological space $|X_{\lambda}|$ is Hausdorff by [18, Proposition 4.4]. For all $\nu \leq \lambda$, there is the equality $X_{\nu}^{0} = X^{0}$. Denote by

$$c_{\nu} = |\operatorname{Glob}^{top}(\mathbf{D}^{n_{\nu}})| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n_{\nu}-1})|$$

the ν -th cell of X_{λ} . It is called a globular cell. Like in the usual setting of CW-complexes, $\widehat{g_{\nu}}$ induces a homeomorphism from c_{ν} to $\widehat{g_{\nu}}(c_{\nu})$ equipped with the relative topology. The map $\widehat{g_{\nu}}$: Glob^{top}($\mathbf{D}^{n_{\nu}}$) $\rightarrow X_{\lambda}$ is called the *attaching map* of the globular cell c_{ν} . The state $\widehat{g_{\nu}}(0) \in X^0$ ($\widehat{g_{\nu}}(1) \in X^0$ resp.) is called the *initial (final resp.) state* of c_{ν} and is denoted by c_{μ}^- (c_{μ}^+ resp.). The integer $n_{\nu} + 1$ is called the *dimension* of the globular cell c_{ν} . It is denoted by dim c_{ν} . The states of X^0 are also called the globular cells of dimension 0. By convention, a state of X^0 viewed as a globular cell of dimension 0 is equal to its initial state and to its final state. Thus, for $\alpha \in X^0$, one has $\alpha = \alpha^+ = \alpha^-$. The set of globular cells of X_{λ} is denoted by $\mathcal{C}(X_{\lambda})$. The set of globular cells of dimension $n \ge 0$ of X_{λ} is denoted by $\mathcal{C}_n(X_{\lambda})$. In particular, $\mathcal{C}_0(X_{\lambda}) = X^0$. The closure $\widehat{g_{\nu}}(c_{\nu})$ of c_{ν} in $|X_{\lambda}|$ is denoted by $\widehat{c_{\nu}}$.

5.1. Notation. Denote by cell_f the class of finite compositions of pushouts of the inclusions $\{\mathbf{S}^{n-1} \subset \mathbf{D}^n \mid n \ge 0\}$. Let

$$\mathbf{S}_{0}^{n-1} = \{(x_{1}, \dots, x_{n}, 0) \mid \sum_{i} x_{i}^{2} = 1\},\$$
$$\mathbf{D}_{0}^{n} = \{(x_{1}, \dots, x_{n}, 0) \mid \sum_{i} x_{i}^{2} \leq 1.$$

There are the homeomorphisms $\mathbf{S}_0^{n-1} \cong \mathbf{S}^{n-1}$ and $\mathbf{D}_0^n \cong \mathbf{D}^n$ for all $n \ge 0$.

5.2. **Proposition.** Let $n \ge 1$. Let F be a finite subset of the interior of \mathbf{D}^n . Then the inclusion $\mathbf{S}^{n-1} \cup F \subset \mathbf{D}^n$ belongs to cell_f .

Proof. For n = 1, write $F = \{u_1 < \cdots < u_p\}$ with $p \ge 1$. One has

$$\mathbf{D}^1 = [-1, 1] = \bigcup_{1 \le k \le p+1} [u_{k-1}, u_k]$$

with $u_0 = -1$ and $u_{p+1} = 1$. This implies that $\mathbf{S}^0 \cup F \subset \mathbf{D}^1$ belongs to \mathbf{cell}_f . We prove now by induction on $n \ge 1$

 $\mathcal{E}(n): \forall u \in \mathbf{D}^n \setminus \mathbf{S}^{n-1}, \mathbf{S}^{n-1} \cup \{u\} \subset \mathbf{D}^n \in \operatorname{cell}_f.$

The case $\mathcal{E}(1)$ is treated above. Assume $\mathcal{E}(n)$ for $n \ge 1$. We want to prove $\mathcal{E}(n+1)$. Using a homeomorphism, we can suppose that u is the center of \mathbf{D}^{n+1} . Consider the commutative diagram of topological spaces



Since the square is a pushout, $\mathcal{E}(n)$ implies that the inclusion $\mathbf{S}^n \cup \{u\} \subset \mathbf{D}_0^n \cup \mathbf{S}^n$ belongs to **cell**_f. The inclusion $\mathbf{D}_0^n \cup \mathbf{S}^n \subset \mathbf{D}^{n+1}$ belongs to **cell**_f as well since \mathbf{D}^{n+1} is obtained from $\mathbf{D}_0^n \cup \mathbf{S}^n$ by using two pushouts along the inclusion $\mathbf{S}^n \subset \mathbf{D}^{n+1}$. We have proved $\mathcal{E}(n+1)$. We proceed now by induction on $p \ge 0$ to prove

$$\mathcal{E}'(p): \forall q \leq p, \forall n \geq 2, \mathbf{S}^{n-1} \cup \{u_1, \dots, u_q\} \subset \mathbf{D}^n \in \operatorname{cell}_f.$$

There is nothing to prove for $\mathcal{E}'(0)$ and $\mathcal{E}'(1)$ is already proved above. Assume $\mathcal{E}'(p)$ for $p \ge 1$. We want to prove $\mathcal{E}'(p+1)$. Using a homeomorphism, one can suppose that u_{p+1} is the center of \mathbf{D}^n and that $\mathbf{D}_0^{n-1} \cap \{u_1, \ldots, u_p\} = \emptyset$. Consider the commutative

diagram of topological spaces

Since the square is a pushout, $\mathcal{E}(n-1)$ implies that the inclusion $\mathbf{S}^{n-1} \cup \{u_1, \ldots, u_{p+1}\} \subset$ $\mathbf{D}_0^{n-1} \cup \mathbf{S}^{n-1} \cup \{u_1, \dots, u_p\}$ belongs to **cell**_f. The inclusion $\mathbf{D}_0^{n-1} \cup \mathbf{S}^{n-1} \cup \{u_1, \dots, u_p\} \subset \mathbf{D}^n$ belongs to **cell**_f as well since \mathbf{D}^n is obtained from $\mathbf{D}_0^{n-1} \cup \mathbf{S}^{n-1} \cup \{u_1, \dots, u_p\}$ by using two pushouts along two inclusions of the form $\mathbf{S}^{n-1} \cup F \subset \mathbf{D}^n$ with $F \subset \mathbf{D}^n \setminus \mathbf{S}^{n-1}$ and Fof cardinal lower than p. We have proved $\mathcal{E}'(p+1)$.

5.3. **Proposition.** Let $n \ge 1$. Consider a finite set

$$F \subset |\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|$$

Then the following data assemble into a multipointed d-space denote by $\operatorname{Glob}^{top}(\mathbf{D}^n)_F$:

- The set of states is $\{0, 1\} \cup F$.
- The underlying space is $|\operatorname{Glob}^{top}(\mathbf{D}^n)|$;
- For all $\alpha \neq \beta \in \{0,1\} \cup F$, $\mathbb{P}^{top}_{\alpha,\beta} \operatorname{Glob}^{top}(\mathbf{D}^n)_F = \overrightarrow{P}(\overrightarrow{\operatorname{Sp}}(\operatorname{Glob}^{top}(\mathbf{D}^n)))(\alpha,\beta)$. For all $\alpha \in \{0,1\} \cup F$, $\mathbb{P}^{top}_{\alpha,\alpha} \operatorname{Glob}^{top}(\mathbf{D}^n)_F = \varnothing$.

In particular, there is the isomorphism of multipointed d-spaces

$$\operatorname{Glob}^{top}(\mathbf{D}^n) \cong \operatorname{Glob}^{top}(\mathbf{D}^n)_{\varnothing}.$$

Proof. The composition of two execution paths is an execution path and the set of execution paths is closed under reparametrization by \mathcal{M} .

5.4. Notation. Let $\ell < \ell'$ be two real numbers. The multipointed *d*-space $\overrightarrow{[\ell, \ell']}$ is defined as follows: the underlying space is the segment $[\ell, \ell']$, the set of states is $\{\ell, \ell'\}$ and the set of execution paths is the set of nondecreasing surjective maps from [0, 1] to $[\ell, \ell']$. For all topological spaces Z, the unique map $Z \to \{0\}$ induces a map of multipointed d-spaces $\pi: \operatorname{Glob}^{top}(Z) \to \operatorname{Glob}^{top}(\{0\}) \cong \overrightarrow{[0,1]}.$

5.5. **Proposition.** Let $n \ge 0$. Consider a finite set

 $F \subset |\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|.$

Then the map of multipointed d-spaces $\operatorname{Glob}^{top}(\mathbf{S}^{n-1}) \subset \operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is a finite composition of pushouts of the maps $C: \varnothing \to \{0\}$ and $\operatorname{Glob}^{top}(\mathbf{S}^{k-1}) \subset \operatorname{Glob}^{top}(\mathbf{D}^k)$ for $k \ge 0.$

Proof. The case n = 0 is trivial. Assume that $n \ge 1$. We consider the pushout diagram of multipointed *d*-spaces (the poset structure is defined in Proposition 9.1)



Since the functor $X \mapsto X^0$ from multipointed *d*-spaces to sets is colimit-preserving, we have $X_F^0 = \{0, 1\} \cup F$. Note that the two sums in the pushout diagram above are taken on all pairs (u, v) such that u < v, $]u, v[= \emptyset$ and $(u, v) \neq (0, 1)$. The latter condition implies that the sums are empty in the case $F = \emptyset$. This implies that $X_{\emptyset} = \text{Glob}^{top}(\mathbf{S}^{n-1})$. Consider the equivalence relation on F induced by $u \sim v$ if and only if u < v. Since u < v implies that there is a directed path from u to v, every element of the quotient set F/\sim corresponds to a unique element of the interior of \mathbf{D}^n . Thus, there is a canonical inclusion $F/\sim \subset \mathbf{D}^n \setminus \mathbf{S}^{n-1}$. By induction on the cardinal of F, we can verify that the multipointed d-space $X_F \upharpoonright_{\{0,1\}}$ is isomorphic to $\text{Glob}^{top}(\mathbf{S}^{n-1} \sqcup F/\sim)$. There is a pushout diagram of multipointed d-spaces



We can now conclude the proof. The inclusion $\operatorname{Glob}^{top}(\mathbf{S}^{n-1}) \subset \operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is the composite of the three inclusions

$$\operatorname{Glob}^{top}(\mathbf{S}^{n-1}) \subset \operatorname{Glob}^{top}(\mathbf{S}^{n-1}) \sqcup \coprod_{\substack{(u,v) \in (\{0,1\} \cup F)^2 \\ u < v,]u,v [= \varnothing, (u,v) \neq (0,1)}} \overrightarrow{[\pi(u), \pi(v)]} \xrightarrow{[\pi(u), \pi(v)]}$$

$$\operatorname{Glob}^{top}(\mathbf{S}^{n-1}) \sqcup \coprod_{\substack{(u,v) \in (\{0,1\} \cup F)^2 \\ u < v,]u,v [= \varnothing, (u,v) \neq (0,1)}} \overrightarrow{[\pi(u), \pi(v)]} \subset X_F$$

$$X_F \subset \operatorname{Glob}^{top}(\mathbf{D}^n)_F$$

Thanks to Proposition 5.2, we deduce that the inclusion $\operatorname{Glob}^{top}(\mathbf{S}^{n-1}) \subset \operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is a finite composition of pushouts of the maps $R : \{0,1\} \to \{0\}, C : \emptyset \to \{0\}$ and $\operatorname{Glob}^{top}(\mathbf{S}^{k-1}) \subset \operatorname{Glob}^{top}(\mathbf{D}^k)$ for $k \ge 0$. Since the inclusion $\operatorname{Glob}^{top}(\mathbf{S}^{n-1}) \subset$ $\operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is one-to-one on states, the map $R : \{0,1\} \to \{0\}$ can be removed. And the proof is complete.

5.6. Corollary. Let $n \ge 0$. Consider a finite set

$$F \subset |\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|.$$

The multipointed d-space $\operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is cellular.

5.7. Notation. Let $\operatorname{Glob}(\mathbf{D}^n)_F = \operatorname{cat}(\operatorname{Glob}^{top}(\mathbf{D}^n)_F)$. We have $\operatorname{Glob}(\mathbf{D}^n)_{\varnothing} = \operatorname{Glob}(\mathbf{D}^n)$ since $\operatorname{Glob}(\mathbf{D}^n) = \operatorname{cat}(\operatorname{Glob}^{top}(\mathbf{D}^n)).$

5.8. Corollary. Let $n \ge 0$. Consider a finite set

 $F \subset |\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|.$

The map of flows $\operatorname{Glob}(\mathbf{S}^{n-1}) \subset \operatorname{Glob}(\mathbf{D}^n)_F$ is a finite composition of pushouts of the generating cofibrations of the q-model structure of flows. In particular, it is a q-cofibration.

Proof. It is a consequence of Proposition 3.7.

6. GLOBULAR SUBDIVISION

6.1. Definition. [20, Definition 4.10] A map of multipointed d-spaces $f: X \to Y$ is a globular subdivision if both X and Y are cellular and if f induces a homeomorphism between the underlying topological spaces of X and Y. We say that Y is a *qlobular* subdivision of X when there exists such a map. This situation is denoted by

$$f: X \xrightarrow{\mathrm{sbd}} Y.$$

6.2. **Proposition.** The categorical localization of the full subcategory of cellular multipointed d-spaces by the globular subdivisions is locally small.

Proof. It is an adaptation of the proof of [20, Theorem 4.12].

6.3. **Proposition.** Let $f: X \xrightarrow{\text{sbd}} Y$ be a globular subdivision. For all $\alpha, \beta \in X^0$, there is the homeomorphisms $\mathbb{P}^{top}_{\alpha,\beta}X \cong \mathbb{P}^{top}_{f(\alpha),f(\beta)}Y$ and $\overrightarrow{T}(X)(\alpha,\beta) \cong \overrightarrow{T}(Y)(f(\alpha),f(\beta))$.

Proof. By [19, Theorem 9.3], there is the isomorphism of directed spaces $\overrightarrow{\mathrm{Sp}}(f) : \overrightarrow{\mathrm{Sp}}(X) \cong$ $\overrightarrow{\mathrm{Sp}}(Y)$. We obtain the homeomorphisms $\overrightarrow{T}(X)(\alpha,\beta) \cong \overrightarrow{T}(Y)(f(\alpha),f(\beta))$ for all $\alpha,\beta \in X^0$. The homeomorphisms $\mathbb{P}^{top}_{\alpha,\beta}X \cong \mathbb{P}^{top}_{f(\alpha),f(\beta)}Y$ for all $\alpha,\beta \in X^0$ are a consequence of Proposition 4.6.

6.4. **Definition.** Let $f: X \xrightarrow{\text{sbd}} Y$ be a globular subdivision. Choose a cellular decomposition of X. Every point $\alpha \in Y^0$ is in a unique globular cell c_α of X. Let $\alpha^- = c_\alpha^-$ and $\alpha^+ = c_{\alpha}^+$. Note that when $\alpha \in X^0$, then $\alpha^- = \alpha^+ = \alpha$. We obtain two set maps $\alpha \mapsto \alpha^$ and $\alpha \mapsto \alpha^+$ from Y^0 to X^0 .

Proposition 6.5 proves that the set maps $\alpha \mapsto \alpha^-$ and $\alpha \mapsto \alpha^+$ of Definition 6.4 from Y^0 to X^0 do not depend of the choice of the cellular decomposition of X.

6.5. **Proposition.** Let $f: X \xrightarrow{\text{sbd}} Y$ be a globular subdivision. Consider two cellular decompositions $\mathcal{C}_0(X)$ and $\mathcal{C}_1(X)$ of X and the four associated set maps $\alpha \mapsto \alpha_0^-, \alpha \mapsto \alpha_1^-$, $\alpha \mapsto \alpha_0^+$ and $\alpha \mapsto \alpha_1^+$ from Y^0 to X^0 . Then for all $\alpha \in Y^0$, one has $\alpha_0^- = \alpha_1^-$ and $\alpha_0^+ = \alpha_1^+.$

Proof. We just have to consider the case $\alpha \in Y^0 \setminus X^0$. The point $\alpha \in |X| = |Y|$ belongs to a unique globular cell c_i of $\mathcal{C}_i(X)$ for i = 0, 1 with $\dim(c_0) \ge 1$ and $\dim(c_1) \ge 1$. There exists a unique execution path γ_i of X up to reparametrization from c_i^- to c_i^+ for i = 0, 1with $\gamma_i(]0,1[) \cap X^0 = \emptyset$. Thus $\alpha_0^- = c_0^- = c_1^- = \alpha_1^-$ and $\alpha_0^+ = c_0^+ = c_1^+ = \alpha_1^+$.

6.6. **Definition.** Let $f: X \xrightarrow{\text{sbd}} Y$ be a globular subdivision. Then the maps $\alpha \mapsto \alpha^-$ and $\alpha \mapsto \alpha^+$ from Y^0 to X^0 , which depend only on f by Proposition 6.5, are called the *connection maps* of the globular subdivision f.

Consider the identity Id_X of a cellular multipointed *d*-space *X*. It is a globular subdivision. The connection maps of Id_X are the identity of X^0 . The notations $(-)^-, (-)^+$: $X^0 \to X^0$ are therefore consistent with the other meaning of the notations $(-)^-$ and $(-)^+$ given in Section 5.

6.7. **Proposition.** Let $f: X \xrightarrow{\text{sbd}} Y$ be a globular subdivision. Let $\alpha, \beta \in Y^0 \setminus X^0$. The following statements are equivalent:

- (1) There exists a directed path γ of Y from α to β such that $\gamma([0,1]) \subset |X| \setminus X^0$.
- (2) For every cellular decomposition of X, α and β are in the same globular cell and there exists a unique directed path from α to β up to reparametrization (the corresponding trace of $\overrightarrow{T}(X)(\alpha, \beta) = \overrightarrow{T}(Y)(\alpha, \beta)$ is denoted by $\tau_{\alpha,\beta}$).

Proof. By [19, Theorem 9.3], and the map $f: X \to Y$ being a globular subdivision, there is the isomorphism of directed spaces $\overrightarrow{\operatorname{Sp}}(f) : \overrightarrow{\operatorname{Sp}}(X) \cong \overrightarrow{\operatorname{Sp}}(Y)$, which implies $\overrightarrow{T}(X)(\alpha,\beta) = \overrightarrow{T}(Y)(\alpha,\beta)$. The implication (2) \Rightarrow (1) is obvious. If (1) holds, then α and β belongs by to the same globular cell of X; if \widehat{g} is the attaching map, the existence of γ implies that $\widehat{g}(z,t) = \alpha$ and $\widehat{g}(z,u) = \beta$ for some $t \leq u$. Hence we obtain (2). \Box

6.8. **Theorem.** Let $f : X \xrightarrow{\text{sbd}} Y$ be a globular subdivision. Let $\alpha, \beta \in Y^0$. Then either there exists a directed path γ of Y from α to β such that $\gamma([0,1]) \subset |X| \setminus X^0$ and there is the homeomorphism

$$\overrightarrow{T}(Y)(\alpha,\beta) \cong \{\tau_{\alpha,\beta}\} \cup \overrightarrow{T}(X)(\alpha^+,\beta^-),$$

or there is no such directed path and there is the homeomorphism

$$\overrightarrow{T}(Y)(\alpha,\beta) \cong \overrightarrow{T}(X)(\alpha^+,\beta^-).$$

Proof. Choose a cellular decomposition of X. Assume that there is a directed path γ of $\overrightarrow{Sp}(Y)$ such that $\gamma([0,1]) \subset |X| \setminus X^0$. Then $\alpha = \gamma(0)$ and $\beta = \gamma(1)$ belongs to the same globular cell of X of dimension greater than 1. The homeomorphism $\overrightarrow{T}(Y)(\alpha,\beta) \cong \{\tau_{\alpha,\beta}\} \cup \overrightarrow{T}(Y)(\alpha^+,\beta^-)$ is a consequence of [19, Proposition 8.12] $(\tau_{\alpha,\beta})$ is defined in Proposition 6.7). Now assume that there is not such a directed path. There are five mutually exclusive possibilities.

- (1) $\alpha, \beta \in X^0$: $\alpha = \alpha^+$ and $\beta = \beta^-$ implies the equality $\overrightarrow{T}(Y)(\alpha, \beta) = \overrightarrow{T}(Y)(\alpha^+, \beta^-)$.
- (2) $\alpha \in X^0$ and $\beta \in |X| \setminus X^0$: $\alpha = \alpha^+$ and [19, Proposition 8.8] implies the homeomorphism $\overrightarrow{T}(Y)(\alpha, \beta) \cong \overrightarrow{T}(Y)(\alpha^+, \beta^-)$.
- (3) $\alpha \in |X| \setminus X^0$ and $\beta \in X^0$: $\beta = \beta^-$ and [19, Proposition 8.7] implies the homeomorphism $\overrightarrow{T}(Y)(\alpha, \beta) \cong \overrightarrow{T}(Y)(\alpha^+, \beta^-)$.
- (4) $\alpha, \beta \in |X| \setminus X^0$ and belonging to the same globular cell c of X: it is a variant of [19, Proposition 8.12] whose proof is left to the reader; intuitively, every trace from α to β has to go out from c by the unique trace going from α to c^+ and to return to c by c^- followed by the unique trace from c^- to β .

(5) $\alpha, \beta \in |X| \setminus X^0$ and belonging to two different globular cells of X: [19, Proposition 8.9] implies the homeomorphism $\overrightarrow{T}(Y)(\alpha, \beta) \cong \overrightarrow{T}(Y)(\alpha^+, \beta^-)$. The proof is complete thanks to Proposition 6.3.

A globular subdivision $f: X \xrightarrow{\text{sbd}} Y$ is not necessarily well-behaved with respect to the chosen cellular decompositions of X and Y. More precisely, the image $f(\hat{c})$ of the closure \hat{c} of a globular cell c of X is not necessarily a cell subcomplex of Y. To see that, let us start from the homeomorphism $[0, 1/3] \sqcup_{1/3} [1/3, 1] \cong [0, 2/3] \sqcup_{2/3} [2/3, 1]$. This gives rise to two cellular decompositions of $\text{Glob}^{top}([0, 1])$ which can be depicted as follows (the red lines must be identified to a point):



The identity of $\text{Glob}^{top}([0, 1])$ is a globular subdivision which is such a pathological example. Theorem 6.9 proves that it is always possible to choose another cellular decomposition of Y to avoid this kind of issue.

6.9. **Theorem.** Let $f: X \xrightarrow{\text{sbd}} Y$ be a globular subdivision. Let $\widetilde{X} : \lambda \to \mathcal{M}d\mathbf{Top}$ be a cellular decomposition of X. There exists a transfinite tower of cellular multipointed d-spaces $\widetilde{Y} : \lambda \to \mathcal{M}d\mathbf{Top}$ and a map of transfinite towers $\widetilde{X} \to \widetilde{Y}$ such that the colimit is the globular subdivision $X \to Y$ and such that for all $\nu < \lambda$, there is the commutative diagrams of multipointed d-spaces of the form



Moreover, the connection maps of the globular subdivision $f: X_{\nu} \xrightarrow{\text{sbd}} Y_{\nu}$ are the restrictions to Y_{ν}^{0} of the connection maps of the globular subdivision $f: X_{\nu+1} \xrightarrow{\text{sbd}} Y_{\nu+1}$ for all $\nu < \lambda$.

The hypothesis that Y is cellular is used to guarantee that Y^0 is discrete and at the very end of the proof for using [19, Theorem 9.3(2)]. Without this hypothesis, and even by assuming Y^0 discrete, there is no guarantee for the map $\tilde{Y}_{\lambda} \to Y$, which induces a homeomorphism between the set of states and between the underlying spaces, to be

an isomorphism: the multipointed d-space Y could have more execution paths than \tilde{Y}_{λ} indeed.

Proof. There is a bijection of sets

$$|X| = \bigsqcup_{c \in \mathcal{C}(X)} c$$

where $\mathcal{C}(X)$ is the set of globular cells of \widetilde{X} . We obtain a bijection of sets

$$Y| = \bigsqcup_{c \in \mathcal{C}(X)} f(c).$$

Each cell c_{ν} of X for $\nu < \lambda$ corresponds to a pushout diagram of the form



for some $n_{\nu} \ge 0$. Consider a globular cell c of X with $\dim(c) \ge 1$. The closure $\widehat{f(c)}$ of each f(c) is a compact in the Hausdorf space |X| = |Y|. Therefore, the set $f(c) \cap Y^0 \subset \widehat{f(c)} \cap Y^0$ is finite, Y^0 being discrete because Y is cellular by hypothesis. Let $F(c) = f(c) \cap Y^0$ for c running over the set $\mathcal{C}(X)$ of globular cells of X of dimension greater than 1. We are going to construct by transfinite induction a transfinite tower $\widetilde{Y} : \lambda \to \mathcal{M} d\mathbf{Top}$ and a map of transfinite towers $\widetilde{X} \to \widetilde{Y}$ such that for all $\nu \le \lambda$, there is a homeomorphism $\widetilde{X}_{\nu} \cong \widetilde{Y}_{\nu}$ as follows. Let $\widetilde{Y}_0 = X^0$. Assume that the map of transfinite towers $\widetilde{X} \to \widetilde{Y}$ is constructed until some ordinal $\nu < \lambda$. Consider the commutative diagram of solid arrows of multipointed d-spaces



The universal property of the pushout yields a (unique) map $X_{\nu+1} \to \tilde{Y}_{\nu+1}$. Since the functor $X \mapsto (|X|, X^0)$ from \mathcal{M} **dTop** to **mTop** is topological, it is colimit-preserving. Therefore, we obtain from the homeomorphism $|\widetilde{X}_{\nu}| \cong |\widetilde{Y}_{\nu}|$ the homeomorphism $|\widetilde{X}_{\nu+1}| \cong |\widetilde{Y}_{\nu+1}|$. For a limit ordinal $\nu \leq \lambda$, let us define the map of multipointed *d*-space $\widetilde{X}_{\nu} \to \widetilde{Y}_{\nu}$ as a colimit. Since the functor $X \mapsto |X|$ is colimit-preserving, we obtain the homeomorphism

 $|\widetilde{X}_{\nu}| \cong |\widetilde{Y}_{\nu}|$ also for the limit ordinals $\nu \leq \lambda$. By Proposition 5.5, each multipointed *d*-space \widetilde{Y}_{ν} for $\nu \leq \lambda$ is cellular. We deduce that the maps $\widetilde{X}_{\nu} \to \widetilde{Y}_{\nu}$ are globular subdivisions for all $\nu \leq \lambda$. By reorganizing the cube above, we obtain the planar commutative diagram of multipointed *d*-spaces of the statement of the theorem for all $\nu < \lambda$. The map of multipointed *d*-spaces $\widetilde{X}_0 \to Y$ factors uniquely as a composite $\widetilde{X}_0 \cong \widetilde{Y}_0 \to Y$. Assume that for an ordinal $\nu < \lambda$, the map of multipointed *d*-spaces $\widetilde{X}_{\nu} \to Y$ factors uniquely as a composite $\widetilde{X}_{\nu} \cong \widetilde{Y}_{\nu} \to Y$. From the commutative square of multipointed *d*-spaces



we deduce that the map of multipointed *d*-spaces $\widetilde{X}_{\nu+1} \to Y$ factors uniquely as a composite $\widetilde{X}_{\nu+1} \cong \widetilde{Y}_{\nu+1} \to Y$. We obtain by transfinite induction that the globular subdivision $X \to Y$ factors uniquely as a composite of globular subdivisions $X \to \widetilde{Y}_{\lambda} \to Y$. It remains to prove that every execution path of Y is an execution path of \widetilde{Y}_{λ} to complete the proof. Consider an execution path γ of Y. It is a directed path of the directed space $\overrightarrow{Sp}(Y)$ by definition of the functor $\overrightarrow{Sp} : \mathcal{M}d\mathbf{Top} \to d\mathbf{Top}$. By [19, Theorem 9.3(2)], one has $\overrightarrow{Sp}(X) = \overrightarrow{Sp}(Y)$. Thus, γ is a directed path of the directed space $\overrightarrow{Sp}(X)$. By [19, Theorem 4.9], X being cellular, there exists an execution path γ' of X and $\phi \in \mathcal{I}$ such that $\gamma = \gamma' \phi$. In plain English, every execution path of Y is a piece of an execution path of X between two points of Y^0 . Therefore it is an execution path of \widetilde{Y}_{λ} , which means that $\widetilde{Y}_{\lambda} = Y$. Finally, the location of the q-cofibrations is a consequence of Proposition 5.5. \Box

7. The Reedy category $\mathcal{P}^{C}(S)$

Let S be a nonempty set. Let $C : P \to S$ be a set map from the underlying set P of a nonempty poset (P, \leq) to the set S. We introduce a small category $\mathcal{P}^{C}(S)$ defined by generators and relations as follows:

• The objects are the tuples of the form

$$\underline{m} = ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n))$$

with $n \ge 1, u_0, \ldots, u_n \in S, \epsilon_1, \ldots, \epsilon_n \in \{0\} \sqcup \{(u, v) \in P \times P \mid u < v\}$ and

 $\forall i \text{ such that } 1 \leq i \leq n, \epsilon_i \neq 0 \Rightarrow (C, C)(\epsilon_i) = (u_{i-1}, u_i).$

The integer n is the *length* of the tuple. The integer $\sum_i h(\epsilon_i)$ with h(0) = 0 and h(u, v) = 1 for all $u < v \in P$ is the *height* of the tuple <u>m</u>.

• There is an arrow

$$c_{n+1}: (\underline{m}, (x, 0, y), (y, 0, z), \underline{n}) \to (\underline{m}, (x, 0, z), \underline{n})$$

for every tuple $\underline{m} = ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n))$ with $n \ge 1$ and every tuple $\underline{n} = ((u'_0, \epsilon'_1, u'_1), (u'_1, \epsilon'_2, u'_2), \dots, (u'_{n'-1}, \epsilon'_{n'}, u'_{n'}))$ with $n' \ge 1$. It is called a *composition map*.

• There is an arrow

$$I_{n+1}^{\epsilon}: (\underline{m}, (u, 0, v), \underline{n}) \to (\underline{m}, (u, \epsilon, v), \underline{n})$$

with $\epsilon \neq 0$ for every tuple $\underline{m} = ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n))$ with $n \geq 1$ and every tuple $\underline{n} = ((u'_0, \epsilon'_1, u'_1), (u'_1, \epsilon'_2, u'_2), \dots, (u'_{n'-1}, \epsilon'_{n'}, u'_{n'}))$ with $n' \geq 1$. It is called an *inclusion map*.

- There are the relations (group A) $c_i c_j = c_{j-1} c_i$ if i < j (which means since c_i and c_j may correspond to several maps that if c_i and c_j are composable, then there exist c_{j-1} and c_i composable satisfying the equality).
- There are the relations (group B) $I_i^{\epsilon} I_j^{\epsilon'} = I_j^{\epsilon'} I_i^{\epsilon}$ if $i \neq j$. By definition of these maps, I_i^{ϵ} is never composable with $I_i^{\epsilon'}$.
- There are the relations (group C)

$$c_i \cdot I_j^{\epsilon} = \begin{cases} I_{j-1}^{\epsilon} \cdot c_i & \text{if } j \ge i+2\\ I_j^{\epsilon} \cdot c_i & \text{if } j \leqslant i-1 \end{cases}$$

By definition of these maps, c_i and I_i^{ϵ} are never composable as well as c_i and I_{i+1}^{ϵ} .

Assume that C consists of a set map from $\{0,1\}$ to S with the ordering 0 < 1. The map C is determined by the choice of a pair $(C(0), C(1)) = (u, v) \in S \times S$. The only element of $\{(u, v) \in P \times P \mid u < v\}$ is in this case the pair (0, 1). In a tuple $\underline{m} = ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \ldots, (u_{n-1}, \epsilon_n, u_n))$, if $\epsilon_i \neq 0$ for some $i \in \{1, \ldots, n\}$, then $\epsilon_i = (0, 1)$ necessarily and (C, C)(0, 1) = (u, v). This implies that $u_{i-1} = u$ and $u_i = v$. We therefore recover the definition of the small category $\mathcal{P}^{u,v}(S)$ of [16, Section 3].

7.1. **Definition.** Denote by $\mathcal{P}^{C}(S)_{+}$ the subcategory of $\mathcal{P}^{C}(S)$ generated by all objects of $\mathcal{P}^{C}(S)$ and by the inclusion maps. Denote by $\mathcal{P}^{C}(S)_{-}$ the subcategory of $\mathcal{P}^{C}(S)$ generated by all objects of $\mathcal{P}^{C}(S)$ and by the composition maps.

There is the obvious proposition

7.2. **Proposition.** With the notations above. The mapping

$$\Phi : ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n)) \mapsto ((u_0, h(\epsilon_1), u_1), (u_1, h(\epsilon_2), u_2), \dots, (u_{n-1}, h(\epsilon_n), u_n))$$

induces a functor

$$\Phi: \mathcal{P}^C(S) \to \mathcal{P}^{u,v}(S).$$

Moreover, for all objects $\underline{m}, \underline{n}$ of $\mathcal{P}^{C}(S)$, the functor Φ induces a bijection of sets

$$\Phi: \mathcal{P}^C(S)(\underline{m}, \underline{n}) \cong \mathcal{P}^{u, v}(S)(\Phi(\underline{m}), \Phi(\underline{n}))$$

which takes a map of $\mathcal{P}^{C}(S)_{+}$ to a map of $\mathcal{P}^{u,v}(S)_{+}$ and which takes a map $\mathcal{P}^{C}(S)_{-}$ to a map of $\mathcal{P}^{u,v}(S)_{-}$. Finally the functor Φ preserves the height and the length.

7.3. **Proposition.** The small categories $\mathcal{P}^{C}(S)_{-}$, $\mathcal{P}^{C}(S)_{+}$ and $\mathcal{P}^{C}(S)$ are posets.

Proof. A small category is a poset if and only if the set of maps between any pair of objects is almost one. By [16, Proposition 3.2], the category $\mathcal{P}^{u,v}(S)_{-}$ is a poset. By [16, Proposition 3.5], the category $\mathcal{P}^{u,v}(S)_{+}$ is a poset. Finally, by [16, Corollary 3.8], the category $\mathcal{P}^{u,v}(S)$ is a poset. The proof is complete thanks to Proposition 7.2.

7.4. **Definition.** An object <u>n</u> of the small category $\mathcal{P}^{C}(S)$ is simplifiable if the matching category $\partial(\underline{n}\downarrow\mathcal{P}^{C}(S)_{-})$ is nonempty.

7.5. **Proposition.** Let \underline{n} be an object of $\mathcal{P}^{C}(S)$. Then either \underline{n} is not simplifiable (in this case, let $\mathbb{S}(\underline{n}) := \underline{n}$) or the matching category $\partial(\underline{n}\downarrow\mathcal{P}^{C}(S)_{-})$ has a terminal object denoted by $\mathbb{S}(\underline{n})$ and the latter is not simplifiable.

Proof. By [16, Proposition 3.4], the proposition holds for $\mathcal{P}^{u,v}(S)$. The proof is complete thanks to Proposition 7.2.

7.6. **Proposition.** The pair $(\mathcal{P}^{C}(S)_{+}, \mathcal{P}^{C}(S)_{-})$ endows the small category $\mathcal{P}^{C}(S)$ with a structure of Reedy category with the \mathbb{N} -valued degree map defined by

$$d((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n)) = n + \sum_i h(\epsilon_i).$$

Moreover, in the canonical decomposition $f = f_+ f_-$ with $f_+ \in \operatorname{Mor}(\mathcal{P}^C(S)_+)$ and $f_- \in \operatorname{Mor}(\mathcal{P}^C(S)_-)$, the source of f_+ , which is the target of f_- , is uniquely determined by the source and the target of f.

The minimal value of the degree map is 1 and it is reached for the objects $((u_0, 0, u_1))$ for (u_0, u_1) running over $S \times S$.

Proof. The composition maps decrease the degree by one, the inclusion maps increase the degree by one. So every map of $\mathcal{P}^{C}(S)_{+}$ increases the degree and every map of $\mathcal{P}^{C}(S)_{-}$ decreases the degree. Let

$$f: ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n)) \to ((u'_0, \epsilon'_1, u'_1), (u'_1, \epsilon'_2, u'_2), \dots, (u'_{n-1}, \epsilon'_{n'}, u'_{n'}))$$

be a map of $\mathcal{P}^{C}(S)$. By definition of the small category $\mathcal{P}^{C}(S)$, f is a composite of composition maps and of inclusion maps. Using the relations of group C, we obtain a factorization $f = f_{+}.f_{-}$ with $f_{+} \in \operatorname{Mor}(\mathcal{P}^{C}(S)_{+})$ and $f_{-} \in \operatorname{Mor}(\mathcal{P}^{C}(S)_{-})$. By definition of the inclusion maps, the source of f_{+} , which is the target of f_{-} , is of the form

$$((u'_0, \epsilon''_1, u'_1), (u'_1, \epsilon''_2, u'_2), \dots, (u'_{n-1}, \epsilon''_{n'}, u'_{n'}))$$

with $h(\epsilon_j') \leq h(\epsilon_j)$ for $1 \leq j \leq n'$. And by definition of the composition maps, there is the equality $(u'_0, u'_1, \ldots, u'_{n'}) = (u_{i_0}, u_{i_1}, \ldots, u_{i_{n'}})$ where $0 = i_0 < i_1 < \cdots < i_{n'} = n$ and with

$$\epsilon''_{j} = \begin{cases} 0 & \text{if } i_{j} - i_{j-1} > 1\\ \epsilon_{i_{j}} & \text{if } i_{j} - i_{j-1} = 1. \end{cases}$$

In other terms, there is only one possibility for the source of f_+ which is the target of f_- . The proof is complete thanks to Proposition 7.3.

7.7. Theorem. Let \mathcal{K} be a model category. Let $\mathbf{CAT}(\mathcal{P}^{C}(S), \mathcal{K})$ be the category of functors and natural transformations from $\mathcal{P}^{C}(S)$ to \mathcal{K} . Then there exists a unique model structure on $\mathbf{CAT}(\mathcal{P}^{C}(S), \mathcal{K})$ such that the weak equivalences are the pointwise weak equivalences and such that a map of diagrams $f: \mathcal{D} \to \mathcal{E}$ is a cofibration (called a Reedy cofibration) if for all objects \underline{n} of $\mathcal{P}^{C}(S)$, the canonical map $L_{\underline{n}}\mathcal{E} \sqcup_{L_{\underline{n}}\mathcal{D}} \mathcal{D}(\underline{n}) \to \mathcal{E}(\underline{n})$ is a cofibration of \mathcal{K} . Moreover the colimit functor

$$\varinjlim: \mathbf{CAT}(\mathcal{P}^C(S), \mathcal{K}) \longrightarrow \mathcal{K}$$

is a left Quillen adjoint.

Proof. A model structure is characterized by its class of weak equivalences and its class of cofibrations. Hence the uniqueness. The existence is explained e.g. in [22, Theorem 15.3.4]. The matching category of an object is either empty or connected by Proposition 7.5. The last assertion is then the consequence of [22, Proposition 15.10.2] and [22, Theorem 15.10.8]. \Box

The following additional facts are worth being mentioned. However, they are not used in the paper.

7.8. **Proposition.** Let \underline{n} be an object of $\mathcal{P}^{C}(S)$. Then either $\partial(\mathcal{P}^{C}(S)_{+}\downarrow\underline{n})$ is empty (in this case, let $\mathbb{I}(\underline{n}) := \underline{n}$) or it has an initial object denoted by $\mathbb{I}(\underline{n})$.

Proof. Let $\underline{n} = ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n))$. Then we have necessarily $\mathbb{I}(\underline{n}) = ((u_0, 0, u_1), (u_1, 0, u_2), \dots, (u_{n-1}, 0, u_n)).$

The proposition is then a consequence of Proposition 7.3.

We deduce that the limit functor $\lim_{K \to \infty} : \mathbf{CAT}(\mathcal{P}^{C}(S), \mathcal{K}) \to \mathcal{K}$ is also a right Quillen adjoint by Proposition 7.8, [22, Proposition 15.10.2] and [22, Theorem 15.10.8].

8. PUSHOUT ALONG A GENERATING SUBDIVISION

8.1. **Definition.** A poset (P, \leq) is *bounded* if there exist $\hat{0} \in P$ and $\hat{1} \in P$ such that $P = [\hat{0}, \hat{1}]$ and such that $\hat{0} \neq \hat{1}$. Let $\hat{0} = \min P$ (the bottom element) and $\hat{1} = \max P$ (the top element).

8.2. Notation. [11, Definition 4.4] Let \mathcal{T} be the class of inclusions of finite bounded posets $P_1 \subset P_2$ preserving the bottom element and the top element. The class \mathcal{T} is essentially small.

8.3. **Definition.** A generating subdivision is a q-cofibration of flows $f^{cof}: P_1^{cof} \to P_2^{cof}$ between q-cofibrant flows such that there exists a commutative square of flows



with $f: P_1 \subset P_2 \in \mathcal{T}$ and such that the horizontal maps are weak equivalences of the q-model structure of flows.

In [10–12], such a map is called a generating T-homotopy equivalence. For the same reason as in [19, Definition 9.1] where the T-homotopy equivalence terminology is abandoned to the more appropriate globular subdivision terminology, we want to forget the old terminology which was a bit naive.

8.4. Notation. Let \mathcal{T}^{cof} be an *arbitrary* choice of generating subdivisions f^{cof} for f running over the class of maps \mathcal{T} . The class \mathcal{T}^{cof} is essentially small.

8.5. Theorem. Consider a pushout diagram of flows of the form



where the left vertical map is a generating subdivision. Then for all $(\alpha, \beta) \in A^0 \times A^0$, the map f induces a trivial h-cofibration of spaces $\mathbb{P}_{\alpha,\beta}A \to \mathbb{P}_{f(\alpha),f(\beta)}X$.

Proof. For each pair (u, v) of P_1 with u < v, consider the topological space $T_{u,v}$ defined by the pushout diagram of spaces



Since the map $P_1^{cof} \to P_2^{cof}$ is a q-cofibration between q-cofibrant flows by hypothesis, the left vertical map $\mathbb{P}_{u,v}P_1^{cof} \to \mathbb{P}_{u,v}P_2^{cof}$ is a q-cofibration of spaces between q-cofibrant spaces by [16, Theorem 5.7]. Since both $\mathbb{P}_{u,v}P_1^{cof}$ and $\mathbb{P}_{u,v}P_2^{cof}$ are contractible (since P_1 and P_2 are posets and u < v), the map $\mathbb{P}_{u,v}P_1^{cof} \to \mathbb{P}_{u,v}P_2^{cof}$ is therefore a weak homotopy equivalence between q-cofibrant spaces, which means that it is a homotopy equivalence. This implies that the left vertical map $\mathbb{P}_{u,v}P_1^{cof} \to \mathbb{P}_{u,v}P_2^{cof}$ is a trivial h-cofibration, i.e. a trivial cofibration for the h-model structure of **Top**. This implies that all maps $\mathbb{P}_{g(u),g(v)}A \to T_{u,v}$ are trivial h-cofibrations of spaces as well for all pairs (u, v) of $P_1 \times P_1$ such that u < v, being pushouts of trivial h-cofibrations.

Consider the set map $C: P_1 \subset A^0$. Let $\mathcal{D}^f: \mathcal{P}^C(A^0) \to \mathbf{Top}$ be the diagram of spaces defined by:

- $\mathcal{D}^{f}((u_{0}, \epsilon_{1}, u_{1}), (u_{1}, \epsilon_{2}, u_{2}), \dots, (u_{n-1}, \epsilon_{n}, u_{n})) = Z_{u_{0}, u_{1}} \times Z_{u_{1}, u_{2}} \times \dots \times Z_{u_{n-1}, u_{n}}$ with $Z_{u_{i-1}, u_{i}} = \begin{cases} \mathbb{P}_{u_{i-1}, u_{i}} A & \text{if } \epsilon_{i} = 0\\ T_{\epsilon_{i}} & \text{if } \epsilon_{i} \neq 0 \text{ (in this case, } (C, C)(\epsilon_{i}) = (u_{i-1}, u_{i})). \end{cases}$
- The composition maps $c'_i s$ are induced by the compositions of paths of A.
- The inclusion maps I_i^{ϵ} are induced by the q-cofibrations $f : \mathbb{P}_{u,v}A \to T_{\epsilon}$ with $(C, C)(\epsilon) = (u, v).$

Let $\underline{n} \in \operatorname{Obj}(\mathcal{P}^{C}(A^{0}))$ with $\underline{n} = ((u_{0}, \epsilon_{1}, u_{1}), (u_{1}, \epsilon_{2}, u_{2}), \ldots, (u_{n-1}, \epsilon_{n}, u_{n}))$. Then the continuous map $L_{\underline{n}}\mathcal{D}^{f} \longrightarrow \mathcal{D}^{f}(\underline{n})$ is the pushout product of the maps $\emptyset \to \mathbb{P}_{u_{i-1}, u_{i}}A$ for *i* running over $\{i \in [1, n] | \epsilon_{i} = 0\}$ and of the maps $\mathbb{P}_{u_{i}, v_{i}}A \to T_{\epsilon_{i}}$ for *i* running over $\{i \in [1, n] | \epsilon_{i} \neq 0\}$. Moreover, if for all $i \in [1, n]$, we have $\epsilon_{i} = 0$, then $L_{\underline{n}}\mathcal{D}^{f} = \emptyset$. The argument is explained in [16, Proposition 5.2]. We deduce as in the proof of [16, Theorem 5.4] that the map of diagrams $\mathcal{D}^{\mathrm{Id}_{A}} \longrightarrow \mathcal{D}^{f}$ is a trivial Reedy h-cofibration. Therefore by passing to the colimit which is a left Quillen adjoint by Theorem 7.7, we deduce that the map $\varinjlim \mathcal{D}^{\mathrm{Id}_{A}} \longrightarrow \varinjlim \mathcal{D}^{f}$ is a trivial h-cofibration of **Top**. It turns out that $\mathbb{P}A \cong \varinjlim \mathcal{D}^{\mathrm{Id}_A}$ and that $\mathbb{P}X \cong \varinjlim \mathcal{D}^f$ by an analogue of [16, Theorem 4.8]. Thus, for all $(\alpha, \beta) \in \overline{A^0} \times A^0$, the map f induces a trivial h-cofibration $\mathbb{P}_{\alpha,\beta}A \to \mathbb{P}_{f(\alpha),f(\beta)}X$. \Box

8.6. Corollary. Let $f: A \to X$ be a map of $cof(\mathcal{T}^{cof})$. Then the continuous maps

$$I^+(A)(\alpha,\beta) \longrightarrow I^+(X)(f(\alpha),f(\beta))$$

are trivial h-cofibrations for all $(\alpha, \beta) \in A^0 \times A^0$, and in particular a homotopy equivalences.

Proof. If $\alpha \neq \beta$, then the map

$$I^{+}(A)(\alpha,\beta) \cong \mathbb{P}_{\alpha,\beta}A \to \mathbb{P}_{f(\alpha),f(\beta)}X \cong I^{+}(X)(f(\alpha),f(\beta))$$

is a trivial h-cofibration, being a retract of a transfinite composition of trivial h-cofibrations. If $\alpha = \beta$, then the map

$$\mathrm{I}^+(A)(\alpha,\beta) \cong \{\mathrm{Id}_\alpha\} \sqcup \mathbb{P}_{\alpha,\beta}A \to \{\mathrm{Id}_{f(\alpha)}\} \sqcup \mathbb{P}_{f(\alpha),f(\beta)}X \cong \mathrm{I}^+(X)(f(\alpha),f(\beta))$$

is also a trivial h-cofibration, being also a retract of a transfinite composition of trivial h-cofibrations. $\hfill \Box$

In Theorem 8.5 as well as in Corollary 8.6, the q-cofibrancy of the flows A and X is not required. If A and X are q-cofibrant, then the maps $\emptyset \to \mathbb{P}_{u_{i-1},u_i}A$ for i running over $\{i \in [1,n] | \epsilon_i = 0\}$ are also q-cofibrations, the topological spaces $\mathbb{P}_{u_{i-1},u_i}A$ being qcofibrant by [16, Theorem 5.7]. This implies that the map of diagrams $\mathcal{D}^{\mathrm{Id}_A} \longrightarrow \mathcal{D}^f$ is a trivial Reedy q-cofibration and that, for all $(\alpha, \beta) \in A^0 \times A^0$, the map f therefore induces a trivial q-cofibration $\mathbb{P}_{\alpha,\beta}A \to \mathbb{P}_{f(\alpha),f(\beta)}X$. Thus, if $f : A \to X$ is a map of $\operatorname{cof}(\mathcal{T}^{cof})$ between q-cofibrant flows, then the continuous maps $\mathrm{I}^+(A)(\alpha,\beta) \longrightarrow \mathrm{I}^+(X)(f(\alpha),f(\beta))$ are trivial q-cofibrations and homotopy equivalences for all $(\alpha,\beta) \in A^0 \times A^0$.

9. The set of maps \mathcal{T}^{gl}

We want to introduce in this section a very specific family \mathcal{T}^{gl} of generating subdivisions which makes the computations easy for Theorem 9.5. We will prove in Proposition 10.1 that this arbitrary choice has no consequence for the sequel.

9.1. **Proposition.** Let $n \ge 0$. Consider a finite set

 $F \subset |\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|.$

The finite set $\{0,1\} \cup F$ can be equipped with a poset structure as follows: u < v if and only if $\mathbb{P}_{u,v}^{top} \text{Glob}^{top}(\mathbf{D}^n)_F$ is nonempty (which implies contractible in this particular case). Moreover, the identity of $\text{Glob}^{top}(\mathbf{D}^n)_F^0 = \{0,1\} \cup F$ induces a trivial q-fibration of flows

$$\operatorname{Glob}(\mathbf{D}^n)_F \xrightarrow{\simeq} (\{0,1\} \cup F, \leqslant).$$

Proof. The map of flows $\operatorname{Glob}(\mathbf{D}^n)_F \to (\{0,1\} \cup F, \leqslant)$ is a q-fibration since all spaces of execution paths of the poset $(\{0,1\} \cup F, \leqslant)$ are discrete. It is a weak equivalence since $\mathbb{P}^{top}_{u,v}\operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is either for $u \ge v$ empty or homeomorphic to \mathcal{M} for u < v which is contractible.

The poset $(\{0,1\} \cup F, \leq)$ of Proposition 9.1 looks as follows: a smallest element 0, a biggest element 1 and a finite number of finite strictly increasing chains of the form $0 < a_1 < \cdots < a_p < 1$ with $p \ge 1$. One of these posets is depicted in Figure 1.



FIGURE 1. The poset $(\{0,1\} \cup \{a_1, b_1, c_1, c_2, d_1\}, \leqslant)$ with $0 < a_1 < 1, 0 < b_1 < 1, 0 < c_1 < c_2 < 1$ and $0 < d_1 < 1$

9.2. **Proposition.** Let $n \ge 0$. Consider a finite set $F \subset |\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|$. There exists a (unique) multipointed d-space W_F^n and a factorization

$$\operatorname{Glob}^{top}(\mathbf{D}^n) \xrightarrow{i_F^n} W_F^n \xrightarrow{\simeq} \operatorname{Glob}^{top}(\mathbf{D}^n)_F$$

of the globular subdivision $\operatorname{Glob}^{top}(\mathbf{D}^n) \to \operatorname{Glob}^{top}(\mathbf{D}^n)_F$ such that

 $i_F^n \in \operatorname{cell}(I^{gl,top} \cup \{C\})$

and such that there are the homeomorphisms

$$\mathbb{P}^{top}_{\alpha,\beta}W^n_F \cong \begin{cases} \mathbf{D}^{n+1} & \text{if } (\alpha,\beta) = (0,1), \\ \mathbb{P}^{top}_{\alpha,\beta}\text{Glob}^{top}(\mathbf{D}^n)_F & \text{if } (\alpha,\beta) \neq (0,1). \end{cases}$$

In particular, the map $W_F^n \to \operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is a weak equivalence between cellular multipointed d-spaces. Finally, the map $\operatorname{cat}(W_F^n) \to \operatorname{Glob}(\mathbf{D}^n)_F$ is a weak equivalence between cellular flows.

Proof. The multipointed d-space W_F^n is obtained in two steps. The first step is the pushout diagram of multipointed d-spaces



The multipointed d-space V_F^n is cellular by Proposition 5.5. It satisfies

$$\mathbb{P}^{top}_{\alpha,\beta}V_F^n \cong \begin{cases} \mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n & \text{if } (\alpha,\beta) = (0,1) \\ \mathbb{P}^{top}_{\alpha,\beta} \text{Glob}^{top}(\mathbf{D}^n)_F & \text{if } (\alpha,\beta) \neq (0,1) \end{cases}$$

After choosing a homeomorphism $\mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n \cong \mathbf{S}^n$, the second step is the pushout diagram of multipointed *d*-spaces



By Theorem 3.8, the map $\operatorname{cat}(i_F^n)$: $\operatorname{Glob}(\mathbf{D}^n) \to \operatorname{cat}(W_F^n)$ belongs to $\operatorname{cell}(I^{gl} \cup \{C\})$, i.e. it is a q-cofibration of flows. Since both W_F^n and $\operatorname{Glob}^{top}(\mathbf{D}^n)_F$ are q-cofibrant, using Theorem 3.6, the image by the functor $\operatorname{cat} : \mathcal{M}d\mathbf{Top} \to \mathbf{Flow}$ of the weak equivalence $W_F^n \to \operatorname{Glob}^{top}(\mathbf{D}^n)_F$ is a weak equivalence of the q-model structure of flows between two q-cofibrant flows, and even between cellular flows by Theorem 3.8.

9.3. Proposition. With the notations of Proposition 9.1. The q-cofibration

 $\operatorname{cat}(i_F^n):\operatorname{Glob}(\mathbf{D}^n)\longrightarrow\operatorname{cat}(W_F^n)$

is a generating subdivision; more precisely, there exists a commutative diagram of flows

$$\begin{array}{c} \operatorname{Glob}(\mathbf{D}^n) & \stackrel{\simeq}{\longrightarrow} & \{0 < 1\} \\ \\ \underset{\operatorname{cat}(i_F^n)}{\overset{}{\longrightarrow}} & & \downarrow^{\subset} \\ \\ \underset{\operatorname{cat}(W_F^n) & \stackrel{\simeq}{\longrightarrow} & (\{0,1\} \cup F, \leqslant) \end{array}$$

such that the horizontal maps are weak equivalences of the q-model structure of flows and where the map of posets $\{0 < 1\} \subset (\{0, 1\} \cup F, \leqslant)$ is defined in Proposition 9.1.

)

Proof. It is a consequence of Proposition 9.1 and Proposition 9.2. The two horizontal maps are q-fibrations since the space of execution paths of the flows $\{0 < 1\}$ and $(\{0, 1\} \cup F, \leq)$ are discrete.

9.4. Notation. Consider the set of generating subdivisions

$$\mathcal{T}^{gl} = \{ \operatorname{cat}(i_F^n) : \operatorname{Glob}(\mathbf{D}^n) \to \operatorname{cat}(W_F^n) \}$$

with $n \ge 0$ and F running over all finite subsets of $|\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|$.

Theorem 9.5 is the analogue for the maps of \mathcal{T}^{gl} of Theorem 6.8 for the globular subdivisions.

9.5. Theorem. Consider a pushout diagram of multipointed d-spaces of the form



where A is a cellular multipointed d-spaces. Consider the set maps $(-)^+, (-)^- : X^0 \to A^0$ defined by $\alpha^+ = \alpha^- = \alpha$ if $\alpha \in A^0$ and $\alpha^+ = g(1) = \hat{g}(1)$ and $\alpha^- = g(0) = \hat{g}(0)$ if $\alpha \in X^0 \setminus A^0$. Let $\alpha, \beta \in X^0$. There are two mutually exclusive cases:

(1) α, β in $X^0 \setminus A^0$ and $I^+(cat(W_F^n))(\alpha, \beta) \neq \emptyset$; in this case, there is the homotopy equivalence

 $I^{+}(\operatorname{cat}(X))(\alpha,\beta) \simeq \{\tau_{\alpha,\beta}\} \sqcup I^{+}(\operatorname{cat}(A))(\alpha^{+},\beta^{-}).$

(2) There is the homotopy equivalence

 $I^+(cat(X))(\alpha,\beta) \simeq I^+(cat(A))(\alpha^+,\beta^-)$

otherwise.

Proof. By Proposition 5.5 and Theorem 3.8, there is the pushout diagram of cellular flows



Thanks to Proposition 9.3, Corollary 8.6 can be used, the inclusion of posets $\{0 < 1\} \subset (\{0, 1\} \cup F, \leq)$ belonging to \mathcal{T} (see Notation 8.2). The functor $I^+ : \mathbf{Flow} \to \mathbf{Cat_{Top}}$ being a left adjoint, there is the pushout diagram of enriched small categories

$$\begin{array}{c|c} \mathrm{I}^{+}(\mathrm{Glob}(\mathbf{D}^{n})) & \stackrel{\mathrm{I}^{+}(g)}{\longrightarrow} & \mathrm{I}^{+}(\mathrm{cat}(A)) \\ & & & & \downarrow \\ & & & \downarrow \\ \mathrm{I}^{+}(\mathrm{cat}(W_{F}^{n})) & \stackrel{\mathrm{I}^{+}(\widehat{g})}{\longrightarrow} & \mathrm{I}^{+}(\mathrm{cat}(X)) \end{array}$$

Assume first that $\alpha, \beta \in X^0 \setminus A^0$ and that $\mathbb{P}^{top}_{\alpha,\beta} W_F^n \neq \emptyset$ or $\alpha = \beta$, i.e. $I^+(\operatorname{cat}(W_F^n))(\alpha,\beta)$ is not empty. By Proposition 9.2 (i.e. by construction of the multipointed *d*-space W_F^n), there is the homeomorphism

$$I^{+}(\operatorname{cat}(X))(\alpha,\beta) \cong I^{+}(\operatorname{cat}(W_{F}^{n}))(\alpha,\beta) \sqcup$$
$$I^{+}(\operatorname{cat}(W_{F}^{n}))(\alpha,g(1)) \times I^{+}(X)(g(1),g(0)) \times I^{+}(\operatorname{cat}(W_{F}^{n}))(g(0),\beta).$$

Since the map $\operatorname{cat}(W_F^n) \to \operatorname{Glob}(\mathbf{D}^n)_F$ is a weak equivalence of the q-model structure of flows between q-cofibrant flows by Proposition 9.2, there are the homotopy equivalences

$$I^{+}(\operatorname{cat}(W_{F}^{n}))(\alpha,\beta) \simeq I^{+}(\operatorname{Glob}(\mathbf{D}_{F}^{n}))(\alpha,\beta)$$
$$I^{+}(\operatorname{cat}(W_{F}^{n}))(\alpha,g(1)) \simeq I^{+}(\operatorname{Glob}(\mathbf{D}_{F}^{n}))(\alpha,g(1))$$
$$I^{+}(\operatorname{cat}(W_{F}^{n}))(g(0),\beta) \simeq I^{+}(\operatorname{Glob}(\mathbf{D}_{F}^{n}))(g(0),\beta)$$

by Theorem 4.7. Thus the topological spaces $I^+(\operatorname{cat}(W_F^n))(\alpha,\beta)$, $I^+(\operatorname{cat}(W_F^n))(\alpha,g(1))$ and $I^+(\operatorname{cat}(W_F^n))(g(0),\beta)$ are contractible. Using Corollary 8.6, we obtain the homotopy equivalence

$$I^{+}(\operatorname{cat}(X))(\alpha,\beta) \simeq \{\tau_{\alpha,\beta}\} \sqcup I^{+}(\operatorname{cat}(A))(g(1),g(0)) \cong \{\tau_{\alpha,\beta}\} \sqcup I^{+}(\operatorname{cat}(A))(\alpha^{+},\beta^{-}).$$

The case $\alpha, \beta \in X^0 \setminus A^0$ and $I^+(cat(W_F^n))(\alpha, \beta) = \emptyset$ leads to the homotopy equivalence

$$I^{+}(\operatorname{cat}(X))(\alpha,\beta) \simeq I^{+}(\operatorname{cat}(A))(g(1),g(0)) \cong I^{+}(\operatorname{cat}(A))(\alpha^{+},\beta^{-})$$

for similar reasons. The other cases are similar to the one treated in the proof of Theorem 6.8. It remains three mutually exclusive cases.

(1) $\alpha, \beta \in A^0$: Corollary 8.6 implies the homotopy equivalence

$$I^+(cat(X))(\alpha,\beta) \simeq I^+(cat(A))(\alpha,\beta).$$

(2) $\alpha \in A^0$ and $\beta \in X^0 \setminus A^0$: by Proposition 9.2, we obtain the homeomorphism

 $I^{+}(\operatorname{cat}(X))(\alpha,\beta) \cong I^{+}(\operatorname{cat}(X))(\alpha,g(0))) \times I^{+}(\operatorname{cat}(W_{F}^{n}))(g(0),\beta),$

and by Corollary 8.6 the homotopy equivalence

 $I^{+}(\operatorname{cat}(X))(\alpha,\beta) \simeq I^{+}(\operatorname{cat}(A))(\alpha,g(0)),$

the space $I^+(\operatorname{cat}(W_F^n))(g(0),\beta)$ being contractible by Proposition 9.2. (3) $\alpha \in X^0 \setminus A^0$ and $\beta \in A^0$: by Proposition 9.2, we obtain the homeomorphism

 $I^{+}(\operatorname{cat}(X))(\alpha,\beta) \cong I^{+}(\operatorname{cat}(W_{F}^{n}))(\alpha,g(1))) \times I^{+}(\operatorname{cat}(X))(g(1),\beta),$

and by Corollary 8.6 the homotopy equivalence

$$I^{+}(\operatorname{cat}(X))(\alpha,\beta) \simeq I^{+}(\operatorname{cat}(A))(g(1),\beta),$$

the space $I^+(cat(W_F^n))(\alpha, g(1)))$ being contractible by Proposition 9.2.

10. Globular subdivision and maps of \mathcal{T}^{cof}

Proposition 10.1 is required before proving Theorem 10.2.

10.1. **Proposition.** Let $n \ge 0$. Let F be a finite subset of $|\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|$. Consider a pushout diagram of flows



with A and X q-cofibrant. Then the q-cofibration $f: A \to X$ factors as a composite

 $f: A \xrightarrow{\in \mathbf{cell}(\mathcal{T}^{cof})} X' \xrightarrow{\simeq} X$

of a map of $\operatorname{cell}(\mathcal{T}^{cof})$ followed by a weak equivalence of flows.

Proof. By Proposition 9.3, there is a commutative diagram of solid arrows of q-cofibrant flows



Using Proposition 2.5, we deduce the existence of h_1 and h_2 making the diagram above commutative. By the two-out-of-three property, the maps h_1 and h_2 are weak equivalences of flows. Consider the commutative diagram of solid arrows (the existence of the flow X'

and of the map $s: X' \to X$ comes from the universal property of the pushout)



By the cube lemma [23, Lemma 5.2.6], the map s is a weak equivalence of flows. \Box

10.2. **Theorem.** Let $n \ge 0$. Let F a finite subset of $|\operatorname{Glob}^{top}(\mathbf{D}^n)| \setminus |\operatorname{Glob}^{top}(\mathbf{S}^{n-1})|$. Consider a commutative diagram of cellular multipointed d-spaces



such that $f: X \to Y$ and $\overline{f}: \overline{X} \to \overline{Y}$ are globular subdivisions. Assume that $\operatorname{cat}(f)$ factors as a composite $\operatorname{cat}(f) = w.i$ where $i: \operatorname{cat}(X) \to Z$ belongs to $\operatorname{cell}(\mathcal{T}^{cof})$ and where $w: Z \to \operatorname{cat}(Y)$ is a weak equivalence of the q-model structure of flows for some flow Z. Then there exists a commutative diagram of flows



such that $\overline{\imath}$: cat $(\overline{X}) \to \overline{Z}$ belongs to cell (\mathcal{T}^{cof}) and $\overline{w}: \overline{Z} \to cat(\overline{Y})$ is a weak equivalence of the q-model structure of flows, and finally such that the canonical map

$$U = \operatorname{cat}(\overline{X}) \sqcup_{\operatorname{cat}(X)} Z \to \overline{Z}$$



FIGURE 2. Adding and subdividing the globular cell $\operatorname{Glob}(\mathbf{D}^n)$ to $\operatorname{cat}(X)$

induced by the universal property of the pushout belongs to $\mathbf{cell}(\mathcal{T}^{cof})$.

 $\mathit{Proof.}$ Let us introduce the multipointed d-space \widehat{Y} characterized by the following commutative diagram



We obtain the commutative diagram of multipointed d-spaces



By Corollary 2.2, all squares of the diagram above are pushout squares. This implies that the maps $\overline{X} \to \widehat{Y}$ and $S : \widehat{Y} \to \overline{Y}$ are globular subdivisions. Thanks to the universal

property of the pushout, we obtain the commutative diagram of multipointed d-spaces



By Corollary 2.2, all squares of the diagram above are pushout squares as well. Using Proposition 5.5 and Theorem 3.8, we obtain the commutative diagram of flows



Thanks to the universal property of the pushout, with $V = \operatorname{cat}(W_F^n) \sqcup_{\operatorname{Glob}(\mathbf{D}^n)} U$, we obtain the commutative diagram of q-cofibrant flows (in fact, they are even all cellular) depicted in Figure 2. The maps $U \to \operatorname{cat}(\widehat{Y})$ and $V \to \operatorname{cat}(T)$ of the diagram of Figure 2 are weak equivalences of the q-model structure of flows since all flows of the diagram above are q-cofibrant and since they are pushouts of the weak equivalence $Z \to \operatorname{cat}(Y)$ along q-cofibrations (we could also invoke the left properness of the q-model structure of flows [16, Theorem 5.6]). By Proposition 10.1, the map $U \to V$ factors as a composite

$$U \xrightarrow{\in \mathbf{cell}(\mathcal{T}^{cof})} \overline{Z} \xrightarrow{\simeq} V$$

of a map of $\operatorname{cell}(\mathcal{T}^{cof})$ followed by a weak equivalence of flows. Thus the composite map map

$$\overline{\imath}: \operatorname{cat}(\overline{X}) \longrightarrow U \longrightarrow \overline{Z}$$

belongs to $\operatorname{cell}(\mathcal{T}^{cof})$. It remains to prove that the composite map

 \overline{w}

$$: \overline{Z} \xrightarrow{\simeq} V \xrightarrow{\simeq} \operatorname{cat}(T) \longrightarrow \operatorname{cat}(\overline{Y})$$

is a weak equivalence of flows. It therefore remains to prove that the map of flows $\operatorname{cat}(T) \to \operatorname{cat}(\overline{Y})$ is a weak equivalence of flows. The commutative square of flows



is a pushout square by Corollary 2.2. Thus the map $\operatorname{cat}(T) \to \operatorname{cat}(\overline{Y})$ of the diagram above induces a bijection on the states, the functor $X \mapsto X^0$ from flows to sets being colimitpreserving and since $\operatorname{cat}(W_F^n)^0 = \operatorname{Glob}(\mathbf{D}^n)_F^0 = \{0,1\} \cup F$. We obtain the bijections

$$\operatorname{cat}(T)^0 \cong \operatorname{cat}(\overline{Y})^0 \cong \operatorname{cat}(\widehat{Y})^0 \sqcup F.$$

Consider the set maps $(-)^-, (-)^+ : \operatorname{cat}(T)^0 \cong \operatorname{cat}(\overline{Y})^0 \to \operatorname{cat}(\widehat{Y})^0$ defined by

$$\begin{split} \alpha^- &= \begin{cases} \alpha & \text{ if } \alpha \in \operatorname{cat}(\widehat{Y})^0 \\ k(0) & \text{ if } \alpha \in F \end{cases} \\ \alpha^+ &= \begin{cases} \alpha & \text{ if } \alpha \in \operatorname{cat}(\widehat{Y})^0 \\ k(1) & \text{ if } \alpha \in F \end{cases} \end{split}$$

with $k : \operatorname{Glob}(\mathbf{D}^n) \to \operatorname{cat}(\overline{X}) \to U \to \operatorname{cat}(\widehat{Y})$ (note that it is possible that k(0) = k(1)). To complete the proof, it suffices to prove that for all $\alpha, \beta \in T^0$, the induced continuous map $\mathbb{P}_{\alpha,\beta}\operatorname{cat}(T) \to \mathbb{P}_{\alpha,\beta}\operatorname{cat}(\overline{Y})$ is a weak homotopy equivalence. Note that it is not possible to invoke the left properness of the q-model category of flows since the map $\operatorname{cat}(W_F^n) \to V \to \operatorname{cat}(T)$ has no reason to be a q-cofibration. It then suffices to prove that, for all $\alpha, \beta \in T^0$, the induced continuous map

$$I^+(cat(T))(\alpha,\beta) \to I^+(cat(\overline{Y}))(\alpha,\beta)$$

is a weak homotopy equivalence to complete the proof. Assume that $\alpha, \beta \in F$ and that $I^+(\operatorname{cat}(W_F^n))(\alpha, \beta) \neq \emptyset$. Then one has

$$I^{+}(\operatorname{cat}(T))(\alpha,\beta) \simeq \{\tau_{\alpha,\beta}\} \sqcup I^{+}(\operatorname{cat}(Y))(\alpha^{+},\beta^{-})$$
$$\cong \{\tau_{\alpha,\beta}\} \sqcup \overrightarrow{T}(\widehat{Y})(\alpha^{+},\beta^{-})$$
$$\cong \{\tau_{\alpha,\beta}\} \sqcup \overrightarrow{T}(\overline{Y})(\alpha^{+},\beta^{-})$$
$$\cong \overrightarrow{T}(\overline{Y})(\alpha,\beta)$$
$$\cong I^{+}(\operatorname{cat}(\overline{Y}))(\alpha,\beta),$$

the homotopy equivalence by Theorem 9.5, the first and fourth homeomorphisms by definition of the space of traces, the second homeomorphism by Theorem 6.8, the map $S: \hat{Y} \longrightarrow \overline{Y}$ being a globular subdivision, and finally the third homeomorphism by [19, Proposition 8.12]. In all other cases, using Theorem 9.5 and Theorem 6.8 in a similar way, we have

$$I^{+}(\operatorname{cat}(T))(\alpha,\beta) \simeq I^{+}(\operatorname{cat}(\widehat{Y}))(\alpha^{+},\beta^{-})$$
$$\cong \overrightarrow{T}(\widehat{Y})(\alpha^{+},\beta^{-})$$

$$\cong \overrightarrow{T}(\overline{Y})(\alpha^{+},\beta^{-})$$
$$\cong \overrightarrow{T}(\overline{Y})(\alpha,\beta)$$
$$\cong \mathrm{I}^{+}(\mathrm{cat}(\overline{Y}))(\alpha,\beta),$$

the third homeomorphism by [19, Proposition 8.7, Proposition 8.8 and Proposition 8.9]. $\hfill\square$

10.3. **Theorem.** Let $f: X \xrightarrow{\text{sbd}} Y$ be a globular subdivision of multipointed d-spaces. Then $\operatorname{cat}(f)$ factors as a composite

$$\operatorname{cat}(f) = w.i$$

where i belongs to $\operatorname{cell}(\mathcal{T}^{cof})$ and where w is a weak equivalence of the q-model structure of flows.

Using the language of [10], the map of flows cat(f) is a dihomotopy equivalence.

Proof. We consider a cellular decomposition of X. Using Theorem 6.9, there exists a transfinite tower of cellular multipointed d-spaces $\tilde{Y} : \lambda \to \mathcal{M}\mathbf{dTop}$ and a map of transfinite towers $\tilde{X} \to \tilde{Y}$ such that the colimit is the globular subdivision $X \to Y$ and such that for all $\nu < \lambda$, there is the commutative diagrams of multipointed d-spaces of the form



Moreover, the connection maps of the globular subdivision $f: X_{\nu} \xrightarrow{\text{sbd}} Y_{\nu}$ are the restrictions to Y_{ν}^{0} of the connection maps of the globular subdivision $f: X_{\nu+1} \xrightarrow{\text{sbd}} Y_{\nu+1}$ for all $\nu < \lambda$.

We then have to prove by a transfinite induction that, for all $\nu \leq \lambda$, the globular subdivision $X_{\nu} \xrightarrow{\text{sbd}} Y_{\nu}$ factors as a composite $X_{\nu} \to Z_{\nu} \to Y_{\nu}$ where the left-hand map belongs to $\text{cell}(\mathcal{T}^{cof})$ and where the right-hand map is a weak equivalence of the q-model structure of flows. The case $\nu = 0$ is trivial: $X_0 = Y_0 = X^0$ indeed. The passage from $\nu < \lambda$ to $\nu + 1$ is a consequence of Theorem 10.2. It remains the case where $\nu \leq \lambda$ is a limit ordinal. The fact that $X_{\nu} \to Z_{\nu}$ belongs to $\text{cell}(\mathcal{T}^{cof})$ is a consequence of Theorem 10.2 and Proposition 2.4. The fact that $Z_{\nu} \to Y_{\nu}$ is a weak equivalence of the q-model structure of flows is a consequence of Proposition 2.3. 10.4. Corollary. Let X and Y be two cellular multipointed d-spaces related by a finite zigzag sequence of globular subdivisions. Then the associated flows cat(X) and cat(Y) are related by a finite zigzag of maps of $cell(\mathcal{T}^{cof})$ and of weak equivalences of flows.

Using the language of [10], the two flows cat(X) and cat(Y) are dihomotopy equivalent.

Proof. It is a consequence of Theorem 10.3.

Corollary 10.4 implies that the map of flows $\operatorname{cat}(f) : \operatorname{cat}(X) \to \operatorname{cat}(Y)$ yields an isomorphism between the underlying homotopy types of $\operatorname{cat}(X)$ and $\operatorname{cat}(Y)$ [12, Theorem 9.1]. The latter point is also a consequence of the fact that f induces a homeomorphism between the underlying spaces of X and Y and that, for any cellular multipointed d-space Z, the underlying homotopy type of the flow $\operatorname{cat}(Z)$ is the homotopy type of |Z|. This obvious fact is not proved in [12, 15, 18]: this is clearly an omission. We prove it now using the material of [18] recalled in Theorem 3.6.

10.5. **Proposition.** Let Z be q-cofibrant multipointed d-space. Then the underlying homotopy type of the flow cat(Z) is the homotopy type of |Z|.

Proof. After [15, Proposition 8.16], the underlying homotopy type of the flow $\operatorname{cat}(Z)$ is the homotopy type of the topological space $|\mathbb{M}_{!}^{top}(\mathbb{M}(\operatorname{cat}(Z))^{cof})|$ where $(-)^{cof}$ is some q-cofibrant replacement of the q-model structure of Moore flows. Since $\mathbb{M}_{!} \to \mathbb{M}$ is a Quillen equivalence by Theorem 3.6, from the isomorphism of flows $\operatorname{cat}(Z) \cong \operatorname{cat}(Z)$ and from the isomorphism of functors $\operatorname{cat} \cong \mathbb{M}_{!}\mathbb{M}$, we obtain the weak equivalence of Moore flows $\mathbb{M}^{top}(Z) \to \mathbb{M}(\operatorname{cat}(Z))$. Factor the map $\emptyset \to \mathbb{M}(\operatorname{cat}(Z))$ as a composite $\emptyset \to (\mathbb{M}(\operatorname{cat}(Z)))^{cof} \xrightarrow{\simeq} \mathbb{M}(\operatorname{cat}(Z))$. The Moore flow $\mathbb{M}^{top}(Z)$ being q-cofibrant by Theorem 3.6, Z being q-cofibrant, the weak equivalence $\mathbb{M}^{top}(Z) \to \mathbb{M}(\operatorname{cat}(Z))$ factors as a composite of weak equivalences

$$\mathbb{M}^{top}(Z) \xrightarrow{\simeq} (\mathbb{M}(\operatorname{cat}(Z)))^{cof} \xrightarrow{\simeq} \mathbb{M}(\operatorname{cat}(Z)).$$

The functor $\mathbb{M}_{!}^{top}$ being a left Quillen functor, we obtain a weak equivalence of multipointed *d*-spaces $\mathbb{M}_{!}^{top}\mathbb{M}^{top}(Z) \simeq \mathbb{M}_{!}^{top}(\mathbb{M}(\operatorname{cat}(Z))^{cof})$. Since *Z* is q-cofibrant by hypothesis, there is the isomorphism of multipointed *d*-spaces $Z \cong \mathbb{M}_{!}^{top}\mathbb{M}^{top}(Z)$ by Theorem 3.6. Thus there is a weak equivalence of multipointed *d*-spaces $Z \simeq \mathbb{M}_{!}^{top}(\mathbb{M}(\operatorname{cat}(Z))^{cof})$. Hence the proof is complete. \Box

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