

CORRECTIONS TO THE HOMOTOPY THEORY OF MOORE FLOWS AND ITS APPLICATIONS (October 11, 2026)

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ABSTRACT. We correct the papers on Moore flows and their applications to natural systems and globular subdivisions. The Quillen equivalences between Moore flows, multipointed d -spaces, and ordinary flows, and the equivalences induced by derived categorization, remain valid. The unit of the adjunction between Moore $\mathcal{M}$ -flows and $\mathcal{M}$ -multipointed d -spaces need not be an isomorphism on a q -cofibrant object. We replace the affected proofs and specify the loop-free hypotheses required for the comparison of natural systems.

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Part 1. General introduction

This paper collects the errata to [13, 14, 17–19]. We first state the conclusions available to a reader wishing to use these papers. The subsequent parts give the corrections to their proofs and to the auxiliary statements.

1. MOORE I, MOORE II, AND MOORE III

The category $\mathcal{G}$ of reparametrizations uses increasing homeomorphisms of intervals; the category $\mathcal{M}$ uses continuous nondecreasing surjections and therefore allows stop intervals. **All the Quillen equivalences of [13, 14, 17] remain valid.** In particular, for $\mathcal{P} \in \{\mathcal{G}, \mathcal{M}\}$ the adjunctions

$$\mathcal{P}\mathbf{dTop} \begin{array}{c} \xrightarrow{\mathbb{M}^{\mathcal{P}}} \\ \xleftarrow{\mathbb{M}_!^{\mathcal{P}}} \end{array} \mathcal{P}\mathbf{Flow} \begin{array}{c} \xrightarrow{\mathbb{M}_!} \\ \xleftarrow{\mathbb{M}} \end{array} \mathbf{Flow}$$

are Quillen equivalences between $\mathcal{P}$ -multipointed d -spaces, Moore $\mathcal{P}$ -flows, and ordinary flows for their q-model structures, with $\mathbb{M}_!^{\mathcal{P}}$ and $\mathbb{M}_!$ the left adjoints; see Theorems 12.3,

18.3, and 25.4. The inclusion $\mathcal{G} \subset \mathcal{M}$ and the saturation reflector induce further Quillen equivalences, displayed in Corollary 29.4; see Theorems 28.2 and 29.3.

The assertion that the unit is an isomorphism must be removed for $\mathcal{M}$. For $\mathcal{G}$, the unit on a q-cofibrant Moore $\mathcal{G}$ -flow and the counit on a q-cofibrant $\mathcal{G}$ -multipointed d -space are isomorphisms, as stated in Theorems 18.1 and 18.2. For $\mathcal{M}$, the unit $Y \rightarrow \mathbb{M}^{\mathcal{M}}\mathbb{M}_!^{\mathcal{M}}Y$ is a q-weak equivalence for every q-cofibrant Moore $\mathcal{M}$ -flow Y by Theorem 25.3, but need not be an isomorphism by Proposition 22.3. The counit $\mathbb{M}_!^{\mathcal{M}}\mathbb{M}^{\mathcal{M}}X \rightarrow X$ remains an isomorphism for every q-cofibrant $X \in \mathcal{M}\mathbf{dTop}$ by Theorem 25.2. This correction leaves the Quillen equivalence intact. The failure of the unit isomorphism also applies to the composite adjunction between Moore $\mathcal{M}$ -flows and saturated $\mathcal{M}$ -multipointed d -spaces.

The total left derived functor of categorization is an equivalence for both $\mathcal{G}$ and $\mathcal{M}$. Write $\text{cat}_{\mathcal{P}} = \mathbb{M}_!\mathbb{M}^{\mathcal{P}}$ for categorization, whose path spaces are the execution paths modulo reparametrization. For a functorial q-cofibrant replacement Q_d , the functor

$$\mathbf{Lcat}_{\mathcal{P}} : \mathbf{Ho}(\mathcal{P}\mathbf{dTop}) \xrightarrow{\sim} \mathbf{Ho}(\mathbf{Flow}), \quad X \mapsto \text{cat}_{\mathcal{P}}(Q_d X)$$

is the equivalence asserted in [14, Theorem 8.14] and [17, Theorem 15]. Sections 19 and 26 establish this conclusion with the corrected proofs.

For $\mathcal{G}$ and $\mathcal{M}$, categorization has the following properties.

- (i) It preserves a globular attachment to a cellular $\mathcal{P}$ -multipointed d -space. More precisely, if A is such a multipointed d -space, the following two squares are pushouts in $\mathcal{P}\mathbf{dTop}$ and $\mathbf{Flow}$, respectively:

$$\begin{array}{ccc} \text{Glob}^{\mathcal{P}}(S^{n-1}) & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ \text{Glob}^{\mathcal{P}}(D^n) & \longrightarrow & X. \end{array} \quad \begin{array}{ccc} \text{Glob}(S^{n-1}) & \longrightarrow & \text{cat}_{\mathcal{P}} A \\ \downarrow & \lrcorner & \downarrow \\ \text{Glob}(D^n) & \longrightarrow & \text{cat}_{\mathcal{P}} X. \end{array}$$

For $\mathcal{G}$, this follows from Theorem 17.2 because $\mathbb{M}_!$ is a left adjoint and preserves colimits. For $\mathcal{M}$, it is Theorem 24.2.

- (ii) It preserves the colimits at limit ordinals in a globular cellular presentation. For $\mathcal{G}$, combine Proposition 17.3 with preservation of colimits by $\mathbb{M}_!$; for $\mathcal{M}$, use Proposition 24.3.
- (iii) It takes q-cofibrant $\mathcal{P}$ -multipointed d -spaces to q-cofibrant ordinary flows; see Proposition 19.2 and Corollary 24.4.
- (iv) It preserves q-weak equivalences between q-cofibrant $\mathcal{P}$ -multipointed d -spaces; see Propositions 19.2 and 26.1.

For a q-cofibrant $\mathcal{P}$ -multipointed d -space, where $\mathcal{P} = \mathcal{G}, \mathcal{M}$, the quotient from execution paths to traces is a trivial Hurewicz fibration with a continuous section; its source is m-cofibrant and its target is q-cofibrant by Theorem 26.2. These assertions, together with the four properties above, are the Moore-flow results needed in the two applications below.

2. NATURAL HOMOTOPY OF MULTIPOINTED d -SPACES

The main results of [19] remain valid with the loop-free hypothesis. A cellular multipointed d -space is loop-free when it has no execution loop at a state. This hypothesis is

required in [19, Theorem 4.10 and Section 8] and for both cellular $\mathcal{M}$ -multipointed d -spaces in [19, Theorem 9.4]. After localization at weak homotopy equivalences, there is an open morphism from the natural system associated with a loop-free cellular $\mathcal{M}$ -multipointed d -space to its discrete natural system in $\text{Diag}(\mathbf{Ho}(\mathbf{Top}))$. This gives bisimilarity up to homotopy. For every globular subdivision $f : X \rightarrow Y$ between loop-free cellular $\mathcal{M}$ -multipointed d -spaces, the four natural systems $\overrightarrow{\text{NT}}(X)$, $\overrightarrow{\text{NT}}(Y)$, $\overrightarrow{\text{NT}}_d(X)$, and $\overrightarrow{\text{NT}}_d(Y)$ are bisimilar up to homotopy; see Theorem 32.2.

Loop-freeness is required for this open morphism between localized natural systems and is independent of the corrections to Moore III. The cellular pushout statement used in [19, Proposition 7.1] follows from the preservation property of $\text{cat}_{\mathcal{M}}$ above, with its original hypotheses. The conclusion that q-weak equivalence need not preserve bisimilarity up to homotopy also remains valid: Proposition 34.1 supplies a complete finite loop-free counterexample.

3. GLOBULAR SUBDIVISIONS ARE DIHOMOTOPY EQUIVALENCES

The main result of [18] remains valid. Every globular subdivision $f : X \rightarrow Y$ between cellular $\mathcal{M}$ -multipointed d -spaces has the factorization

$$\text{cat}_{\mathcal{M}}(f) = wi, \quad i \in \text{cell}(\mathcal{T}^{\text{cof}}), \quad w \text{ a q-weak equivalence,}$$

asserted in [18, Theorems 9.2–9.3]. Here $\mathcal{T}^{\text{cof}}$ is the class of generating subdivisions defined in that paper. The consequences for dihomotopy equivalence, branching and merging homology, and underlying homotopy types remain valid. Execution loops and stopped attaching paths are permitted.

The proof uses the preservation by categorization of cellular globe attachments, colimits at limit ordinals, q-cofibrancy, and q-weak equivalences stated above. It does not use the false assertion that the unit is an isomorphism for $\mathcal{M}$. Its calculation of traces retains the summand of paths that leave a cell and return to it, so the loop-free restriction needed for discrete natural systems does not enter this argument. Sections 35–38 give the corrected trace decomposition, the verification that the particular map in the factorization is a q-weak equivalence, and the repaired references to Moore III.

4. ORGANIZATION OF THE ERRATUM

Parts 2, 3, and 4 correspond to [13, 14, 17]. **We state the valid published results used as inputs with their references and omit their proofs when no correction is needed.** We prove the replacements needed for the tensor of constant presheaves, the cellular word diagrams, and the topology of execution paths. A table at the end of Part 4 records the Quillen equivalences and the quotients from execution paths to traces, together with their inputs.

For $\mathcal{M}$, stopped boundary meridians invalidate the closedness argument for characteristic chains and the resulting assertion that $\mathbb{M}^{\mathcal{M}}$ preserves cellular globe attachments. Section 22 gives a finite loop-free example. Continuous normalization from [20] supplies representatives of traces; compact families of these representatives have only finitely many carriers. Finite products of characteristic disks then prove that categorization preserves globe attachments to cellular $\mathcal{M}$ -multipointed d -spaces. Comparing the unit for a Moore $\mathcal{M}$ -flow with the two quotient maps to traces proves that it is a q-weak equivalence on

q-cofibrant Moore $\mathcal{M}$ -flows. This proves the Quillen equivalence with $\mathcal{M}$ -multipointed d -spaces without asserting that the unit is an isomorphism.

Parts 5 and 6 give the remaining details of the errata to [19] and [18]. They include the corrected comparison of natural systems, the full replacement counterexample, the trace calculations for subdivisions, and the auxiliary corrections used in the main proofs. A reference accompanied by one of these citations uses the numbering of the cited paper. All other numbered references refer to the present document.

5. SPACES, COLIMITS, AND MODEL CATEGORIES

5.1. Execution paths and Moore composition. An ordinary flow records states, spaces of execution paths between states, and an associative composition of execution paths. A multipointed d -space records the states as points of an underlying space and the execution paths as continuous maps. The q-model structures distinguish the state set and the weak homotopy types of the execution-path spaces; see [9, 11, 16]. The relation between the two approaches was first established by categorization of globular complexes [10]. The adjunctions of Moore flows with ordinary flows and with multipointed d -spaces express it by Quillen equivalences for the q-model structures.

Moore composition retains the length of each path. If $x : [0, a] \rightarrow Z$ and $y : [0, b] \rightarrow Z$ satisfy $x(a) = y(0)$, their composite has domain $[0, a + b]$, and composition is strictly associative. Changes of parameter act contravariantly on the spaces of paths of each positive length. With the compact-open topology on the changes of parameter, these spaces form enriched presheaves. A Moore flow is a semicategory enriched over the resulting category of presheaves.

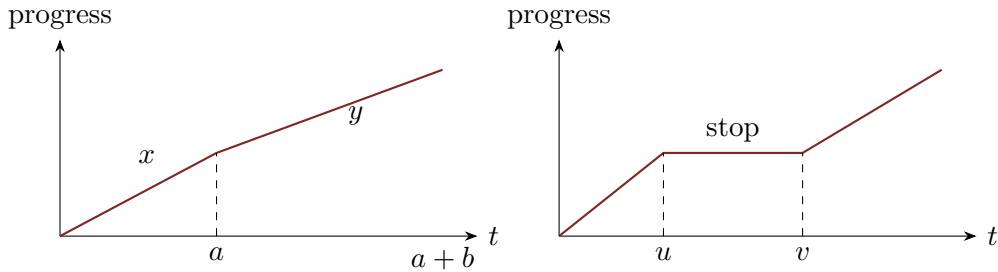


FIGURE 1. Moore composition retains the joining time. A nondecreasing surjection may replace a joining time by a stop interval.

5.2. The category of spaces. Throughout, **Top** denotes either the category of Δ -generated spaces or its full subcategory of Δ -Hausdorff spaces, with one choice fixed. A space is Δ -generated when its topology is final with respect to its continuous maps from standard topological simplices. A Δ -generated space is Δ -Hausdorff when the image of every continuous curve $\mathbb{I} \rightarrow Z$ is closed. For an ordinary topological space Z , its Δ -kelleyfication is denoted by $k_\Delta Z$ and has the same underlying set. Both choices of **Top** are locally presentable and cartesian closed; see [7] and [15, Section 2 and Appendix B]. Colimits in the Δ -Hausdorff category include the reflection to Δ -Hausdorff Δ -generated spaces. When a word presentation is calculated first in Δ -generated spaces, we verify that it is already Δ -Hausdorff before using the same presentation in the Δ -Hausdorff category.

We write $\mathbb{I} = [0, 1]$, D^n for the closed unit disk, and $S^{n-1} = \partial D^n$, with $D^0 = \{0\}$ and $S^{-1} = \emptyset$. Products, coproducts, limits, and colimits without another qualification are taken in the chosen category **Top**. We explicitly write an ordinary product or an ordinary relative topology when that distinction is needed. For $A, B \in \mathbf{Top}$, the internal hom is

$$\mathbf{TOP}(A, B) = k_\Delta C_{co}(A, B).$$

Here C_{co} is the compact-open mapping space. A subspace used as an object of **Top** receives the Δ -kelleyfication of its ordinary relative topology.

Every Δ -generated space is the coproduct of its nonempty path components, and these components are open. The component set $\mathbf{CC}(Z)$ is discrete; see [11, Proposition 2.8] and [15, Lemma 5.8]. Continuous maps from $\mathbb{I}$ suffice to test continuity between Δ -generated spaces, in the form used in [14, Corollary 2.3]: a continuous bijection $U \rightarrow V$ is a homeomorphism if its inverse carries every continuous curve $\mathbb{I} \rightarrow V$ to a continuous curve $\mathbb{I} \rightarrow U$.

5.1. Proposition. *Let Z be the underlying space of a cellular multipointed d -space. Suppose that Z is Hausdorff and its state subset is closed and discrete. If every compact subset meets only finitely many open cells, every compact subset is contained in a finite closed cellular subspace. Such a subspace is compact and metrizable.*

Proof. The compact discrete space $K \cap Z^0$ is finite. Start with these states and the finitely many cells met by K , and successively include the cells and states met by their attaching boundaries. Compactness of each boundary makes the resulting tree finitely branching; cell ordinals strictly decrease along every branch. König's infinity lemma [4, Lemma 8.1.2] therefore makes this tree finite.

The resulting cellular subspace is the image of a finite coproduct D of characteristic disks and points. It is compact, hence closed in Z , and its relative topology is the quotient topology. For the quotient $q : D \rightarrow Y$, let $\mathcal{B}$ be a countable basis of D . Closedness of q gives the countable basis

$$\{Y \setminus q(D \setminus U) : U \text{ is a finite union of members of } \mathcal{B}\}$$

of Y : each compact fiber $q^{-1}(y)$ is covered by finitely many basis elements inside $q^{-1}(V)$ for any neighborhood V of y . Urysohn's metrization theorem [29, Section 34] now applies. $\square$

5.2. Lemma. *Let $f : U \rightarrow V$ be a continuous bijection of topological spaces, with V Hausdorff. Suppose continuous maps $p_i : K_i \rightarrow V$ and $w_i : K_i \rightarrow U$ from compact spaces satisfy $fw_i = p_i$. Then $f^{-1}|_{p_i(K_i)}$ is continuous for the ordinary relative topology. If U, V are Δ -generated and every continuous curve in V has image in a finite union of the $p_i(K_i)$, then f is a homeomorphism.*

Proof. The closed surjection $p_i : K_i \rightarrow p_i(K_i)$ is a quotient map. Injectivity of f gives the continuous factorization

$$\begin{array}{ccc} K_i & \xrightarrow{w_i} & U \\ p_i \downarrow & \nearrow f^{-1} & \downarrow f \\ p_i(K_i) & \hookrightarrow & V. \end{array}$$

For a curve $\xi : \mathbb{I} \rightarrow V$, the finitely many sets $\xi^{-1}(p_i(K_i))$ form a closed cover. The pasting lemma makes $f^{-1}\xi$ continuous; the curve test gives the conclusion. The quotient and pasting lemmas are recalled in [29, Sections 22 and 18]. $\square$

5.3. The q-model structure. The q-model structure of **Top** has weak homotopy equivalences as weak equivalences and Serre fibrations as fibrations. It is generated by

$$I_{\mathbf{Top}} = \{S^{n-1} \hookrightarrow D^n : n \geq 0\}, \quad J_{\mathbf{Top}} = \{D^n \times \{0\} \hookrightarrow D^n \times \mathbb{I} : n \geq 0\}.$$

It is a cartesian monoidal model structure. We use the standard closure properties of cofibrations under pushouts, transfinite compositions, and retracts; see [22, 23]. All spaces are fibrant. The mixed model structure has weak homotopy equivalences and Hurewicz fibrations. Its cofibrant objects are precisely the spaces homotopy equivalent to q-cofibrant spaces [3, Corollary 3.7]. We use the term m-cofibrant for this class.

For later retract arguments, let $i : A \rightarrow X$ be a q-cofibration in one of the combinatorial model categories used below. Factor it by the small object argument as $i = pj$, where $j : A \rightarrow B$ is a relative complex of generating q-cofibrations and $p : B \rightarrow X$ is a trivial q-fibration. The lifting axiom supplies

$$(5.1) \quad \begin{array}{ccc} A & \xrightarrow{j} & B \\ i \downarrow & \nearrow s & \downarrow p \\ X & \xrightarrow{\text{id}_X} & X. \end{array}$$

The equations $si = j$ and $ps = \text{id}_X$ make i a retract of j in the arrow category. For a q-cofibrant object X , take $A = \emptyset$ to obtain the cellular replacement and its section:

$$(5.2) \quad \begin{array}{ccc} \emptyset & \xrightarrow{\quad} & B \\ \downarrow & \nearrow s & \downarrow p \\ X & \xrightarrow{\text{id}_X} & X. \end{array}$$

Here B is cellular and $ps = \text{id}_X$. These are the retract constructions used in the standard closure arguments for cofibrations [23].

5.3. Proposition. *Suppose $D \sqcup E$ is projectively q-cofibrant in a category of enriched presheaves with objectwise weak equivalences and fibrations. Then D is projectively q-cofibrant. The corresponding assertion holds for spaces.*

Proof. Let $p : A \rightarrow B$ be a trivial projective q-fibration and let $f : D \rightarrow B$ be a map. To prove cofibrancy of D , we must solve

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & A \\ \downarrow & \nearrow \tilde{f} & \downarrow p \\ D & \xrightarrow{f} & B. \end{array}$$

We first check that $p \sqcup \text{id}_E : A \sqcup E \rightarrow B \sqcup E$ is an objectwise trivial q-fibration. At an object ℓ of the indexing category, the generating lifting problems are

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{a} & A(\ell) \sqcup E(\ell) \\ \downarrow & \nearrow h & \downarrow p(\ell) \sqcup \text{id} \\ D^n & \xrightarrow{b} & B(\ell) \sqcup E(\ell). \end{array}$$

Connectedness of D^n puts b in one summand. In the $E(\ell)$ summand take $h = b$; in the $B(\ell)$ summand use the trivial q-fibration $p(\ell)$. This includes $n = 0$, and proves that $p \sqcup \text{id}_E$ is an objectwise trivial q-fibration.

Cofibrancy of $D \sqcup E$ now provides the diagonal in

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \sqcup E \\ \downarrow & \nearrow H & \downarrow p \sqcup \text{id}_E \\ D \sqcup E & \xrightarrow{f \sqcup \text{id}_E} & B \sqcup E. \end{array}$$

The equation $(p \sqcup \text{id}_E)H = f \sqcup \text{id}_E$ gives $H(D) \subset A$, so $\tilde{f} = H|_D$ solves the original square. For spaces, omit evaluation at ℓ . $\square$

5.4. Proposition ([23, Proposition 1.3.13]). *Let $F \dashv G : \mathcal{C} \rightleftarrows \mathcal{D}$ be a Quillen adjunction and suppose all objects of $\mathcal{D}$ are fibrant. If G reflects weak equivalences and the unit $X \rightarrow GFX$ is a weak equivalence for every cofibrant X , then the adjunction is a Quillen equivalence.*

5.4. State sets and cellular presentations. A discrete set with no execution paths is denoted by $S^\flat$. The two maps that add and identify states are

$$C : \emptyset \rightarrow \{0\}^\flat, \quad R : \{0, 1\}^\flat \rightarrow \{0\}^\flat.$$

They occur among the generating cofibrations of both Moore flows and multipointed d -spaces. For cellular Moore flows and cellular multipointed d -spaces starting from the empty object, their effect can be incorporated into the initial state set.

5.5. Proposition. *Let X be a cellular Moore flow or a cellular multipointed d -space constructed from C , R , and globe attachments. There is a cellular presentation of X starting with $(X^0)^\flat$ and using only globe attachments thereafter.*

Proof. Let X_ν be the original cellular presentation and form the set colimit $S = \text{colim}_\nu X_\nu^0$. Replace each stage by the pushout

$$\begin{array}{ccc} (X_\nu^0)^\flat & \longrightarrow & X_\nu \\ \downarrow & \lrcorner & \downarrow \\ S^\flat & \longrightarrow & \widehat{X}_\nu. \end{array}$$

The functor $S \mapsto S^\flat$ is left adjoint to the state-set functor, so it preserves colimits. Commutation of pushouts with colimits gives

$$\text{colim}_\nu \widehat{X}_\nu \cong X \sqcup_{S^\flat} S^\flat \cong X.$$

In the new presentation, additions and identifications of states become identities of S^b , while globe attachments keep their form. For Moore flows, they preserve the state set by Proposition 9.3. For multipointed d -spaces, the underlying stages are obtained by ordinary disk attachments to S . They are Δ -generated and Hausdorff by Proposition 23.1, so they belong to either chosen category of spaces; the Δ -Hausdorff reflection identifies no points. Every stage therefore has state set S , and the displayed isomorphism identifies S with X^0 . This gives the fixed state set used in [17, Section 4]. $\square$

Part 2. Moore I: reparametrization categories and flows

6. REPARAMETRIZATION CATEGORIES

6.1. Definition ([13, Definition 4.3 and Appendix]). A reparametrization category is a nonempty small topologically enriched strict semimonoidal category $(\mathcal{P}, \otimes)$ with the following properties. Every hom space $\mathcal{P}(\ell, \ell')$ is contractible. If $\ell' = \ell'_1 \otimes \ell'_2$ and $\varphi : \ell \rightarrow \ell'$, there are objects ℓ_1, ℓ_2 and maps $\varphi_i : \ell_i \rightarrow \ell'_i$ such that $\ell = \ell_1 \otimes \ell_2$ and $\varphi = \varphi_1 \otimes \varphi_2$.

We write $\ell + \ell'$ for the tensor of objects. The tensor of morphisms retains the symbol $\otimes$. Write $\mathcal{P}_0$ for the ordinary category obtained by forgetting the topologies of the hom spaces. No commutativity or symmetry is assumed. Nonemptiness is needed for the Quillen equivalence between enriched presheaves and spaces in Theorem 12.2. The terminal enriched category **1**, with its unique semimonoidal structure, is an example.

6.2. Proposition ([13, Propositions 4.9 and 4.11]).

The following categories are reparametrization categories. Their objects are the positive real numbers. The hom spaces are

$$\mathcal{G}(a, b) = \{\varphi : [0, a] \xrightarrow{\cong} [0, b] : \varphi \text{ is increasing}\},$$

$$\mathcal{M}(a, b) = \{\varphi : [0, a] \rightarrow [0, b] : \varphi \text{ is continuous and nondecreasing}\}.$$

They receive the Δ -kelleyfication of the relative compact-open topology. The tensor is concatenation of changes of parameter:

$$(\varphi_1 \otimes \varphi_2)(t) = \begin{cases} \varphi_1(t), & 0 \leq t \leq a_1, \\ b_1 + \varphi_2(t - a_1), & a_1 \leq t \leq a_1 + a_2, \end{cases} \quad \varphi_i \in \mathcal{P}(a_i, b_i).$$

For either category, a contraction is

$$H(\varphi, s)(t) = (1 - s)\varphi(t) + s\frac{b}{a}t.$$

For $0 < c < b$, the set $\varphi^{-1}(c) \subset]0, a[$ is a singleton for $\mathcal{G}$ and a nonempty compact interval for $\mathcal{M}$. Restriction at any $r \in \varphi^{-1}(c)$ gives

$$\varphi = \varphi_1 \otimes \varphi_2, \quad \varphi_1 \in \mathcal{P}(r, c), \quad \varphi_2 \in \mathcal{P}(a - r, b - c),$$

as required by the factorization axiom. Neither category is symmetric with respect to concatenation of morphisms.

6.3. Notation. For $\ell > 0$, let $\mu_\ell : [0, \ell] \rightarrow \mathbb{I}$ be the increasing homeomorphism $t \mapsto t/\ell$. Thus a unit-length path x gives the path $x\mu_\ell$ of length ℓ .

6.4. Proposition ([14, Proposition 2.5 and Lemma 6.2]; [17, Proposition 3]). *The compact-open topology on $\mathcal{G}(a, b)$ and on $\mathcal{M}(a, b)$ is the topology of uniform convergence. These spaces are metrizable and locally path-connected, and their ordinary topology is already Δ -generated. Inversion is continuous on $\mathcal{G}$.*

7. ENRICHED PRESHEAVES AND THEIR TENSOR

A $\mathcal{P}$ -space is a topologically enriched functor $D : \mathcal{P}^{op} \rightarrow \mathbf{Top}$. A morphism is an enriched natural transformation. We denote the underlying category by $[\mathcal{P}^{op}, \mathbf{Top}]_0$. The action has the form

$$\mathcal{P}(\ell, \ell') \times D(\ell') \longrightarrow D(\ell), \quad (\varphi, x) \longmapsto x\varphi.$$

The space of natural transformations is an end. The enriched Yoneda lemma and the density presentation are

$$(7.1) \quad [\mathcal{P}^{op}, \mathbf{Top}]_0(\mathcal{P}(-, \ell) \times U, D) \cong \mathbf{Top}(U, D(\ell)),$$

$$(7.2) \quad D \cong \int^\ell \mathcal{P}(-, \ell) \times D(\ell).$$

The corresponding statements with mapping spaces are homeomorphisms [25, Chapter 3].

7.1. Proposition ([12, Propositions 5.1, 5.3 and 5.5]). *The category $[\mathcal{P}^{op}, \mathbf{Top}]_0$ is locally presentable. Limits and colimits of enriched presheaves are calculated objectwise. For every ℓ , the functor $\mathbb{F}_\ell^{\mathcal{P}^{op}} : U \mapsto \mathcal{P}(-, \ell) \times U$ is left adjoint to evaluation at ℓ .*

7.2. Definition ([13, Definition 5.7]). The tensor of $\mathcal{P}$ -spaces is

$$(7.3) \quad (D \otimes E)(L) = \int^{\ell_1, \ell_2} \mathcal{P}(L, \ell_1 + \ell_2) \times D(\ell_1) \times E(\ell_2).$$

The coend is formed in $\mathbf{Top}$. Its representatives satisfy

$$(7.4) \quad [\psi, x\varphi_1, y\varphi_2] = [(\varphi_1 \otimes \varphi_2)\psi, x, y].$$

The coend retains the order of the factors and the topology of the hom spaces.

7.3. Theorem ([13, Theorem 5.14 and Proposition 5.16]). *The tensor makes $[\mathcal{P}^{op}, \mathbf{Top}]_0$ a biclosed semimonoidal category. It preserves colimits separately in each variable. There are natural isomorphisms*

$$\mathbb{F}_\ell^{\mathcal{P}^{op}}(U) \otimes \mathbb{F}_{\ell'}^{\mathcal{P}^{op}}(V) \cong \mathbb{F}_{\ell+\ell'}^{\mathcal{P}^{op}}(U \times V).$$

The internal homs and their adjunctions are

$$\begin{aligned} [E, F]_R(\ell) &= [\mathcal{P}^{op}, \mathbf{Top}](E, F(\ell + -)) = \int_a \mathbf{TOP}(E(a), F(\ell + a)), \\ [D, F]_L(\ell) &= [\mathcal{P}^{op}, \mathbf{Top}](D, F(- + \ell)) = \int_a \mathbf{TOP}(D(a), F(a + \ell)), \\ [D \otimes E, F] &\cong [D, [E, F]_R] \cong [E, [D, F]_L]. \end{aligned}$$

The functors $F(\ell + -)$ and $F(- + \ell)$ need not agree on morphisms.

7.4. Proposition ([13, Proposition 5.18]). *There is a natural associative homeomorphism*

$$\mathrm{colim}(D \otimes E) \cong \mathrm{colim} D \times \mathrm{colim} E.$$

For a representable presheaf, $\mathrm{colim} \mathcal{P}(-, \ell) \cong \{*\}$. Consequently $\mathrm{colim} \mathbb{F}_\ell^{\mathcal{P}^{op}}(U) \cong U$.

Density (7.2) reduces the proposition to

$$\operatorname{colim}(\mathbb{F}_\ell^{\mathcal{P}^{op}}(U) \otimes \mathbb{F}_{\ell'}^{\mathcal{P}^{op}}(V)) \cong \operatorname{colim} \mathbb{F}_{\ell+\ell'}^{\mathcal{P}^{op}}(U \times V) \cong U \times V.$$

The comparison $[\psi, x, y] \mapsto ([x], [y])$ is compatible with associativity.

8. CONSTANT PRESHEAVES AND JOINING TIMES

The constant presheaf on U is denoted by $\Delta_{\mathcal{P}^{op}}U$. There is always a canonical map

$$(8.1) \quad m_{U,V} : \Delta_{\mathcal{P}^{op}}U \otimes \Delta_{\mathcal{P}^{op}}V \longrightarrow \Delta_{\mathcal{P}^{op}}(U \times V), \quad [\psi, u, v] \longmapsto (u, v).$$

It is continuous, natural, and associative. It is the lax semimonoidal structure on the right adjoint of the strong semimonoidal colimit functor. Proposition 5.17 of [13] asserts that this map is always an isomorphism. The following calculation identifies the domain of this map for $\mathcal{P} = \mathcal{G}$.

8.1. Proposition. *For $L > 0$, there is a natural homeomorphism*

$$(\Delta_{\mathcal{G}^{op}}U \otimes \Delta_{\mathcal{G}^{op}}V)(L) \cong]0, L[\times U \times V.$$

For $\theta \in \mathcal{G}(L, L')$, the presheaf action on the interval factor is $]0, L'[-\rightarrow]0, L[, a \mapsto \theta^{-1}(a)$. The map in (8.1) is the projection onto $U \times V$.

Proof. Products preserve colimits, so take $U = V = \{*\}$ and write Q_L for the coend formed in Δ -generated spaces. Its summands are Δ -generated by Proposition 6.4, so Q_L has the ordinary quotient topology. The cut map is well defined by

$$c : Q_L \longrightarrow]0, L[, \quad c[\psi] = \psi^{-1}(\ell_1), \quad ((\varphi_1 \otimes \varphi_2)\psi)^{-1}(\ell'_1) = \psi^{-1}(\ell_1).$$

It is continuous on each summand by continuity of inversion and evaluation (Proposition 6.4).

For $a = c[\psi]$, restriction gives $\psi = \psi_1 \otimes \psi_2$ with $\psi_1 \in \mathcal{G}(a, \ell_1)$ and $\psi_2 \in \mathcal{G}(L - a, \ell_2)$. Relation (7.4) gives

$$[\psi]_{(\ell_1, \ell_2)} = [\operatorname{id}_L]_{(a, L-a)}, \quad [\psi] = [\psi'] \iff c[\psi] = c[\psi'].$$

In the fixed summand $(1, 1)$, the section $q(a) = \mu_a \otimes \mu_{L-a}$ solves

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathcal{G}(L, 2) \\ \downarrow & \nearrow q & \downarrow \psi \mapsto \psi^{-1}(1) \\]0, L[& \xrightarrow{\operatorname{id}} &]0, L[. \end{array}$$

The family $(a, t) \mapsto (\mu_a \otimes \mu_{L-a})(t)$ is continuous, so its adjoint q is continuous in the compact-open topology. Its composite with $\mathcal{G}(L, 2) \rightarrow Q_L$ is c^{-1} . Hence $Q_L \cong]0, L[$; this space is already Δ -generated and Hausdorff, so Δ -Hausdorffization makes no further identifications. Finally,

$$c[\psi\theta] = \theta^{-1}(c[\psi]), \quad (\Delta_{\mathcal{G}^{op}}U \otimes \Delta_{\mathcal{G}^{op}}V)(L) \cong Q_L \times U \times V.$$

□

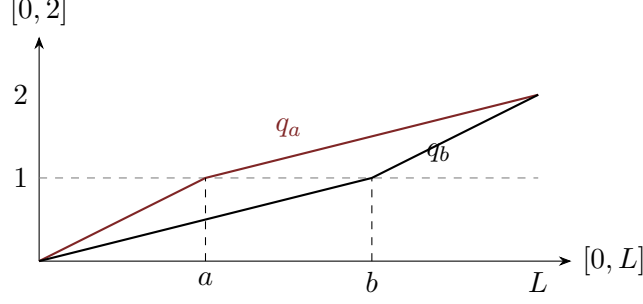


FIGURE 2. The sections $q_a = \mu_a \otimes \mu_{L-a}$ and $q_b = \mu_b \otimes \mu_{L-b}$ represent distinct coend classes when $a \neq b$.

8.2. Proposition. *For $\mathcal{M}$, the canonical map $\Delta_{\mathcal{M}^{op}}U \otimes \Delta_{\mathcal{M}^{op}}V \rightarrow \Delta_{\mathcal{M}^{op}}(U \times V)$ is an isomorphism.*

Proof. Reduce to $U = V = \{*\}$. Factoring $\psi \in \mathcal{M}(L, \ell_1 + \ell_2)$ at $a \in \psi^{-1}(\ell_1)$ gives $[\psi] = [\text{id}_L]_{(a, L-a)}$. For $0 < a < b < L$, the map

$$\psi_{a,b}(t) = \min \left\{ \frac{t}{a}, 1 \right\} + \max \left\{ \frac{t-b}{L-b}, 0 \right\} \quad \text{satisfies} \quad \psi_{a,b}^{-1}(1) = [a, b].$$

Its factorizations at a and b give

$$[\text{id}_L]_{(a, L-a)} = [\psi_{a,b}]_{(1,1)} = [\text{id}_L]_{(b, L-b)}.$$

The point coend is therefore a singleton. Products preserve colimits, giving the assertion for U, V . $\square$

The general theory requires only the lax semimonoidal structure (8.1); Proposition 8.1 shows that it is not strong for $\mathcal{G}$.

9. MOORE FLOWS AND LOCAL PRESENTABILITY

9.1. Definition ([13, Definition 6.1]). A *Moore $\mathcal{P}$ -flow*, also called a $\mathcal{P}$ -flow, is a small semicategory enriched over $([\mathcal{P}^{op}, \mathbf{Top}]_0, \otimes)$. It has a set X^0 of states, a $\mathcal{P}$ -space $\mathbb{P}_{\alpha,\beta}X$ for each ordered pair of states, and associative composition maps

$$\mathbb{P}_{\alpha,\beta}X \otimes \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X.$$

There are no specified identities. A morphism consists of a state map and compatible maps of path presheaves. We denote their category by $\mathcal{P}\mathbf{Flow}$.

At individual lengths this means continuous maps

$$\mathbb{P}_{\alpha,\beta}^{\ell_1}X \times \mathbb{P}_{\beta,\gamma}^{\ell_2}X \longrightarrow \mathbb{P}_{\alpha,\gamma}^{\ell_1+\ell_2}X.$$

For $\varphi_i : \ell'_i \rightarrow \ell_i$, naturality says

$$(x\varphi_1) * (y\varphi_2) = (x * y)(\varphi_1 \otimes \varphi_2).$$

The total path presheaf is $\mathbb{P}X = \coprod_{\alpha,\beta \in X^0} \mathbb{P}_{\alpha,\beta}X$.

9.2. Definition. For $D \in [\mathcal{P}^{op}, \mathbf{Top}]_0$, the globe $\text{Glob}(D)$ has states 0, 1, only possibly nonempty hom object $\mathbb{P}_{0,1} \text{Glob}(D) = D$, and no nonempty composable pairs.

9.3. Proposition ([13, Propositions 6.4 and 6.10]). *The state functor preserves limits and colimits. There is a natural bijection*

$$\mathcal{P}\mathbf{Flow}(\mathbf{Glob}(D), X) \cong \coprod_{\alpha, \beta \in X^0} [\mathcal{P}^{op}, \mathbf{Top}]_0(D, \mathbb{P}_{\alpha, \beta} X).$$

9.1. Limits and the solution set. For $X : I \rightarrow \mathcal{P}\mathbf{Flow}$, the limit is given by [13, Theorem 6.8]:

$$(\lim_i X_i)^0 = \lim_i X_i^0, \quad \mathbb{P}_{\alpha, \beta}(\lim_i X_i) = \lim_i \mathbb{P}_{\alpha_i, \beta_i} X_i, \quad \alpha = (\alpha_i), \beta = (\beta_i).$$

Composition is induced by the compositions in X_i , since products at each length commute with limits.

9.4. Proposition. *The category $\mathcal{P}\mathbf{Flow}$ is bicomplete.*

Proof. Completeness follows from the displayed construction. For cocompleteness, apply the adjoint functor theorem to the constant-diagram functor $\mathcal{P}\mathbf{Flow} \rightarrow \mathcal{P}\mathbf{Flow}^I$. For a fixed small diagram $X : I \rightarrow \mathcal{P}\mathbf{Flow}$, choose an infinite cardinal

$$\kappa \geq \max(\aleph_0, |\mathbf{Obj} \mathcal{P}|, |\mathbf{Mor} \mathcal{P}_0|, \sum_i \#X_i),$$

where $\#X_i = |X_i^0| + \sum_{\ell, \alpha, \beta} |\mathbb{P}_{\alpha, \beta}^\ell X_i|$. Given a cocone into Y , let Z_n consist of its images after n successive closures under reparametrization and binary composition. Then

$$|Z_0| \leq \kappa, \quad |Z_{n+1}| \leq \kappa + \kappa|Z_n| + |Z_n|^2 \leq \kappa, \quad \#Z = \left| \bigcup_{n \geq 0} Z_n \right| \leq \kappa.$$

The generated subflow $Z \subset Y$, with the Δ -kelleyfication of the relative topology on each hom subset, inherits continuous actions and compositions. The cocone factors through Z .

There is a set of such bounded structures and cocones up to isomorphism. This is a solution set for the limit-preserving constant-diagram functor; its left adjoint is colimit. The bound $\#Z < \sup_i \#X_i$ in [13, proof of Theorem 6.8] already fails for a one-state identity cocone. Even $\#Z \leq \sup_i \#X_i$ fails for a discrete diagram of one-state flows whose images are distinct. $\square$

9.2. The free semicategory. An enriched graph consists of a state set S and hom objects $D_{\alpha, \beta} \in \mathcal{V} = [\mathcal{P}^{op}, \mathbf{Top}]_0$. Its free semicategory has the same states and hom objects

$$(9.1) \quad T(D)_{\alpha, \beta} = \coprod_{n \geq 1} \coprod_{\substack{\alpha_0 = \alpha, \alpha_n = \beta \\ \alpha_1, \dots, \alpha_{n-1} \in S}} D_{\alpha_0, \alpha_1} \otimes \cdots \otimes D_{\alpha_{n-1}, \alpha_n}.$$

Composition and monad multiplication concatenate words; the monad unit includes words of length one. No identity paths are added.

9.5. Proposition. *For a filtered diagram $(X_i)_{i \in I}$ of Moore $\mathcal{P}$ -flows with common state set S and identity state maps, its colimit has state set S and hom presheaves*

$$\mathbb{P}_{\alpha, \beta}(\operatorname{colim}_i X_i) \cong \operatorname{colim}_i \mathbb{P}_{\alpha, \beta} X_i.$$

Proof. Set $H_{\alpha, \beta} = \operatorname{colim}_i \mathbb{P}_{\alpha, \beta} X_i$. The diagonal $I \rightarrow I \times I$ is final, and tensor preserves colimits separately. Thus

$$\begin{aligned} H_{\alpha, \beta} \otimes H_{\beta, \gamma} &\cong \operatorname{colim}_{(i, j) \in I \times I} (\mathbb{P}_{\alpha, \beta} X_i \otimes \mathbb{P}_{\beta, \gamma} X_j) \\ &\cong \operatorname{colim}_i (\mathbb{P}_{\alpha, \beta} X_i \otimes \mathbb{P}_{\beta, \gamma} X_i) \longrightarrow H_{\alpha, \gamma}. \end{aligned}$$

The same calculation with three factors proves associativity. The universal property of the hom colimits gives the universal property in $\mathcal{P}\mathbf{Flow}$. This is the fixed-state case of [13, Theorem 6.14], without the additional topological hypotheses used there. $\square$

9.6. Proposition. *The category $\mathcal{P}\mathbf{Flow}$ is locally presentable. For a sufficiently large regular cardinal λ , the globes on a dense family of λ -presentable $\mathcal{P}$ -spaces and the one-state pathless flow form a strong generator of λ -presentable objects.*

Proof. The category $\mathcal{V}\text{-}\mathbf{Gph}$ of enriched graphs with variable state set is locally presentable [26, Proposition 4.4].

The free semicategory monad (9.1) preserves filtered colimits. Indeed, each finite state list and each finite list of equalities among its representatives occurs at a common stage. The category J of representatives of a state list is therefore filtered. For its n hom factors, finality of $J \rightarrow J^n$ gives

$$\mathrm{colim}_{j \in J} (D_{j,1} \otimes \cdots \otimes D_{j,n}) \cong (\mathrm{colim}_{j \in J} D_{j,1}) \otimes \cdots \otimes (\mathrm{colim}_{j \in J} D_{j,n}).$$

Taking the coproduct over state lists and $n \geq 1$ gives $\mathrm{colim}_i T(D_i) \cong T(\mathrm{colim}_i D_i)$, including diagrams whose state maps identify states. Consequently T is accessible and its algebra category $\mathcal{P}\mathbf{Flow}$ is locally presentable [1]. This is the argument of [26, Theorem 3.3], with words of length zero omitted.

Choose a dense family (G_i) of presentable objects of $\mathcal{V}$. Enlarge λ so that the graph category is locally λ -presentable, T preserves λ -filtered colimits, and the two-state graphs with sole hom object G_i are λ -presentable [26, Lemma 4.3]. Their free algebras are $\mathrm{Glob}(G_i)$. The free-algebra adjunction makes these globes λ -presentable, since the forgetful functor creates λ -filtered colimits. The one-state pathless flow is presentable for the same reason. Proposition 9.3 shows that these objects detect isomorphisms on states and on every hom presheaf, so they form a strong generator. $\square$

This supplies the presentability step left implicit in [13, Theorem 6.11].

9.7. Proposition ([13, Theorem 6.13]). *The total path-presheaf functor is a right adjoint. Its left adjoint assigns to D the coproduct of the globes on its nonempty component presheaves.*

Proof. Contractibility of the hom spaces implies that parallel morphisms induce the same map on path components, and a morphism in the opposite direction induces its inverse. Thus the canonical maps

$$\mathrm{CC}(D(\ell)) \xrightarrow{\cong} \mathrm{CC}(D) := \mathrm{colim}_{\mathcal{P}_0^{op}} \mathrm{CC}(D(\ell))$$

identify the components intrinsically. For $c \in \mathrm{CC}(D)$, let $D_c(\ell) \subset D(\ell)$ be the component labelled by c , with the relative topology. These components are open, and the actions preserve their labels. Hence $D \cong \coprod_c D_c$ in $[\mathcal{P}^{op}, \mathbf{Top}]_0$.

Since every $D_c(\ell)$ is nonempty and path-connected, a map $D_c \rightarrow \mathbb{P}X$ has its image in one hom presheaf. Proposition 9.3 gives

$$\begin{aligned} [\mathcal{P}^{op}, \mathbf{Top}]_0(D, \mathbb{P}X) &\cong \coprod_{c \in \mathrm{CC}(D)} \coprod_{\alpha, \beta \in X^0} [\mathcal{P}^{op}, \mathbf{Top}]_0(D_c, \mathbb{P}_{\alpha, \beta} X) \\ &\cong \mathcal{P}\mathbf{Flow} \left(\coprod_{c \in \mathrm{CC}(D)} \mathrm{Glob}(D_c), X \right). \end{aligned}$$

$\square$

10. THE Q-MODEL STRUCTURE OF MOORE FLOWS

10.1. Theorem ([13, Theorems 8.8–8.10 and 8.16]). *There is a combinatorial q -model structure on $\mathcal{P}\mathbf{Flow}$. Its generating cofibrations are*

$$\{\mathrm{Glob}(\mathbb{F}_\ell^{\mathcal{P}^{op}} S^{n-1}) \rightarrow \mathrm{Glob}(\mathbb{F}_\ell^{\mathcal{P}^{op}} D^n) : n \geq 0, \ell \in \mathrm{Obj} \mathcal{P}\} \cup \{C, R\}.$$

A morphism $f : X \rightarrow Y$ is a weak equivalence if its state map is bijective and each $\mathbb{P}_{\alpha,\beta} X \rightarrow \mathbb{P}_{f\alpha,f\beta} Y$ is an objectwise weak homotopy equivalence. It is a fibration if these maps are objectwise q -fibrations. All objects are fibrant.

The construction applies the theorem on model categories of fibrant objects in [24], using Proposition 9.6. Its path object is

$$\begin{aligned} \mathrm{Path}(X)^0 &= X^0, & \mathbb{P}_{\alpha,\beta}^\ell \mathrm{Path}(X) &= \mathrm{TOP}(\mathbb{I}, \mathbb{P}_{\alpha,\beta}^\ell X), \\ (p * q)(t) &= p(t) * q(t). \end{aligned}$$

The category $[\mathcal{P}^{op}, \mathbf{Top}]_0$ has the projective q -model structure, with objectwise weak homotopy equivalences and q -fibrations and generating cofibrations $\mathbb{F}_\ell^{\mathcal{P}^{op}}(I_{\mathbf{Top}})$ [12, 28]. The free presheaves on disks and spheres are projectively q -cofibrant. Their values are m -cofibrant because the hom spaces of $\mathcal{P}$ are contractible; objectwise q -cofibrancy need not hold.

10.2. Theorem ([13, Theorem 9.4]). *If f and g are projective q -cofibrations of $\mathcal{P}$ -spaces, their pushout product for $\otimes$ is a projective q -cofibration. It is trivial if one factor is trivial.*

On free presheaves the calculation is

$$\mathbb{F}_\ell^{\mathcal{P}^{op}}(f) \square \mathbb{F}_{\ell'}^{\mathcal{P}^{op}}(g) \cong \mathbb{F}_{\ell+\ell'}^{\mathcal{P}^{op}}(f \square g).$$

The cartesian pushout-product property in $\mathbf{Top}$, closure under pushouts and transfinite compositions, and the cellular retract (5.1) give the theorem and its iterated version. The factors remain in their prescribed order.

11. CELLULAR WORD DIAGRAMS

Fix a nonempty state set S and $u, v \in S$.

11.1. Definition. Let $\mathcal{W}^{u,v}(S)$ have objects

$$\mathbf{n} = ((u_0, \varepsilon_1, u_1), \dots, (u_{n-1}, \varepsilon_n, u_n)), \quad n \geq 1,$$

where $\varepsilon_i \in \{0, 1\}$ and $\varepsilon_i = 1$ implies $(u_{i-1}, u_i) = (u, v)$. Its generators replace two adjacent unmarked factors by their composite or replace an unmarked $(u, 0, v)$ by $(u, 1, v)$:

$$\begin{aligned} c_i &: (\mathbf{a}, (x, 0, y), (y, 0, z), \mathbf{b}) \longrightarrow (\mathbf{a}, (x, 0, z), \mathbf{b}), \\ I_i &: (\mathbf{a}, (u, 0, v), \mathbf{b}) \longrightarrow (\mathbf{a}, (u, 1, v), \mathbf{b}). \end{aligned}$$

The prefix $\mathbf{a}$ and suffix $\mathbf{b}$ may have length zero. The relations are associativity of adjacent compositions, commutation of operations on disjoint factors, and the interchange of an inclusion with a composition away from that factor. An inclusion cannot commute through a composition using that factor.

These are the generators and relations of [15, Section 3], with empty prefixes and suffixes. The Reedy degree is

$$d(\mathbf{n}) = n + \sum_i \varepsilon_i.$$

The inverse maps are composites of c_i ; the direct maps are composites of I_i . Every morphism factors uniquely as an inverse map followed by a direct map. The latching category is the punctured cube of marked positions. A nonempty matching category has a terminal object, obtained by composing every maximal unmarked block. Thus $\mathcal{W}^{u,v}(S)$ has fibrant constants, and colimit is left Quillen for the Reedy q-model structure [30, Corollary 9.4 and Lemma 9.6].

11.2. Remark. If both the prefix and the suffix were required to be nonempty, all words of length one or two would be isolated. In the flow with paths $x : a \rightarrow b$, $y : b \rightarrow c$, and $z = x * y : a \rightarrow c$, the length-two element (x, y) and the length-one element z would not be identified. Thus the word category with that restriction cannot present the path space even when the globe morphism being pushed out is an identity. Allowing empty prefixes and suffixes is necessary in all three parts.

Consider a pushout of Moore flows and the induced pushout of presheaves:

$$\begin{array}{ccc} \text{Glob}(\partial Z) & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow f \\ \text{Glob}(Z) & \longrightarrow & X \end{array} \quad \begin{array}{ccc} \partial Z & \longrightarrow & \mathbb{P}_{u,v}A \\ \downarrow & \lrcorner & \downarrow \\ Z & \longrightarrow & T. \end{array}$$

Define $D_f : \mathcal{W}^{u,v}(A^0) \rightarrow [\mathcal{P}^{op}, \mathbf{Top}]_0$ by

$$D_f(\mathbf{n}) = Z_1 \otimes \cdots \otimes Z_n, \quad Z_i = \begin{cases} \mathbb{P}_{u_{i-1}, u_i} A, & \varepsilon_i = 0, \\ T, & \varepsilon_i = 1. \end{cases}$$

The maps c_i use composition in A , and the maps I_i use $\mathbb{P}_{u,v}A \rightarrow T$.

11.3. Proposition ([13, Section 9]; [15, Section 4]). *There is a canonical isomorphism $\text{colim } D_f \cong \mathbb{P}X$. At a word $\mathbf{n}$, the latching map is the ordered pushout product of $\emptyset \rightarrow \mathbb{P}_{u_{i-1}, u_i} A$ at the unmarked positions and $\mathbb{P}_{u,v}A \rightarrow T$ at the marked positions.*

Proof. Let Q have state set A^0 and hom presheaves given by the summands of $\text{colim } D_f$ with the corresponding endpoints. Concatenation descends to associative composition, since tensor preserves colimits separately and respects the defining relations. Those relations and the pushout defining T give a natural pullback of sets:

$$\begin{array}{ccc} \mathcal{P}\mathbf{Flow}(Q, B) & \longrightarrow & \mathcal{P}\mathbf{Flow}(A, B) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{P}\mathbf{Flow}(\text{Glob}(Z), B) & \longrightarrow & \mathcal{P}\mathbf{Flow}(\text{Glob}(\partial Z), B). \end{array}$$

Thus $Q \cong X$ by the pushout universal property.

For $M(\mathbf{n}) = \{i : \varepsilon_i = 1\}$, put $Z_i^J = T$ if $i \in J$ and $Z_i^J = \mathbb{P}_{u_{i-1}, u_i} A$ otherwise. The direct predecessors give

$$L_{\mathbf{n}} D_f \cong \operatorname{colim}_{J \subsetneq M(\mathbf{n})} \bigotimes_{i=1}^n Z_i^J \longrightarrow \bigotimes_{i=1}^n Z_i^{M(\mathbf{n})} = D_f(\mathbf{n}),$$

which is the stated ordered pushout product, including the one-letter case. $\square$

11.4. Theorem ([13, Theorem 9.10]). *If $\partial Z \rightarrow Z$ is a projective q -cofibration and $\mathbb{P}A$ is projectively q -cofibrant, then $\mathbb{P}A \rightarrow \mathbb{P}X$ is a projective q -cofibration. If $\partial Z \rightarrow Z$ is a trivial projective q -cofibration, then so is $\mathbb{P}A \rightarrow \mathbb{P}X$.*

Proof. Compare D_f with the diagram D_{id_A} associated to the identity of A , whose colimit is $\mathbb{P}A$. Write $E = D_{\operatorname{id}_A}$ and $D = D_f$. The relative latching map is $r_{\mathbf{n}}$ in

$$\begin{array}{ccc} L_{\mathbf{n}} E & \longrightarrow & E(\mathbf{n}) \\ \downarrow & \lrcorner & \downarrow \\ L_{\mathbf{n}} D & \longrightarrow & E(\mathbf{n}) \amalg_{L_{\mathbf{n}} E} L_{\mathbf{n}} D \\ & \searrow & \downarrow r_{\mathbf{n}} \\ & & D(\mathbf{n}). \end{array}$$

At an unmarked word, $r_{\mathbf{n}}$ is an identity. At a marked word, $L_{\mathbf{n}} E \rightarrow E(\mathbf{n})$ is a pushout product with an identity factor, hence an isomorphism; $r_{\mathbf{n}}$ identifies with $L_{\mathbf{n}} D \rightarrow D(\mathbf{n})$.

Proposition 5.3, the presheaf pushout defining T , and Theorem 10.2 imply that every $r_{\mathbf{n}}$ is a projective q -cofibration, trivial when $\partial Z \rightarrow Z$ is trivial. Hence $E \rightarrow D$ is a Reedy q -cofibration, and the left Quillen colimit functor gives the assertion. $\square$

11.5. Theorem. *If X is a q -cofibrant Moore $\mathcal{P}$ -flow, then $\mathbb{P}X$ and every $\mathbb{P}_{\alpha, \beta} X$ are projectively q -cofibrant.*

Proof. For a q -cellular flow, Proposition 5.5 puts the states at the initial stage. The initial path presheaf is empty, and Theorem 11.4 handles successors. At a limit ordinal λ , Proposition 9.5 gives

$$\mathbb{P}X_{\lambda} \cong \operatorname{colim}_{\nu < \lambda} \mathbb{P}X_{\nu}.$$

Thus $\emptyset \rightarrow \mathbb{P}X_{\lambda}$ is the transfinite composite of the projective q -cofibrations established before λ . This completes the induction.

For q -cofibrant X , apply $\mathbb{P}$ to the cellular retract (5.2), then apply Proposition 5.3 to the hom presheaves. This proves [13, Theorem 9.11] with the corrected word category and limit step. $\square$

12. THE QUILLEN EQUIVALENCE BETWEEN MOORE FLOWS AND ORDINARY FLOWS

An ordinary flow is a Moore **1**-flow. Equivalently, it is a small topologically enriched semicategory. Its q -model structure has weak equivalences that are bijective on states and weak homotopy equivalences on hom spaces; its fibrations are q -fibrations on hom spaces [9, 16]. For an ordinary flow X , define $\mathbb{M}X$ by $(\mathbb{M}X)^0 = X^0$ and

$$\mathbb{P}_{\alpha, \beta} \mathbb{M}X = \Delta_{\mathcal{P}^{op}}(\mathbb{P}_{\alpha, \beta} X).$$

Its composition is

$$\begin{aligned} \Delta_{\mathcal{P}^{op}}(\mathbb{P}_{\alpha,\beta}X) \otimes \Delta_{\mathcal{P}^{op}}(\mathbb{P}_{\beta,\gamma}X) &\xrightarrow{m} \Delta_{\mathcal{P}^{op}}(\mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X) \\ &\xrightarrow{\Delta_{\mathcal{P}^{op}}(*)} \Delta_{\mathcal{P}^{op}}(\mathbb{P}_{\alpha,\gamma}X). \end{aligned}$$

For a Moore $\mathcal{P}$ -flow Y , define $\mathbb{M}_!Y$ on the same states by

$$\mathbb{P}_{\alpha,\beta}\mathbb{M}_!Y = \operatorname{colim} \mathbb{P}_{\alpha,\beta}Y.$$

Proposition 7.4 supplies its composition.

12.1. Proposition. *These constructions define an adjunction $\mathbb{M}_! \dashv \mathbb{M}$ between Moore $\mathcal{P}$ -flows and ordinary flows. It is Quillen for their q -model structures.*

Proof. For each state map, the adjunctions

$$[\mathcal{P}^{op}, \mathbf{Top}]_0(D, \Delta_{\mathcal{P}^{op}}U) \cong \mathbf{Top}(\operatorname{colim} D, U)$$

preserve compatibility with composition: the comparison of Proposition 7.4 and $m_{U,V}$ are mates under $\operatorname{colim} \dashv \Delta_{\mathcal{P}^{op}}$. They therefore assemble into $\mathbb{M}_! \dashv \mathbb{M}$. The right adjoint preserves fibrations and trivial fibrations, which are defined objectwise on hom presheaves. $\square$

12.2. Theorem ([12, Theorem 7.6]). *For a nonempty small enriched category whose hom spaces are contractible, the adjunction between colimits and constant presheaves is a Quillen equivalence for the projective q -model structure on enriched presheaves and the q -model structure on $\mathbf{Top}$.*

For the Δ -Hausdorff choice of $\mathbf{Top}$, use the extension of this theorem stated in [13, p. 5].

12.3. Theorem. *For every nonempty reparametrization category $\mathcal{P}$, the adjunction $\mathbb{M}_! \dashv \mathbb{M}$ between Moore $\mathcal{P}$ -flows and ordinary flows is a Quillen equivalence for their q -model structures.*

Proof. The right adjoint $\mathbb{M}$ reflects weak equivalences in the q -model structures. For a q -cofibrant Moore flow Y , Theorem 11.5 makes every hom presheaf projectively q -cofibrant. The unit $Y \rightarrow \mathbb{M}\mathbb{M}_!Y$ is bijective on states, and its maps of hom presheaves are

$$\mathbb{P}_{\alpha,\beta}Y \longrightarrow \Delta_{\mathcal{P}^{op}} \operatorname{colim} \mathbb{P}_{\alpha,\beta}Y.$$

These maps are objectwise weak homotopy equivalences by Theorem 12.2. Proposition 5.4 completes the proof. This proves the main conclusion of [13, Theorem 10.9] without assuming that the map $m_{U,V}$ in (8.1) is an isomorphism. $\square$

For later use, put $q_Y^\ell : \mathbb{P}_{\alpha,\beta}^\ell Y \rightarrow \mathbb{P}_{\alpha,\beta}\mathbb{M}_!Y$ for the component at length ℓ of the unit $Y \rightarrow \mathbb{M}\mathbb{M}_!Y$. If Y is q -cofibrant, each q_Y^ℓ is a weak homotopy equivalence. For $\mathcal{P} = \mathcal{G}$ or $\mathcal{M}$, all lengths are isomorphic, and the colimit can be calculated using the one-object full subcategory on length 1. The hom space $\mathbb{P}_{\alpha,\beta}\mathbb{M}_!Y$ is consequently the coequalizer of the action of $\mathcal{P}(1, 1)$ on $\mathbb{P}_{\alpha,\beta}^1 Y$, that is, the quotient in $\mathbf{Top}$ by the equivalence relation generated by $x \sim x\varphi$ for $\varphi \in \mathcal{P}(1, 1)$.

Part 3. Moore II: increasing homeomorphisms

13. MULTIPOINTED d -SPACES AND MOORE PATHS

We give the definitions for $\mathcal{P} \in \{\mathcal{G}, \mathcal{M}\}$. This part treats cellular $\mathcal{G}$ -multipointed d -spaces; Part 4 treats $\mathcal{M}$.

13.1. Definition ([14, Section 2]; [17, Section 3]). For $\mathcal{P} \in \{\mathcal{G}, \mathcal{M}\}$, a $\mathcal{P}$ -multipointed d -space is a triple $X = (|X|, X^0, \mathbb{P}^{\text{top}}X)$. Here $|X| \in \mathbf{Top}$, the state set X^0 is a subset of $|X|$, and $\mathbb{P}^{\text{top}}X$ is a set of continuous maps $\mathbb{I} \rightarrow |X|$. Every such execution path has endpoints in X^0 . The set is closed under the action of $\mathcal{P}(1, 1)$ by precomposition and under normalized composition

$$(x *_N y)(t) = \begin{cases} x(2t), & 0 \leq t \leq \frac{1}{2}, \\ y(2t - 1), & \frac{1}{2} \leq t \leq 1, \end{cases} \quad x(1) = y(0).$$

A morphism is a continuous map on underlying spaces preserving states and execution paths. Constants are not required to be execution paths.

For fixed endpoints α, β , the space $\mathbb{P}_{\alpha, \beta}^{\text{top}}X$ has the Δ -kelleyfication of the relative compact-open topology. The total path space is the coproduct of these spaces over ordered pairs of states. This definition of the total space also applies when the state subset is not discrete in the underlying space.

A path of length $\ell > 0$ is an execution path after rescaling its domain to $\mathbb{I}$. Explicitly,

$$\mathbb{P}_{\alpha, \beta}^{\ell}X = \{x\mu_{\ell} : x \in \mathbb{P}_{\alpha, \beta}^{\text{top}}X\}.$$

Its topology is the Δ -kelleyfication of the relative compact-open topology on maps $[0, \ell] \rightarrow |X|$. Precomposition and Moore composition are continuous. Moore composition satisfies

$$(13.1) \quad (x * y)(t) = \begin{cases} x(t), & 0 \leq t \leq \ell_1, \\ y(t - \ell_1), & \ell_1 \leq t \leq \ell_1 + \ell_2, \end{cases}$$

for composable paths of lengths ℓ_1, ℓ_2 . The concatenation laws and their compatibility with reparametrization are [14, Propositions 2.7 and 3.4–3.8]. In Proposition 4.10 there, the second factor must be $y\mu_{\ell_2}$.

13.2. Proposition ([14, Theorem 3.9]; [17, Theorem 2]). *For a colimit of multipointed d -spaces, the execution-path set is generated by the images of the execution paths of the input objects, under finite Moore composition and the permitted changes of parameter. The underlying multipointed space is the corresponding colimit of underlying multipointed spaces.*

This statement concerns generated sets of paths. It does not assert that a colimit of the associated path presheaves always has the relative mapping-space topology. That topology requires the additional cellular arguments below.

13.3. Definition. For $Z \in \mathbf{Top}$, the globe $\text{Glob}^{\mathcal{P}}(Z)$ has underlying space

$$(\{0, 1\} \sqcup Z \times \mathbb{I}) / ((z, 0) \sim 0, (z, 1) \sim 1),$$

states $0, 1$, and paths $\delta_z \varphi$ with $z \in Z$, $\varphi \in \mathcal{P}(1, 1)$, where $\delta_z(t) = [z, t]$.

13.4. **Proposition** ([14, Propositions 2.12, 4.13 and 4.15]). *For $\mathcal{G}$, the path presheaf of a globe is $\mathcal{G}(-, 1) \times Z$. There is a natural bijection*

$$\mathcal{G}\mathbf{dTop}(\mathbf{Glob}^{\mathcal{G}}(Z), X) \cong \coprod_{\alpha, \beta \in X^0} \mathbf{Top}(Z, \mathbb{P}_{\alpha, \beta}^{\text{top}} X).$$

14. THE Q-MODEL STRUCTURES AND THE ADJUNCTION FOR $\mathcal{G}$

14.1. **Theorem** ([14, Theorem 4.12 and Appendix B]). *The spaces $\mathbb{P}_{\alpha, \beta}^{\ell} X$ and Moore composition define a right adjoint $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{Flow}$, with left adjoint $\mathbb{M}_!^{\mathcal{G}}$.*

The relations for states, reparametrization, and Moore composition determine $\mathbb{M}_!^{\mathcal{G}}$. The globe adjunction and enriched Yoneda give

$$(14.1) \quad \mathbb{M}_!^{\mathcal{G}} \mathbf{Glob}(\mathcal{G}(-, \ell) \times Z) \cong \mathbf{Glob}_{\ell}^{\mathcal{G}}(Z),$$

where $\mathbf{Glob}_{\ell}^{\mathcal{G}}(Z)$ has interval $[0, \ell]$ and is isomorphic to $\mathbf{Glob}^{\mathcal{G}}(Z)$ by rescaling. In particular, (14.1) determines $\mathbb{M}_!^{\mathcal{G}}$ on the generating cofibrations.

14.2. **Theorem** ([16, Theorem 6.16]). *The category $\mathcal{G}\mathbf{dTop}$ has a combinatorial q -model structure. Its generating cofibrations are*

$$\{\mathbf{Glob}^{\mathcal{G}}(\mathbb{S}^{n-1}) \rightarrow \mathbf{Glob}^{\mathcal{G}}(\mathbb{D}^n) : n \geq 0\} \cup \{C, R\}.$$

A map is a weak equivalence when its state map is bijective and its execution-path maps are weak homotopy equivalences. It is a fibration when its execution-path maps are q -fibrations. All objects are fibrant.

14.3. **Proposition.** *The adjunction $\mathbb{M}_!^{\mathcal{G}} \dashv \mathbb{M}^{\mathcal{G}}$ between Moore $\mathcal{G}$ -flows and $\mathcal{G}$ -multipointed d -spaces is Quillen for their q -model structures. The right adjoint reflects weak equivalences.*

Proof. Rescaling gives natural homeomorphisms

$$\mathbb{P}_{\alpha, \beta}^{\ell} X \cong \mathbb{P}_{\alpha, \beta}^{\text{top}} X \quad (\ell > 0).$$

Thus $\mathbb{M}^{\mathcal{G}}$ preserves q -fibrations and trivial q -fibrations. Since it preserves states, the same homeomorphisms show that it reflects weak equivalences. $\square$

The remainder of this part proves that the unit is an isomorphism on q -cofibrant Moore $\mathcal{G}$ -flows. This implies the Quillen equivalence with $\mathcal{G}$ -multipointed d -spaces for the q -model structures. The essential step is that $\mathbb{M}^{\mathcal{G}}$ preserves globular attachments whose old stage is cellular.

15. CELLULAR $\mathcal{G}$ -MULTIPOINTED d -SPACES AND CARRIERS

By Proposition 5.5, a cellular $\mathcal{G}$ -multipointed d -space has a presentation starting from a discrete state set S and successively forming pushouts

$$(15.1) \quad \begin{array}{ccc} \mathbf{Glob}^{\mathcal{G}}(\mathbb{S}^{n_{\nu}-1}) & \xrightarrow{g_{\nu}} & X_{\nu} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Glob}^{\mathcal{G}}(\mathbb{D}^{n_{\nu}}) & \xrightarrow{\widehat{g}_{\nu}} & X_{\nu+1}. \end{array}$$

The new open cell c_ν is parametrized by $\text{int}(\mathbf{D}^{n_\nu}) \times]0, 1[$. Its dimension is $n_\nu + 1$. Its closure is the image of the full characteristic globe. The initial and final states of a cell may coincide.

15.1. Proposition ([14, Propositions 5.2–5.5]). *The underlying space of a cellular $\mathcal{G}$ -multipointed d -space is Hausdorff. Its states form a closed discrete subspace. Every stage inclusion is a closed q -fibration on underlying spaces. A compact subspace meets only finitely many open cells. The execution paths valued in an earlier stage are exactly the execution paths of that stage.*

15.2. Theorem ([14, Theorems 5.9 and 5.20; Proposition 5.13]). *Every execution path x of a cellular $\mathcal{G}$ -multipointed d -space is regular: it has no nontrivial constant interval. It has a unique expression*

$$x = (\widehat{g}_{\nu_1} \delta_{z_1} \varphi_1 \mu_{\ell_1}) * \cdots * (\widehat{g}_{\nu_m} \delta_{z_m} \varphi_m \mu_{\ell_m}),$$

$$z_i \in \text{int } \mathbf{D}^{n_{\nu_i}}, \quad \varphi_i \in \mathcal{G}(1, 1), \quad \ell_i > 0, \quad \sum_i \ell_i = 1.$$

In a single globular attachment, an old block is an execution path of the preceding stage. Composing adjacent old blocks gives a unique decomposition into old blocks and interior traversals of the new cell, with no adjacent old blocks.

The carrier is $\text{Carrier}(x) = [c_{\nu_1}, \dots, c_{\nu_m}]$. A cell may occur several times; regular execution paths need not be injective.

The join of $x *_N y$ occurs at $1/2$. For cellular X , the correction to [14, Proposition 5.12] is

$$x([0, 1]) \cap X^0 \neq \emptyset \iff \exists y, z \in \mathbb{P}^{\text{top}} X, \varphi \in \mathcal{G}(1, 1) : x = (y *_N z) \varphi.$$

Without φ , the right-hand condition would require $x(1/2) \in X^0$.

15.3. Lemma. *Let (Y, d) be a metric space and let $r_j : K_j \times \mathbb{I} \rightarrow Y$, $1 \leq j \leq N$ with $N \geq 1$, be continuous, with each K_j nonempty and compact and each $r_j(z, -)$ nonconstant. Then*

$$a := \min_{j, z} \text{diam } r_j(z, \mathbb{I}) > 0.$$

Suppose a path $x : \mathbb{I} \rightarrow Y$ and intervals satisfy

$$0 \leq s_1 < t_1 \leq \cdots \leq s_m < t_m \leq 1, \quad r_{j_i}(z_i, \mathbb{I}) \subseteq x([s_i, t_i]).$$

If, for some $\delta > 0$,

$$t - s < \delta \implies \text{diam } x([s, t]) < a,$$

then $m \leq \lfloor 1/\delta \rfloor$.

Proof. Continuity and compactness give the positive minimum through

$$\left| \text{diam } u(\mathbb{I}) - \text{diam } v(\mathbb{I}) \right| \leq 2 \sup_t d(u(t), v(t)) \quad (u, v \in C_{co}(\mathbb{I}, Y)).$$

Every full traversal has diameter at least a , hence duration at least δ . Summing the successive durations gives $m\delta \leq 1$. $\square$

15.4. Proposition. *Let X be a cellular multipointed d -space whose paths have carriers given by finite full traversals of its open globular cells. Suppose its underlying space has the properties in Proposition 15.1, and every characteristic meridian, including those on the attaching boundary, is nonconstant. Let K be a compact space and $k \mapsto x_k$ a continuous map $K \rightarrow C_{co}(\mathbb{I}, |X|)$ whose values are execution paths. Only finitely many carriers occur in this family.*

Proof. The compact evaluation image $\{x_k(t) : k \in K, t \in \mathbb{I}\}$ meets only finitely many open cells, say $c_1, \dots, c_N$. The assertion is immediate if $K = \emptyset$ or $N = 0$. By Proposition 5.1, the image and these cell closures lie in a finite compact metrizable cellular subspace Y . Fix a compatible metric d on Y and put $r_{j,z}(t) = \hat{g}_j(z, t)$.

Let $a > 0$ be the lower diameter bound of Lemma 15.3 for these characteristic families. The compact family in $C_{co}(\mathbb{I}, Y)$ is equicontinuous [29, Chapter 7]. Hence some $\delta > 0$ satisfies

$$t - s < \delta \implies \text{diam } x_k([s, t]) < a \quad (k \in K).$$

Lemma 15.3 gives

$$\text{Carrier}(x_k) \in \bigcup_{0 \leq m \leq \lfloor 1/\delta \rfloor} \{c_1, \dots, c_N\}^m,$$

a finite set. □

15.5. Corollary. *For a continuous curve $\xi : \mathbb{I} \rightarrow \mathbb{P}_{\alpha, \beta}^{\text{top}} X$ in a cellular $\mathcal{G}$ -multipointed d -space, the set of carriers of $\xi(t)$ is finite.*

Proof. The family is compact, and its characteristic boundary meridians are execution paths of an earlier cellular stage. They are regular and nonconstant by Theorem 15.2. Apply Proposition 15.4. □

Corollary 15.5 replaces the uses of [14, Proposition 5.16]. The assertion there about intersections with slices is false: a slice, the image of a characteristic disk at a fixed interval coordinate, may meet the same execution path at different points contributed by different boundary meridians.

16. CHARACTERISTIC CHAINS AND COMPACT-OPEN LIMITS

Fix a carrier $\mathbf{c} = [c_{\nu_1}, \dots, c_{\nu_m}]$, with $m \geq 1$. Put

$$B_{\mathbf{c}} = \prod_{i=1}^m D^{n_{\nu_i}}, \quad r_z = \hat{g}_{\nu_1} \delta_{z_1} * \dots * \hat{g}_{\nu_m} \delta_{z_m} : [0, m] \rightarrow |X|.$$

The characteristic map for this chain of globes is

$$(16.1) \quad F_{\mathbf{c}} : \mathcal{G}(1, m) \times B_{\mathbf{c}} \longrightarrow \mathbb{P}_{\alpha, \beta}^{\text{top}} X, \quad (\varphi, z) \longmapsto r_z \varphi.$$

Boundary parameters are allowed, so the image may contain other carriers. The domain is the path space of the chain of globes [14, Proposition 6.3].

16.1. Proposition. *Let Y be a metric space, $L > 0$, and $r_k : [0, L] \rightarrow Y$ continuous. Suppose $r_k \rightarrow r$ and $r_k \varphi_k \rightarrow x$ uniformly, where $\varphi_k \in \mathcal{G}(1, L)$ and r, x have no nontrivial constant intervals. Then (φ_k) has a subsequence converging uniformly to $\varphi \in \mathcal{G}(1, L)$, with $x = r\varphi$.*

Proof. For $0 < \varepsilon \leq L$, regularity of r and compactness give

$$A_\varepsilon := \min_{0 \leq u \leq L-\varepsilon} \text{diam } r([u, u + \varepsilon]) > 0,$$

$$\min_{0 \leq u \leq L-\varepsilon} \text{diam } r_k([u, u + \varepsilon]) \geq A_\varepsilon - 2 \sup_u d(r_k(u), r(u)) \geq \frac{A_\varepsilon}{2}$$

for all sufficiently large k . The uniformly convergent family $x_k = r_k \varphi_k$ is equicontinuous [29, Chapter 7]. Choose $\delta > 0$ with $\text{diam } x_k([s, t]) < A_\varepsilon/2$ whenever $t - s < \delta$. For all sufficiently large k ,

$$\begin{aligned} t - s < \delta, \quad \varphi_k(t) - \varphi_k(s) \geq \varepsilon \\ \implies \quad \frac{A_\varepsilon}{2} \leq \text{diam } r_k(\varphi_k([s, t])) = \text{diam } x_k([s, t]) < \frac{A_\varepsilon}{2}, \end{aligned}$$

a contradiction. Shrinking δ for the finitely many remaining φ_k proves equicontinuity. By Arzelà–Ascoli, a subsequence converges uniformly to a continuous nondecreasing surjection $\varphi : \mathbb{I} \rightarrow [0, L]$.

Uniform convergence gives $x = r\varphi$. Since x has no nontrivial constant interval, φ is strictly increasing, hence $\varphi \in \mathcal{G}(1, L)$. $\square$

16.2. Proposition. *The map F_c is closed onto its image, and its image is closed in $\mathbb{P}_{\alpha, \beta}^{\text{top}} X$. More precisely, for any sequence $r_{z_k} \varphi_k$ converging compact-open to an execution path x , the lifts (φ_k, z_k) have a convergent subsequence with limit mapping to x .*

Proof. By Proposition 5.1, all relevant cell closures lie in a finite compact metrizable cellular subspace Y . Since Y is closed in $|X|$ and has its relative topology, x is valued in Y and the convergence is uniform for a compatible metric on Y . Compactness of B_c gives a subsequence with

$$z_k \longrightarrow z, \quad r_{z_k} \longrightarrow r_z \text{ uniformly.}$$

The paths r_z and x are regular by Theorem 15.2. Proposition 16.1 therefore gives a further subsequence with $(\varphi_k, z_k) \rightarrow (\varphi, z)$ and $x = r_z \varphi$.

If $A \subset \mathcal{G}(1, m) \times B_c$ is closed, the same argument applied to lifts in A shows that $F_c(A)$ is sequentially closed. The target is Δ -generated, hence sequential, so $F_c(A)$ is closed.

With its ordinary relative topology, $E_c = \text{Im } F_c$ is consequently the quotient of $\mathcal{G}(1, m) \times B_c$ by F_c . This domain is metrizable and locally path-connected by Proposition 6.4, hence Δ -generated. Its ordinary quotient is Δ -generated as well, so the relative topology on E_c is also its subspace topology in **Top**. $\square$

Proposition 16.2 replaces [14, Theorem 6.11] by using compact-open convergence. Pointwise convergence is insufficient when loops occur.

16.3. Example. Let $p(t) = \exp(2\pi it)$ be the execution loop in the circle obtained by attaching one directed segment to one state. For $0 < s \leq 1/2$, set

$$\lambda_s(t) = \begin{cases} t/(1-s), & 0 \leq t \leq 1-s, \\ 1 + (t-1+s)/s, & 1-s \leq t \leq 1. \end{cases}$$

Then $\lambda_s \in \mathcal{G}(1, 2)$ and $x_s = (p * p)\lambda_s$ converges pointwise to p as $s \rightarrow 0$. Compact-open convergence fails, since

$$t_s = 1 - \frac{s}{2}, \quad \sup_t |x_s(t) - p(t)| \geq |x_s(t_s) - p(t_s)| = |-1 - e^{-\pi i s}| \longrightarrow 2.$$

Every path in the image of $F_{[c,c]}$, where c is the sole open cell, has winding number 2; p has winding number 1.

17. PRESERVATION OF CELLULAR GLOBULAR ATTACHMENTS BY $\mathbb{M}^{\mathcal{G}}$

Let A be a cellular $\mathcal{G}$ -multipointed d -space and attach one globe to obtain X :

$$\begin{array}{ccc} \text{Glob}^{\mathcal{G}}(S^{n-1}) & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ \text{Glob}^{\mathcal{G}}(D^n) & \longrightarrow & X. \end{array}$$

The induced pushout in Moore $\mathcal{G}$ -flows gives the comparison morphism ψ :

$$\begin{array}{ccccc} \mathbb{M}^{\mathcal{G}} \text{Glob}^{\mathcal{G}}(S^{n-1}) & \longrightarrow & \mathbb{M}^{\mathcal{G}} A & & \\ \downarrow & & \downarrow & & \\ \mathbb{M}^{\mathcal{G}} \text{Glob}^{\mathcal{G}}(D^n) & \longrightarrow & \bar{X} & \xrightarrow{\psi} & \mathbb{M}^{\mathcal{G}} X. \end{array}$$

The word diagram of Section 11 describes $\mathbb{P}\bar{X}$.

17.1. Proposition. *The map ψ is bijective on states and on path elements at every length.*

Proof. First form the word colimit in Δ -generated spaces. Both state sets are A^0 , and Proposition 13.2 gives surjectivity on paths. For injectivity, use the word relations to replace boundary new factors by their attaching images and compose adjacent old factors. By Theorem 15.2, a realized path x of length one uniquely determines the remaining interior new traversals and maximal old blocks. If there is no new traversal, its word is the element x of $\mathbb{M}^{\mathcal{G}} A$.

Otherwise let $0 = t_0 < t_1 < \dots < t_r = 1$ be the joining times of these factors and normalize each restriction to a path y_i of length one. Let e_i be $y_i \in \mathbb{P}^1 \mathbb{M}^{\mathcal{G}} A$ for an old block, or its uniquely determined representative in the attached free globe for a new traversal. Cutting the original reparametrization at the joining times and using the coend relations gives the word

$$(e_1 * \dots * e_r) \kappa, \quad \kappa(t) = i - 1 + \frac{t - t_{i-1}}{t_i - t_{i-1}} \quad (t_{i-1} \leq t \leq t_i),$$

where $\kappa \in \mathcal{G}(1, r)$. Its factors and joining times depend only on x , proving injectivity. This is the normal-form argument of [14, Theorem 7.2], with the corrected word category.

The resulting continuous injections $\mathbb{P}_{\alpha, \beta}^{\ell} \bar{X} \rightarrow \mathbb{P}_{\alpha, \beta}^{\ell} X$ imply that the source spaces are Hausdorff. Thus the pushout also belongs to the full subcategory of Δ -Hausdorff spaces and has the same universal property there. $\square$

17.2. Theorem. *The morphism $\psi : \bar{X} \rightarrow \mathbb{M}^{\mathcal{G}} X$ is an isomorphism. Thus $\mathbb{M}^{\mathcal{G}}$ preserves the displayed globular attachment to the cellular $\mathcal{G}$ -multipointed d -space A .*

Proof. By rescaling, it suffices to consider length 1. Put $T = \mathbb{P}_{\alpha, \beta}^1 \bar{X}$ and $E = \mathbb{P}_{\alpha, \beta}^{\text{top}} X$. Proposition 17.1 gives a continuous bijection $h : T \rightarrow E$. The curve test reduces continuity

of h^{-1} to the lifting problem

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & T \\ \downarrow & \nearrow \tilde{\xi} & \downarrow h \\ \mathbb{I} & \xrightarrow{\quad \xi \quad} & E. \end{array}$$

For every carrier $\mathbf{c}$ occurring along ξ , put $E_{\mathbf{c}} = \text{Im } F_{\mathbf{c}}$ and $K_{\mathbf{c}} = \xi^{-1}(E_{\mathbf{c}})$. By Corollary 15.5 and Proposition 16.2, the $K_{\mathbf{c}}$ form a finite closed cover of $\mathbb{I}$. Use the ordinary relative topologies on $E_{\mathbf{c}}$ and $K_{\mathbf{c}}$.

Represent each characteristic meridian in $\mathbb{M}^{\mathcal{G}}A$ or in the attached free globe. Moore composition and reparametrization give a continuous map $s_{\mathbf{c}}$ with $hs_{\mathbf{c}} = F_{\mathbf{c}}$. Writing $q_{\mathbf{c}}$ for $F_{\mathbf{c}}$ with codomain $E_{\mathbf{c}}$ and $j_{\mathbf{c}}$ for the inclusion, quotient descent gives the dashed map:

$$\begin{array}{ccc} \mathcal{G}(1, m) \times B_{\mathbf{c}} & \xrightarrow{s_{\mathbf{c}}} & T \\ q_{\mathbf{c}} \downarrow & \nearrow \sigma_{\mathbf{c}} & \downarrow h \\ E_{\mathbf{c}} & \xrightarrow{j_{\mathbf{c}}} & E. \end{array}$$

Indeed, injectivity of h makes $s_{\mathbf{c}}$ constant on the fibers of the quotient map $q_{\mathbf{c}}$. Thus $\sigma_{\mathbf{c}} = h^{-1}j_{\mathbf{c}}$ is continuous. The maps $\sigma_{\mathbf{c}}\xi|_{K_{\mathbf{c}}}$ agree on intersections; finite closed pasting gives a continuous lift $\tilde{\xi} = h^{-1}\xi$. This proves the conclusion of [14, Theorem 7.3 and Corollary 7.4]. $\square$

17.3. Proposition ([14, Theorems 5.6 and 5.7]). *For a globular cellular tower of $\mathcal{G}$ -multipointed d -spaces, the functors $\mathbb{P}^{\text{top}}$ and $\mathbb{M}^{\mathcal{G}}$ preserve its limit-ordinal colimits. The execution-path topology at a limit is the final topology of the preceding path spaces.*

Theorem 17.2 and Proposition 17.3 show that $\mathbb{M}^{\mathcal{G}}$ takes cellular $\mathcal{G}$ -multipointed d -spaces to cellular Moore $\mathcal{G}$ -flows. Passing to retracts gives:

17.4. Corollary. *If X is q -cofibrant in $\mathcal{G}\mathbf{dTop}$, then $\mathbb{M}^{\mathcal{G}}X$ is q -cofibrant in $\mathcal{G}\mathbf{Flow}$.*

Proof. Apply $\mathbb{M}^{\mathcal{G}}$ to the cellular retract (5.2) in $\mathcal{G}\mathbf{dTop}$. Its image is a retract of a cellular Moore $\mathcal{G}$ -flow, hence q -cofibrant. $\square$

18. THE UNIT, COUNIT, AND QUILLEN EQUIVALENCE FOR $\mathcal{G}$

18.1. Theorem. *The unit $\eta_Y : Y \rightarrow \mathbb{M}^{\mathcal{G}}\mathbb{M}_!^{\mathcal{G}}Y$ is an isomorphism for every q -cofibrant Moore $\mathcal{G}$ -flow Y .*

Proof. First let Y be cellular, with the globular presentation of Proposition 5.5. The unit is an isomorphism on pathless state sets and free globes by Proposition 13.4 and (14.1). The left adjoint $\mathbb{M}_!^{\mathcal{G}}$ preserves the presentation; $\mathbb{M}^{\mathcal{G}}$ preserves its realized globe attachments by Theorem 17.2 and its limit colimits by Proposition 17.3. Naturality and transfinite induction prove that η_Y is an isomorphism.

For q -cofibrant Y , naturality makes η_Y a retract of the unit of a cellular Moore flow, using the cellular retract (5.2) in $\mathcal{G}\mathbf{Flow}$. A retract of an isomorphism is an isomorphism. This proves the conclusion of [14, Theorem 7.6]. $\square$

18.2. Theorem. *The counit $\varepsilon_X : \mathbb{M}_!^{\mathcal{G}} \mathbb{M}^{\mathcal{G}} X \rightarrow X$ is an isomorphism for every q -cofibrant $\mathcal{G}$ -multipointed d -space X .*

Proof. The counit is an isomorphism on pathless state sets and globes. Theorem 17.2 and Proposition 17.3 give the same transfinite induction as for the unit, now on a cellular presentation in $\mathcal{G}\mathbf{dTop}$. For q -cofibrant X , naturality and the cellular retract (5.2) in $\mathcal{G}\mathbf{dTop}$ make ε_X a retract of an isomorphism. This proves the conclusion of [14, Corollary 7.9]. $\square$

18.3. Theorem. *For the q -model structures on Moore $\mathcal{G}$ -flows and $\mathcal{G}$ -multipointed d -spaces, the adjunction $\mathbb{M}_!^{\mathcal{G}} \dashv \mathbb{M}^{\mathcal{G}}$ is a Quillen equivalence.*

Proof. Apply Proposition 5.4 to Proposition 14.3 and Theorem 18.1. This proves the conclusion of [14, Theorem 8.1]. $\square$

The proof uses the unique joining times of increasing homeomorphisms and the regularity of characteristic boundary paths. For $\mathcal{M}$, stop intervals prevent the application of Proposition 16.1; Part 4 supplies a different argument.

19. CATEGORIZATION AND THE HOMOTOPY CATEGORY

19.1. Definition ([14, Definition 8.7 and Theorem 8.11]). The categorization functor is $\text{cat}_{\mathcal{G}} = \mathbb{M}_! \mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \rightarrow \mathbf{Flow}$. It preserves states. Its path space is the quotient

$$\mathbb{P}_{\alpha,\beta} \text{cat}_{\mathcal{G}} X = \mathbb{P}_{\alpha,\beta}^{\text{top}} X / \mathcal{G}(1, 1).$$

The composition is induced by normalized composition, or equivalently by Moore composition after forgetting lengths.

The quotient is formed in **Top**. The inclusion of the full subcategory on length 1 is an equivalence with $\mathcal{G}$, so the colimit defining $\mathbb{M}_!$ is this orbit quotient. In particular, $\text{cat}_{\mathcal{G}} \text{Glob}^{\mathcal{G}}(Z) \cong \text{Glob}(Z)$.

19.2. Proposition. *The functor $\text{cat}_{\mathcal{G}}$ takes q -cofibrant $\mathcal{G}$ -multipointed d -spaces to q -cofibrant ordinary flows and preserves weak equivalences between them.*

Proof. Corollary 17.4 and the left Quillen functor $\mathbb{M}_!$ give cofibrancy. All objects of $\mathcal{G}\mathbf{dTop}$ are fibrant, so $\mathbb{M}^{\mathcal{G}}$ preserves weak equivalences; $\mathbb{M}_!$ preserves weak equivalences between cofibrant objects. $\square$

Let Q_d and Q_G be functorial q -cofibrant replacements in $\mathcal{G}\mathbf{dTop}$ and $\mathcal{G}\mathbf{Flow}$, respectively. The functors

$$\Phi_G(X) = \text{cat}_{\mathcal{G}}(Q_d X), \quad \Psi_G(Z) = \mathbb{M}_!^{\mathcal{G}} Q_G(\mathbb{M} Z)$$

represent the composites of the derived equivalences

$$\begin{array}{ccccc} \text{Ho}(\mathcal{G}\mathbf{dTop}) & \xrightarrow{\mathbf{RM}^{\mathcal{G}}} & \text{Ho}(\mathcal{G}\mathbf{Flow}) & \xrightarrow{\mathbf{LM}_!} & \text{Ho}(\mathbf{Flow}). \\ & \xleftarrow{\mathbf{LM}_!^{\mathcal{G}}} & & \xleftarrow{\mathbf{RM}} & \end{array}$$

Here all objects are fibrant, and $\mathbb{M}^{\mathcal{G}} Q_d X$ is q -cofibrant by Corollary 17.4. Thus Φ_G and Ψ_G induce mutually inverse equivalences, proving the conclusion of [14, Theorem 8.14].

Categorization is neither a left nor a right adjoint [14, Theorem 8.8]; its total derived functor is described by cofibrant replacement as in [5].

Part 4. Moore III: nondecreasing surjections and saturation

20. MOORE PATHS WITH STOP INTERVALS

We retain all positive lengths and replace increasing homeomorphisms by continuous nondecreasing surjections. In particular, a change of parameter is allowed to be constant on a nontrivial interval. These intervals are the stop intervals of the resulting execution path. The tensor in $\mathcal{M}$ still concatenates the changes of parameter in their specified order.

20.1. Proposition ([17, Propositions 12–14 and Theorem 3]).

The functor $\mathbb{M}^{\mathcal{M}} : \mathcal{MdTop} \rightarrow \mathcal{MFlow}$ defined by the execution-path spaces at each positive length is a right adjoint, with left adjoint $\mathbb{M}_!^{\mathcal{M}}$. There are natural homeomorphisms

$$\mathbb{P}_{0,1}^{\ell} \text{Glob}^{\mathcal{M}}(Z) \cong \mathcal{M}(\ell, 1) \times Z,$$

and natural isomorphisms

$$\mathbb{M}_!^{\mathcal{M}} \text{Glob}(\mathcal{M}(-, \ell) \times Z) \cong \text{Glob}_{\ell}^{\mathcal{M}}(Z).$$

The globe mapping bijection holds with $\mathcal{M}$ in place of $\mathcal{G}$.

For the realization formula, the statement with compact Z in [17, Proposition 14] suffices for all generating disks and spheres. The same formula for arbitrary Z follows from the globe mapping bijection and enriched Yoneda. No inference about cellular pushouts of the right adjoint is part of these formulas.

20.2. Theorem ([17, Section 4]; [16, Theorems 6.14 and 6.16]). *The category $\mathcal{MdTop}$ has a combinatorial q -model structure. Its generating cofibrations are the maps $\text{Glob}^{\mathcal{M}}(\mathbb{S}^{n-1}) \rightarrow \text{Glob}^{\mathcal{M}}(\mathbb{D}^n)$ together with C and R . A morphism is a weak equivalence if it is bijective on states and a weak homotopy equivalence on each execution-path space. It is a fibration if each execution-path map is a q -fibration. All objects are fibrant.*

20.3. Proposition. *For the q -model structures on Moore $\mathcal{M}$ -flows and $\mathcal{M}$ -multipointed d -spaces, the adjunction $\mathbb{M}_!^{\mathcal{M}} \dashv \mathbb{M}^{\mathcal{M}}$ is Quillen. Its right adjoint reflects weak equivalences.*

Proof. As in Proposition 14.3, rescaling identifies every positive-length path space with the unit-length space. The right adjoint therefore preserves fibrations and trivial fibrations. Its state map and its length-1 maps detect the criterion for weak equivalences of Theorem 20.2. $\square$

21. CELLULAR PATHS AND REMOVAL OF STOPS

Let X be a cellular $\mathcal{M}$ -multipointed d -space. We use its presentation with a fixed state set from Proposition 5.5. On underlying spaces, each globe attachment is a disk attachment. The following topological and set-theoretic results of Moore III will be used without their proofs.

21.1. Proposition ([17, Propositions 15–18]). *The underlying space $|X|$ is Hausdorff, its states form a closed discrete subspace, and stage inclusions are closed q -cofibrations. A*

compact subspace meets finitely many open globular cells. An execution path of X valued in an earlier cellular stage is an execution path of that stage. The interior meridians of each attached cell are regular.

21.2. Theorem ([17, Theorem 6 and Proposition 20]). *Every execution path x has a carrier $[c_{\nu_1}, \dots, c_{\nu_m}]$ and interior parameters $z_i \in \text{int } D^{n_{\nu_i}}$. Its globular naturalization is the regular Moore path*

$$r_x = \underline{\text{nat}}^{gl}(x) = \widehat{g}_{\nu_1} \delta_{z_1} * \dots * \widehat{g}_{\nu_m} \delta_{z_m} : [0, m] \longrightarrow |X|.$$

The carrier, the interior parameters, and r_x are uniquely determined by x . There is a nondecreasing surjection $\eta_x : \mathbb{I} \rightarrow [0, m]$ with $x = r_x \eta_x$. The rescaled path $r_x \mu_m^{-1}$ is an execution path of X .

The uniqueness required here is uniqueness of the carrier and of the regular naturalization. It does not require a unique placement of a join within a stop interval. Two adjacent traversals may meet along an entire interval on which the path is constant at their common state. Choosing a point of that interval is harmless for a particular Moore decomposition, but such choices need not vary continuously in a family.

21.3. Proposition ([17, Proposition 19]; [6, Lemma 3.9]). *Let $r : \mathbb{I} \rightarrow Z$ be a regular nonconstant path in a Hausdorff space. If a continuous path $x : \mathbb{I} \rightarrow Z$ and $\varphi, \psi \in \mathcal{M}(1, 1)$ satisfy $x\varphi = r\psi$, then there is a unique $\rho \in \mathcal{M}(1, 1)$ with $x = r\rho$. If $r\varphi = r\psi$, then $\varphi = \psi$.*

The factorization assertion is the regular-representative result of Fahrenberg and Raussen. It gives the unique diagonal in the commutative square

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\psi} & \mathbb{I} \\ \varphi \downarrow & \nearrow \rho & \downarrow r \\ \mathbb{I} & \xrightarrow{x} & Z. \end{array}$$

Indeed, $r\rho\varphi = x\varphi = r\psi$, so cancellation gives $\rho\varphi = \psi$. The cancellation assertion is the special case also recorded in [17, Proposition 19]. These statements concern parametrized paths and do not identify paths merely because their images agree. For example, the once-around and twice-around traversals of a directed loop have different traces.

21.4. Definition ([17, Definition 18]). An $\mathcal{M}$ -multipointed d -space is saturated if, for every continuous path x and every $\varphi \in \mathcal{M}(1, 1)$, the implication $x\varphi \in \mathbb{P}^{\text{top}} X \implies x \in \mathbb{P}^{\text{top}} X$ holds.

21.5. Proposition. *Every q -cofibrant $\mathcal{M}$ -multipointed d -space is saturated. None of its execution paths is constant.*

Proof. First let X be cellular. Every path has at least one interior-cell traversal by Theorem 21.2, so its naturalization, and hence the path itself, is nonconstant. Suppose $x\varphi$ is an execution path. Write

$$x\varphi = r\eta = (r\mu_m^{-1})(\mu_m\eta)$$

using its globular naturalization $r : [0, m] \rightarrow |X|$. The path $r\mu_m^{-1}$ is regular and is an execution path by Theorem 21.2. Proposition 21.3 gives $x = (r\mu_m^{-1})\rho$ for some $\rho \in \mathcal{M}(1, 1)$. Closure under nondecreasing surjections makes x an execution path. Thus X is saturated.

For q -cofibrant A , the lifting problem (5.2) in $\mathbf{MdTop}$ gives a cellular X and maps $A \xrightarrow{s} X \xrightarrow{p} A$ with $ps = \text{id}_A$. If $x\varphi$ is an execution path of A , then $(sx)\varphi$ is one of X . Saturation of X implies that sx , and therefore $psx = x$, is an execution path. This is the retract argument of [17, Proposition 30]. A constant execution path of A would give a constant execution path of X , which has just been excluded. $\square$

This proves the assertion of [17, Proposition 31] directly. Reflectivity of the saturated subcategory alone would not prove that an ambient cellular colimit is already saturated; the regular factorization supplies the missing step.

21.6. Corollary. *Every compact family of execution paths in a cellular $\mathcal{M}$ -multipointed d -space has finitely many carriers.*

Proof. Proposition 21.1 supplies the underlying-space hypotheses of Proposition 15.4. A characteristic boundary meridian is an execution path of an earlier stage and is nonconstant by Proposition 21.5. The interior meridians are regular and nonconstant. Apply Proposition 15.4. $\square$

In particular, a compact-open convergent sequence together with its limit is a compact family, so it has finitely many carriers. This recovers [17, Theorem 8] with compact-open convergence as hypothesis.

22. A STOPPED ATTACHING MAP

The following attachment explains why a different topology argument is necessary. It uses two distinct states and no loops. Put $A = \text{Glob}^{\mathcal{M}}(\{*\})$, identify $|A|$ with $\mathbb{I}$, and let $e(t) = t$ denote its regular meridian. Define

$$\sigma(t) = \min(2t, 1).$$

Attach $\text{Glob}^{\mathcal{M}}(\mathbb{D}^1)$ to A by sending both boundary meridians to $e\sigma$:

$$\begin{array}{ccc} \text{Glob}^{\mathcal{M}}(\{-1, 1\}) & \xrightarrow{g} & A \\ \downarrow & & \downarrow \\ \text{Glob}^{\mathcal{M}}([-1, 1]) & \xrightarrow[\widehat{g}]{} & X. \end{array} \quad \begin{array}{c} \uparrow \\ \uparrow \end{array}$$

There are no composable pairs of execution paths, since every path starts at 0 and ends at 1. On either boundary parameter, $\widehat{g}(\pm 1, u) = \sigma(u)$.

22.1. Lemma. *There is a continuous family $(\theta_s)_{0 < s \leq 1/2}$ in $\mathcal{G}(1, 1)$ such that $\rho_s = \sigma\theta_s$ extends continuously to $\rho_0 = \text{id}_{\mathbb{I}}$ in $\mathcal{M}(1, 1)$, while $(\theta_{1/k})_{k \geq 3}$ has no convergent subsequence in $\mathcal{M}(1, 1)$.*

Proof. Set

$$\theta_s(t) = \begin{cases} \frac{t}{2(1-s)}, & 0 \leq t \leq 1-s, \\ \frac{1}{2} + \frac{t-1+s}{2s}, & 1-s \leq t \leq 1. \end{cases}$$

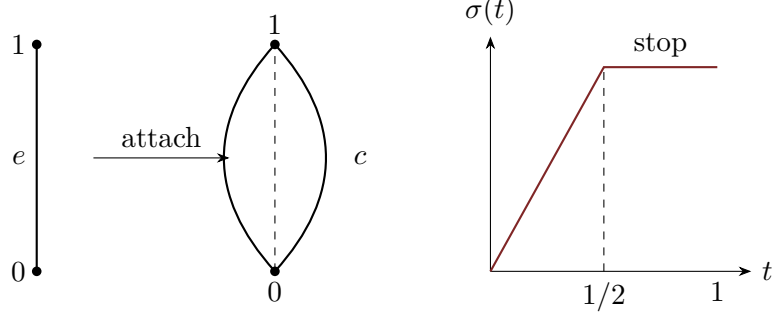


FIGURE 3. Both boundary meridians are attached to the same stopped path. The diagram on the left records the globe and its states; the graph on the right records the boundary parameter.

These increasing homeomorphisms satisfy

$$\rho_s(t) = \min\left(\frac{t}{1-s}, 1\right), \quad \|\rho_s - \text{id}_{\mathbb{I}}\|_{\infty} = \max\left\{\sup_{t \leq 1-s} \frac{st}{1-s}, \sup_{t \geq 1-s} (1-t)\right\} = s,$$

$$\lim_{s \downarrow 0} \theta_s(t) = \begin{cases} t/2, & t < 1, \\ 1, & t = 1. \end{cases}$$

The pointwise limit is discontinuous, so no subsequence converges uniformly. $\square$

22.2. Proposition. *The characteristic map*

$$F : \mathcal{M}(1, 1) \times [-1, 1] \longrightarrow \mathbb{P}_{0,1}^{\text{top}} X, \quad F(\theta, z) = \hat{g}\delta_z\theta,$$

does not have closed image.

Proof. An interior parameter gives a path meeting the new open cell. At a boundary parameter the path is $e\sigma\theta$ and has a nontrivial final stop. Thus $e \notin \text{Im } F$. Lemma 22.1 and the old-stage inclusion give

$$F(\theta_s, -1) = e\rho_s \longrightarrow e \notin \text{Im } F$$

in $\mathbb{P}_{0,1}^{\text{top}} X$. $\square$

22.3. Proposition. *There is a cellular, hence q -cofibrant, Moore $\mathcal{M}$ -flow Y for which $Y \rightarrow \mathbb{M}^{\mathcal{M}}\mathbb{M}_1^{\mathcal{M}}Y$ is a continuous bijection on paths but is not an isomorphism.*

Proof. First form the pushout of Moore flows of the free globes realizing the preceding attachment in Δ -generated spaces. Its old stage is a free 0-globe on two states, and its new stage attaches a free 1-globe along its boundary, so the resulting Moore flow Y is cellular. Its length-1 path space is

$$\begin{aligned} T &= \mathcal{M}(1, 1) \coprod_{\mathcal{M}(1,1) \times \{-1,1\}} \left(\mathcal{M}(1, 1) \times [-1, 1] \right) \\ &= \left(\mathcal{M}(1, 1) \sqcup (\mathcal{M}(1, 1) \times [-1, 1]) \right) / ((\theta, \pm 1) \sim \sigma\theta). \end{aligned}$$

There is no composition to add. Objectwise colimits and the globe adjunction justify this quotient. The summands are metrizable and locally path-connected, hence Δ -generated, so their ordinary quotient is Δ -generated. Proposition 20.1 and preservation of colimits by $\mathbb{M}_1^{\mathcal{M}}$ give $\mathbb{M}_1^{\mathcal{M}}Y \cong X$ in that setting.

The length-1 unit $h : T \rightarrow \mathbb{P}_{0,1}^{\text{top}} X$ maps an old parameter ρ to $e\rho$ and an attached parameter (θ, z) to $F(\theta, z)$. It is a continuous bijection: an interior meridian determines z and θ , and the boundary paths have exactly the displayed old-stage identifications. The target is Hausdorff because $|X|$ is Hausdorff. Thus the continuous injection h makes the ordinary quotient T Hausdorff and in particular Δ -Hausdorff. The same pushout and unit are consequently obtained in Δ -Hausdorff Δ -generated spaces, with no further identifications. Affine rescaling gives the asserted bijection at every positive length.

Let $z_s = -1 + s$ and $a_s = [(\theta_s, z_s)]$. The family $h(a_s)$ extends continuously at $s = 0$ by e . To check continuity into $\mathbb{P}_{0,1}^{\text{top}} X$ with its topology in **Top**, give the underlying space $|X|$ of this finite cellular attachment a compatible metric d . Uniform continuity of its characteristic globe gives

$$\sup_{u \in \mathbb{I}} d(\widehat{g}(z_s, u), \widehat{g}(-1, u)) \rightarrow 0.$$

Consequently

$$\sup_{t \in \mathbb{I}} d(h(a_s)(t), e(t)) \leq \sup_{u \in \mathbb{I}} d(\widehat{g}(z_s, u), \widehat{g}(-1, u)) + \sup_{t \in \mathbb{I}} d(e\rho_s(t), e(t)) \rightarrow 0.$$

For positive s the formulas are continuous. The displayed bound makes the adjoint map continuous also along $s = 0$. Its parameter interval is Δ -generated, so it is a continuous curve in $\mathbb{P}_{0,1}^{\text{top}} X$.

Let $J = [0, 1/2]$ and denote this curve by $\gamma : J \rightarrow \mathbb{P}_{0,1}^{\text{top}} X$, with $\gamma(0) = e$ and $\gamma(s) = h(a_s)$ for $s > 0$. The lifting problem

$$\begin{array}{ccc} \{0\} & \xrightarrow{0 \mapsto [\text{id}_{\mathbb{I}}]} & T \\ \downarrow & & \downarrow h \\ J & \xrightarrow{\gamma} & \mathbb{P}_{0,1}^{\text{top}} X \end{array}$$

has no continuous diagonal. Since h is bijective, any diagonal would have to take the value a_s at every positive s . Consider

$$C = \{(\theta_{1/k}, z_{1/k}) : k \geq 3\} \subset \mathcal{M}(1, 1) \times [-1, 1].$$

Lemma 22.1 makes C closed in the metrizable product. It misses the attaching boundary, so the quotient inverse images of $[C]$ are C in the attached summand and $\emptyset$ in the old summand. Hence $[C]$ is closed in T , and

$$\gamma^{-1}(h(T \setminus [C])) = J \setminus \{1/k : k \geq 3\}$$

is not a neighborhood of 0. The square has no continuous diagonal, and h^{-1} is discontinuous. $\square$

This example disproves the assertion in [17, Theorem 10] that characteristic maps have closed image, the isomorphism assertion for cellular pushouts in [17, Theorem 13 and Corollary 8], and the assertion that the unit is an isomorphism in [17, Theorem 14]. The argument below is therefore needed to prove that the unit is a weak equivalence on q-cofibrant Moore $\mathcal{M}$ -flows and that $\mathbb{M}_1^{\mathcal{M}} \dashv \mathbb{M}^{\mathcal{M}}$ is a Quillen equivalence for the q-model structures. The obstruction is already present in a single finite attachment.

23. CONTINUOUS REPRESENTATIVES OF TRACES

We use the normalization theorem of [20]. Normalized representatives depend on metric data and summable positive weights; the trace quotient is intrinsic.

A space is submetrizable if its topology admits a coarser metrizable topology, equivalently if it admits a continuous injection into a metric space.

23.1. Proposition ([20, Lemma 7.1]). *Every cellular space obtained by transfinite attachments of finite-dimensional disks to a discrete space admits a continuous injective map into a Hilbert space. Every retract of such a space is therefore submetrizable.*

23.1. The progress function. Fix a continuous injection $\iota : Z \rightarrow (Y, d)$ into a metric space. For $\varepsilon > 0$ and $t \in \mathbb{I}$, following [27, p. 122], define the truncated variation of a path x by

$$V_\varepsilon(x, t) = \sup_{0=t_0 < \dots < t_k=t} \sum_{j=1}^k \left(d(\iota x(t_{j-1}), \iota x(t_j)) - \varepsilon \right)_+, \quad r_+ = \max(r, 0).$$

Set $V_\varepsilon(x, 0) = 0$. Finiteness, continuity, and equivariance under reparametrization are proved in [20, Lemma 4.1 and Proposition 4.2].

Choose positive numbers $\varepsilon_n \rightarrow 0$, positive summable weights (w_n) , and a continuous strictly increasing function $\theta : [0, \infty[\rightarrow [0, 1[$ with $\theta(0) = 0$. Set

$$(23.1) \quad L_x(t) = \sum_{n \geq 1} w_n \theta(V_{\varepsilon_n}(x, t)), \quad \lambda_x(t) = \frac{L_x(t)}{L_x(1)}$$

for nonconstant x . The hypotheses on the weights give the uniform tail bound

$$0 \leq \sum_{n > N} w_n \theta(V_{\varepsilon_n}(x, t)) \leq \sum_{n > N} w_n \rightarrow 0.$$

The progress theorem [20, Proposition 4.4] states that L_x is continuous, nondecreasing, and satisfies

$$(23.2) \quad L_x(a) = L_x(b) \iff x \text{ is constant on } [a, b] \quad (0 \leq a \leq b \leq 1).$$

Thus $L_x(1) > 0$, $\lambda_x \in \mathcal{M}(1, 1)$, and x is constant on each fiber of λ_x . Quotient descent gives the unique continuous factorization

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\lambda_x} & \mathbb{I} \\ & \searrow x & \downarrow N(x) \\ & & Z. \end{array}$$

By (23.2), $N(x)$ is regular. The identity $L_{x\rho} = L_x\rho$ gives equivariance, and [20, Lemma 5.2 and Proposition 5.4] proves continuity of N in the ordinary compact-open topology.

23.2. The normalization theorem.

23.2. Theorem ([20, Sections 4–5 and Corollary 5.10]). *Let Z be submetrizable, and let $E \subset C_{co}(\mathbb{I}, Z)$ be a family of nonconstant paths with fixed endpoints. Assume either that E is saturated and stable under $\mathcal{M}(1, 1)$, or that E consists of regular paths and is stable under $\mathcal{G}(1, 1)$. After choosing a continuous metric injection, there are continuous*

normalization maps $N : E \rightarrow E$ and $x \mapsto \lambda_x$ with

$$(23.3) \quad \begin{aligned} x &= N(x)\lambda_x, & \lambda_{N(x)} &= \text{id}_{\mathbb{I}}, & N(N(x)) &= N(x), \\ N(x\rho) &= N(x), & \lambda_{x\rho} &= \lambda_x\rho. \end{aligned}$$

Here ρ ranges over $\mathcal{M}(1,1)$ in the first case and over $\mathcal{G}(1,1)$ in the second. Correspondingly $\lambda_x \in \mathcal{M}(1,1)$ or $\lambda_x \in \mathcal{G}(1,1)$, and $N(x)$ is regular. The maps are continuous in the relative ordinary compact-open topology, and after Δ -kelleyfication. The deformation

$$(23.4) \quad H(x, s) = N(x) \left((1-s)\lambda_x + s \text{id}_{\mathbb{I}} \right)$$

stays in E , preserves trace classes, and is a strong deformation onto $N(E)$. The quotient in **Top** admits the section induced by N and is a trivial Hurewicz fibration.

The saturation assumption makes the regular factor $N(x)$ belong to E . The interpolation belongs to the allowed reparametrization space, so stability keeps H in E . When E consists of regular paths, λ_x and its interpolation with $\text{id}_{\mathbb{I}}$ are strictly increasing. These observations check the invariance hypotheses in Corollary 5.10 of [20]. The references to this preprint use the numbering of its first arXiv version.

23.3. Definition. For $\mathcal{P} \in \{\mathcal{G}, \mathcal{M}\}$, set $\text{cat}_{\mathcal{P}} = \mathbb{M}_! \mathbb{M}^{\mathcal{P}}$. For fixed states, write

$$T_{\alpha, \beta}(X) = \mathbb{P}_{\alpha, \beta} \text{cat}_{\mathcal{P}} X, \quad q_X : \mathbb{P}_{\alpha, \beta}^{\text{top}} X \longrightarrow T_{\alpha, \beta}(X).$$

Thus $T_{\alpha, \beta}(X)$ is the quotient by the equivalence relation generated by $x \sim x\rho$, $\rho \in \mathcal{P}(1,1)$, with its quotient topology in **Top**.

For $\mathcal{M}$ the quotient is not an orbit space of a group. Its equivalence relation is generated by a monoid action. On the saturated nonconstant family in Theorem 23.2, two paths are equivalent exactly when they have the same normalized representative: invariance gives one implication, while $x = N(x)\lambda_x$ gives the other. This also agrees with the regular-representative description of traces in [6].

23.4. Proposition. *Let X be q -cofibrant in $\mathcal{M}\mathbf{dTop}$ or $\mathcal{G}\mathbf{dTop}$. For every fixed pair of states, the quotient q_X has a continuous section j_X and is a trivial Hurewicz fibration. After a choice of normalization, there is a homeomorphism*

$$(23.5) \quad \mathbb{P}_{\alpha, \beta}^{\text{top}} X \cong T_{\alpha, \beta}(X) \times \mathcal{P}(1,1), \quad x \longmapsto (q_X x, \lambda_x),$$

under which q_X is the first projection.

Proof. The cellular retract (5.2) makes $|X|$ submetrizable by Proposition 23.1. Its execution paths are saturated and nonconstant for $\mathcal{M}$ by Proposition 21.5, and regular for $\mathcal{G}$ by Theorem 15.2; regularity passes to retracts through the section. Theorem 23.2, followed by Δ -kelleyfication, therefore gives a continuous normalization on $E = \mathbb{P}_{\alpha, \beta}^{\text{top}} X$ with

$$q_X x = q_X y \iff N(x) = N(y), \quad N^2 = N.$$

The retraction $N : E \rightarrow N(E)$ is a quotient map. Its image, with the ordinary relative topology, is a Hausdorff retract of the Δ -generated Hausdorff space E , so it belongs to either chosen **Top**. Hence $T = T_{\alpha,\beta}(X) \cong N(E)$, and the inclusion of $N(E)$ gives the section

$$\begin{array}{ccc} \emptyset & \longrightarrow & E \\ \downarrow & \nearrow j_X & \downarrow q_X \\ T & \xrightarrow{\text{id}_T} & T. \end{array}$$

The inverse of (23.5) is

$$(\tau, \rho) \longmapsto j_X(\tau)\rho.$$

Precomposition is continuous, and (23.3) gives the inverse identities

$$\begin{aligned} q_X(j_X(\tau)\rho) &= \tau, & \lambda_{j_X(\tau)\rho} &= \rho, \\ j_X(q_X x)\lambda_x &= N(x)\lambda_x = x. \end{aligned}$$

For the Hurewicz lifting property, consider

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ \downarrow i_0 & \nearrow \tilde{F} & \downarrow q_X \\ A \times \mathbb{I} & \xrightarrow{F} & T, \end{array} \quad i_0(a) = (a, 0).$$

The product decomposition supplies the continuous diagonal

$$\tilde{F}(a, s) = j_X(F(a, s))\lambda_{f(a)}, \quad \tilde{F}i_0 = f, \quad q_X\tilde{F} = F.$$

The contraction $(\rho, s) \mapsto (1-s)\rho + s\text{id}_{\mathbb{I}}$ of $\mathcal{P}(1, 1)$ makes the projection a homotopy equivalence. $\square$

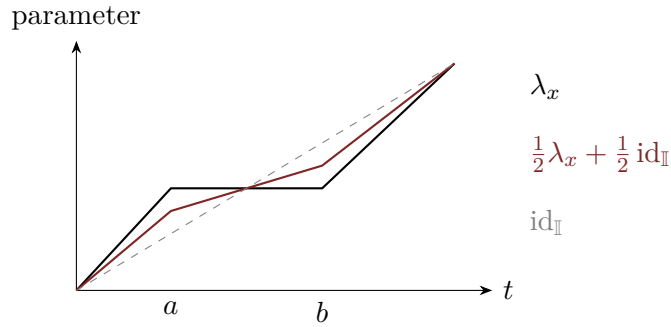


FIGURE 4. Interpolation removes a stop interval without changing the trace. Normalization separates the trace coordinate from the reparametrization coordinate.

The quotient q_X is natural in X . The homeomorphism (23.5) depends on the metric injection used for normalization.

For q -cofibrant X and a fixed trace τ , its fiber is described by the pullback square

$$\begin{array}{ccc} \mathcal{P}(1, 1) & \xrightarrow{\rho \mapsto j_X(\tau)\rho} & \mathbb{P}_{\alpha, \beta}^{\text{top}} X \\ \downarrow & \lrcorner & \downarrow q_X \\ \{\tau\} & \hookrightarrow & T_{\alpha, \beta}(X). \end{array}$$

This is a pullback in **Top** by (23.5).

24. CELLULAR PUSHOUTS AFTER PASSING TO TRACES

Consider a cellular attachment in **MdTop** and the corresponding pushout of flows:

$$(24.1) \quad \begin{array}{ccc} \text{Glob}^{\mathcal{M}}(\mathbb{S}^{n-1}) & \xrightarrow{g} & A \\ \downarrow & \lrcorner & \downarrow \\ \text{Glob}^{\mathcal{M}}(\mathbb{D}^n) & \longrightarrow & X \end{array} \quad \begin{array}{ccc} \text{Glob}(\mathbb{S}^{n-1}) & \xrightarrow{\text{cat}_{\mathcal{M}} g} & \text{cat}_{\mathcal{M}} A \\ \downarrow & \lrcorner & \downarrow \\ \text{Glob}(\mathbb{D}^n) & \longrightarrow & W. \end{array}$$

The trace of a meridian of a globe is determined by its disk parameter, so categorization gives the displayed globe maps and a canonical continuous map $c : W \rightarrow \text{cat}_{\mathcal{M}} X$. Both W and $\text{cat}_{\mathcal{M}} X$ have state set A^0 .

24.1. Proposition. *The map c is bijective on states and paths.*

Proof. In Δ -generated spaces, Proposition 11.3 with $\mathcal{P} = \mathbf{1}$ presents W by words of old traces and new disk parameters. Reduce a word by replacing boundary parameters with their attaching traces and composing adjacent old factors. The remaining factors are old traces and interior new meridians, with no adjacent old factors; consecutive new factors are allowed.

By Theorem 21.2, every trace of X is the composite of the interior meridians in its carrier, proving surjectivity. For injectivity, expand each old factor of a reduced word into its carrier, which is preserved in X by Proposition 21.1. Equality of image traces gives equal carriers and interior parameters by Proposition 21.3 and Theorem 21.2. The new-cell occurrences and the intervening maximal old blocks therefore agree, so the reduced words are equal.

Each hom space of W thus admits a continuous injection into the Hausdorff trace space of X from Proposition 23.4. It is itself Hausdorff, so the same word presentation computes the pushout in the Δ -Hausdorff category. $\square$

24.2. Theorem. *The canonical map $c : W \rightarrow \text{cat}_{\mathcal{M}} X$ is an isomorphism of flows. Thus $\text{cat}_{\mathcal{M}}$ preserves a globe attachment whose old stage is cellular.*

Proof. Fix states α, β and write $T = \mathbb{P}_{\alpha, \beta} \text{cat}_{\mathcal{M}} X$. Proposition 24.1 gives a continuous bijection $c_{\alpha, \beta} : \mathbb{P}_{\alpha, \beta} W \rightarrow T$. Take a continuous curve $\xi : \mathbb{I} \rightarrow T$. Proposition 23.4 supplies

its continuous lift in the square

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathbb{P}_{\alpha,\beta}^{\text{top}} X \\ \downarrow & \nearrow j_X \xi & \downarrow q_X \\ \mathbb{I} & \xrightarrow{\xi} & T. \end{array}$$

Corollary 21.6 gives finitely many carriers $\mathbf{c}_1, \dots, \mathbf{c}_N$ for $j_X \xi$.

For a carrier $\mathbf{c} = [c_{\nu_1}, \dots, c_{\nu_m}]$, let

$$B_{\mathbf{c}} = \prod_{i=1}^m D^{n_{\nu_i}}, \quad p_{\mathbf{c}}(z) = [\hat{g}_{\nu_1} \delta_{z_1} * \dots * \hat{g}_{\nu_m} \delta_{z_m}] \in T.$$

Rescale the Moore composite to length 1 before taking its trace. All characteristic meridians, including boundary meridians, are execution paths, so $p_{\mathbf{c}}$ is continuous. Sending each old meridian to its trace in $\text{cat}_{\mathcal{M}} A$ and each new meridian to its disk parameter gives the continuous map

$$\begin{array}{ccc} B_{\mathbf{c}} & \xrightarrow{w_{\mathbf{c}}} & \mathbb{P}_{\alpha,\beta} W \\ & \searrow p_{\mathbf{c}} & \downarrow c_{\alpha,\beta} \\ & & T. \end{array}$$

The pushout relation ensures commutativity also on boundary parameters. The source $B_{\mathbf{c}}$ is compact, T is Hausdorff, and

$$\xi(\mathbb{I}) \subset \bigcup_{i=1}^N p_{\mathbf{c}_i}(B_{\mathbf{c}_i}).$$

Lemma 5.2 proves that $c_{\alpha,\beta}$ is a homeomorphism for every pair of states. $\square$

Normalization supplies a compact family of representatives; the descent argument uses only compact products of characteristic disks.

24.3. Proposition. *The functor $\text{cat}_{\mathcal{M}}$ preserves limit-ordinal colimits of globular cellular towers with a fixed state set.*

Proof. Let $X_{\lambda} = \text{colim}_{\nu < \lambda} X_{\nu}$ at a limit ordinal and form $D = \text{colim}_{\nu < \lambda} \text{cat}_{\mathcal{M}} X_{\nu}$ first in Δ -generated spaces. Proposition 9.5 with $\mathcal{P} = \mathbf{1}$ computes its hom spaces. Each finite carrier and its cell closures lie in some X_{ν} . Proposition 21.1 and uniqueness of carriers therefore make the canonical map

$$d : D \longrightarrow \text{cat}_{\mathcal{M}} X_{\lambda}$$

a bijection on paths. It is continuous by construction. Each hom space of D therefore admits a continuous injection into the corresponding Hausdorff trace space of X_{λ} , and is itself Hausdorff. In particular, it is already Δ -Hausdorff. The same filtered colimit consequently computes the colimit in the Δ -Hausdorff setting; the reflection makes no identifications. It remains to prove that this continuous bijection is a homeomorphism on hom spaces.

Fix a continuous curve $\xi : \mathbb{I} \rightarrow T_{\alpha,\beta}(X_\lambda)$. Its normalized representatives solve the lifting problem

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathbb{P}_{\alpha,\beta}^{\text{top}} X_\lambda \\ \downarrow & \nearrow j_{X_\lambda} \xi & \downarrow q_{X_\lambda} \\ \mathbb{I} & \xrightarrow{\xi} & T_{\alpha,\beta}(X_\lambda). \end{array}$$

Their compact evaluation image in $|X_\lambda|$ lies in the closure of finitely many cells and states by Proposition 21.1 and Proposition 5.1. There is $\nu < \lambda$ containing this finite cellular subspace. The adjoint map of $j_{X_\lambda} \xi$ takes values in the closed embedded underlying space $|X_\nu|$ and is continuous there. Every member is an execution path of X_ν by Proposition 21.1. The internal-hom adjunction in **Top** therefore gives a continuous curve in $\mathbb{P}_{\alpha,\beta}^{\text{top}} X_\nu$. Applying q_{X_ν} and the colimit inclusion gives $d^{-1}\xi$. This is continuous. The curve test proves that d is an isomorphism. $\square$

24.4. Corollary. *The functor $\text{cat}_{\mathcal{M}}$ takes q -cofibrant $\mathcal{M}$ -multipointed d -spaces to q -cofibrant flows. Every trace space $T_{\alpha,\beta}(X)$ of such an object is q -cofibrant in **Top**.*

Proof. Categorization of a discrete $\mathcal{M}$ -multipointed d -space consisting solely of states is the flow with the same states and no paths, and categorization of a globe is its flow globe. Theorem 24.2 and Proposition 24.3 consequently turn a globular cellular presentation into a cellular flow presentation. A q -cofibrant $\mathcal{M}$ -multipointed d -space is a retract of a cellular $\mathcal{M}$ -multipointed d -space by the lifting problem (5.2) in **MdTop**, and the functor preserves that retract. This proves the first assertion. Apply Theorem 11.5 with $\mathcal{P} = \mathbf{1}$ for the second. $\square$

25. THE Q-QUILLEN EQUIVALENCE BETWEEN MOORE $\mathcal{M}$ -FLOWS AND $\mathcal{M}$ -MULTIPOINTED d -SPACES

Applying $\mathbb{M}_!$ to the unit $Y \rightarrow \mathbb{M}^{\mathcal{M}} \mathbb{M}_!^{\mathcal{M}} Y$ gives a natural morphism of flows

$$(25.1) \quad \nu_Y : \mathbb{M}_! Y \longrightarrow \text{cat}_{\mathcal{M}} \mathbb{M}_!^{\mathcal{M}} Y.$$

The morphism ν_Y fixes states and sends a path class to the trace of its realized execution path.

25.1. Proposition. *For every q -cofibrant $\mathcal{M}$ -flow Y , the map ν_Y is an isomorphism of flows.*

Proof. Both functors $\mathbb{M}_!$ and $\text{cat}_{\mathcal{M}} \mathbb{M}_!^{\mathcal{M}}$ send a Moore $\mathcal{M}$ -flow with no paths to the ordinary flow with the same states and no paths. On a free globe $\text{Glob}(\mathcal{M}(-, \ell) \times Z)$, enriched Yoneda, Proposition 7.4, and Proposition 20.1 identify both sides with $\text{Glob}(Z)$. The morphism ν_Y is the identity on the parameter space Z .

Let Y be a cellular Moore $\mathcal{M}$ -flow. The functors $\mathbb{M}_!$ and $\mathbb{M}_!^{\mathcal{M}}$ are left adjoints and preserve colimits. The realization of a generating Moore $\mathcal{M}$ -flow globe attachment is a generating globe attachment in **MdTop**. Theorem 24.2 and Proposition 24.3 show that $\text{cat}_{\mathcal{M}}$ preserves every globe attachment and colimit at a limit ordinal in the cellular presentation of $\mathbb{M}_!^{\mathcal{M}} Y$. Naturality and transfinite induction prove that ν_Y is an isomorphism. Proposition 5.5 allows the fixed-state presentations used in that induction. Finally, an

arbitrary q -cofibrant Y is a retract of a cellular Moore $\mathcal{M}$ -flow by the lifting problem (5.2) in $\mathcal{M}\mathbf{Flow}$. The natural map ν_Y is then a retract of an isomorphism and is itself an isomorphism. $\square$

25.2. Theorem (The counit conclusion of [17, Theorem 14]). *For every q -cofibrant $X \in \mathcal{M}\mathbf{dTop}$, the counit*

$$\varepsilon_X : \mathbb{M}_!^{\mathcal{M}} \mathbb{M}^{\mathcal{M}} X \longrightarrow X$$

is an isomorphism.

Proof. First suppose X cellular. Its realization is the quotient of its states and the cylinders $\mathbb{P}_{\alpha,\beta}^a \mathbb{M}^{\mathcal{M}} X \times [0, a]$. The counit sends $[x, t]$ to $x(t)$ and fixes states. For $x = r_x \eta_x$, Theorem 21.2 and the defining relations give

$$r_x = y_1 * \cdots * y_m, \quad y_i = \hat{g}_{\nu_i} \delta_{z_i}, \quad u = \eta_x(t),$$

$$[x, t] = [r_x, u] = \begin{cases} \alpha_i, & u = i \in \{0, \dots, m\}, \\ [y_i, u - i + 1], & i - 1 < u < i, \end{cases}$$

where α_i are the successive states of the carrier. Uniqueness of the open cell and its coordinates proves that evaluation is bijective. The realization quotient is consequently Hausdorff, so it also computes the realization in the Δ -Hausdorff category.

For each characteristic globe, the continuous family $z \mapsto \hat{g}_\nu \delta_z$ supplies

$$\begin{array}{ccc} |\mathrm{Glob}^{\mathcal{M}}(\mathbf{D}^{n_\nu})| & \xrightarrow{(z,t) \mapsto [\hat{g}_\nu \delta_z, t]} & |\mathbb{M}_!^{\mathcal{M}} \mathbb{M}^{\mathcal{M}} X| \\ & \searrow \hat{g}_\nu & \downarrow \varepsilon_X \\ & & |X|. \end{array}$$

The final topology defined by the states and characteristic globes makes ε_X^{-1} continuous. It sends an execution path x to the execution path $t \mapsto [x, t]$, hence is a morphism of multipointed d -spaces. For q -cofibrant X , apply naturality to the cellular retract (5.2) in $\mathcal{M}\mathbf{dTop}$. $\square$

25.3. Theorem. *For every q -cofibrant Moore $\mathcal{M}$ -flow Y , the unit $\eta_Y : Y \rightarrow \mathbb{M}^{\mathcal{M}} \mathbb{M}_!^{\mathcal{M}} Y$ is a weak equivalence.*

Proof. Put $X = \mathbb{M}_!^{\mathcal{M}} Y$. The left Quillen functor $\mathbb{M}_!^{\mathcal{M}}$ makes X q -cofibrant. Fix states α, β and a positive length ℓ . The definitions give a commutative triangle

$$\begin{array}{ccc} \mathbb{P}_{\alpha,\beta}^\ell Y & \xrightarrow{\eta_Y^\ell} & \mathbb{P}_{\alpha,\beta}^\ell \mathbb{M}^{\mathcal{M}} X \\ & \searrow q_Y^\ell & \downarrow q_X^\ell \\ & & \mathbb{P}_{\alpha,\beta} \mathbb{M}_! Y \cong T_{\alpha,\beta}(X), \end{array}$$

where the isomorphism is ν_Y . Theorem 12.3 and Proposition 23.4 make q_Y^ℓ and q_X^ℓ weak homotopy equivalences. Two-out-of-three gives the assertion on paths, and the state map is a bijection. $\square$

25.4. Theorem. *The adjunction $\mathbb{M}_!^{\mathcal{M}} \dashv \mathbb{M}^{\mathcal{M}}$ between Moore $\mathcal{M}$ -flows and $\mathcal{M}$ -multipointed d -spaces is a Quillen equivalence for the q -model structures.*

Proof. It is Quillen and its right adjoint reflects weak equivalences by Proposition 20.3. Every object of $\mathcal{M}\mathbf{dTop}$ is fibrant. Theorem 25.3 supplies the unit condition of Proposition 5.4. $\square$

This establishes the conclusion about Quillen equivalence of [17, Theorem 14] by a different argument. Proposition 22.3 disproves its stronger assertion that the unit is an isomorphism. The counit isomorphism on q -cofibrant $\mathcal{M}$ -multipointed d -spaces remains valid by Theorem 25.2. For arbitrary $X \in \mathcal{M}\mathbf{dTop}$, a derived counit is a weak equivalence: for a q -cofibrant replacement $Q\mathbb{M}^{\mathcal{M}}X \rightarrow \mathbb{M}^{\mathcal{M}}X$, the adjunct $\mathbb{M}_!^{\mathcal{M}}Q\mathbb{M}^{\mathcal{M}}X \rightarrow X$ is a weak equivalence by the Quillen-equivalence criterion.

26. CATEGORIZATION, COFIBRANCY, AND DERIVED FUNCTORS

26.1. Proposition. *The categorization functor $\text{cat}_{\mathcal{M}} : \mathcal{M}\mathbf{dTop} \rightarrow \mathbf{Flow}$ preserves weak equivalences between q -cofibrant $\mathcal{M}$ -multipointed d -spaces.*

Proof. Let $f : X \rightarrow X'$ be such a weak equivalence. Its state map is bijective. For every pair of states, naturality of the trace quotient gives

$$\begin{array}{ccc} \mathbb{P}_{\alpha,\beta}^{\text{top}}X & \xrightarrow{\mathbb{P}^{\text{top}}f} & \mathbb{P}_{f\alpha,f\beta}^{\text{top}}X' \\ q_X \downarrow & & \downarrow q_{X'} \\ T_{\alpha,\beta}(X) & \xrightarrow{\mathbb{P}_{\text{cat}_{\mathcal{M}}f}} & T_{f\alpha,f\beta}(X'). \end{array}$$

The upper arrow is a weak homotopy equivalence by hypothesis. Both vertical arrows are homotopy equivalences by Proposition 23.4. The two-out-of-three property gives the required weak homotopy equivalence on traces. $\square$

26.2. Theorem. *For $\mathcal{P} \in \{\mathcal{G}, \mathcal{M}\}$ and q -cofibrant $X \in \mathcal{P}\mathbf{dTop}$, the quotient*

$$q_X : \mathbb{P}_{\alpha,\beta}^{\text{top}}X \longrightarrow T_{\alpha,\beta}(X)$$

is a trivial Hurewicz fibration with a continuous section. Its source is m -cofibrant, and its target is q -cofibrant.

Proof. The fibration and section are Proposition 23.4. The flow $\text{cat}_{\mathcal{P}}X$ obtained by categorization is q -cofibrant by Proposition 19.2 for $\mathcal{G}$ and Corollary 24.4 for $\mathcal{M}$. Its individual hom spaces are q -cofibrant by Theorem 11.5 with $\mathcal{P} = \mathbf{1}$. Since q_X is a homotopy equivalence, its source has the homotopy type of a q -cofibrant space and is m -cofibrant by Cole's criterion [3, Corollary 3.7]. $\square$

This proves the conclusion of [17, Theorem 16] without using the false claim that $\mathbb{M}^{\mathcal{M}} : \mathcal{M}\mathbf{dTop} \rightarrow \mathcal{M}\mathbf{Flow}$ preserves every cellular globe attachment. The normalization product also identifies every fiber with $\mathcal{G}(1, 1)$ or $\mathcal{M}(1, 1)$. Both are contractible. The assertion that this quotient is a trivial Hurewicz fibration is stronger than the assertion that it merely induces isomorphisms on homotopy groups.

26.3. Proposition. *Let X be q -cofibrant in $\mathcal{M}\mathbf{dTop}$ and let $p : Y \rightarrow \mathbb{M}^{\mathcal{M}}X$ be a cofibrant replacement in $\mathcal{M}\mathbf{Flow}$. The natural morphism*

$$\mathbb{M}_!Y \xrightarrow{\mathbb{M}_!p} \text{cat}_{\mathcal{M}}X$$

is a weak equivalence of flows.

Proof. Its state map is bijective. At length 1 and fixed endpoints, naturality gives the commutative square

$$\begin{array}{ccc} \mathbb{P}_{\alpha,\beta}^1 Y & \xrightarrow{p^1} & \mathbb{P}_{p\alpha,p\beta}^{\text{top}} X \\ q_Y^1 \downarrow & & \downarrow q_X \\ \mathbb{P}_{\alpha,\beta} \mathbb{M}_! Y & \xrightarrow{\mathbb{P}\mathbb{M}_! p} & T_{p\alpha,p\beta}(X). \end{array}$$

The upper arrow is a weak homotopy equivalence because p is a weak equivalence. The two quotient maps are weak homotopy equivalences by Theorem 12.3 and Proposition 23.4. The square gives the result by two-out-of-three. $\square$

Let Q_d and Q_M be functorial q-cofibrant replacements in $\mathcal{M}\mathbf{dTop}$ and $\mathcal{M}\mathbf{Flow}$. The functors

$$\Phi_M(X) = \text{cat}_{\mathcal{M}}(Q_d X), \quad \Psi_M(Z) = \mathbb{M}_!^{\mathcal{M}} Q_M(\mathbb{M} Z)$$

induce mutually inverse equivalences between the homotopy categories of the q-model structures on $\mathcal{M}$ -multipointed d -spaces and ordinary flows. Proposition 26.1 shows that Φ_M preserves weak equivalences in the q-model structures. Proposition 26.3 identifies the functor induced by Φ_M on homotopy categories with the composite of the right derived functor of $\mathbb{M}^{\mathcal{M}}$ and the left derived functor of $\mathbb{M}_!$. Theorems 25.4 and 12.3 then give its inverse. This recovers the homotopy-category conclusion of [17, Theorem 15] without a cofibrancy assertion for $\mathbb{M}^{\mathcal{M}}(Q_d X)$.

The same equivalence of homotopy categories holds for saturated $\mathcal{M}$ -multipointed d -spaces with the q-model structure constructed below. The categorization functor remains the quotient by the permitted reparametrizations; saturation changes which execution-path families are admitted, rather than the definition of a trace of an admitted path.

27. LOCALLY FINITE CELLULAR MULTIPOINTED d -SPACES

We also recover the assertion about the ordinary topology in [17, Theorem 11]. This result is useful when execution paths are studied by uniform convergence. Its proof must take account of the stopped attaching map in Section 22, because the images of the characteristic maps for $\mathcal{M}$ need not be closed.

27.1. Definition ([17, Definition 14]). A cellular multipointed d -space is locally finite if every point of its underlying space has an open neighborhood meeting only finitely many open cells.

27.2. Proposition ([17, Proposition 24 and Corollary 7]). *The underlying space of a locally finite cellular multipointed d -space is metrizable.*

The metric may be chosen separately on its open path components. This permits an arbitrary set of components. No countability assumption on the whole state set is required.

27.3. Proposition. *Let X be a locally finite cellular $\mathcal{P}$ -multipointed d -space, where $\mathcal{P} \in \{\mathcal{G}, \mathcal{M}\}$. Every execution path x_0 has an ordinary compact-open neighborhood in its execution-path family on which only finitely many carriers occur.*

Proof. The compact image $x_0(\mathbb{I})$ has an open neighborhood U meeting finitely many cells: cover it by the neighborhoods supplied by local finiteness and take a finite subcover. The compact-open neighborhood $[\mathbb{I}, U]$ restricts nearby paths to these cells. Include their cell closures and the finitely many endpoint states in a finite compact cellular subspace Y , using Proposition 5.1. Every nearby execution path is valued in Y , because a path is a union of full carrier traversals and its endpoints.

Use the restriction of a compatible metric d on $|X|$. Regularity for $\mathcal{G}$ and Proposition 21.5 for $\mathcal{M}$ make every characteristic meridian nonconstant, including boundary meridians. Lemma 15.3 gives a lower diameter bound $a > 0$ for the characteristic families in Y . Choose $\delta > 0$ such that $v - u < \delta$ implies $\text{diam } x_0([u, v]) < a/2$, and restrict to the neighborhood

$$\Omega = [\mathbb{I}, U] \cap \left\{ x : \sup_t d(x(t), x_0(t)) < a/4 \right\}.$$

For $x \in \Omega$,

$$v - u < \delta \implies \text{diam } x([u, v]) \leq \text{diam } x_0([u, v]) + 2 \sup_t d(x(t), x_0(t)) < a.$$

If N cells occur in Y , Lemma 15.3 gives

$$\#\{\text{Carrier}(x) : x \in \Omega\} \leq \sum_{m=1}^{\lfloor 1/\delta \rfloor} N^m < \infty.$$

□

27.4. Theorem. *For a locally finite cellular $\mathcal{P}$ -multipointed d -space X , where $\mathcal{P} \in \{\mathcal{G}, \mathcal{M}\}$, the ordinary relative compact-open topology of every $\mathbb{P}_{\alpha, \beta}^{\text{top}} X$ is metrizable and locally path-connected. It is already Δ -generated. The same holds for the total execution-path space.*

Proof. Write E_{co} for the ordinary execution-path space with the fixed endpoints. Metrizability follows from Proposition 27.2, since compact-open convergence of paths on $\mathbb{I}$ is uniform convergence for a compatible metric. Let $q_{co} : E_{co} \rightarrow T_{co}$ be its ordinary trace quotient. The ordinary continuity in Theorem 23.2 gives the section in the square

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & E_{co} \\ \downarrow & \nearrow j_{co} & \downarrow q_{co} \\ T_{co} & \xrightarrow{\text{id}_{T_{co}}} & T_{co}. \end{array}$$

It also gives the product homeomorphism

$$E_{co} \cong T_{co} \times \mathcal{P}(1, 1).$$

In particular, T_{co} embeds in the metrizable Hausdorff space E_{co} . Let $T = \mathbb{P}_{\alpha, \beta} \text{cat}_{\mathcal{P}} X$ be the trace space with its quotient topology in **Top**. Its canonical bijection to T_{co} is continuous: the ordinary quotient map is continuous on the finer convenient execution-path topology, and the quotient in **Top** descends to $k_{\Delta} T_{co}$ and then to T_{co} .

For each carrier $\mathbf{c}$, use the continuous map $p_{\mathbf{c}} : B_{\mathbf{c}} \rightarrow T$ from the proof of Theorem 24.2; its formula also applies for $\mathcal{G}$. Lemma 5.2 makes the inverse $T_{co} \rightarrow T$ continuous on each compact carrier image. Near any $x_0 \in E_{co}$, Proposition 27.3 supplies finitely many such images. Their inverse images form a finite closed cover of the chosen neighborhood, so

the pasting lemma makes $E_{co} \rightarrow T$ continuous there. Quotient descent gives

$$T_{co} \xrightarrow{\cong} T.$$

The flow $\text{cat}_{\mathcal{P}} X$ is q-cofibrant by Proposition 19.2 or Corollary 24.4. Its hom spaces are q-cofibrant, hence retracts of quotients of coproducts of disks and points, and therefore locally path-connected [29, Section 25]. Proposition 6.4 and the product decomposition give local path-connectedness of E_{co} . A metrizable locally path-connected space is Δ -generated [2, Proposition 3.11].

Finally, the states are a closed discrete subspace. Endpoint evaluation makes the fixed-endpoint execution-path families open in the ordinary total execution-path family. Its topology is consequently their coproduct topology. A coproduct of metrizable locally path-connected spaces is again metrizable and locally path-connected, so the total-space conclusion follows. $\square$

28. CHANGING THE REPARAMETRIZATION CATEGORY

The inclusion $i : \mathcal{G} \hookrightarrow \mathcal{M}$ is enriched and semimonoidal. Restriction of an $\mathcal{M}$ -space along i^{op} is a $\mathcal{G}$ -space. Its canonical lax semimonoidal map comes from the same coend representatives, viewed with fewer relations. Applying it to composition defines restriction $i^* : \mathcal{M}\mathbf{Flow} \rightarrow \mathcal{G}\mathbf{Flow}$. There is also restriction $j : \mathcal{M}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{dTop}$, retaining the same underlying space, states, and execution paths.

28.1. Proposition ([17, Proposition 26 and the adjoint construction in Theorem 17]). *Both i^* and j are right adjoints and right Quillen functors. The left adjoint of j keeps the underlying space and states and closes execution paths under nondecreasing surjections. There are natural identities*

$$i^* \mathbb{M}_{\mathcal{M}} = \mathbb{M}_{\mathcal{G}}, \quad i^* \mathbb{M}^{\mathcal{M}} = \mathbb{M}^{\mathcal{G}} j.$$

The functor i^* is accessible and preserves limits, so it has a left adjoint by the adjoint functor theorem for locally presentable categories [1, Theorem 1.66]. Its lax semimonoidal structure map supplies the composition used above. The fibration criteria are objectwise, so restricting the action preserves fibrations and trivial fibrations. For j , the path spaces themselves are unchanged. These facts establish the right Quillen statements.

28.2. Theorem. *The right Quillen functors $i^* : \mathcal{M}\mathbf{Flow} \rightarrow \mathcal{G}\mathbf{Flow}$ and $j : \mathcal{M}\mathbf{dTop} \rightarrow \mathcal{G}\mathbf{dTop}$ are Quillen equivalences.*

Proof. The identity $i^* \mathbb{M}_{\mathcal{M}} = \mathbb{M}_{\mathcal{G}}$ gives the commuting diagram of right adjoints

$$\begin{array}{ccc} \mathbf{Flow} & \xrightarrow{\mathbb{M}_{\mathcal{M}}} & \mathcal{M}\mathbf{Flow} \\ & \searrow \mathbb{M}_{\mathcal{G}} & \downarrow i^* \\ & & \mathcal{G}\mathbf{Flow}. \end{array}$$

Theorem 12.3 and two-out-of-three for Quillen equivalences [23, Corollary 1.3.15] prove the assertion for i^* .

For j , use the commuting square

$$\begin{array}{ccc} \mathcal{M}\mathbf{dTop} & \xrightarrow{j} & \mathcal{G}\mathbf{dTop} \\ \mathbb{M}^{\mathcal{M}} \downarrow & & \downarrow \mathbb{M}^{\mathcal{G}} \\ \mathcal{M}\mathbf{Flow} & \xrightarrow{i^*} & \mathcal{G}\mathbf{Flow}. \end{array}$$

Theorems 18.3 and 25.4 make the vertical adjunctions Quillen equivalences. Passing to derived right functors and applying two-out-of-three proves the assertion for j , recovering [17, Proposition 26 and Theorem 17]. $\square$

The left adjoint of j preserves q-cofibrant objects because it is left Quillen. In particular, closing the execution paths of a q-cofibrant $\mathcal{G}$ -multipointed d -space under $\mathcal{M}$ gives a q-cofibrant $\mathcal{M}$ -multipointed d -space with the same underlying space. The theorem concerns the q-model structures, whose weak equivalences induce weak homotopy equivalences on execution-path spaces; it does not remove the distinction between regular paths and paths with stops at the level of individual parametrizations.

29. THE Q-MODEL STRUCTURE ON SATURATED $\mathcal{M}$ -MULTIPOINTED d -SPACES

29.1. Proposition ([17, Propositions 27–29]). *For $X \in \mathcal{M}\mathbf{dTop}$, let $\widehat{X}$ have the same underlying multipointed space and execution paths*

$$\mathbb{P}^{\text{top}}\widehat{X} = \{x : \mathbb{I} \rightarrow |X| : \text{there is } \varphi \in \mathcal{M}(1, 1) \text{ with } x\varphi \in \mathbb{P}^{\text{top}}X\}.$$

This defines the left adjoint of the full inclusion $U : \mathcal{M}\mathbf{dTop}^{\text{sat}} \hookrightarrow \mathcal{M}\mathbf{dTop}$. The category $\mathcal{M}\mathbf{dTop}^{\text{sat}}$ is locally presentable. Every $\mathcal{G}$ -multipointed d -space is saturated with respect to increasing homeomorphisms, whereas an $\mathcal{M}$ -multipointed d -space need not be saturated.

For example, on the underlying interval with states 0 and 1, admit only the paths $\sigma\rho$, $\rho \in \mathcal{M}(1, 1)$, for the stopped map σ of Section 22. This is stable under $\mathcal{M}(1, 1)$, and there are no composable pairs. The identity becomes an execution path after composition with σ , but is not admitted itself. Thus it is not saturated. The reflector adds the missing regular representative and all paths trace-equivalent to it.

29.2. Proposition. *If X is saturated, its usual path object is saturated. The category $\mathcal{M}\mathbf{dTop}^{\text{sat}}$ consequently has the right-induced q-model structure along U .*

Proof. Write $\text{Path}(X)$ for the usual path object, with underlying space $\text{TOP}(\mathbb{I}, |X|)$ and states the constant maps at states of X :

$$\begin{array}{ccc} X & \xrightarrow[\sim]{c} & \text{Path}(X) \\ & \searrow \Delta_X & \downarrow (\text{ev}_0, \text{ev}_1) \\ & & X \times X. \end{array}$$

The exponential law identifies its execution-path spaces with $\text{TOP}(\mathbb{I}, \mathbb{P}_{\alpha, \beta}^{\text{top}}X)$. For a continuous path u in its underlying space, put $u_s(t) = u(t)(s)$. Saturation of X gives

$$u\varphi \in \mathbb{P}^{\text{top}}\text{Path}(X) \implies u_s\varphi \in \mathbb{P}^{\text{top}}X \quad (s \in \mathbb{I}) \implies u_s \in \mathbb{P}^{\text{top}}X \quad (s \in \mathbb{I}).$$

The compact-open exponential law and Δ -kelleyfication make $s \mapsto u_s$ continuous into $\mathbb{P}_{\alpha,\beta}^{\text{top}} X$, so $u \in \mathbb{P}^{\text{top}} \text{Path}(X)$.

The reflector and inclusion form an adjunction between locally presentable categories by Proposition 29.1. All $\mathcal{M}$ -multipointed d -spaces are fibrant in the q -model structure. The Quillen path-object transfer argument therefore applies; see [16, Theorem 2.1] and [8, 21], as in [17, Theorem 18]. The weak equivalences and fibrations are those created by the inclusion. $\square$

29.3. Theorem. *The adjunction $(\widehat{-}) \dashv U : \mathcal{M}\mathbf{dTop} \rightleftarrows \mathcal{M}\mathbf{dTop}^{\text{sat}}$ is a Quillen equivalence. The q -cofibrant objects of the two categories agree under U , and its unit and counit are isomorphisms on q -cofibrant objects.*

Proof. The right-induced model structure makes U right Quillen and makes it reflect weak equivalences. Globes are saturated: if $x\varphi = \delta_z\rho$, surjectivity of φ puts the image of x in the meridian δ_z , and projection to its interval gives $x = \delta_z\eta$. The equality $\eta\varphi = \rho$ forces η to be a nondecreasing surjection. Thus x is an execution path of the globe. The objects consisting solely of states are saturated as well, because they have no execution paths. Consequently the reflector fixes every generating q -cofibration of Theorem 20.2, including C and R .

These reflected maps generate the cofibrations in the right-induced structure. Indeed, for a generating q -cofibration $i : A_i \rightarrow B_i$ and a morphism $p : Y \rightarrow Z$ in $\mathcal{M}\mathbf{dTop}^{\text{sat}}$, the adjunction identifies the lifting problems

$$\begin{array}{ccc} \widehat{A}_i & \longrightarrow & Y \\ \widehat{i} \downarrow & \nearrow & \downarrow p \\ \widehat{B}_i & \longrightarrow & Z \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} A_i & \longrightarrow & UY \\ i \downarrow & \nearrow & \downarrow Up \\ B_i & \longrightarrow & UZ. \end{array}$$

Thus the right lifting property against all the reflected generators is exactly the ambient trivial-fibration condition. Their ambient cellular colimits are saturated by Proposition 21.5, so these colimits are also their colimits in the saturated subcategory. The same holds for retracts by the retract proof in that proposition. Thus the two classes of q -cofibrant objects agree.

For a q -cofibrant X in $\mathcal{M}\mathbf{dTop}$, the unit $X \rightarrow U\widehat{X}$ is an isomorphism by saturation. The counit is an isomorphism on every saturated object because U is full and faithful. Proposition 5.4 gives the Quillen equivalence. This establishes [17, Theorems 18–19] after supplying the cellular saturation proof. $\square$

29.4. **Corollary.** *The following commuting diagram consists of right adjoints of Quillen equivalences for the q -model structures:*

$$\begin{array}{ccccc}
 \mathcal{M}\mathbf{dTop}^{\text{sat}} & \xrightarrow{U} & \mathcal{M}\mathbf{dTop} & \xrightarrow{\mathbb{M}^{\mathcal{M}}} & \mathcal{M}\mathbf{Flow} \\
 & & \downarrow j & & \downarrow i^* \\
 & & \mathcal{G}\mathbf{dTop} & \xrightarrow{\mathbb{M}^{\mathcal{G}}} & \mathcal{G}\mathbf{Flow} \\
 & & & & \uparrow \mathbb{M}_{\mathcal{G}} \\
 & & & & \mathbf{Flow}
 \end{array}
 \quad \begin{array}{c} \curvearrowright \\ \mathbb{M}_{\mathcal{M}} \end{array}$$

In particular, $\mathbb{M}^{\mathcal{M}}U : \mathcal{M}\mathbf{dTop}^{\text{sat}} \rightarrow \mathcal{M}\mathbf{Flow}$ is a right Quillen equivalence.

Proof. Compose the Quillen equivalences of Theorems 29.3 and 25.4. All the other arrows were established in Theorems 12.3 and 28.2. $\square$

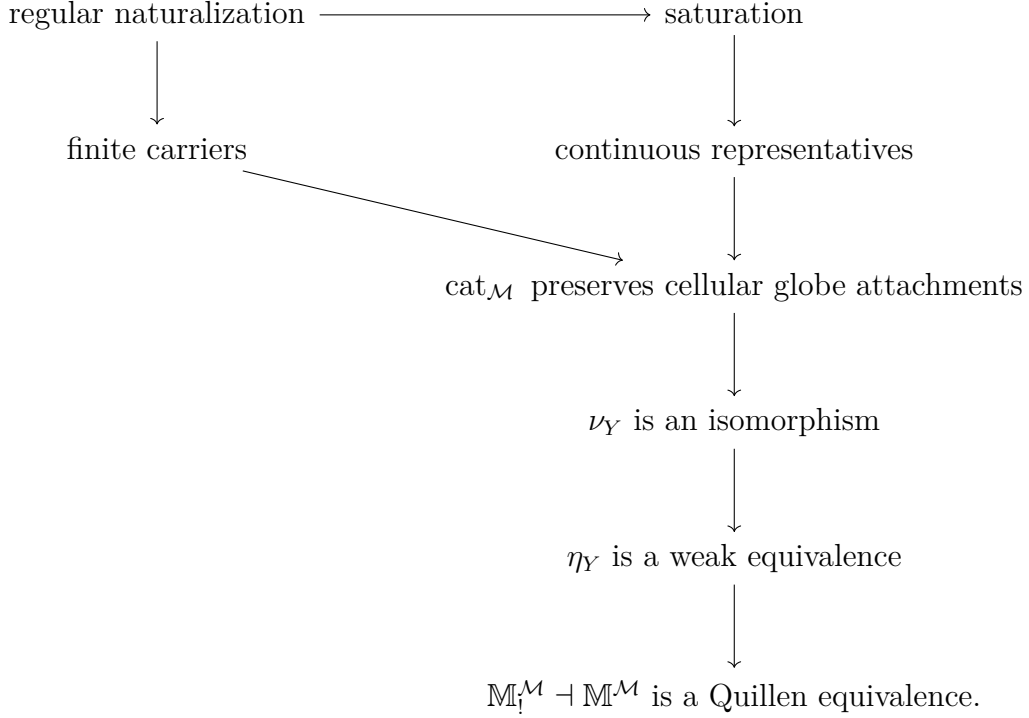
The unit of this composite adjunction need not be an isomorphism on a q -cofibrant Moore $\mathcal{M}$ -flow Y . The $\mathcal{M}$ -multipointed d -space $\mathbb{M}_!^{\mathcal{M}}Y$ is already saturated, so the example of Proposition 22.3 also applies after the saturation reflector. Accordingly, the conclusion about Quillen equivalence of [17, Corollary 10] survives, while its assertion that the composite unit is an isomorphism does not.

30. SUMMARY OF THE QUILLEN EQUIVALENCES AND TRACE QUOTIENTS

The table records the main conclusions and their hypotheses. All Quillen equivalences concern the q -model structures.

Adjunction, functor, or quotient	Statement established here	Required input
$\mathbb{M}_! \dashv \mathbb{M}$	Quillen equivalence between Moore $\mathcal{P}$ -flows and ordinary flows for nonempty $\mathcal{P}$	Cofibrant hom presheaves and the Quillen equivalence $\text{colim} \dashv \Delta_{\mathcal{P}op}$
$\mathbb{M}_!^{\mathcal{G}} \dashv \mathbb{M}^{\mathcal{G}}$	Quillen equivalence; unit is an isomorphism on q-cofibrant Moore $\mathcal{G}$ -flows; counit is an isomorphism on q-cofibrant $\mathcal{G}$ -multipointed d -spaces	Regularity and closed images of characteristic maps in the compact-open topology
$\mathbb{M}_!^{\mathcal{M}} \dashv \mathbb{M}^{\mathcal{M}}$	Quillen equivalence; unit is a weak equivalence on q-cofibrant Moore $\mathcal{M}$ -flows; counit is an isomorphism on q-cofibrant $\mathcal{M}$ -multipointed d -spaces	Normalization, preservation of cellular globe attachments by $\text{cat}_{\mathcal{M}}$, and evaluation by the counit
Execution paths of q-cofibrant $\mathcal{G}$ - or $\mathcal{M}$ -multipointed d -spaces to traces	Trivial Hurewicz fibration; source m-cofibrant and target q-cofibrant	Submetrizability, saturation or regularity, and q-cofibrancy of $\text{cat}_{\mathcal{P}} X$
Restriction i^* and j	Right Quillen equivalences between $\mathcal{M}\mathbf{Flow}$ and $\mathcal{G}\mathbf{Flow}$, and between $\mathcal{M}\mathbf{dTop}$ and $\mathcal{G}\mathbf{dTop}$	Commuting diagrams of right adjoints and two-out-of-three
Saturation reflector	Quillen equivalence between $\mathcal{M}\mathbf{dTop}$ and $\mathcal{M}\mathbf{dTop}^{\text{sat}}$; unit is an isomorphism on q-cofibrant $\mathcal{M}$ -multipointed d -spaces	Regular factorization and saturated path objects

The following diagram records which assertions are used to prove others in the $\mathcal{M}$ case. Its arrows denote logical dependencies.



The normalization theorem is applied after cellular saturation has been proved. The preservation of globe attachments by $\text{cat}_{\mathcal{M}}$ then establishes the Quillen equivalence.

Part 5. Erratum to Natural homotopy of multipointed d-spaces

31. NATURAL SYSTEMS AND LOCALIZATION

This part corrects the comparison with discrete natural systems and the counterexample to bisimilarity in [19]. Section 33 replaces the use of the false assertion that $\mathbb{M}^{\mathcal{M}}$ preserves cellular globe attachments.

31.1. Directed spaces and their traces. A directed space in the sense of Grandis is a space equipped with continuous paths containing the constant paths and closed under concatenation and continuous nondecreasing changes of parameter. The directed reflection $\vec{\text{Sp}}(X)$ of an $\mathcal{M}$ -multipointed d -space X has underlying space $|X|$ and the directed structure generated by its execution paths. It is left adjoint to the full and faithful functor which makes every point a state and every directed path an execution path [19, Propositions 3.6–3.7 and Theorems 3.8–3.9].

Write $\vec{T}_X(x, y) = \vec{T}(\vec{\text{Sp}}(X))(x, y)$ for the space of directed paths from x to y modulo the equivalence relation generated by nondecreasing surjective changes of parameter. The topology is the quotient in **Top** of the convenient compact-open topology. Concatenation induces continuous composition of traces, and the constant trace at x is an identity. Thus the traces form a category $\vec{T}(\vec{\text{Sp}}(X))$ whose objects are all points of $|X|$. The execution traces of $\text{cat}_{\mathcal{M}} X$ are formed from the specified execution paths and have endpoints in X^0 ; the notation $T_{\alpha, \beta}(X)$ from Part 4 refers to these execution traces. For cellular X ,

execution paths are nonconstant by Theorem 21.2, so these execution traces exclude the constant identities of the directed reflection.

31.1. Proposition. *For a cellular $\mathcal{M}$ -multipointed d -space X and $\alpha, \beta \in X^0$, the canonical maps give homeomorphisms*

$$\vec{T}_X(\alpha, \beta) \cong \begin{cases} T_{\alpha, \beta}(X), & \alpha \neq \beta, \\ \{\text{id}_\alpha\} \sqcup T_{\alpha, \alpha}(X), & \alpha = \beta. \end{cases}$$

Proof. For a nonconstant directed path $\gamma : \alpha \rightarrow \beta$, [19, Theorem 4.9] and Theorem 21.2 give $\gamma = r\psi$, where $r : [0, m] \rightarrow |X|$ is a globular naturalization and $\psi : \mathbb{I} \rightarrow [0, m]$ is continuous and nondecreasing. Since $r^{-1}(X^0) = \{0, \dots, m\}$,

$$k = \psi(0), \quad l = \psi(1), \quad k, l \in \{0, \dots, m\}, \quad k < l,$$

$$r_{k,l}(u) = r(k + (l - k)u), \quad \rho(u) = \frac{\psi(u) - k}{l - k}, \quad \rho \in \mathcal{M}(1, 1), \quad \gamma = r_{k,l}\rho.$$

The normalized composite $r_{k,l}$, and hence γ , is an execution path. Thus the nonconstant directed paths with these endpoints are exactly $\mathbb{P}_{\alpha, \beta}^{\text{top}} X$.

Consider a continuous curve ξ in the space of directed paths from α to itself. Its compact evaluation image lies in a finite compact metrizable cellular subspace Y by Propositions 21.1 and 5.1. Fix a compatible metric d on Y . If Y has no positive-dimensional cells, every member is constant. Otherwise, Proposition 21.5 makes all its characteristic meridians nonconstant, and Lemma 15.3 gives $a > 0$ such that

$$\xi(s) \neq c_\alpha \implies \text{diam } \xi(s)(\mathbb{I}) \geq a.$$

At a parameter s_0 with $\xi(s_0) = c_\alpha$, continuity in the uniform topology gives

$$\sup_t d(\xi(s)(t), \alpha) < a/2 \implies \text{diam } \xi(s)(\mathbb{I}) < a \implies \xi(s) = c_\alpha$$

for all s near s_0 . The curve test makes $\{c_\alpha\}$ open; Hausdorffness makes it closed. Its complement has the convenient relative compact-open topology of $\mathbb{P}_{\alpha, \alpha}^{\text{top}} X$. Surjective reparametrizations preserve both summands, so the trace quotient preserves this coproduct. This proves the topological assertion of [18, Proposition 5.7]. $\square$

The objects of the factorization category $\text{Fact}(\mathcal{C})$ are the morphisms of $\mathcal{C}$. A morphism $(p, q) : u \rightarrow u'$ is a commutative diagram

$$\begin{array}{ccc} x' & \xrightarrow{p} & x \\ u' \downarrow & & \downarrow u \\ y' & \xleftarrow{q} & y. \end{array}$$

Write $*$ for composition in the order of execution, so that $u' = p * u * q$. The natural system associated with X is

$$\vec{\text{NT}}(X) : \text{Fact}(\vec{T}(\vec{\text{Sp}}(X))) \longrightarrow \mathbf{Top}, \quad \vec{\text{NT}}(X)(u : x \rightarrow y) = \vec{T}_X(x, y).$$

It sends the extension (p, q) to $v \mapsto p * v * q$. These definitions and the continuity of the extension maps are those of [19, Section 6].

31.2. Bisimilarity of diagrams. For a category $\mathcal{C}$, an object of $\text{Diag}(\mathcal{C})$ is a functor $F : \mathcal{I} \rightarrow \mathcal{C}$ with $\mathcal{I}$ small. A morphism $(f, \mu) : F \rightarrow G$ consists of $f : \mathcal{I} \rightarrow \mathcal{J}$ and $\mu : F \Rightarrow Gf$. It is open if μ is a natural isomorphism and f has the right lifting property in both diagrams below [19, Definition 5.2].

Write $[1] = (0 \rightarrow 1)$ and let $\iota_0 : \{0\} \rightarrow [1]$ be the inclusion. The lifting problems in the category of small categories are

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & \mathcal{I} \\ \downarrow & \nearrow \tilde{j} & \downarrow f \\ \{0\} & \xrightarrow{j} & \mathcal{J} \end{array} \qquad \begin{array}{ccc} \{0\} & \xrightarrow{i} & \mathcal{I} \\ \downarrow \iota_0 & \nearrow \tilde{v} & \downarrow f \\ [1] & \xrightarrow{v} & \mathcal{J}. \end{array}$$

Thus f is surjective on objects, and every $v : f(i) \rightarrow j'$ has a lift $\tilde{v} : i \rightarrow i'$ with $f(i') = j'$ and $f(\tilde{v}) = v$.

Two diagrams $F : \mathcal{I} \rightarrow \mathcal{C}$ and $G : \mathcal{J} \rightarrow \mathcal{C}$ are bisimilar if they admit a bisimulation: a collection of triples (i, η, j) with $\eta : F(i) \cong G(j)$, covering the objects of both indexing categories, such that every outgoing morphism on either side has a matching outgoing morphism on the other side. More precisely, an arrow $a : i \rightarrow i'$ has a matching arrow $b : j \rightarrow j'$ and a triple (i', η', j') for which

$$\begin{array}{ccc} F(i) & \xrightarrow[\cong]{\eta} & G(j) \\ F(a) \downarrow & & \downarrow G(b) \\ F(i') & \xrightarrow[\cong]{\eta'} & G(j') \end{array}$$

commutes; the same condition is required starting with an arrow from j . Equivalently, the diagrams are related by a span of open morphisms [19, Definition 5.1 and Proposition 5.3]. Write $\mathbf{h} : \mathbf{Top} \rightarrow \mathbf{Ho}(\mathbf{Top})$ for localization at weak homotopy equivalences. Diagrams of spaces are bisimilar up to homotopy when their images under $\mathbf{h}$ are bisimilar in $\text{Diag}(\mathbf{Ho}(\mathbf{Top}))$.

For a morphism (f, μ) of diagrams of spaces, being open up to homotopy means that $(f, \mathbf{h}\mu)$ is open. An open morphism between the localized diagrams need not lift to a morphism of diagrams of spaces. The comparison below is asserted in $\text{Diag}(\mathbf{Ho}(\mathbf{Top}))$, as required for bisimilarity up to homotopy in [19, Definition 5.4].

32. LOOP-FREENESS AND DISCRETE NATURAL SYSTEMS

32.1. Definition. A cellular $\mathcal{M}$ -multipointed d -space X is loop-free if $\mathbb{P}_{\alpha, \alpha}^{\text{top}} X = \emptyset$ for every $\alpha \in X^0$.

Equivalently, every directed path of $\overrightarrow{\text{Sp}}(X)$ with equal endpoints is constant. Indeed, [19, Theorems 4.7 and 4.9] express a nonconstant directed path as a segment of a globular naturalization. A nonconstant loop based in the interior of a cell must leave that cell through its final state and return through its initial state. Composing that return with a meridian gives an execution loop at a state. The reverse implication follows directly because an execution path is directed and cellular execution paths are nonconstant by Theorem 21.2.

Add loop-freeness to [19, Theorem 4.10] and throughout Section 8 of that paper. In [19, Theorem 9.4], require both cellular $\mathcal{M}$ -multipointed d -spaces to be loop-free. The corresponding statements in the introduction require the same hypotheses. These changes concern the comparison with discrete natural systems, independently of the stopped-boundary obstruction in Part 4.

To see the need for the hypothesis in the proof, let c be a positive-dimensional globular cell, with initial and final states c^- and c^+ . A trace leaving c and returning to c has a portion from c^+ to c^- . Together with a meridian from c^- to c^+ , that portion produces a nonconstant loop. In the loop-free case,

$$\vec{T}_X(c^+, c^-) = \emptyset.$$

Consequently, for $x, y \in c$,

$$\vec{T}_X(x, y) = \begin{cases} \{\tau_{x,y}\}, & x, y \text{ lie on one fiber in increasing order,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

This is the calculation required in [19, Propositions 8.12–8.13]. Without loop-freeness, a trace may leave c and return to a different fiber, or to an earlier point of the same fiber.

Loop-freeness also excludes the positive-dimensional cell visits of zero duration that invalidate the characterization of discrete traces in [19, Theorem 4.10]. Under the corrected hypothesis, the discrete trace $\text{dt}(u)$ of a directed trace records its finite sequence of visited globular cells, including states and with consecutive repetitions suppressed. These words form the category $\mathcal{C}_X$ of discrete traces. Its elementary nonidentity morphisms go from c^- to c and from c to c^+ ; composition concatenates the words at their common endpoint. A point is sent to its containing cell. The discrete trace functor is

$$\text{dt} : \vec{T}(\vec{\text{Sp}}(X)) \longrightarrow \mathcal{C}_X.$$

For each positive-dimensional cell c , choose its center $x_c = \widehat{g}_c(0, 1/2)$; for a state c , put $x_c = c$. The discrete natural system of [19, Definition 8.15] has values

$$\vec{\text{NT}}_d(X)([c_1, \dots, c_m]) = \vec{T}_X(x_{c_1}, x_{c_m}).$$

Its extension maps concatenate the fixed fiber segments joining these centers to the incident states.

32.2. Theorem (Natural systems and discrete natural systems). *Let X be a loop-free cellular $\mathcal{M}$ -multipointed d -space. There is an open morphism*

$$\underline{\text{h}}\vec{\text{NT}}(X) \longrightarrow \underline{\text{h}}\vec{\text{NT}}_d(X) \quad \text{in } \text{Diag}(\mathbf{Ho}(\mathbf{Top})).$$

In particular, $\vec{\text{NT}}(X)$ and $\vec{\text{NT}}_d(X)$ are bisimilar up to homotopy. If $f : X \rightarrow Y$ is a globular subdivision between loop-free cellular $\mathcal{M}$ -multipointed d -spaces, the four natural systems $\vec{\text{NT}}(X)$, $\vec{\text{NT}}(Y)$, $\vec{\text{NT}}_d(X)$, and $\vec{\text{NT}}_d(Y)$ are bisimilar up to homotopy.

Proof. The homeomorphisms and elementary extension calculations of [19, Section 8, proof of Theorem 8.16], with loop-freeness imposed throughout, give

$$\underline{\text{h}}\vec{\text{NT}}(X) \cong (\underline{\text{h}}\vec{\text{NT}}_d(X)) \text{Fact}(\text{dt}).$$

For endpoints in one cell, use $\vec{T}_X(c^+, c^-) = \emptyset$. The extension squares commute strictly or up to homotopy, hence commute after localization.

Put $\mathcal{I}_X = \text{Fact}(\overrightarrow{T}(\overrightarrow{\text{Sp}}(X)))$, $\mathcal{J}_X = \text{Fact}(\mathcal{C}_X)$, and $f_X = \text{Fact}(\text{dt})$. The functor f_X is surjective on objects. Its extension-lifting condition is the following problem in the category of small categories:

$$\begin{array}{ccc} \{0\} & \xrightarrow{u} & \mathcal{I}_X \\ \downarrow \iota_0 & \nearrow \tilde{e} & \downarrow f_X \\ [1] & \xrightarrow{e} & \mathcal{J}_X. \end{array}$$

For $e : f_X(u) \rightarrow w$, the argument with prescribed endpoints in [19, Section 8, proof of Theorem 8.16] supplies

$$(p, q) : u \longrightarrow u', \quad f_X(u') = w, \quad f_X(p, q) = e.$$

This defines $\tilde{e}$. The comparison is therefore open, and [19, Proposition 5.3] gives bisimilarity.

For a globular subdivision $f : X \rightarrow Y$, the isomorphism $\overrightarrow{\text{Sp}}(f)$ of [19, Theorem 9.3] induces the middle isomorphism in

$$\underline{\mathbf{hNT}}_d(X) \longleftarrow \underline{\mathbf{hNT}}(X) \xrightarrow{\cong} \underline{\mathbf{hNT}}(Y) \longrightarrow \underline{\mathbf{hNT}}_d(Y).$$

Composites of open morphisms are open. The diagram gives a span of open morphisms between any two of its objects, proving the corrected assertion of [19, Theorem 9.4]. $\square$

For completeness, [19, Proposition 2.11] should assume that the regular path γ is nonconstant. Cancellation fails for a constant path of positive length. Its applications use nonconstant globular naturalizations, so this correction does not alter those arguments.

33. THE CELLULAR PUSHOUT USED BY THE NATURAL SYSTEMS

The proof of [19, Proposition 7.1] invokes [17, Corollary 8] to assert preservation of a cellular attachment by $\mathbb{M}^{\mathcal{M}}$. Proposition 22.3 disproves that intermediate assertion, even in a finite loop-free example. The statement of the proposition nevertheless remains valid: if

$$\begin{array}{ccc} \text{Glob}^{\mathcal{M}}(\mathbb{S}^{n-1}) & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ \text{Glob}^{\mathcal{M}}(\mathbb{D}^n) & \longrightarrow & B \end{array} \quad \text{with } A \text{ cellular,}$$

then Theorem 24.2 gives the pushout square

$$\begin{array}{ccc} \text{Glob}(\mathbb{S}^{n-1}) & \longrightarrow & \text{cat}_{\mathcal{M}} A \\ \downarrow & \lrcorner & \downarrow \\ \text{Glob}(\mathbb{D}^n) & \longrightarrow & \text{cat}_{\mathcal{M}} B. \end{array}$$

Theorem 24.2 therefore replaces the original proof, including the required homeomorphisms of trace spaces.

This supplies the homeomorphisms used in [19, Propositions 8.7–8.9 and 8.12], under the hypotheses of Section 32. The naturalization and cellular topology used in [19, Section 4]

rely on the valid [17, Theorem 6 and Propositions 15, 18–19]. Section 26 recovers the equivalence induced by derived categorization used in the final discussion citing [17, Theorem 15].

34. A REPLACEMENT COUNTEREXAMPLE TO BISIMILARITY

Failure of openness of one map does not exclude a bisimulation. Replace the example of [19, Section 10] by the following one.

34.1. Proposition. *There is a trivial q -cofibration $i : B \rightarrow A$ between finite loop-free cellular $\mathcal{M}$ -multipointed d -spaces such that $\overrightarrow{\text{NT}}(A)$ and $\overrightarrow{\text{NT}}(B)$ are not bisimilar up to homotopy.*

Proof. Let B be the cellular $\mathcal{M}$ -multipointed d -space with states $\{a, b, c\}$ and four globular cells of dimension 1:

$$a \begin{array}{c} \xrightarrow{e_0} \\ \xrightarrow{e_1} \end{array} b \begin{array}{c} \xrightarrow{f_0} \\ \xrightarrow{f_1} \end{array} c.$$

The nonempty execution-trace spaces are

$$T_{a,b}(B) = \{e_0, e_1\}, \quad T_{b,c}(B) = \{f_0, f_1\}, \quad T_{a,c}(B) = \{e_i * f_j \mid i, j \in \{0, 1\}\},$$

with the discrete topology. Put $\tau = e_0 *_N f_0$ and form the pushout

$$\begin{array}{ccc} \text{Glob}^{\mathcal{M}}(\{0\}) & \xrightarrow{\tau} & B \\ j \downarrow & & \downarrow i \\ \text{Glob}^{\mathcal{M}}([0, 1]) & \xrightarrow{\quad \quad} & A. \end{array}$$

The top map sends the unique meridian to τ and its endpoints to a, c . The fiber at 1 supplies a new globular cell of dimension 1, and $]0, 1[\times]0, 1[$ supplies a new globular cell d of dimension 2. Both $\mathcal{M}$ -multipointed d -spaces are finite cellular and loop-free.

The map j is a q -cofibration: first add the fiber at 1, then attach the globular cell of dimension 2 with its boundary fibers at 0 and 1. It is also a q -weak equivalence, since its only nonempty map of execution-path spaces is

$$\mathcal{M}(1, 1) \longrightarrow [0, 1] \times \mathcal{M}(1, 1), \quad \rho \longmapsto (0, \rho),$$

a homotopy equivalence. Thus its pushout i is a trivial q -cofibration. Applying Theorem 24.2 to the two cellular attachments gives

$$\overrightarrow{T}_A(a, c) \cong [0, 1] \sqcup \{1, 2, 3\}.$$

The interval contains the trace of τ ; the isolated points are the other three composites.

Choose $x \in d$ and let $u = \text{id}_x$. A path reaching x begins at a or at an earlier point of its fiber; a path leaving x ends at c or at a later point of that fiber. Hence

$$\overrightarrow{\text{NT}}(A)(u) = \{\text{id}_x\}, \quad u \longrightarrow u' \implies \#\pi_0 \overrightarrow{\text{NT}}(A)(u') = \begin{cases} 4, & u' : a \rightarrow c, \\ 1, & \text{otherwise.} \end{cases}$$

Conversely, every singleton-valued object v of $\overrightarrow{\text{NT}}(B)$ has an extension $v \rightarrow v'$ whose value is a discrete two-point space. If both endpoints of v lie on one closed edge, including the case of a constant trace at a state, extend to the endpoints a, b or b, c of the

corresponding edge family. Otherwise its initial endpoint is in an e_i and its final endpoint is in an f_j . Extend the initial endpoint to a and keep the final endpoint inside f_j . The two choices of the first edge give exactly two traces.

Suppose that the localized natural systems were bisimilar. Coverage of the objects of $\underline{\mathbf{hNT}}(A)$ would relate u to an object v whose value is isomorphic in $\mathbf{Ho}(\mathbf{Top})$ to a singleton. All values of $\overrightarrow{\mathbf{NT}}(B)$ are finite discrete spaces, so v is singleton-valued. Write $e_B : v \rightarrow v'$ for its two-point extension. The back condition of a bisimulation would supply an extension $e_A : u \rightarrow u'$ and an isomorphism η' making the following square commute in $\mathbf{Ho}(\mathbf{Top})$:

$$\begin{array}{ccc} \underline{\mathbf{hNT}}(A)(u) & \xrightarrow[\cong]{\eta} & \underline{\mathbf{hNT}}(B)(v) \\ \underline{\mathbf{hNT}}(A)(e_A) \downarrow & & \downarrow \underline{\mathbf{hNT}}(B)(e_B) \\ \underline{\mathbf{hNT}}(A)(u') & \xrightarrow[\cong]{\eta'} & \underline{\mathbf{hNT}}(B)(v'). \end{array}$$

Since π_0 factors through $\mathbf{Ho}(\mathbf{Top})$, the lower isomorphism would give

$$\# \pi_0 \overrightarrow{\mathbf{NT}}(A)(u') = \# \pi_0 \overrightarrow{\mathbf{NT}}(B)(v') = 2,$$

contrary to $\# \pi_0 \overrightarrow{\mathbf{NT}}(A)(u') \in \{1, 4\}$. □

Part 6. Erratum to Globular subdivisions are dihomotopy equivalences

35. THE TRACE DECOMPOSITION USED IN SUBDIVISION

The factorization theorem of [18] and its consequences remain valid. The trace decompositions used there include traces which leave a cell and return to it, so no loop-free hypothesis is needed.

In [18, Proposition 7.7(2)], uniqueness is to be understood among directed paths avoiding X^0 . In [18, Proposition 7.8(3)], impose $\gamma(\mathbb{I}) \subset c$. Put $x = \gamma(0)$ and $y = \gamma(1)$. The corrected decomposition is

$$(35.1) \quad \overrightarrow{T}_X(x, y) \cong \{\tau_{x,y}\} \sqcup \overrightarrow{T}_X(c^+, c^-),$$

where $\tau_{x,y}$ is the unique trace inside c . The second summand is mapped by

$$\rho \longmapsto \tau_{x,c^+} * \rho * \tau_{c^-,y}.$$

The condition $\gamma(\mathbb{I}) \subset c$ places x, y on one fiber in increasing order, so $\tau_{x,y}$ exists; $\tau_{x,x} = \text{id}_x$. These statements apply to every positive-dimensional cell, including dimension 1.

The hypotheses in [18, Theorems 7.9 and 9.2] ensure that $\tau_{x,y}$ exists whenever its singleton summand is used. Otherwise the decomposition is $\overrightarrow{T}_X(x, y) \cong \overrightarrow{T}_X(c^+, c^-)$.

36. THE COLLAPSE AND THE UNDERLYING GLOBE FUNCTOR

36.1. The collapse in Proposition 8.5. Replace the last assertion of [18, Proposition 8.5] by the statement that

$$\text{cat}_{\mathcal{M}}(W_F^n) \longrightarrow \text{Glob}(\mathbf{D}^n)_F$$

is a q-weak equivalence between cellular flows. Delete the corresponding cofibration assertion at the end of its proof. Here W_F^n and $\text{Glob}(\mathbf{D}^n)_F$ retain the notation of that

proposition: the first is its cellular replacement of the subdivided globe, and the displayed map is its collapse.

For $n = 0$ and $F = \emptyset$, the map on its nonempty path space is $\mathbf{D}^1 \rightarrow \mathbf{D}^0$. This map is not a q-cofibration, since it is not injective. The proposition separately constructs a q-cofibration $\text{cat}_{\mathcal{M}}(i_F^n)$. That q-cofibration and the q-weak equivalence of the collapse are all the properties required in the subsequent argument.

36.2. The underlying globe functor. In [18, Proposition A.1], replace the left-adjoint assertion by preservation of q-cofibrations and trivial q-cofibrations. Indeed,

$$|\text{Glob}^{\mathcal{M}}(\emptyset)| = \{0, 1\} \neq \emptyset,$$

so $Z \mapsto |\text{Glob}^{\mathcal{M}}(Z)|$ is not a left adjoint $\mathbf{Top} \rightarrow \mathbf{Top}$. The diagram functor $\mathcal{D}$ in that proof also fails to preserve all colimits. Its Reedy calculation, followed by the left Quillen colimit functor, still proves the required preservation assertion.

There is also a direct proof. Put $S = \{0, 1\}$ and $\Gamma(Z) = (S \rightarrow |\text{Glob}^{\mathcal{M}}(Z)|)$. The adjunction is

$$\begin{aligned} \Gamma : \mathbf{Top} &\rightleftarrows (S \downarrow \mathbf{Top}) : P_S, \\ P_S(X; x_0, x_1) &= \{\gamma \in \mathbf{TOP}(\mathbb{I}, X) \mid \gamma(0) = x_0, \gamma(1) = x_1\}. \end{aligned}$$

For $u : A \rightarrow B$, the underlying globe map is the pushout

$$\begin{array}{ccc} (A \times \mathbb{I}) \sqcup_{A \times \partial \mathbb{I}} (B \times \partial \mathbb{I}) & \longrightarrow & |\text{Glob}^{\mathcal{M}}(A)| \\ \downarrow u \sqcup (\partial \mathbb{I} \hookrightarrow \mathbb{I}) & \ulcorner & \downarrow |\text{Glob}^{\mathcal{M}}(u)| \\ B \times \mathbb{I} & \longrightarrow & |\text{Glob}^{\mathcal{M}}(B)|. \end{array}$$

The upper map is the globe quotient on $A \times \mathbb{I}$ and the endpoint quotient $B \times \partial \mathbb{I} \rightarrow S$ on the other summand. The pushout-product axiom for the cartesian q-model structure [23, Chapter 4] gives

$$\begin{aligned} u \text{ a (trivial) q-cofibration} &\implies u \sqcup (\partial \mathbb{I} \hookrightarrow \mathbb{I}) \text{ a (trivial) q-cofibration} \\ &\implies \Gamma(u) \text{ a (trivial) q-cofibration.} \end{aligned}$$

Thus Γ is left Quillen, proving the preservation used in [18, Propositions A.2–A.3].

37. THE WEAK EQUIVALENCE INDUCED BY THE COLLAPSE

The last step of [18, Theorem 9.2] requires a q-weak equivalence $\text{cat}_{\mathcal{M}}(T) \rightarrow \text{cat}_{\mathcal{M}}(\overline{Y})$. Let $g : \text{Glob}^{\mathcal{M}}(\mathbf{S}^{n-1}) \rightarrow U$ attach a globe to a cellular $\mathcal{M}$ -multipointed d -space, and put

$$\widehat{Y} = U \sqcup_{\text{Glob}^{\mathcal{M}}(\mathbf{S}^{n-1})} \text{Glob}^{\mathcal{M}}(\mathbf{D}^n).$$

Let F be a finite set in the interior of this new globe, and denote by $\text{Glob}^{\mathcal{M}}(\mathbf{D}^n)_F$ the globe with these additional states. Its directed reflection agrees with that of the original globe. The construction in [18, Proposition 8.5] gives the factorization

$$\text{Glob}^{\mathcal{M}}(\mathbf{D}^n) \xrightarrow{i_F^n} W_F^n \xrightarrow{c_F} \text{Glob}^{\mathcal{M}}(\mathbf{D}^n)_F$$

of the globe subdivision, where i_F^n is a cellular q-cofibration and c_F is the collapse considered in Section 36. Form the two pushout squares

$$\begin{array}{ccc} \text{Glob}^{\mathcal{M}}(\mathbb{D}^n) & \xrightarrow{k} & \hat{Y} \\ i_F^n \downarrow & \lrcorner & \downarrow j \\ W_F^n & \longrightarrow & T \end{array} \quad \begin{array}{ccc} \text{Glob}^{\mathcal{M}}(\mathbb{D}^n) & \xrightarrow{k} & \hat{Y} \\ \text{subdivision} \downarrow & \lrcorner & \downarrow S \\ \text{Glob}^{\mathcal{M}}(\mathbb{D}^n)_F & \longrightarrow & \bar{Y}. \end{array}$$

The identity of $\hat{Y}$ and c_F induce $r : T \rightarrow \bar{Y}$, with $rj = S$. The map S is a globular subdivision. In [18, Theorem 9.2], $U = Y$ and $\hat{Y} = \bar{X} \sqcup_X Y$.

For an $\mathcal{M}$ -multipointed d -space Z , put

$$C_Z(p, q) = \mathbf{I}^+(\text{cat}_{\mathcal{M}} Z)(p, q),$$

where $\mathbf{I}^+$ freely adjoins identities to a flow. Thus $C_Z(p, q) = T_{p,q}(Z)$ when $p \neq q$, and $C_Z(p, p) = \{\text{id}_p\} \sqcup T_{p,p}(Z)$. For cellular Z , Proposition 31.1 identifies these spaces with $\vec{T}_Z(p, q)$. Identify the old states with $O = \hat{Y}^0$ and the new states with F , so that $T^0 = \bar{Y}^0 = O \sqcup F$. Write $s = k(0)$ and $t = k(1)$; these two old states need not be distinct.

For $\alpha, \beta \in F$, put

$$R_\alpha = C_{W_F^n}(\alpha, 1), \quad L_\beta = C_{W_F^n}(0, \beta), \quad K_{\alpha, \beta} = C_{W_F^n}(\alpha, \beta).$$

37.1. Lemma. *The spaces R_α, L_β are contractible. The space $K_{\alpha, \beta}$ is empty if the subdivided globe has no internal trace from α to β , and contractible otherwise.*

Proof. The collapse c_F of [18, Proposition 8.5] is a q-weak equivalence between cellular multipointed d -spaces. Proposition 26.1 makes $\text{cat}_{\mathcal{M}}(c_F)$ a q-weak equivalence. For $\alpha, \beta \in F$, its target spaces at $(\alpha, 1)$ and $(0, \beta)$ are singletons, and its target at (α, β) is empty or a singleton. Its source spaces at these pairs are q-cofibrant by Theorem 26.2; adjoining an isolated identity preserves q-cofibrancy. Every nonempty source is therefore homotopy equivalent to a point by [23, Section 1.2]. $\square$

37.2. Proposition. *The induced map $\text{cat}_{\mathcal{M}}(r) : \text{cat}_{\mathcal{M}}(T) \rightarrow \text{cat}_{\mathcal{M}}(\bar{Y})$ is a q-weak equivalence.*

Proof. For $p, q \in O$, consider

$$\begin{array}{ccc} C_{\hat{Y}}(p, q) & \xrightarrow[\simeq]{a_{p,q}} & C_T(p, q) \\ & \searrow C_S & \downarrow b_{p,q} \\ & & C_{\bar{Y}}(p, q). \end{array}$$

Here $a_{p,q} = C_j$ is a weak homotopy equivalence by [18, Proposition 8.8], quantified over $O \times O$, and $b_{p,q} = C_r$. Since $b_{p,q}a_{p,q} = C_S$ is a homeomorphism by [18, Proposition 7.3], $b_{p,q}$ is a weak homotopy equivalence.

The word decompositions of [18, Theorem 8.9] give

$$\begin{aligned} C_T(\alpha, \beta) &\cong K_{\alpha, \beta} \sqcup (R_\alpha \times C_T(t, s) \times L_\beta), \\ C_T(p, \beta) &\cong C_T(p, s) \times L_\beta, \quad p \in O, \\ C_T(\alpha, q) &\cong R_\alpha \times C_T(t, q), \quad q \in O. \end{aligned}$$

The two summands separate internal traces from traces passing through the old states t, s .

Let $E_{\alpha, \beta}$ be the singleton internal trace when that trace exists, and the empty space otherwise. The corresponding target decompositions are

$$\begin{aligned} C_{\overline{Y}}(\alpha, \beta) &\cong E_{\alpha, \beta} \sqcup C_{\overline{Y}}(t, s), \\ C_{\overline{Y}}(p, \beta) &\cong C_{\overline{Y}}(p, s), \quad p \in O, \\ C_{\overline{Y}}(\alpha, q) &\cong C_{\overline{Y}}(t, q), \quad q \in O. \end{aligned}$$

These are [18, Theorem 7.9], with (35.1). Under these identifications, $b = C_r$ is

$$\begin{aligned} K_{\alpha, \beta} &\longrightarrow E_{\alpha, \beta}, \\ R_\alpha \times C_T(t, s) \times L_\beta &\longrightarrow C_{\overline{Y}}(t, s), & (u, v, w) &\longmapsto b_{t, s}(v), \\ C_T(p, s) \times L_\beta &\longrightarrow C_{\overline{Y}}(p, s), & (v, w) &\longmapsto b_{p, s}(v), \\ R_\alpha \times C_T(t, q) &\longrightarrow C_{\overline{Y}}(t, q), & (u, v) &\longmapsto b_{t, q}(v). \end{aligned}$$

Lemma 37.1, the equivalences $b_{p, q}$ for $p, q \in O$, and preservation of weak homotopy equivalences by products and coproducts imply that C_r is a weak homotopy equivalence for every pair of states. At equal endpoints it has the form

$$C_r(p, p) = \text{id}_{\{\text{id}_p\}} \sqcup (\text{cat}_{\mathcal{M}}(r))_{p, p} : \{\text{id}_p\} \sqcup T_{p, p}(T) \longrightarrow \{\text{id}_p\} \sqcup T_{p, p}(\overline{Y}).$$

Hence its restriction to execution traces is a weak homotopy equivalence. Since r^0 is bijective, $\text{cat}_{\mathcal{M}}(r)$ is a q-weak equivalence.

If $s = t$, the factor $\text{id}_s \in C_T(s, s)$ represents passage through the old state, so its summand remains disjoint from $K_{\alpha, \beta}$. The same decompositions therefore apply. $\square$

Proposition 37.2 supplies the q-weak equivalence needed in [18, Theorem 9.2] and at successor stages of [18, Theorem 9.3].

The resulting factorization remains

$$\text{cat}_{\mathcal{M}}(f) = wi, \quad i \in \text{cell}(\mathcal{T}^{\text{cof}}), \quad w \text{ a q-weak equivalence,}$$

for every globular subdivision $f : X \rightarrow Y$ between cellular $\mathcal{M}$ -multipointed d -spaces. The class $\mathcal{T}^{\text{cof}}$ is the class of generating subdivisions fixed in [18, Definition 8.3]. The consequences for dihomotopy equivalence, branching and merging homology, and underlying homotopy types are unchanged.

38. THE MOORE-FLOW RESULTS USED IN SUBDIVISION

The subdivision argument uses categorization and the quotient to traces. Part 4 supplies the required results with their original hypotheses.

- (1) The derived equivalence and preservation of q-weak equivalences in [18, Theorem 4.2(1)–(2)] cite [17, Theorem 15]. The replacements are Corollary 24.4, Propositions 26.1 and 26.3, and the derived equivalence of Section 26.

- (2) The globe attachments in [18, Proposition 4.4 and Theorem 4.5] use [19, Proposition 7.1]; apply Theorem 24.2 as in Section 33. Adding discrete states gives a coproduct with the pathless space on those states.
- (3) Preservation of cellular colimits at limit ordinals by $\mathbb{M}^{\mathcal{M}}$, cited in [18, Proposition 4.1], is the valid [17, Theorem 5]. Proposition 24.3 supplies preservation by $\text{cat}_{\mathcal{M}}$.
- (4) The path quotient in [18, Proposition 5.7 and Theorem 5.8] uses [17, Theorem 16]. Theorem 26.2 gives its homotopy equivalence, m-cofibrant source, and q-cofibrant target, and proves that it is a trivial Hurewicz fibration. Proposition 31.1 supplies the isolated identity at equal endpoints.

Thus [18, Theorems 9.2–9.3] still allow stopped attaching paths and execution loops.

39. THE INTRODUCTORY COUNTEREXAMPLE AND REMAINING CORRECTIONS

For the introductory counterexample in [18], use $i : B \rightarrow A$ from Proposition 34.1 in place of [19, Proposition 10.1]. Write Q_{Σ} for localization of cellular $\mathcal{M}$ -multipointed d -spaces at globular subdivisions and Q_{dih} for localization of flows at dihomotopy equivalences. Since $\overrightarrow{\text{Sp}}$ sends globular subdivisions to isomorphisms [19, Theorem 9.3],

$$Q_{\Sigma}(i) \text{ invertible} \implies \overrightarrow{\text{Sp}}(i) \text{ invertible} \implies \overrightarrow{\text{NT}}(B) \cong \overrightarrow{\text{NT}}(A),$$

contrary to Proposition 34.1. On the other hand, Proposition 26.1 gives

$$\text{cat}_{\mathcal{M}}(i) \text{ a q-weak equivalence} \implies Q_{\text{dih}} \text{cat}_{\mathcal{M}}(i) \text{ invertible.}$$

Thus the functor induced by categorization between these localizations does not reflect isomorphisms and cannot be an equivalence.

For completeness, [18, Proposition 8.8] is quantified over $A^0 \times A^0$. The proof of [18, Theorem 7.10] uses that Y^0 is closed and discrete; this follows from the cellular topology statement in Proposition 21.1. The unused cardinal bound following [18, Theorem 7.2] should be deleted. These corrections, together with Sections 35–38, supply the assertions used in the main proofs.

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