Proof Construction and Non-Commutativity: a Cluster Calculus

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ABSTRACT

An increasing interest is directed at the extension of the "proof search as computation" paradigm, already successfully applied to Linear Logic, to a logic that is not only resource-aware but also order-sensitive. This paper is a contribution to proof search in Non-Commutative Logic.

Our key result is to give a simple method for propagating the order structure during proof search. Such a method is general, in that it can be applied to n-ary connectives. This enables us to define a *cluster calculus,* which analyses clusters of synchronous and of asynchronous connectives in a single step, with a single n-ary rule.

Keywords

Linear Logic, Non-Commutative Logic, Logic Programming, Proof Search, Focalization.

1. INTRODUCTION

"Proof search as computation" is a programming paradigm (cf. [9]) that can fit features such as coordinations of entities, interaction, non-determinism, features that are all gaining importance in computer science. One can think of operating systems, flight schedulers, the Internet... These examples are all situations where a notion of *order* or *priority* arises in a natural way. For this reason, while the logic programming interpretation of formulas-as-instructions and proofs-as-states has been successfully applied to Linear Logic, in recent years an increasing amount of interest has been directed to proof search in a logical system that is not only resource-sensitive but also order-aware (cf. [4, 8, 6, 12], and recent work by W.O'Hearn and D.Pym).

Let us mention some of the different insights motivating research in this direction: dealing with situations where computation is naturally subject to ordering constraints (cf. [12], which gives program examples dealing with natural language parsing and sorting), encoding of concurrency problems [5], managing coordination with priority, e.g. in the access to resources and information (recent work by Andreoli).

Most of this research is carried out in either of two main frameworks that incorporate in a conservative way the notion of order into Linear Logic: Non-Commutative Logic (NL) by Abrusci and Ruet [1, 13], and Ordered Linear Logic (OLL) by Polakow and Pfenning [11]. This paper is a contribution to proof search in Non-Commutative Logic.

NL is characterized by a notion of sequent enriched with a structure of order, namely an order variety [13] (cf. Section 3 for a review). Proof search in this setting has not yet a satisfactory status, in particular concerning the effective managing of context splitting, but important progresses have been made recently with the extension of *focalization* to NL.

Focalization [2] is a central tool for proof search in (classical) Linear Logic, for it enables us to reduce the non-determinism of the choices to be performed. Focalization relies on a distinction between two homogeneous families of connectives: positive, or *synchronous* (timing is relevant), and negative, or *asynchronous* (timing is irrelevant). The very meaning of focalization is that connectives of the same family can be grouped and dealt with as a single connective.

Dealing with focalization in NL raises several new questions. As developed in [8] (cf. Section 3.2), all questions play on the "bottom-up" application of positive rules and may be reduced to: (1.) Under which conditions is a partition of the context compatible with the order on the given conclusion (*coherence conditions*)? (2.) Once given a partition, *which order* is associated to the premises? (3.) Since there are, in general, several possible solutions to (2), which solutions are *optimal*? We address these same questions with a new approach, which allows us to extend the results.

Our key result is to give a simple way to propagate the order structure when decomposing a non-commutative conjunction (Next). Such a solution is optimal. On the binary connectives, our calculus is equivalent to the one by Maieli and Ruet. We then develop the results in two ways:

– we define a *cluster calculus* where packs of all synchronous (Tensor, Next) connectives or all asynchronous (Par, Sequential) connectives are decomposed "at once", as a single n-ary rule (Section 6);

– we are (eventually) able to reduce both the space and the conditions to be tested in order to build an NL proof (Section 8).

The ability to deal with clusters of homogeneous connectives is a necessary step towards the extension to NL of *constraint based techniques* of proof search (cf. [3]). This is a method that allows us to deal with contexts splitting in Linear Logic. Implementations will make the object of future investigation.

Ludics

Our results issue from a study directed at the extension of Ludics [7] to encompass Non-Commutative Logic. On one side, this confirms the intuition that the dynamic underlying Ludics is an interactive proof-search. On the other side, while this work is self-standing, it is an interesting fact that the cluster calculus comes with a semantic counterpart.

Note

This work is carried out in multiplicative NL, where all difficulties specific to the non-commutative setting are concentrated. The extension of the framework to the additives ($\oplus, \&$) is immediate, and brings no surprise. As for the exponentials – for which the focalized approach has never been as satisfactory as for the multiplicative-additive fragment – we prefer to postpone the discussion to future work. This is motivated by recent discoveries in Ludics ([7]), based on a finer decomposition of the exponentials: in contrast with the traditional approach, "Bang" turns out to be (essentially) negative, whereas "Why Not" positive.

2. FOCALIZATION

At the basis of focalization [2] is the distinction of Linear Logic connectives into two families: positives ($\otimes, \oplus$) and negatives ($\parr, \&$). These two families correspond to a distinction of the non-determinism involved by the timing of the choices during proof construction: the timing may or may not be relevant.

Negative connectives carry *asynchronous* non-determinism: their rules are reversible, and in the proof search they are to be performed as soon as possible.

Positive connectives introduce *synchronous* (true) non-determinism. Such connectives enjoy the *focalization property*: given a sequent of positive formulas which is the conclusion of a certain proof, there exists a formula, the *"focus,"* that may be selected as principal and entirely decomposed up to its first negative subformulas.

In a focalized sequent calculus negative rules are applied immediately, and positive rules, once chosen a focus, are persistently applied up to their negative subformulas. This is exactly the sense of our *cluster calculus.* Rather than keeping on applying positive rules on the same focus in a threat, we can do it as a single step.

3. NON-COMMUTATIVE LOGIC AND ORDER VARIETIES

Non-Commutative Logic is a conservative extension of Linear Logic that unifies commutative and cyclic linear logic. To the formulas of the sequent is associated a structure of order that is an *order variety* [13].

DEFINITION 3.1 (ORDER VARIETY). *Let Λ be a set. An* order variety *on Λ is a ternary relation $\boldsymbol{\alpha}$ which is :*

cyclic: $\forall x, y, z \in \Lambda, \boldsymbol{\alpha}(x, y, z) \Rightarrow \boldsymbol{\alpha}(y, z, x)$;

anti-reflexive: $\forall x, y \in \Lambda, \neg\boldsymbol{\alpha}(x, x, y)$;

transitive : $\forall x, y, z, t \in \Lambda, \boldsymbol{\alpha}(x, y, z)$ *and* $\boldsymbol{\alpha}(y, x, t) \Rightarrow \boldsymbol{\alpha}(x, t, z)$;

spreading: $\forall x, y, z, t \in \Lambda, \boldsymbol{\alpha}(x, y, z) \Rightarrow \begin{cases} \boldsymbol{\alpha}(t, y, z) \text{ or} \\ \boldsymbol{\alpha}(x, t, z) \text{ or} \\ \boldsymbol{\alpha}(x, y, t) \end{cases}$

The main property of an order variety is that it can be presented as a (strict) partial order as soon as we take out a point.One can think of a circle, that becomes a segment as soon as one fix a point (a point of view), and in fact the notion of order variety corresponds to the notion of partial order in the same way as the oriented circle corresponds to the oriented segment. The interest of this operation is that there is no privileged point of view. That is, an order variety is at the same time an order with respect to any of its point. This is what makes the order variety a very synthetic tool.

Next two definitions express the one-to-one correspondence between order varieties and orders.

DEFINITION 3.2. *Let $\boldsymbol{\alpha}$ be an order variety on Λ and $x \in \Lambda$. Define the binary relation $\boldsymbol{\alpha}_x$ on $\Lambda \setminus \{x\}$ by : $\boldsymbol{\alpha}_x(y, z)$ iff $\boldsymbol{\alpha}(x, y, z)$.*

PROPOSITION 3.3. *If $\boldsymbol{\alpha}$ is an order variety on Λ and $x \in \Lambda$, then $\boldsymbol{\alpha}_x$ is a strict partial order on $\Lambda \setminus \{x\}$. It is called the order* induced by $\boldsymbol{\alpha}$ and x.

Conversely, each strict partial order defines an order variety on the same domain:

DEFINITION 3.4. *Let $\boldsymbol{\omega} = (\Lambda, <)$ be a strict partial order on Λ. Define the ternary relation $\overline{\omega}$ on Λ by : $\overline{\omega}(x, y, z)$ iff for a cyclic permutation (x', y', z') of (x, y, z), we have $(x' < y' < z')$ or $(x' < y'$ and z' is comparable with neither x' nor $y')$.*

PROPOSITION 3.5. *If $(\Lambda, \boldsymbol{\omega})$ is a strict partial order, then $\overline{\omega}$ is an order variety on Λ.*

The two operations commute. To make this more precise, we need to use an other essential feature of order varieties, the "gluing". First recall the notion of serial and parallel composition of orders.

Definition 3.6 (Serial/parallel composition). *Let $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ be partial orders on disjoint sets Λ_1, Λ_2. Their* serial *and* parallel *composition $\boldsymbol{\omega}_1 < \boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_1 \parallel \boldsymbol{\omega}_2$ respectively are the two partial orders on $\Lambda_1 \cup \Lambda_2$ defined by:*
$(\boldsymbol{\omega}_1 < \boldsymbol{\omega}_2)(x, y)$ iff either $\boldsymbol{\omega}_1(x, y)$ or $\boldsymbol{\omega}_2(x, y)$ or ($x \in \Lambda_1$ and $y \in \Lambda_2$);
$(\boldsymbol{\omega}_1 \parallel \boldsymbol{\omega}_2)(x, y)$ iff either $\boldsymbol{\omega}_1(x, y)$ or $\boldsymbol{\omega}_2(x, y)$.

Given two orders, to build an order variety out of either their serial composition or their parallel composition yields to the same result:

Proposition 3.7. *If $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$ are two partial orders on disjoint sets, then the following order varieties are equal: $\overline{\boldsymbol{\omega} < \boldsymbol{\tau}} = \overline{\boldsymbol{\omega} \parallel \boldsymbol{\tau}} = \overline{\boldsymbol{\tau} < \boldsymbol{\omega}}$*

This naturally leads to the definition of a more general operation, named gluing:

Definition 3.8 (Gluing). *The order variety of the previous Proposition is indicated by $\boldsymbol{\omega} * \boldsymbol{\tau}$:*

$$\boldsymbol{\omega} * \boldsymbol{\tau} \;=\; \overline{\boldsymbol{\omega} < \boldsymbol{\tau}} \;=\; \overline{\boldsymbol{\omega} \parallel \boldsymbol{\tau}} \;=\; \overline{\boldsymbol{\tau} < \boldsymbol{\omega}}$$

Then we have

Proposition 3.9. *Let $\boldsymbol{\alpha}$ be an order variety on a set Λ, $x \in \Lambda$ and $\boldsymbol{\omega}$ a (strict) partial order on $\Lambda \setminus \{x\}$. Then :*

$$\boldsymbol{\alpha}_x * x \;=\; \boldsymbol{\alpha} \quad \textit{and} \quad (\boldsymbol{\omega} * x)_x \;=\; \boldsymbol{\omega}$$

where by x we mean the unique strict partial order on $\{x\}$.

In the next Section we will review the sequent calculus for NL, which is based on order varieties, and more precisely on *series-parallel* order varieties.

Definition 3.10 (Series-Parallel Order Variety). *Series-parallel order varieties are those order vareties that can be presented by a series-parallel order.*

We recall that the class of *series-parallel orders* is the least class of finite orders containing empty orders on singletons and closed under serial and parallel composition (cf. [10] for a survey on the subject). A fact that we will use to prove our main Theorem is that series-parallel orders admit a negative characterization as those orders on finite sets whose restriction to any 4-elements subset $\{a, b, c, d\}$ is different from the order $\mathbf{P_4} = \{(a, b), (c, b), (c, d)\}$.

3.1 NL calculus

A Linear Logic sequent is a structure $\vdash \Lambda$, where Λ is a multiset of formula occurrences. In Non-Commutative Logic a sequent has the form $\vdash \Lambda \ \langle \boldsymbol{\alpha} \rangle$, where $\boldsymbol{\alpha}$ is a *series-parallel order variety* on the support Λ ($|\boldsymbol{\alpha}| = \Lambda$). Since $\boldsymbol{\alpha}$ can be presented as an order w.r.t. any of its points, it is always possible to fix a point $\xi \in \Lambda$ and present $\boldsymbol{\alpha}$ as $\boldsymbol{\alpha}_\xi * \xi$ (cf. Proposition 3.9).

All along the paper, when it is not ambiguous we will not explicitly mention the support, writing simply $\vdash \boldsymbol{\alpha}$ (resp. $\vdash \boldsymbol{\alpha}_\xi * \xi$) for $\vdash \Lambda \ \langle \boldsymbol{\alpha} \rangle$ (resp. $\vdash \Lambda \ \langle \boldsymbol{\alpha}_\xi * \xi \rangle$).

The calculus for Non-Commutative Logic (NL) [13] is given in the Appendix. Given the orders $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ on the premises, the Tensor rule composes them in parallel:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 \parallel \boldsymbol{\tau}_2,$$

where $\boldsymbol{\tau}$ is the order on the conclusion. The Next rule instead composes the two orders serially:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_2 < \boldsymbol{\tau}_1.$$

The two special cases of empty and total order varieties respectively correspond to commutative and cyclic linear logic. What allows the two levels – commutative and non-commutative – to interleave is the notion of entropy.

Definition 3.11 (Entropy). *$\trianglelefteq$ is the relation between partial orders on the same set defined by :*

$$\boldsymbol{\omega} \trianglelefteq \boldsymbol{\sigma} \quad \textit{iff} \quad \boldsymbol{\omega} \subseteq \boldsymbol{\sigma} \quad \textit{and} \quad \overline{\boldsymbol{\omega}} \subseteq \overline{\boldsymbol{\sigma}}$$

Note that entropy does not correspond to inclusion of orders, but it does correspond to inclusion of order varieties:

Proposition 3.12. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be order varieties on Λ, and $x \in \Lambda$. Then*

$$\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x \quad \textit{iff} \quad \boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$$

Intuitively, entropy corresponds to a loss of information on the order. In the case of series-parallel orders, it is performed by replacing some serial ($<$) with parallel compositions ($\parallel$). For instance:

$$((a < b) \parallel c) \trianglelefteq ((a < b) < c)$$

3.2 Focalized NL

In the original NL calculus, entropy was given as a structural rule. Read bottom-up, the entropy rule enables to *increase the order* underlying a sequent, and such an operation is highly non-deterministic. For this reason, entropy represents the main difficulty for the definition of a focalized sequent calculus for NL.

In the focalized calculus designed by Maieli and Ruet this problem is addressed by making entropy implicit and pushing it to the $\odot$ rule. The Next rule is therefore expressed as

$$\frac{\vdash \boldsymbol{\tau}_1 * \xi_1 \quad \vdash \boldsymbol{\tau}_2 * \xi_2}{\vdash \boldsymbol{\tau} * \xi_1 \odot \xi_2} \ \odot$$

where

$$\textbf{(Entropy equation)} \qquad \boldsymbol{\tau} \trianglelefteq (\boldsymbol{\tau}_2 < \boldsymbol{\tau}_1)$$

This is the formulation we will follow from now on.

This Next Rule (read bottom-up) brings to the proof search several new critical points, developed in [8]:
(1.) *To split* the context in such a way that the equation $\boldsymbol{\tau} \trianglelefteq (\boldsymbol{\tau}_2 < \boldsymbol{\tau}_1)$ admits a solution.
(2.) To *calculate* the orders $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ associated to the premises.

Also, in order to commit ourselves as little as possible in the proof search, the "quantity" of entropy should be minimized. Hence:
(3.) *Optimal solutions* $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2$. For any pair of orders $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ on Λ_1, Λ_2 respectively, such that $\boldsymbol{\tau} \trianglelefteq (\boldsymbol{\tau}_2 < \boldsymbol{\tau}_1)$, we have $\boldsymbol{\mu}_i \trianglelefteq \boldsymbol{\tau}_i$.

The solutions given in [8] are of combinatorial nature, and exploit the representation of the orders as special binary trees (so, in particular, they do not easily generalize to n-ary connectives). In the next section we intend to address the same questions with a more synthetic approach.

3.3 Conventions

All along the paper, when we speak of orders we always mean *strict partial orders*.

We will indicate by ξ the focus of a rule (the principal formula); we indicate by ξ_i its subformulas (the secondary formulas). If the context of ξ is Λ, then Λ_i is the context of ξ_i. If a context formula is annotated with an index i, then it belongs to Λ_i, e.g. $x_i \in \Lambda_i$.

We will always indicate by $\boldsymbol{\alpha}$ an order variety. We will make frequent use of the operations defined in Section 3. In particular, note that, by Definition 3.2, $\boldsymbol{\alpha}_\xi$ is a partial order (the order induced by $\boldsymbol{\alpha}$ fixing ξ). By Definition 3.8, if $\boldsymbol{\tau}$ is an order, $\boldsymbol{\tau} * \xi$ is an order variety, the one obtained by gluing.

Because of the definition of order variety, all the triples are taken modulo cyclic permutation. We also adopt the following notations: $(\Lambda_1, \Lambda_2, \Lambda_3) = \{(x_1, x_2, x_3) : x_i \in \Lambda_i\}$; $\xi_I = \bigcup_{i \in I}\{\xi_i\}$; $\boldsymbol{\alpha}[\xi_I/\xi] = \bigcup_{i \in I} \boldsymbol{\alpha}[\xi_i/\xi]$.

4. BINARY CALCULUS: THE NEXT RULE

We start our investigation from the binary calculus, and in particular from the study of the Next rule, which gathers all difficulties specific to the non-commutative setting.

A simple example is enough to realize that NL imposes constraints on the possible ways of splitting the context:

$$\dfrac{\dfrac{\vdash a,\ ?\quad \vdash b,\ ?}{\vdash a^\perp < b^\perp * a \otimes b}\ \otimes}{\vdash a^\perp \nabla b^\perp, a \otimes b}\ \nabla$$

It is not possible to split the context of $a \otimes b$, because the Tensor rule would give $a^\perp \parallel b^\perp$ rather than $a^\perp < b^\perp$. The compatibility of the context splitting with the order is the question addressed by the *"coherence conditions"*. To this question is strictly associated another one: how to propagate the order structure from the conclusion to the premises. If this kind of questions are clear for Tensor, it is much less so for Next, because of entropy.

We give a few examples to make clear what we want to achieve, and what the problems are.

4.1 Motivating examples

Let us consider a generic positive rule, where ξ can be either $\xi_1 \odot \xi_2$ or $\xi_1 \otimes \xi_2$, and where $|\boldsymbol{\tau}| = \Lambda$ and $|\boldsymbol{\tau}_i| = \Lambda_i$

$$\frac{\vdash \boldsymbol{\tau}_1 * \xi_1 \quad \vdash \boldsymbol{\tau}_2 * \xi_2}{\vdash \boldsymbol{\tau} * \xi}$$

The rule for Tensor just asks that

"*Λ_1, Λ_2 is a bi-partition of Λ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_1 \parallel \boldsymbol{\tau}_2$.*"

We would like to have something so clear and simple for Next too.

We could be tempted to ask:

"*Λ_1, Λ_2 is a bi-partition of Λ such that $x_1 \in \Lambda_1, x_2 \in \Lambda_2 \Rightarrow x_2 < x_1$ in $\boldsymbol{\tau}$.*"

But such a condition is too strong. In fact, it does not take into account entropy. Let us see this with an example.
Example 1.

$$\frac{\vdash b < c * \xi_1 \quad \vdash a, \xi_2}{\vdash (a < b) \parallel c * \xi_1 \odot \xi_2}\ \odot$$

This derivation is sound since $(a < b) \parallel c$ can be obtained as entropy of the serial composition of the orders indicated on the premises, which is $(a) < (b < c) = (a < b) < (c)$, where by entropy we weaken the most external $<$ into $\parallel$.

The natural condition to ask for would be:

(Coh.1) "*$x_1 \in \Lambda_1, x_2 \in \Lambda_2 \Rightarrow x_1 \not< x_2$ in $\boldsymbol{\tau}$.*"

But this condition is not sufficient to guarantee that we can associate an opportune order variety on the premises [1]. Let us examine the following case:
Example 2.

$$\frac{\vdash \xi_1, y, y' \quad \vdash \xi_2, x, x'}{\vdash (x < y) \parallel (x' < y') * \xi}\ \odot$$

(Coh.1) is respected, but studying the various possibilities, one realizes that there is no way of giving an order on the premises in such a way that the order on the conclusion is obtained as entropy of the serial composition. This is exactly the situation that in [8] is explicitly prohibited by the second condition of "Admissibility."

One more example will illustrate the importance of the propagation of the order.
Example 3.

$$\frac{\vdash b \parallel c * \xi_1 \quad \vdash a, \xi_2}{\vdash (a < b) \parallel c * \xi_1 \odot \xi_2}\ \odot$$

[1] We will discuss this issue again in Section 8, once we have more tools.

The order on the conclusion satisfies (Coh.1), but we have $(a < b) \parallel c \not\trianglelefteq a < (b \parallel c)$.

Next section refines the condition of "coherent partition." Once chosen a coherent partition, the order associated to each premise is well determined, and we give an immediate way to produce it.

4.2 The Next rule

Our approach is based on the following construction. We will prove (Theorem 4.5) that the binary relation so defined is a partial order in the cases we are interested in.

DEFINITION 4.1 ($\boldsymbol{\mu}$). *Let $\boldsymbol{\alpha}$ be an order variety with support $|\boldsymbol{\alpha}| = \{\Lambda, \xi\}$. Given $\Lambda_i \subseteq \Lambda$, we define a binary relation on Λ_i as:*

$$\boxed{\boldsymbol{\mu}_{\Lambda_i}(a, b) \quad \textit{iff} \quad \boldsymbol{\alpha}(a, b, z), \textit{for some } z \notin \Lambda_i}$$

REMARK 4.2. *The definition of $\boldsymbol{\mu}_{\Lambda_i}$ is better understood as a generalization of Definition 3.2 ($\boldsymbol{\mu}_\Lambda$ corresponding to $\boldsymbol{\alpha}_\xi$). In fact the definition of the order $\boldsymbol{\alpha}_\xi$ can be rephrased as:*

$$\boldsymbol{\alpha}_\xi(a, b) \quad \textit{iff} \quad \boldsymbol{\alpha}(z, a, b), \textit{for some } z \notin |\boldsymbol{\alpha}_\xi| = \Lambda$$

We can now define our Next rule (to be read bottom-up) as follows:

$$\boxed{\frac{\vdash \boldsymbol{\mu}_{\Lambda_1} * \xi_1 \quad \vdash \boldsymbol{\mu}_{\Lambda_2} * \xi_2}{\vdash \xi_1 \odot \xi_2, \Lambda \ \langle \boldsymbol{\alpha} \rangle} \odot}$$

where:

– $|\boldsymbol{\mu}_{\Lambda_1}| = \Lambda_i$ (when the partition is obvious, we will often write $\boldsymbol{\mu}_i$ for $\boldsymbol{\mu}_{\Lambda_i}$);

– Λ_1, Λ_2 is a bi-partition of Λ that respects the following *coherence conditions:*

DEFINITION 4.3 (COHERENCE). *Let Λ_1, Λ_2 be a bi-partition of Λ. Such a partition is* coherent on $\boldsymbol{\alpha}$ with respect to *$\xi = \xi_1 \odot \xi_2$ if it satisfies the following conditions:*

(Coh.1) *If $x_1 \in \Lambda_1, x_2 \in \Lambda_2$, then $\neg\boldsymbol{\alpha}(x_1, x_2, \xi)$.*

(Coh.2) *If $a, b \in \Lambda_i$ and $z, z' \notin \Lambda_i$, then $\boldsymbol{\alpha}(a, b, z) \Rightarrow \neg\boldsymbol{\alpha}(b, a, z')$.*

The above definition is better understood when translated into the following terms:

REMARK 4.4. *(Coh.1) and (Coh.2) are equivalent to:*

(Coh.1) $x_1 \not< x_2$ *in* $\boldsymbol{\alpha}_\xi$;

(Coh.2) $\boldsymbol{\mu}_{\Lambda_i}(a, b) \Rightarrow \neg\boldsymbol{\mu}_{\Lambda_i}(b, a)$.

Section 4.3 will prove that

THEOREM 4.5 (INDUCED ORDERS). *Given the above definition of the Next rule, if the partition is coherent then the binary relations $\boldsymbol{\mu}_i$ induced on the premises are partial orders.*

4.2.1 Examples

Let us make this approach work on our previous examples.

Example 1.

$$\frac{\vdash b, c, \xi_1 \ \langle \boldsymbol{\mu}_1 * \xi_1 \rangle \quad \vdash a, \xi_2}{\vdash (a < b) \parallel c * \xi_1 \odot \xi_2} \odot$$

since $\boldsymbol{\alpha} = \{(a, b, c), (a, b, \xi)\}$, thus $\boldsymbol{\mu}_1 = \{(b, c)\}$. That is the order variety on the first premise is $(b < c) * \xi_1$, as it should be.

Example 2. Let us now try the configuration

$$\frac{\vdash y, y', \xi_1 \quad \vdash x, x', \xi_2}{\vdash (x < y) \parallel (x' < y') * \xi_1 \odot \xi_2} \odot$$

The order variety associated to the conclusion is

$$\{(\xi xy), (x'xy), (y'xy), (\xi x'y'), (xx'y'), (yx'y')\}$$

To have $(x'xy)$ and $(xx'y')$ is enough to reject the candidate partition $\Lambda_2 = \{x, x'\}$, $\Lambda_1 = \{y, y'\}$. Any other partition is fine w.r.t. coherence.

4.2.2 Remarks

We give two other ways to characterize $\boldsymbol{\mu}_i$, which may help intuition:

REMARK 4.6. *Let $\xi = \xi_1 \odot \xi_2$. The definition of $\boldsymbol{\mu}_i$ is equivalent to the following two:*

(i): $\boldsymbol{\mu}_i = \bigcup_{z \notin \Lambda_i} \boldsymbol{\alpha}_z \lceil \Lambda_i$

(ii): $\boldsymbol{\mu}_1 = (\boldsymbol{\alpha}_1)_{\xi_1}$ *and* $\boldsymbol{\mu}_2 = (\boldsymbol{\alpha}_2)_{\xi_2}$

where
$\boldsymbol{\alpha}_1 = \left(\boldsymbol{\alpha} \bigcap (\Lambda_1, \Lambda_1, \xi \cup \Lambda_2)\right)[\xi_1 / \xi \cup \Lambda_2]$,
$\boldsymbol{\alpha}_2 = \left(\boldsymbol{\alpha} \bigcap (\Lambda_2, \Lambda_2, \xi \cup \Lambda_1)\right)[\xi_2 / \xi \cup \Lambda_1]$,

and where $[\xi_i / \Lambda]$ indicates that all $z \in \Lambda$ have been renamed as ξ_i.

The interest of (i) is to show that μ_i is in fact a union (of orders), whereas the construction by Maieli and Ruet [8], on the contrary, is based on an intersection (the *wedge*).

The meaning of (ii) is that in fact we extract $\boldsymbol{\mu}_1$ (resp. $\boldsymbol{\mu}_2$) from $\boldsymbol{\alpha}$ by identifying Λ_2 (resp. Λ_1) with ξ. We are performing a quotient.

We conclude this section with a fact that is of interest in view of the implementation: when checking for coherence, it is enough to check just one of the $i \in \{1, 2\}$:

FACT 4.7. *To check (Coh.2) it is enough to check it for one of the* $i \in \{1,2\}$*: if one of the* $\boldsymbol{\mu}_i (i \in \{1,2\})$ *is anti-symmetric, so is the other one.*

PROOF. Let us fix an i, and assume that $\boldsymbol{\alpha}(a, a', z)$ and $\boldsymbol{\alpha}(a', a, z')$, for $a, a' \in \Lambda_i$ and $z \neq z' \notin \Lambda_i$. Then it follows by transitivity of $\boldsymbol{\alpha}$ that $\boldsymbol{\alpha}(a, z', z)$ and $\boldsymbol{\alpha}(a', z, z')$. $\square$

4.3 Proof of Theorem 4.5

We want to prove that if the partition is coherent then any $\boldsymbol{\mu}_i$ is a partial order, and thus $\boldsymbol{\mu}_i * \xi_i$ is an order variety. It is immediate that (Coh.2) implies the anti-symmetry of $\boldsymbol{\mu}_i$:

PROPOSITION 4.8 (ANTI-SYMMETRY). *(Coh.2) implies, for any* $i \in \{1,2\}$*:*

$$\boldsymbol{\mu}_i(a,b) \Rightarrow \neg \boldsymbol{\mu}_i(b,a)$$

The delicate point is to prove transitivity. Note that a priori there is no reason for $\boldsymbol{\mu}_i(a,b)$ and $\boldsymbol{\mu}_i(b,c)$ implying $\boldsymbol{\mu}_i(a,c)$. Typically, $\boldsymbol{\mu}_i(a,b)$ could come from $\boldsymbol{\alpha}(a,b,z)$ and $\boldsymbol{\mu}_i(b,c)$ from $\boldsymbol{\alpha}(b,c,z')$, with $z \neq z'$ and no (a,c,z'') be in $\boldsymbol{\alpha}$. We can build an example:

Example. Let us take $\Lambda = a,b,c,z$, an order variety $\boldsymbol{\alpha} = \{(\xi,a,b),(z,b,c),(\xi,a,z)(\xi,c,z)\}$ and as partition $\Lambda_1 = \{z\}$, $\Lambda_2 = \{a,b,c\}$. The result is that $\boldsymbol{\mu}_2$ is $\{(a,b),(b,c)\}$, which is not closed under transitivity.

But in fact the above example is a bad one, since we are only concerned with series-parellel order varieties, and the above order variety is not. This can be shown to be true in general. It is the object of the next two propositions.

LEMMA 4.9. *Let* $\boldsymbol{\alpha}$ *be a series-parallel order variety on* Λ, ξ*; if* Λ_1, Λ_2 *is a coherent partition of* Λ *w.r.t.* ξ*, if* $z \in \Lambda_1$ *and* $a,b,c \in \Lambda_2$*, then* $\boldsymbol{\alpha}(a,b,\xi)$ *and* $\boldsymbol{\alpha}(b,c,z)$ *imply either* $\boldsymbol{\alpha}(a,c,\xi)$ *or* $\boldsymbol{\alpha}(a,c,z)$.

PROOF. Let us examine the possible spreading (i) of z on $\boldsymbol{\alpha}(\xi,a,b)$, and (ii) of ξ on $\boldsymbol{\alpha}(b,c,z)$. (i): Discharging (ξ,z,b), which contradicts coherence, we have either $\boldsymbol{\alpha}(z,a,b)$ or $\boldsymbol{\alpha}(\xi,a,z)$, where the first one allows us to conclude $\boldsymbol{\alpha}(a,c,z)$, by transitivity with $\boldsymbol{\alpha}(b,c,z)$. (ii): We have either $\boldsymbol{\alpha}(b,c,\xi)$ or $\boldsymbol{\alpha}(\xi,c,z)$, where, again, the first one is enough to conclude $\boldsymbol{\alpha}(a,c,\xi)$.

Let us suppose the worst possibility in both cases. Thus we have $\{(a,z),(c,z),(a,b)\} \subseteq \boldsymbol{\alpha}_\xi$. Our aim is to show that either we have transitivity, or we are able to reach a contradiction, producing a $\mathbf{P_4}$. Let us examine the restriction $\boldsymbol{\alpha}_\xi \lceil a,b,c,z$. Of the twelve possible combinations, the pairs of type (z,Λ_2) are discharged by (Coh.1). By (Coh.2) we can also discharge $(b,a),(c,b)$ and (c,a), the last one because $\boldsymbol{\alpha}(\xi,c,a)$ with $\boldsymbol{\alpha}(c,\xi,b)$ gives $\boldsymbol{\alpha}(\xi,b,a)$. We are left with $(a,c),(b,c),(b,z),(a,b),(a,z),(c,z)$. Any of the first three alone yields to the result: the first one directly, the second and third ones by transitivity of $\boldsymbol{\alpha}$. In fact $\boldsymbol{\alpha}(a,b,\xi)\boldsymbol{\alpha}(b,c,\xi) \rightarrow \boldsymbol{\alpha}(a,c,\xi)$ and $\boldsymbol{\alpha}(b,z,\xi)\boldsymbol{\alpha}(z,b,c) \rightarrow \boldsymbol{\alpha}(b,c,\xi)$. If we suppose that $\neg(a,c), \neg(b,c)$ and $\neg(b,z)$, we conclude $\boldsymbol{\alpha}_\xi \lceil a,b,c,z \subseteq \{(a,z),(c,z),(a,b)\}$. This gives us a contradiction: since $\boldsymbol{\alpha}$ is series-parallel, the restriction of $\boldsymbol{\alpha}_\xi$ to a,b,c,z cannot be $\{(c,z),(a,z),(a,b)\}$, which is a $\mathbf{P_4}$. $\square$

This Lemma, with its obvious symmetric forms, allows us to prove

PROPOSITION 4.10 (TRANSITIVITY). *Given a coherent partition of* Λ, $\boldsymbol{\mu}_{\Lambda_i} (i \in \{1,2\})$ *is transitive.*

PROOF. Let $i \neq j \in \{1,2\}$, $a,b,c \in \Lambda_i$, $z \in \Lambda_j$. Let us suppose to have $\boldsymbol{\mu}_i(a,b)$ and $\boldsymbol{\mu}_i(b,c)$. This means that we have $\boldsymbol{\alpha}(a,b,z')$ and $\boldsymbol{\alpha}(b,c,z'')$, where $z', z'' \notin \Lambda_i$. We want to show that $\boldsymbol{\mu}_i(a,c)$.

If $z' = z''$ the result is trivial, by transitivity of $\boldsymbol{\alpha}$. Otherwise, let us check the possible spreading of c on $\boldsymbol{\alpha}(a,b,z')$. Since (c,b,z') is incoherent, we have either (a,c,z'), hence the result, or (a,b,c). In the same way, the possible spreading of a on $\boldsymbol{\alpha}(b,c,z'')$ give us (a,c,z''), hence the result, or (b,c,a). If we have (a,b,c), we can reduce the problem to the case treated in the previous Lemma. In fact, the spreading of ξ on $\boldsymbol{\alpha}(a,b,c)$ assures us either $\boldsymbol{\alpha}(\xi,b,c)$ or $\boldsymbol{\alpha}(a,b,\xi)$, the third case being excluded by coherence. $\square$

4.4 Computation of the order varieties

One could expect that

$$\boldsymbol{\mu}_i * \xi_i = \begin{array}{l}\{(a,b,\xi_i) : \boldsymbol{\alpha}(a,b,z), \text{for some } z \notin \Lambda_i\} \cup \\ \{(a,b,c) : \boldsymbol{\alpha}(a,b,c)\}\end{array}$$

Note that this is not true, as the following counterexample shows:

Counterexample.

$$\frac{\vdash \xi_1 * y \quad \vdash \xi_2 * \boldsymbol{\mu}_{a,b,c}}{\vdash \xi, y, a, b, c \ \langle (\xi,b,c),(\xi,y,c),(\xi,a,c),(y,a,c),(\xi,y,a),(y,a,b)\rangle} \odot$$

where we have that $\boldsymbol{\mu}_{a,b,c} = \{(a,b),(b,c),(a,c)\}$ and thus $\boldsymbol{\mu}_{a,b,c} * \xi_2 = \{(\xi,a,b),(\xi,b,c),(\xi,a,c),(a,b,c)\}$.

Nonetheless, when we calculate $\boldsymbol{\mu}_i$ from $\boldsymbol{\alpha}$ we forget some information, which we then compute again when gluing with ξ_i. Indeed, it is possible to retrieve $\boldsymbol{\mu}_i * \xi_i$ directely from $\boldsymbol{\alpha}$, through the following characterization:

PROPOSITION 4.11 (CHARACTERIZATION OF $\boldsymbol{\mu}_i * \xi_i$). *We have* $\boldsymbol{\mu}_i * \xi_i = \mathcal{A}_i \bigcup \mathcal{B}_i \bigcup \mathcal{C}_i$*, where:*

$\mathcal{A}_i = \{(a,b,\xi_i) : \boldsymbol{\alpha}(a,b,z), \textit{for some } z \notin \Lambda_i\}$;

$\mathcal{B}_i = \{(a,b,c) : \boldsymbol{\alpha}(a,b,c), a,b,c \in \Lambda_i\}$;

$\mathcal{C}_i = \{(a,b,c) : \boldsymbol{\alpha}(a,b,z) \textit{ and } \boldsymbol{\alpha}(b,c,z'), a,b,c \in \Lambda_i, z \neq z' \notin \Lambda_i\}$.

Such a characterization relies on the following

PROPOSITION 4.12. *Let* $a,b,c \in \Lambda_i$*. Then:*

(i) $(a, b, \xi_i) \in \boldsymbol{\mu}_i * \xi_i$ *iff* $\boldsymbol{\alpha}(a, b, z), z \notin \Lambda_i$;

(ii) $(a, b, c) \in \boldsymbol{\mu}_i * \xi_i$ *iff*

– *either* $\boldsymbol{\alpha}(a, b, c)$,
– *or* $\boldsymbol{\alpha}(z, a', b')$ *and* $\boldsymbol{\alpha}(u, b', c')$, *where* (a', b', c') *is a cyclic permutation of* (a, b, c), $z \neq u$ *and* $z, u \notin \Lambda_i$.

PROOF. (i) By definition of $\boldsymbol{\mu}_i$.

(ii) *If:* We check that $(a, b, c) \in \boldsymbol{\alpha}$ implies $(a, b, c) \in \boldsymbol{\mu}_i * \xi_i$. The other case is immediate since, by construction, $\boldsymbol{\mu}_i(a, b)$ and $\boldsymbol{\mu}_i(b, c) \Rightarrow \boldsymbol{\mu}_i * \xi_i(a, b, c)$. Given $\boldsymbol{\alpha}(a, b, c)$, let us consider the spreading of ξ on it. Let, say, it is $\boldsymbol{\alpha}(\xi, a, b)$, thus $\boldsymbol{\mu}_i(a, b)$. We now need to study the relation of c with a, b in $\boldsymbol{\mu}_i$. If $\boldsymbol{\mu}_i(c, a)$ or $\boldsymbol{\mu}_i(b, c)$, it is $(a, b, c) \in \boldsymbol{\mu}_i * \xi_i$. All the other cases where c is comparable with either a or b reduce to these two. If c is $\boldsymbol{\mu}_i$-incomparable with a, b, again we have $(a, b, c) \in \boldsymbol{\mu}_i * \xi_i$.

(ii) *Only if:* By construction, for an opportune cyclic permutation (a', b', c') of (a, b, c), it is either $\boldsymbol{\mu}_i(a', b')$ and $\boldsymbol{\mu}_i(b', c')$, or $\boldsymbol{\mu}_i(a', b')$ and c' incomparable with a', b' in $\boldsymbol{\mu}_i$. If we are in the last case, it means in particular that there is a $z \notin \Lambda_i$ such that $\boldsymbol{\alpha}(a', b', z)$ holds and *none* of the following is true: $\boldsymbol{\alpha}(z, a', c')$, $\boldsymbol{\alpha}(z, c', a')$,$\boldsymbol{\alpha}(z, b', c')$, $\boldsymbol{\alpha}(z, c', b')$. This means that $\boldsymbol{\alpha}_z(a', b')$ and c' is incomparable with a', b' in $\boldsymbol{\alpha}_z$. Thus $(a', b', c') \in \boldsymbol{\alpha} = \boldsymbol{\alpha}_z * z$. $\square$

In Section 8 we will show that in fact the information carried by $\mathcal{C}_i$ is "redundant" w.r.t. proof search. Thus, in practice, one can work without it.

4.5 Optimality and Adequacy

We have shown that the $\boldsymbol{\mu}_i$ are partial orders, and thus that the $\boldsymbol{\mu}_i * \xi_i$ are order varieties. What we need now to prove is that we are in fact giving solutions to the Entropy equation of Section 3, and that our solutions are optimal (i.e. minimal).

This will also prove adequacy with respect to Non-Commutative Logic, since as a consequence of Theorem 4.13, our binary calculus turns out to be equivalent to the one by Maieli and Ruet.

THEOREM 4.13. *With the notation defined above, we have:*

(i) $\boldsymbol{\alpha}_\xi \trianglelefteq \boldsymbol{\mu}_2 < \boldsymbol{\mu}_1$;

(ii) **Optimality**. *If* $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ *are two orders respectively on* Λ_1 *and* Λ_2, *such that* $\boldsymbol{\alpha}_\xi \trianglelefteq (\boldsymbol{\sigma}_2 < \boldsymbol{\sigma}_1)$, *then* $\boldsymbol{\mu}_i \trianglelefteq \boldsymbol{\sigma}_i$.

PROOF OF (i). We want to show that $\boldsymbol{\alpha} \subseteq (\boldsymbol{\mu}_2 < \boldsymbol{\mu}_1) * \xi$. Let $\mathbf{t}$ be a triple in $\boldsymbol{\alpha}$. (1.) If $\mathbf{t} \in (\xi, \Lambda_i, \Lambda_i)$ or $\mathbf{t} \in (\xi, \Lambda_2, \Lambda_1)$, then $\mathbf{t} \in (\boldsymbol{\mu}_2 < \boldsymbol{\mu}_1) * \xi$, by definition of serial composition of the orders $\boldsymbol{\mu}_2, \boldsymbol{\mu}_1$.

(2.) $\mathbf{t} \in (\Lambda_j, \Lambda_i, \Lambda_i)$. Let us fix $a, b, c \in \Lambda_i$ and $z \in \Lambda_j$. $\boldsymbol{\alpha}(z, a, b)$ implies $\boldsymbol{\mu}_i(a, b)$ and thus (a, b) is in the serial composition of the two orders; the definition of serial composition of orders also implies $(b, z) \in \boldsymbol{\mu}_i < \boldsymbol{\mu}_j$ and $(z, a) \in \boldsymbol{\mu}_j < \boldsymbol{\mu}_i$, thus in both cases (either $i < j$ or $j < i$), $(a, b, z) \in (\boldsymbol{\mu}_2 < \boldsymbol{\mu}_1) * \xi$, by construction.

(3.) $\mathbf{t} \in (\Lambda_i, \Lambda_i, \Lambda_i)$. As in Characterization 4.11 (ii,"If"). To be precise, we have to remark that if we have $\boldsymbol{\mu}_i(a, b)$ and c is incomparable with a, b in $\boldsymbol{\mu}_i$, then c is also incomparable with respect to the serial composition of the two orders. Thus we have $(a, b, c) \in (\boldsymbol{\mu}_2 < \boldsymbol{\mu}_1) * \xi$. $\square$

Before proving (ii) of the Theorem, we note that

LEMMA 4.14. $\boldsymbol{\alpha}(a, b, z)$, *where* $z \in \Lambda_j$ *and* $a, b \in \Lambda_i$, *implies* $(a, b) \in \boldsymbol{\sigma}_i$.

PROOF. $(\boldsymbol{\sigma}_2 < \boldsymbol{\sigma}_1) * \xi$ contains $\boldsymbol{\alpha}$, by the hypothesis of entropy, and $(\xi, \Lambda_2, \Lambda_1)$, by definition of serial composition of orders. Let say $j = 1, i = 2$. Since (a, b, z) and (a, z, ξ) are in $(\boldsymbol{\sigma}_2 < \boldsymbol{\sigma}_1) * \xi$, then (by transitivity) so is (a, b, ξ). This entails $\boldsymbol{\sigma}_2(a, b)$. $\square$

PROOF OF (ii). To prove that $\boldsymbol{\mu}_i * \xi_i \subseteq \boldsymbol{\sigma}_i * \xi_i$, let us examine the triples $\mathbf{t}$ in $\boldsymbol{\mu}_i * \xi_i$, following Characterization 4.11.

$(\mathcal{A}_i)$. If $\mathbf{t} = (a, b, \xi_i)$, then we have $\boldsymbol{\alpha}(a, b, z), z \in \{\xi, \Lambda_j\}$. The previous Lemma allows us to conclude $\boldsymbol{\sigma}_i(a, b)$.

$(\mathcal{B}_i)$. $\mathbf{t} = (a, b, c)$ and $\boldsymbol{\alpha}(a, b, c)$, thus $(a, b, c) \in (\boldsymbol{\sigma}_2 < \boldsymbol{\sigma}_1) * \xi$, hence the result.

$(\mathcal{C}_i)$. $\mathbf{t} = (a, b, c)$ and, for (a', b', c') a cyclic permutation of (a, b, c), we have $\boldsymbol{\alpha}(a', b', z)$ and $\boldsymbol{\alpha}(b', c', u)$, $z, u \in \{\Lambda_j, \xi\}$. This implies (Lemma 4.14) $\boldsymbol{\sigma}_i(a', b')$ and $\boldsymbol{\sigma}_i(b', c')$, hence $(a', b', c') \in \boldsymbol{\sigma}_i * \xi_i$. $\square$

5. THE BINARY CALCULUS

5.1 Binary positive rules

Figure 1 sums up the positive binary rules.

Looking at the table, we see that the treatment of Tensor and Next may be unified, for both coherence and propagation of the order. To this end we associate to each formula ξ an order $\boldsymbol{\omega} = \boldsymbol{\omega}(\xi)$ on its direct subformulas:

$$\boldsymbol{\omega}(\xi_1 \odot \xi_2) = (\xi_2 < \xi_1) \text{ and } \boldsymbol{\omega}(\xi_1 \otimes \xi_2) = (\xi_1 \parallel \xi_2)$$

Then any binary positive rule may be expressed as:

$$\frac{\vdash \boldsymbol{\mu}_{\Lambda_1} * \xi_1 \quad \vdash \boldsymbol{\mu}_{\Lambda_2} * \xi_2}{\vdash \xi, \Lambda \ \langle \boldsymbol{\alpha} \rangle} \ \xi, \boldsymbol{\omega}$$

where

$$\boldsymbol{\mu}_{\Lambda_i}(a, b) \ \text{ iff } \ \boldsymbol{\alpha}(a, b, z), \text{for some } z \notin \Lambda_i$$

and Λ_1, Λ_2 is a bi-partition of Λ that satisfies the *coherence conditions:*

DEFINITION 5.1 (COHERENCE). *A partition* Λ_1, Λ_2 *is coherent if:*

Commutative

$$\frac{\vdash \xi_1 * (\boldsymbol{\alpha}_\xi \lceil \Lambda_1) \quad \vdash \xi_2 * (\boldsymbol{\alpha}_\xi \lceil \Lambda_2)}{\vdash \xi_1 \otimes \xi_2, \Lambda \ \langle \boldsymbol{\alpha} \rangle} \otimes$$

where:
Λ_1, Λ_2 is a bi-partition of Λ such that $x_1 \in \Lambda_1, x_2 \in \Lambda_2 \Rightarrow x_1 \parallel x_2$ *in* $\boldsymbol{\alpha}_\xi$

Non-Commutative

$$\frac{\vdash \xi_1 * \boldsymbol{\mu}_{\Lambda_1} \quad \vdash \xi_2 * \boldsymbol{\mu}_{\Lambda_2}}{\vdash \xi_1 \odot \xi_2, \Lambda \ \langle \boldsymbol{\alpha} \rangle} \odot$$

where:
Λ_1, Λ_2 is a *coherent* bi-partition of Λ

Figure 1: Binary positive rules.

(Coh.1) For any $x_i \in \Lambda_i$, $x_j \in \Lambda_j$, where $i \neq j \in \{1,2\}$: $\boldsymbol{\alpha}(\xi, x_i, x_j) \Rightarrow \boldsymbol{\omega}(\xi_i, \xi_j)$;

(Coh.2) For any $a, b \in \Lambda_i$ and $z, z' \notin \Lambda_i$: $\boldsymbol{\alpha}(a, b, z) \Rightarrow \neg\boldsymbol{\alpha}(b, a, z')$.

PROPOSITION 5.2. *If $\xi = \xi_1 \otimes \xi_2$ we have that*

– $\boldsymbol{\mu}_{\Lambda_i}(a, b) = \boldsymbol{\alpha}_\xi \lceil \Lambda_i$,

– *(Coh.2) is always verified.*

PROOF. (Coh.1) and spreading of $\boldsymbol{\alpha}$ imply that $\boldsymbol{\alpha}(a, b, z) \Rightarrow \boldsymbol{\alpha}(a, b, \xi)$, for $a, b \in \Lambda_i, z \notin \Lambda_i$. Then, in particular, (Coh.2) is ensured by the anti-reflexivity of $\boldsymbol{\alpha}$. □

REMARK 5.3. *(Coh.1) instantiates for Next and Tensor respectively to:*

$\odot$: $\neg\boldsymbol{\alpha}(\xi, x_1, x_2)$, *i.e.* $x_1 \not< x_2$;

$\otimes$: $\neg\boldsymbol{\alpha}(\xi, x_1, x_2) \& \neg\boldsymbol{\alpha}(\xi, x_2, x_1)$, *i.e.* $x_1 \parallel x_2$.

Remember that $x_1 \parallel x_2$ iff ($x_1 \not< x_2$ & $x_2 \not< x_1$).

5.2 Binary negative rules

The negative rules also can be unified into a single one. We define again:

$$\boldsymbol{\omega}(\xi_1 \triangledown \xi_2) = (\xi_1 < \xi_2) \text{ and } \boldsymbol{\omega}(\xi_1 \parr \xi_2) = (\xi_1 \parallel \xi_2)$$

Then:

$$\frac{\vdash \boldsymbol{\alpha}_\xi * \boldsymbol{\omega}}{\vdash \xi, \Lambda \ \langle \boldsymbol{\alpha} \rangle} \ \xi, \boldsymbol{\omega}$$

The order variety $\boldsymbol{\alpha}_\xi * \boldsymbol{\omega}$ admits an explicit characterization (cf. Proposition 6.6), that for Sequential (that for Par, respectively) instantiates to

$\triangledown$: $\boldsymbol{\alpha}_\xi * (\xi_1 < \xi_2) = \boldsymbol{\alpha}[\{\xi_1, \xi_2\}/\xi] \cup (\Lambda, \xi_1, \xi_2)$;

$\parr$: $\boldsymbol{\alpha}_\xi * (\xi_1 \parallel \xi_2) = \boldsymbol{\alpha}[\{\xi_1, \xi_2\}/\xi]$.

5.3 Derivations in the binary calculus

Figure 2 gives an example of a derivation using the binary calculus. All lower-case letters denote atoms. For convenience, the focus is typeset in bold face.

To perform the first positive rule (Next, with focus **F**) with the given partition, we have to check (Coh.1) on the triples in (1). There is only one test: (b, a, F). Since $b \in \Lambda_{b^\perp \otimes c^\perp}$ and $a \in \Lambda_{a^\perp}$, we have to check that we have $(b^\perp \otimes c^\perp) < a^\perp$ in $\boldsymbol{\omega}(F)$, which is the case. The derivation fails on the second positive rule (Tensor, with focus $b^\perp \otimes c^\perp$), because of the test $(b^\perp \otimes c^\perp, c, b)$ in (2).

6. THE CLUSTER CALCULUS

We can now generalize our results to clusters of all positive or all negative connectives. A cluster of all positive (resp. all negative) connectives can be performed at once, as a single connective. We stress two points:

– To propagate the order, it is the same construction $\boldsymbol{\mu}_{\Lambda_i}$ that we already defined to apply.

– For the actual use of the calculus, the characterizations are important results, as, in practice, are the characterizations that one manipulates.

To define the cluster calculus, we exploit again the notion of order associated to the subformulas of a focus.

DEFINITION 6.1. *Given a cluster ξ of positive connectives, let us fix its subformulas $\xi_i (i \in I)$ as either the first negative subformulas of ξ or atoms. We inductively define an order $\boldsymbol{\omega} = \boldsymbol{\omega}(\xi)$ on the subformulas ξ_i :*

$\boldsymbol{\omega}(\xi' \odot \xi'') = \boldsymbol{\omega}(\xi'') < \boldsymbol{\omega}(\xi')$,

$\boldsymbol{\omega}(\xi' \otimes \xi'') = \boldsymbol{\omega}(\xi') \parallel \boldsymbol{\omega}(\xi'')$,

$\boldsymbol{\omega}(\xi_i) = \xi_i$.

The corresponding definition of order associated with a negative cluster is the obvious one.

To allow for compact definitions, it is convenient to extend I with an arbitrary index $\bigstar$, such that $\xi_\bigstar = \xi$. Let $I^* = I \cup \bigstar$. In such a way, we can speak of the order variety $\boldsymbol{\omega} * \xi$ on $\{\xi_i, i \in I^*\}$. As we indicate by Λ_i the context of ξ_i, it is also convenient to define $\Lambda_\bigstar$ as ξ itself.

Let $F = a^\perp \odot (b^\perp \otimes c^\perp)$. As binary formula, its subformulas are $a^\perp$ and $b^\perp \otimes c^\perp$.

$$\dfrac{\dfrac{\vdash a^\perp, a \qquad \dfrac{\vdash b^\perp, b \quad \vdash c^\perp, c}{\vdash \mathbf{b}^\perp \otimes \mathbf{c}^\perp, c, b\ \langle (b^\perp \otimes c^\perp, c, b)\rangle^2}\ \mathbf{b}^\perp \otimes \mathbf{c}^\perp, (b^\perp \parallel c^\perp)}{\vdash b, a, c, \mathbf{F}\ \langle (b,a,c),(b,a,F)\rangle^1}\ \mathbf{F}, ((b^\perp \otimes c^\perp) < a^\perp)}{\vdash \mathbf{b}\nabla\mathbf{a}, c, F\ \langle \emptyset \rangle}\ \mathbf{b}\nabla\mathbf{a}, (b < a)$$

Figure 2: Binary Derivation.

Next two sections discuss the positive and negative cluster rules, which are then resumed in Figure 3.

6.1 Positive rules

The positive cluster rule is

$$\dfrac{\dots\quad \vdash \xi_i, \Lambda_i\ \langle \boldsymbol{\mu}_{\Lambda_i} * \xi_i \rangle \quad \dots}{\vdash \xi, \Lambda\ \langle \boldsymbol{\alpha} \rangle}\ \xi, \boldsymbol{\omega}$$

where:

Definition 6.2 (Induced Orders).

$$\boxed{\boldsymbol{\mu}_{\Lambda_i}(a,b) \ \ \textit{iff}\ \ \boldsymbol{\alpha}(a,b,z), \textit{for some } z \notin \Lambda_i}$$

and $\Lambda_1, \dots \Lambda_n$ is a *coherent* partition of Λ:

Definition 6.3 (Coherence). *For $i, j, k \in I^*$:*

(Coh.1) $\boldsymbol{\alpha}(x_i, x_j, x_k) \Rightarrow (\xi_i, \xi_j, \xi_k)$ *in* $\boldsymbol{\omega} * \xi$.

(Coh.2) *For* $a, b \in \Lambda_i, z, z' \notin \Lambda_i$, $\boldsymbol{\alpha}(a,b,z) \Rightarrow \neg\boldsymbol{\alpha}(b,a,z')$.

Note that

Remark 6.4. *Let Λ_I be a coherent partition of Λ, and $a, b \in \Lambda_i$. The two following sets are equal:*

$\{\boldsymbol{\alpha}(a,b,z) : z = \xi$ *or* $z \in \Lambda_j$, *for j such that $\xi_j > \xi_i$ or $\xi_j < \xi_i$*$\}$,

$\{\boldsymbol{\alpha}(a,b,z) : z \in \Lambda_j$, *for* $j \neq i\}$.

We conclude with

Proposition 6.5 (Characterization). *$\boldsymbol{\mu}_{\Lambda_i} * \xi_i$ admits the same characterization as in the binary case.*

6.2 Negative rules

The negative cluster rule is

$$\dfrac{\vdash \xi_I, \Lambda\ \langle \boldsymbol{\alpha}_\xi * \boldsymbol{\omega} \rangle}{\vdash \xi, \Lambda\ \langle \boldsymbol{\alpha} \rangle}\ \xi, \boldsymbol{\omega}$$

In practice, it is easy to use the following characterization, which is immediate:

Proposition 6.6 (Characterization).

$$\boldsymbol{\alpha}_\xi * \boldsymbol{\omega} = \boldsymbol{\alpha}[\xi_I/\xi] \ \cup\ \boldsymbol{\omega} * \xi \ \cup\ \{(\Lambda, \xi_i, \xi_j) : (\xi_i, \xi_j) \in \boldsymbol{\omega}\}$$

6.3 Decomposition of a cluster in binary steps

As one expects, we have:

Theorem 6.7. *To apply a cluster rule or to decompose it in binary steps give the same result with respect to both coherence and orders induced on the terminal premises.*

We only need to concentrate on the positive case, the negative one being quite immediate. The proof is by induction, with the binary case as evident basis.

Let us fix the setting we need for the inductive step. Let $\xi = \xi' \circ \xi''$, where $\circ$ is either $\odot$ or $\otimes$. Let $\boldsymbol{\omega}(\xi) = \boldsymbol{\omega} = \boldsymbol{\omega}_J \circ \boldsymbol{\omega}_K$, where $\circ$ is either $<$ or $\parallel$, $|\boldsymbol{\omega}| = \xi_I$, $I = J \cup K$, $|\boldsymbol{\omega}_J| = \xi_J$, and $|\boldsymbol{\omega}_K| = \xi_K$.

Let us consider the following application of positive cluster rule:

$$\dfrac{\dots\quad \vdash \boldsymbol{\mu}_i * \xi_i \quad \dots}{\vdash \xi, \Lambda\ \langle \boldsymbol{\alpha} \rangle}\ \xi, \boldsymbol{\omega}$$

where $i \in I$. We now first perform only a $\circ$-step, and then $(\xi', \boldsymbol{\omega}_J)$ and $(\xi'', \boldsymbol{\omega}_K)$ as clusters (the inductive hypothesis applies):

$$\dfrac{\dfrac{\dots\quad \vdash \boldsymbol{\nu}_j * \xi_j \quad \dots}{\vdash \xi', \Lambda_J\ \langle \boldsymbol{\alpha}' \rangle}\ \xi', \boldsymbol{\omega}_J \qquad \dfrac{\dots\quad \vdash \boldsymbol{\nu}_k * \xi_k \quad \dots}{\vdash \xi'', \Lambda_K\ \langle \boldsymbol{\alpha}'' \rangle}\ \xi'', \boldsymbol{\omega}_K}{\vdash \xi, \Lambda\ \langle \boldsymbol{\alpha} \rangle}\ \circ$$

where $j \in J, k \in K$. We need to prove that (i) the cluster step is coherent iff all the steps in the binary derivation are coherent and that (ii) $\boldsymbol{\mu}_i = \boldsymbol{\nu}_i$, for all $i \in I$.

Let us indicate the coherence hypotheses as:
(a.): The partition $\{\Lambda_i, i \in I\}$ is coherent on $\boldsymbol{\alpha}$ w.r.t. $(\xi, \boldsymbol{\omega})$;
(b.): The bi-partition Λ_J, Λ_K is coherent on $\boldsymbol{\alpha}$ w.r.t. the first binary step $(\xi, \xi' \circ \xi'')$;
(c.): The partition Λ_J is coherent on $\boldsymbol{\alpha}'$ w.r.t. $(\xi', \boldsymbol{\omega}_J)$; the partition Λ_K is coherent on $\boldsymbol{\alpha}''$ w.r.t. $(\xi'', \boldsymbol{\omega}_K)$.

Lemma 6.8. *Assuming either of the hypotheses (a.) or (b.)+(c.), if $j_1 \neq j_2 \in J$, $z \notin \Lambda_J$, then $\boldsymbol{\alpha}(x_{j_1}, x_{j_2}, z)$ entails $\boldsymbol{\omega}_J(\xi_{j_1}, \xi_{j_2})$.*

Proof. Let $z \in \Lambda_h, h \in K^*$. Assuming (a.) as hypothesis, $\boldsymbol{\alpha}(x_{j_1}, x_{j_2}, z)$ implies $\boldsymbol{\omega} * \xi(\xi_{j_1}, \xi_{j_2}, \xi_h)$. Since ξ_h is in the same relation ($\parallel$, $<$, or $>$) with both ξ_{j_i}, the only possibility (cf. Definition 3.4) for the order is $\boldsymbol{\omega}(\xi_{j_1}, \xi_{j_2})$. Assuming (b.) and (c.) as hypotheses, $\boldsymbol{\alpha}(x_{j_1}, x_{j_2}, z)$ implies $\boldsymbol{\alpha}'(x_{j_1}, x_{j_2}, \xi')$, hence $\boldsymbol{\omega}_J * \xi'(\xi_{j_1}, \xi_{j_2}, \xi')$. This entails $\boldsymbol{\omega}_J(\xi_{j_1}, \xi_{j_2})$, and thus $\boldsymbol{\omega}(\xi_{j_1}, \xi_{j_2})$. $\square$

Positive Rule (ξ **positive**)	**Negative Rule** (ξ **negative**)
$$\frac{\ldots \quad \vdash \xi_i, \Lambda_i \ \langle \boldsymbol{\mu}_{\Lambda_i} * \xi_i \rangle \quad \ldots}{\vdash \xi, \Lambda \ \langle \boldsymbol{\alpha} \rangle} \ \xi, \boldsymbol{\omega}$$ where $i \in I$, $\Lambda_1, \ldots \Lambda_n$ *coherent* partition of Λ.	$$\frac{\vdash \xi_I, \Lambda \ \langle \boldsymbol{\alpha}_\xi * \boldsymbol{\omega} \rangle}{\vdash \xi, \Lambda \ \langle \boldsymbol{\alpha} \rangle} \ \xi, \boldsymbol{\omega}$$

$$|\boldsymbol{\omega}| = \{\xi_1, \ldots \xi_n\}, \quad I = \{1, \ldots n\}$$

Figure 3: Cluster Rules.

PROPOSITION 6.9 (COHERENCE). *(a.) iff (b.) + (c.).*

PROOF. $(a.) \Rightarrow (b.)$. (Coh.1) is immediate. As for (Coh.2), let assume $\boldsymbol{\alpha}(x_{j_1}, x_{j_2}, z)$, where $j_1, j_2 \in J$ and $z \notin \Lambda_J$. If $j_1 = j_2$ the result is immediate by (a.). Otherwise Lemma 6.8 implies $\boldsymbol{\omega}(\xi_{j_1}, \xi_{j_2})$. If we had $\boldsymbol{\alpha}(x_{j_2}, x_{j_1}, z')$, $z' \notin \Lambda_J$, we would also have $\boldsymbol{\omega}(\xi_{j_2}, \xi_{j_1})$.

$(a.) \Rightarrow (c.)$. (Coh.1). Assume $\boldsymbol{\alpha}'(x_l, x_m, x_n)$, and use Characterization 4.11. If we have $l, m, n \in J$ and $\boldsymbol{\alpha}(x_l, x_m, x_n)$, then we have $\boldsymbol{\omega} * \xi(\xi_l, \xi_m, \xi_n)$ and hence $\boldsymbol{\omega}_J * \xi'(\xi_l, \xi_m, \xi_n)$. If we have (modulo a cyclic permutation) $\boldsymbol{\alpha}(x_l, x_m, z)$ and $\boldsymbol{\alpha}(z', x_m, x_n)$, $z, z' \notin \Lambda_J$, it follows $\boldsymbol{\omega}(\xi_l, \xi_m)$ and $\boldsymbol{\omega}(\xi_m, \xi_n)$, hence $\boldsymbol{\omega}_J(\xi_l, \xi_m)$, $\boldsymbol{\omega}_J(\xi_m, \xi_n)$ and thus $\boldsymbol{\omega}_J * \xi'(\xi_l, \xi_m, \xi_n)$. If $x_n = \xi'$, then we have $\boldsymbol{\alpha}(x_l, x_m, z), z \notin \Lambda_J$, hence $\boldsymbol{\omega}(\xi_l, \xi_m)$, thus $\boldsymbol{\omega}_J(\xi_l, \xi_m)$ and the result. (Coh.2) is immediate.

$(b.) + (c.) \Rightarrow (a.)$. (Coh.1). Assume $\boldsymbol{\alpha}(x_l, x_m, x_n)$. If $l, m, n \in J$, then $\boldsymbol{\alpha}'(x_l, x_m, x_n)$, hence $\boldsymbol{\omega}_J * \xi'(\xi_l, \xi_m, \xi_n)$ and thus $\boldsymbol{\omega} * \xi(\xi_l, \xi_m, \xi_n)$. If $l, m \in J$, and $n \in K^*$, Lemma 6.8 entails $\boldsymbol{\omega}(\xi_l, \xi_m)$, hence $\boldsymbol{\omega} * \xi(\xi_l, \xi_m, \xi_n)$. Finally, if $l \in J, m \in K, x_n = \xi$, the result follows by the hypothesis (b.).

(Coh. 2). Let assume that for $a, b \in \Lambda_i (i \in J)$, we have $\boldsymbol{\alpha}(a, b, z)$ and $\boldsymbol{\alpha}(b, a, z')$. To have both $z, z' \notin \Lambda_J$ is against (b). Assume $z \in \Lambda_j, j \in J$. Thus we have $\boldsymbol{\alpha}'(a, b, z)$. If $j \neq i$, the hypothesis (c.) forces $z' \in \Lambda_i$. In fact, $z' \notin \Lambda_J$ entails $\boldsymbol{\alpha}'(b, a, \xi')$, contradicting (c.); $z' \in \Lambda_J$ entails $\boldsymbol{\alpha}'(b, a, z')$, which contradicts (c.) unless $z' \in \Lambda_i$. ☐

PROPOSITION 6.10 (ORDERS). $\boldsymbol{\mu}_i = \boldsymbol{\nu}_i$.

PROOF. Let $a, b \in |\boldsymbol{\mu}_i| = |\boldsymbol{\nu}_i| = \Lambda_i$.

$\boldsymbol{\mu}_i \subseteq \boldsymbol{\nu}_i$. $(a, b) \in \boldsymbol{\mu}_i$ iff $\boldsymbol{\alpha}(a, b, z)$, where $z \in \Lambda_h, h \neq i$. If $h \in J$ then $\boldsymbol{\alpha}'(a, b, z)$, hence the result. If $h \in K^*$ then $\boldsymbol{\alpha}'(a, b, \xi')$, hence the result.

$\boldsymbol{\nu}_i \subseteq \boldsymbol{\mu}_i$. A cases analysis shows that $\boldsymbol{\alpha}'(a, b, x)$, where $x \notin \Lambda_i$, means that we have $\boldsymbol{\alpha}(a, b, x_h)$, where $h \neq i, h \in I^*$, hence $(a, b) \in \boldsymbol{\mu}_i$. ☐

7. SAMPLE DERIVATIONS

In Figure 4 we give two examples to familiarize ourselves with the cluster calculus. All lower-case letters denote atoms. For convenience, the focus is typeset in bold face, and the rules are annotated with the associated order variety $\boldsymbol{\omega} * \xi$ rather than with $(\xi, \boldsymbol{\omega})$.

To check coherence we have to check the matching of the triples in the sequent order variety $\langle \boldsymbol{\alpha} \rangle$ with those in the order variety $\boldsymbol{\omega}(\xi) * \xi$ induced by the focus.

Example 1: We check the triples in the order variety (1) against those in the order variety $(b^\perp \parallel c^\perp) < a^\perp * F \equiv \{(b^\perp, a^\perp, F), (c^\perp, a^\perp, F)\}$. While (b, a, F) is matched with $(b^\perp, a^\perp, F)$, to (b, a, c) does not correspond the triple $(b^\perp, a^\perp, c^\perp)$. Thus the derivation fails.

Example 2: For the first positive rule: (a, y, F) is matched with $(a^\perp, x, F)$; (a, y, c) is matched with $(a^\perp, x, c^\perp)$, and (a, y, b) is matched with $(a^\perp, x, b^\perp)$. Note that we then have as order variety $\langle (x, y, Z) \rangle$ because of (a, y, Z) in (1): $y, Z \in \Lambda_x$, and $a \notin \Lambda_x$. For the second positive rule, (x, y, Z) is matched with $(x^\perp, y^\perp, Z)$.

Note that in these examples there is no non-determinism in the contexts splitting, because the choice is directed by the atoms.

8. FURTHER RESULTS AND IMPLEMENTATION ISSUES

This section discusses, rather informally, some developments that are oriented towards implementation. The analysis of the coherence conditions and of the way of propagating the orders carried out in this paper brings a better understanding of what is really essential to NL proof search. This now enables us, in particular, (**I**) to get rid of (Coh.2) and (**II**) to reduce the space of the triples we need to test.

(**I**) To use only (Coh.1) as coherence condition is enough to guarantee the correctness of a NL derivation.

PROPOSITION 8.1. *If the search of a (multiplicative) NL proof is successful checking only (Coh.1), then all applications of positive rules also satisfy (Coh.2).*

We give a hint of the proof. Let us consider the case where we have $\boldsymbol{\mu}_i(a, b)$ and $\boldsymbol{\mu}_i(b, a)$. Thus the order variety on the premise contains (a, b, ξ_i) and (b, a, ξ_i). Whatever rule is applied afterwards, there will always be at least one pair of

Example 1: Let $F = a^\perp \odot (b^\perp \otimes c^\perp)$. As ternary formula, its subformulas are $a^\perp$, $b^\perp$, and $c^\perp$.

$$\dfrac{\dfrac{\vdash a^\perp, a \quad \vdash b^\perp, b \quad \vdash c^\perp, c}{\vdash b, a, c, \mathbf{F}\ \langle (b,a,c),(b,a,F)\rangle^1}\ (b^\perp \parallel c^\perp) < a^\perp * \mathbf{F}}{\vdash \mathbf{b}\nabla\mathbf{a}, c, F\ \langle\emptyset\rangle}\ (b < a) * \mathbf{b}\nabla\mathbf{a}$$

Example 2: Let $N = c⅋((a\nabla y)⅋(b⅋Z))$, $Z = (y^\perp \odot x^\perp)$, and $F = (x \odot a^\perp) \odot (b^\perp \otimes c^\perp)$.

$$\dfrac{\dfrac{\dfrac{\vdash y^\perp, y \quad \vdash x^\perp, x}{\vdash x, y, \mathbf{Z}\ \langle (x,y,Z)\rangle^2}\ (x^\perp < y^\perp) * \mathbf{Z} \qquad \vdash a^\perp, a \quad \vdash b^\perp, b \quad \vdash c^\perp, c}{\vdash c, a, y, b, Z, \mathbf{F}\ \langle (a,y,F),(a,y,c),(a,y,b),(a,y,Z)\rangle^1}\ ((b^\perp \parallel c^\perp) < (a^\perp < x)) * \mathbf{F}}{\vdash F, \mathbf{N}\ \langle\emptyset\rangle}\ c \parallel ((a < y) \parallel (b \parallel Z) * \mathbf{N}$$

Figure 4: Sample Derivations.

triples $\{(x,y,z),(y,x,z)\}$. For a multiplicative NL proof to be correct, each branch must terminate either with the unit axiom $\vdash \mathbf{1}$, or with an identity axiom $\vdash p, p^\perp$. Before reaching this, the three points x, y, z must have been separated by a positive rule such as:

$$\dfrac{\vdash x_1, y \quad \vdash x_2, z}{\vdash x, y, z\ \langle \boldsymbol{\alpha} \rangle}$$

Since we have both (x,y,z) and (y,x,z), the above rule can be neither a Tensor nor a Next, which causes the procedure to fail.

Thus in particular, the derivation of Example 2 in Section 4 is not detected as wrong by (Coh.1) just because we stopped too early.

As a consequence we can state:

PROPOSITION 8.2 (COHERENCE). *The following two definitions of coherence are equivalent w.r.t. the search of NL proofs:*

(i) A context partition is coherent if it satisfies (Coh.1) and (Coh.2);

(ii) A context partition is coherent if it satisfies (Coh.1).

A calculus based on (i) is more sensitive than a calculus based on (ii): where (i) fails, the second calculus will go on, but it will still fail before reaching the axioms. On the other hand, to adopt (ii) rather than (i) as definition of coherence presents a clear advantage in terms of the cost of testing coherence. Thus more investigation is necessary to say which method offers more advantages for implementations.

(**II**) The calculus we presented builds a proof propagating order varieties. A way to look at such order varieties is as a set of tests, or a sort of "constraint" that a candidate partition must satisfy. With this perspective, not all the triples in the order variety are necessary to build NL derivations:

PROPOSITION 8.3. *Given a ternary relation $\boldsymbol{\alpha}$ on the conclusion of a positive rule, to propagate the two following ternary relations is equivalent w.r.t. soundness and completeness of the proof search:*

(i) $\boldsymbol{\mu}_i * \xi$ as defined in Section 4, i.e. $\mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{C}_i$, with the notation of Characterization 4.11;

(ii) $\mathcal{A}_i \cup \mathcal{B}_i$, with the notation of Characterization 4.11.

Note that the ternary relation defined in (ii) is not an order variety (cf. Characterization 4.11), but if we wish, we can complete it into an order variety at the end, if the proof search succeeds.

8.1 Future work: towards implementation

We expect to use our calculus in conjunction with constraint based techniques [3]. To extend such techniques to proof construction in NL, the ability of dealing with clusters of connectives, rather than only binary ones, is essential. This is true because the basic objects of the constraint technique, namely the bipoles, are positive clusters built from (positive) atoms and monopoles, where in turn the monopoles are negative clusters of (negative) atoms. The key fact is that bipoles, being two-layered clusters of formulas, can be decomposed in a single step. This makes it possible for the atoms to guide the splitting of the contexts, leading to the progressive instantiation of the partitions.

The advantage of the constraint based approach to proof construction is to unfold all branches of a derivation in parallel, a strategy that suits well the methods of our calculus. We would also be interested in investigating the possibility of using techniques of lazy context splitting.

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APPENDIX

A. NL SEQUENT CALCULUS

We only give the multiplicative fragment, the one we are working with along the paper.

Definition A.1 (Formulas). *The formulas are built from atoms $p, q, .., p^\perp, q^\perp, ...$ and the following connectives:*

commutative: $\otimes$ *(Tensor)*, $⅋$ *(Par);*

non-commutative: $\odot$ *(Next)*, ∇ *(Sequential).*

Definition A.2 (Negation). *Negation is defined by De Morgan rules:*

$$(p)^\perp = p^\perp \qquad (p^\perp)^\perp = p$$
$$(a \odot b)^\perp = b^\perp \nabla a^\perp \qquad (a \nabla b)^\perp = b^\perp \odot a^\perp$$
$$(a \otimes b)^\perp = b^\perp ⅋ a^\perp \qquad (a ⅋ b)^\perp = b^\perp \otimes a^\perp$$

SEQUENT CALCULUS ([13])

Identity

$$\vdash a * a^\perp$$

Positive

$$\frac{\vdash \xi_1 * \boldsymbol{\tau}_1 \quad \vdash \xi_2 * \boldsymbol{\tau}_2}{\vdash \xi_1 \otimes \xi_2 * \boldsymbol{\tau}_1 \parallel \boldsymbol{\tau}_2}\ \otimes \qquad \frac{\vdash \xi_1 * \boldsymbol{\tau}_1 \quad \vdash \xi_2 * \boldsymbol{\tau}_2}{\vdash \xi_2 \odot \xi_1 * \boldsymbol{\tau}_2 < \boldsymbol{\tau}_1}\ \odot$$

Negative

$$\frac{\vdash (\xi_1 \parallel \xi_2) * \boldsymbol{\tau}}{\vdash \xi_1 ⅋ \xi_2 * \boldsymbol{\tau}}\ ⅋ \qquad \frac{\vdash (\xi_1 < \xi_2) * \boldsymbol{\tau}}{\vdash \xi_1 \nabla \xi_2 * \boldsymbol{\tau}}\ \nabla$$

Entropy

$$\frac{\vdash \boldsymbol{\tau} * \xi}{\vdash \boldsymbol{\tau}' * \xi} \qquad \text{where } \boldsymbol{\tau}' \trianglelefteq \boldsymbol{\tau} \quad (*)$$

$(*)$ "$\boldsymbol{\tau}'$ obtained from $\boldsymbol{\tau}$ by replacing some $<$ with $\parallel$"

Remark A.3. *It is easy to have an intuition of why for Next it is $\boldsymbol{\tau}_2 < \boldsymbol{\tau}_1$, rather than $\boldsymbol{\tau}_1 < \boldsymbol{\tau}_2$, if one remembers that $(a \odot b)^\perp = b^\perp \nabla a^\perp$. A derivation of $\vdash (a \odot b)^\perp, (a \odot b)$ is:*

$$\cfrac{\cfrac{\vdash a * a^\perp \quad \vdash b * b^\perp}{\vdash b^\perp < a^\perp * a \odot b}\ \odot}{\vdash b^\perp \nabla a^\perp, (a \odot b)}\ \nabla$$