Ludics with repetitions (Exponentials, Interactive types and Completeness)

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***Abstract*—We prove that is possible to extend Ludics [12] so as to have repetitions (hence exponentials), and still have the results on semantical types which characterize Ludics in the panorama of Game Semantics. The results are obtained by using less structure than in the original paper; this has an interest on its own, and we hope that it will open the way to applying the approach of Ludics to a larger domain.**

***Keywords*-Linear Logic; Ludics; Interactive types; Game Semantics.**

I. Introduction

Ludics is a research program introduced by Girard [12] with the aim of providing a foundation for logic based on interaction. It can be seen as a form of Game Semantics where *first* we have the definition of *interaction* (composition, normalization), and *then* we have semantical *types*, as sets of strategies which ”behave well” w.r.t. composition. This role of interaction in the definition of types is where lies the specificity of Ludics in the panorama of Game Semantics.

Recently, a growing body of work is starting to explore and to develop the potential of this specific approach, and to put at work the more general notion of type offered by Ludics: the notion of type defined through interaction. We mention in particular work by Saurin on interactive proof-search as a logic programming paradigm [18], and work by Terui on computability [20]. Terui gives an especially interesting use of the notion of orthogonality (“to interact well”): if the *strategy* $\mathfrak{D}$ describes an automata, $\mathfrak{D}^\perp$ (the set of all strategies which “interact well” with it) consists of the languages accepted by that automata. Moreover, interactive types seem natural in a process calculus setting, and Faggian-Piccolo ([9]) have shown a close correspondence of Ludics with the Linear Pi-calculus [19].

Interactive types: The computational objects of Ludics — *designs* — can be seen as a linear form of Hyland-Ong (HO) innocent strategies (as pointed out in [8]) or as Curien’s Abstract Böhm Trees.

However, in Game Semantics, we first define the types (arenas), and then the composition of strategies; the type (the arena) guarantees that strategies compose well. In Ludics, *strategies are untyped*, in the sense that all strategies are given on a universal arena (the arena of all possible moves); strategies can always interact with each other, and the interaction may *terminate well* (the two strategies “accept each other”, and are said *orthogonal*) or not (they *deadlock*). An *interactive type* is a set of strategies which “compose well”, and react in the same way to a set of tests (see Section IV). A semantical type $\mathbf{G}$ is any set of strategies which react well to the same set of tests $\mathbf{E}$, which are themselves strategies (counter-strategies),i.e. $\mathbf{G} = \mathbf{E}^\perp$.

Internal completeness: With Ludics, Girard introduces a new notion of completeness, which is called *internal completeness* (see Section V). This is a key, characterizing element of Ludics. We have already mentioned that a semantical type is a set of strategy closed by biorthogonal ($\mathbf{G} = \mathbf{G}^{\perp\perp}$). Internal completeness is the property that *the constructions on semantical types do not require any closure, i.e. are already closed by biorthogonal*. While it is standard in realizability that a semantical type is a set $\mathbf{S}$ of terms closed by biorthogonal ($\mathbf{S} = \mathbf{S}^{\perp\perp}$), when interpreting types one has to perform some kind of closure, and this operation can *introduce new terms*. For example, the interpretation of $A \oplus B$ is $(\mathbf{A} \cup \mathbf{B})^{\perp\perp}$. This set of terms could be in general strictly greater than $\mathbf{A} \cup \mathbf{B}$. We have internal completeness if $\mathbf{A} \cup \mathbf{B}$ is *proven* to be equal to $(\mathbf{A} \cup \mathbf{B})^{\perp\perp}$). Since the closure by biorthogonal does not introduce new terms, we have a *complete description of what inhabits the semantical type.*

In Girard’s paper [12], the semantical types which are interpretation of formulas enjoy internal completeness. This is really the key property (and the one used in [18], [20]). Full completeness (for Multiplicative Additive Linear Logic $\mathbf{MALL}$, in the case of [12]) *follows* from it.

A. Contributions of the paper

The purpose of this paper is two-fold. On the one hand, we show that it is possible to overcome the main limitation of Ludics, namely the constraint of linearity, hence the lack of exponentials, in the sense that internal completeness (and from that full completeness) can be obtained also when having repetitions, if one extends in a rather natural way the setting of Ludics. On the other hand, we provide proofs which use less structure than the original ones by Girard. Not only we believe this improve the understanding of the results, but — more fundamentally — we hope this opens the way to the application of the approach of Ludics to a larger domain. We now give more details on the content of the paper.

Ludics Architecture: A difficulty in [12] is that there is a huge amount of structure. Strategies are an abstraction of $\mathbf{MALL}$ proofs, and enjoy many good properties (analytical theorems). In [12], all proofs of the high level structure of Ludics make essential use of these properties. Since the properties are very specific to the particular nature of the objects, this makes it difficult to extend the — very interesting — approach of Ludics to a different setting, or build the interactive types on different computational objects.

Ludics is introduced in [12] as a construction in several layers.

- At the low level, there is the definition of the *untyped computational structures* (strategies, there called *designs*) and their *dynamics* (interaction). Interaction allows the definition of *orthogonality.*
- The computational objects satisfy certain remarkable properties, called *analytical theorems,* in particular **separation**: *two strategies $\mathfrak{A}, \mathfrak{B}$ are syntactically equal if and only if they are observationally equal (i.e., for each counter-strategy $\mathfrak{C}$, the strategies $\mathfrak{A}, \mathfrak{B}$ react in the same way to $\mathfrak{C}$).*
- At the high level, there is the definition of *interactive types*, which satisfy *internal completeness.*

By relying on less structure, we show that the high level architecture of Ludics is somehow independent from the low level entities (strategies), and in fact could be built on other computational objects.

In particular, separation is a strong property. This makes it a great property to have, but also a property which is not common to have in other settings. However, the fact that computational objects do not enjoy separation does not mean that it is not possible to build the "high level architecture" of Ludics. We show (Section V) in fact that the proofs of internal and full completeness rely on much less structure, namely operational properties of the interaction.

We believe that discriminating between internal completeness and the properties which are specific to the objects is important both to improve understanding of the results, and to make it possible to build the same construction on different entities.

In particular, strategies with repetitions have weaker properties than in the original version. We show that it is still possible to have interactive types, internal completeness, and from this full completeness for polarized $\mathbf{MELL}$ (Multiplicative — Exponentials — Linear Logic). The extension to full polarized Linear Logic [15] is straightforward.

Exponentials in Ludics: Exponentials have been the main open problem in Ludics since [12]. Maurel in [16] proposes a first solution based on the use of probabilistic strategies. This solution is limited by its technical complexity. Therefore, it is not developed till a result of full completeness. Maurel explores also a simpler solution, but does not pursue it further because of the failure of the Separation Theorem. Our work builds on his simpler solution.

B. Our approach

There are two standard branches of Game Semantics: AJM style Game Semantics [1] which is based on Girard's Geometry of Interaction and HO style Game Semantics [14], introducing *innocent strategies.* Strategies in [8] are a linear form of innocent strategies.

The most natural solution to extend Ludics with exponentials is hence to have as strategies standard HO innocent strategies (on an universal arena). To do so, there are two kind of difficulties, which we deal with in this paper.

The first difficulty in extending Ludics with repetitions, is that using HO-style style strategies, separation fails. We deal with this by showing that the proofs of internal completeness and full completeness can be given in a direct way, without relying on Separation (Section 4).

The second difficulty is that one needs to have enough tests. This problem is analogous to the one which has led Girard to the introduction of the *daimon rule*: in Ludics, one typically opposes to an abstract "proof of A" an abstract "counter-proof of A". To have enough tests (that is, to have both proofs of A and proofs of $A^{\perp}$) there is a new rule which allow us to justify any premise. Similarly, when we oppose to a proof of $?\mathbf{A}$ a proof of $!\mathbf{A}^{\perp}$ $(= (?\mathbf{A})^{\perp})$, we need enough counter-strategies. As we illustrate in Section 5, we need non-uniform counter-strategies. We realize this by introducing a *non-deterministic sum* of strategies. Motivations and a sketch of the solution are better detailed in Section VI-C.

C. Related work

AJM style exponentials: A different solution that uses AJM style exponentials is developed by the first author in [3]: $!A$ is interpreted as an infinite tensor product of the interpretation of A, where each copy of the interpretation of A receives a different index. However, the approach we use in this paper is considerably simpler, and we hope more suitable for more applicative uses of Ludics [9], [18], [20].

Game Semantics: We build on the variant of HO strategies introduced in [15]. Moreover, we are interested in connections with the resource modalities of Games Semantics [17].

Abstract Machines: Curien and Herbelin in [6] have studied composition of strategies as sets of views. In particular they have developed the View Abstract Machine (VAM) which is the device we use in this paper.

Non-deterministic innocent strategies: were introduced by Harmer in [13], with the purpose of modeling non-determinism (PCF with erratic choice). In this paper we introduce non-uniform strategies, which are realized by means of *non-deterministic sums* relying on work developed by Faggian and Piccolo [10]. Our purpose here is not to model non-determinism, but to implement *non uniformity* via “formal sums” of strategies, in order to provide enough tests to make possible the interactive approach of Ludics (inside the model), similarly to Girard’s introduction of strategies corresponding to ”incomplete” proofs. The different purpose is reflected in the composition, which in our setting is reduced to deterministic composition. Our strategies could be seen as a ”concrete” implementation of Harmer’s solution, in a simplified setting (see [4] for further details).

II. Calculus

In this section, we introduce a calculus that we call **MELLS**, which is a variant of polarized **MELL** based on synthetic connectives. In section VIII-B, we prove that our model is fully complete for **MELLS**.

A. MELL

Formulas of propositional Multiplicative Exponential Linear Logic **MELL** [11] are finitely generated by the following grammar:

$$F ::= X \mid X^{\perp} \mid \mathbf{0} \mid \top \mid \mathbf{1} \mid \perp \mid F \otimes F \mid F \parr F \mid !F \mid ?F$$

where $X, X^{\perp}$ are propositional variables (also called *atoms*). Linear logic distinguishes formulas into:

- *linear formulas*: $\mathbf{0}, \mathbf{1}, \top, \perp, F \otimes F, F \parr F$;
- *exponential formulas*: $?F, !F$.

Linear formulas can only be used once, while the modalities $!, ?$ allow formulas to be repeated. The possibility of repeating formulas is expressed by the *contraction* rule on $?F$ formulas:

$$\frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}$$

Dually, the modality ! allows proofs to be used several times during Cut-Elimination procedure, once for each duplication of $?F$.

Connectives and constants of **MELL** are also split into two classes, according to their **polarity**

Positive: $\mathbf{0}, \mathbf{1}, \otimes, ?$ | **Negative**: $\top, \perp, \parr, !$

Remark II.1 (Modalities)**.** *Following [19], we write* ! *for the* **negative** *modality, and* ? *for the* **positive** *one, because these symbols are more familiar. However in a polarized setting such as in [15], it is more common to write, resp.,* $\sharp$*(*negative*) and* $\flat$ *(*positive*).*

B. MELLS

We now introduce the calculus **MELLS**. Formulas are here built by *synthetic connectives* [12] i.e. maximal clusters of connectives of the same polarity. The key ingredient that allows for the definition of synthetic connectives is **focalization** [2]. Andreoli has proven that if a sequent is provable, then it is provable with a focusing proof. By using synthetic connectives, formulas are in a canonical form, where immediate subformulas have opposite polarity. Hence in a (cut-free) proof of **MELLS**, there is a positive/negative alternation of rules, which matches the standard Player (positive)/ Opponent (negative) alternation of moves in a strategy (see Section III).

Formulas of **MELLS** split into positive (P) and negative (N) as follows:

$$\begin{array}{rcl} P & ::= & ?\mathbf{0} \mid ?X \mid ?(N \otimes \cdots \otimes N) \\ N & ::= & !T \mid !X^{\perp} \mid !(P \parr \cdots \parr P) \end{array}$$

where X and $X^{\perp}$ are propositional variables.

We use F as a variable for formulas, and indicate the polarity also by writing F^{+} or F^{-}. We often write $F^{+}(N_1, \ldots, N_n)$ and $F^{-}(P_1, \ldots, P_n)$.

Linear negation $^{\perp}$ is defined as usual: $F^{\perp\perp} := F$, $(?\mathbf{0})^{\perp} := !\top$, $(?X)^{\perp} := !X^{\perp}$, $(?(N_1 \otimes \cdots \otimes N_n))^{\perp} := !({N_1}^{\perp} \parr \cdots \parr {N_n}^{\perp})$.

A sequent of **MELLS** is a multi-set of formulas written $\vdash F_1, \ldots, F_n$ such that it contains at most one negative formula.

For Γ multi-set of positive formulas, we have the following rules:

$$\frac{\vdash N_1, F^{+}, \Gamma \ldots \vdash N_n, F^{+}, \Gamma}{\vdash F^{+}, \Gamma}\,\text{Pos} \qquad \frac{\vdash P_1, \ldots, P_n, \Gamma}{\vdash F^{-}, \Gamma}\,\text{Neg}$$

$$\frac{}{\vdash ?X, !X^{\perp}, \Gamma}\,\text{Ax} \qquad \frac{}{\vdash !\top, \Gamma}\,!\top$$

$$\frac{\vdash P, \Xi, \Gamma \qquad \vdash P^{\perp}, \Gamma}{\vdash \Xi, \Gamma}\,\text{Cut}$$

where in the last rule Ξ is either empty or negative.

Notice that usual Linear Logic structural rules (weakening, contraction, promotion and dereliction) are always *implicit* in our calculus.

Proposition II.2. *Cut-Elimination holds for* **MELLS***.*

Remark II.3 (Intuitionistic Logic)**.** *It is easy to check that this calculus is a focalized version of the* $\neg, \wedge$ *fragment of intuitionistic calculus* **LJ** *(The reader can refer to [4] for more details).*

III. HO style Game Semantics

An *innocent strategy* [14] can be described either in terms of all possible interactions for the player (*strategy as set of plays*), or in a compact way, which provides only the minimal information for Player to move (*strategy as set of views*) [13]. It is standard that the two presentations are equivalent. Here we use the “strategy as set of views” description. We revise the definitions, following Harmer’s and Laurent’s presentation.

Let $Pol = \{+, -\}$ be the set of the **polarities** (here, positive and negative). We use $\epsilon \in Pol$ as a variable.

Definition III.1 (Arena)**.** *An* ***arena*** $(A, \vdash_A, \lambda_A)$ *is given by:*

- *a* **directed acyclic graph**, *d.a.g. for short,* $(A, \vdash_A)$ *where:*
  - A *(elements of the d.a.g.) is the set of* **moves***.*
  - $\vdash_A$ *(edges of the d.a.g.) is a well founded* **enabling relation** *on* A*. If there is an edge from* m *to* n*, we write* $m \vdash n$*. If no move enables* m *we write* $\vdash m$*, and call* m **initial***.*
- *a* **function** $\lambda_A : A \to Pol$ *which labels each element with a polarity.*

Enabling relation and polarity satisfy the following:

if $n \vdash m$*, they have opposite polarity.*

If all the initial moves have the same polarity ϵ*, we say that* ϵ *is the polarity of the arena. In this case we say that* A *is a* **polarized arena** *(of polarity* ϵ*).*

Definition III.2 (Strategy)**.** *Let* A *be an arena.*

- *A* **justified sequence** $s = s_0.s_1.\dots.s_n$ *on* A *is a string* $s \in A^*$ *with pointers between the elements in the string which satisfies:*
  - Justification. *For each non-initial move* s_i *of* s*, there is a unique pointer to an earlier occurrence of move* s_j *of* s *(*$j < i$*), called the justifier of* s_i*, such that* $s_j \vdash_A s_i$*.*
  - Alternation. *No two consecutive moves have the same polarity.*
- *A* **view** *(called* chronicle *in [12]) is a justified sequence on* A*, which satisfies:*
  - View. *For each non-initial negative (Opponent) move* s_i*, its justifier is the immediate predecessor* s_{i-1}*.*
- *A* **strategy** $\mathfrak{D}$ *on* A*, denoted by* $\mathfrak{D} : A$ *is a prefix-closed set of views, such that:*
  - Coherence. *If* $s.m, s.n \in \mathfrak{D}$ *and* $s.m \neq s.n$ *then* m, n *are negative.*
  - Maximality. *If* $s.m$ *is maximal in* $\mathfrak{D}$ *(i.e. no other view extends it), then* m *is positive.*

We call *positive* (resp. *negative*) a strategy on a positive (resp. negative) arena.

The polarity of a move in a strategy (i.e. positive/Player or negative/Opponent) is given by the arena. We sometimes put in evidence the polarity of a move x by writing x^+ or x^-, but we omit it when clear from the context. In the same way, we annotate strategies with their polarity, hence writing $\mathfrak{D}^+, \mathfrak{D}^-$.

Tree notation: Emphasizing the tree structure, we also write a strategy whose first move is a as $\mathfrak{D} = a.\mathfrak{D}'$. More precisely, if $\mathfrak{D}$ is a positive strategy, we write it as $\mathfrak{D} = a.\{\mathfrak{E}_1, \dots, \mathfrak{E}_n\}$, instead of $\{a.s : s \in \mathfrak{E}_i, i \in I\}$, where $\mathfrak{E}_i$ are negative strategies; similarly if $\mathfrak{E}$ is a negative strategy.of root a.

Composition of strategies: Composition of strategies as sets of views is well studied by Curien and Herbelin, who introduce the View — Abstract — Machine (VAM) [6] by elaborating on Coquand's Debates machine [5].

IV. Ludics

In this and the next section we give a compact but complete presentation of Ludics, in a language which fits that of Game Semantics.

Let us first stress the peculiarity of Ludics in the panorama of Game Semantics. In Game Semantics, one defines constructions on arenas which correspond to the interpretation of types. A strategy is always "typed", in the sense that it is a strategy on a specific arena. When strategies are opportunely typed, they interact (compose) well.

In the approach of Ludics, strategies are "untyped", in the sense that all strategies are defined on the universal arena. Strategies then interact with each other, and the interaction can *terminate well* (the two strategies "accept" each other) or not (*deadlock*).

Daimon: Ludics provides a homogeneous setting in which live both proofs and tests: proofs of A interact with proofs of $A^\perp$; to this end, it generalizes the notion of proof. To this purpose, a new rule is introduced, called **daimon**: $\dfrac{}{\vdash \Gamma}\,\dagger$. Such a rule allow us to assume any conclusion.

In the semantics, the daimon is a special action which acts as a *termination signal*.

A. Strategies on a Universal Arena

Strategies communicate on names. We can think of names as channels, which can be used to send outputs (if positive) or to receive inputs (if negative). Each strategy $\mathfrak{D}$ has an interface, which provides the names on which $\mathfrak{D}$ can communicate with the rest of the world, and the use (input/output) of each name.

A **name** (called *locus* in [12]) is a string of natural numbers. We use the variables $\xi, \sigma, \tau, \dots$ to range over names. Two names are *disjoint* if neither of them is the prefix of the other one.

An **interface** Γ (called *base* in [12]) is a finite set of pairwise disjoint names, together with a polarity for each name, such that at most one name is negative. If a name ξ has polarity ϵ, we write $\xi^\epsilon \in \Gamma$. We say that an interface Γ is negative if it contains a negative name, positive otherwise.

An **action** x is either the symbol † (called **daimon**) or a pair (ξ, I), where ξ is a name, and I is a finite subset of $\mathbb{N}$.

Given an action (ξ, I) on the name ξ, the set I indicates the names $\{\xi i : i \in I\}$ which are generated from ξ by this action. The prefix relation (written $\xi \sqsubseteq \sigma$) induces a natural relation of dependency on names, which generates an arena.We call **roots** the action † and any action (ξ, I) such that $\xi \in \Gamma$.

Definition IV.1 (Universal Arena on an interface.)**.** *Given an interface Γ, the* ***universal arena*** *$U(\Gamma)$ on Γ is the tuple $(U(\Gamma), \vdash, \lambda)$ where:*

- *The* **set of moves** *is the special action $\dagger$ together with the set of all actions of the form (ξ', I).*
- *The* **polarity** *of the initial actions (ξ, I) is that indicated by the interface for ξ; the polarity of each other action is the one induced by alternation.*
- *The* **enabling relation** *is defined as follows:*

$$(\xi, I) \vdash x, \text{ where:}$$

(i) *either $x = (\xi i, J)$ (with $i \in I$);*
(ii) *or x is a positive root action and (ξ, I) is a negative root action.*

Definition IV.2 (Strategy on a Universal Arena)**.** *Let Γ be an interface. A* **strategy $\mathfrak{D}$ on Γ,** *also written $\mathfrak{D} : \Gamma$ is a strategy (in the sense of Definition III.2) on the universal arena $U(\Gamma)$.*

A strategy which plays a key role is the strategy *daimon* $\{\dagger\}$, which (with a slight abuse of notation) we denote also by $\dagger$.

Dynamics: Composition of strategies is described via the VAM machine. All we need in this paper is Proposition VI.1 in Section VI-A.

B. Designs (linear strategies)

Definition IV.3 (Linear strategy)**.** *We say that a strategy $\mathfrak{D} : \Gamma$ is* **linear** *if in $\mathfrak{D}$ there is no repetition of actions on the same name.*

Linear strategies are essentially the strategies introduced in [12] (there called *designs*). The linearity condition expressed there is actually slightly more complex, because it takes into account also the additive structures (additive duplication is allowed). Since in this paper we do not consider additives, for our discussion it is enough to say that in a strategy *each name is only used once.* Linearity has as consequence that that all pointers are trivial (each move has only one possible justifier), and then can be forgotten.

Composition of linear strategies (see [12], [7]): We can compose two strategies $\mathfrak{D}_1, \mathfrak{D}_2$ when they have compatible interfaces, that is they have a common name, with opposite polarity. For example, $\mathfrak{D}_1 : \sigma^+, \Gamma$ can communicate with $\mathfrak{D}_2 : \sigma^-, \Delta$ through the name σ. The shared name σ, and all names hereditarily generated from σ, are said to be internal.

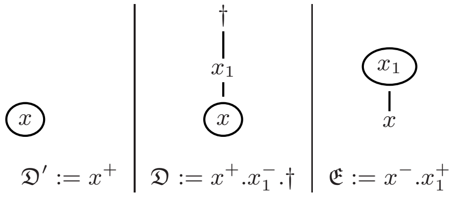
If $\mathfrak{R} = \{\mathfrak{D}_1, \dots, \mathfrak{D}_n\}$ is a finite set of strategies which have pairwise compatible interfaces, we denote by $\llbracket \mathfrak{R} \rrbracket$ the result of the composition, also called *normal form*.

Composition (also called normalization)follows the standard paradigm of *parallel composition (the interaction) plus hiding* of internal communication: $\llbracket \mathfrak{R} \rrbracket$ is obtained from the result of the interaction by hiding all actions on internal names.

The most important case in Ludics is the closed one, when all names in $\mathfrak{R}$ are internal (for example we have $\mathfrak{D} : \xi^+$ and $\mathfrak{E} : \xi^-$). In this case, normal form can be obtained step by step by applying the following rewriting rules:
a) $\llbracket x^+.\{\mathfrak{E}_1, \dots, \mathfrak{E}_n\}, x^-.\mathfrak{D}, \mathfrak{R}' \rrbracket \rightsquigarrow \llbracket \mathfrak{E}_1, \dots, \mathfrak{E}_n, \mathfrak{D}, \mathfrak{R}' \rrbracket$;
b) $\llbracket \dagger, \mathfrak{R}' \rrbracket \rightsquigarrow \dagger$;
c) If none of the case above applies, we have a deadlock (the output is empty).

Since each action only appears once, the dynamics is extremely simple: we match actions of opposite polarity. Let us give an example of how interaction works.

Example IV.4. *Let us consider the following small strategies (think $x = (\xi, I)$ and $x_1 = (\xi 1, K)$).*


x
$\mathfrak{D}' := x^+$
$\dagger$
x_1
x
$\mathfrak{D} := x^+.x_1^-.\dagger$
x_1
x
$\mathfrak{E} := x^-.x_1^+$

Let us have $\mathfrak{D}$ interact with $\mathfrak{E}$. $\mathfrak{D}$ starts by playing the move x^+, $\mathfrak{E}$ checks its answer to that move, which is x_1^+. If $\mathfrak{D}$ receives input x_1, its answer is $\dagger$, which terminates the interaction. Summing up, the interaction produces $x.x_1.\dagger$. If we hide the internal communication, $\dagger$ is the output.

If we have $\mathfrak{E}$ interacting with $\mathfrak{D}'$, we again match x^+ with x^-. Then $\mathfrak{E}$ plays x_1, but $\mathfrak{D}'$ has no answer to the action x_1. Here we have a deadlock.

C. Orthogonality and Interactive types

In the closed case, we only have two possible outcomes: either composition fails (deadlock), or it succeeds by reaching the action $\dagger$, which signals termination.

Definition IV.5 (Orthogonality)**.** *Given two strategies of opposite polarity $\mathfrak{D} : \xi^+, \mathfrak{E} : \xi^-$, they are* ***orthogonal****, written $\mathfrak{D} \perp \mathfrak{E}$, if $\llbracket \mathfrak{D}, \mathfrak{E} \rrbracket = \dagger$.*
If $\mathfrak{D} : \xi^\epsilon$ is a strategy, its ***orthogonal*** *is defined as $\mathfrak{D}^\perp := \{\mathfrak{E} : \xi^{\overline{\epsilon}}, \mathfrak{E} \perp \mathfrak{D}\}$ (where $\overline{\epsilon}$ is the polarity opposite to ϵ). The definition of orthogonality extends to sets of strategies.*

Two strategies are *orthogonal* if at each step any positive action x^+ *finds* its negative dual action x^-, and the computation terminates, that is we eventually meet a $\dagger$ action.

Example IV.6. *In the Example IV.4, $\mathfrak{E} \perp \mathfrak{D}$, while $\mathfrak{E}$ and $\mathfrak{D}'$ are not orthogonal.*

Orthogonality allows the players to agree (or not), without this being guaranteed in advance by the type: $\mathfrak{D}^\perp$ is the set of the counter-strategies which are consensual with $\mathfrak{D}$.

Remark IV.7. *Orthogonality naturally extends to strategies on arbitrary interface. Instead of considering a single counter-strategy, one has to consider a family of counter-strategies (e.g., for $\mathfrak{D} : \xi^+, \sigma^+$, one has to consider $\mathfrak{E} : \xi^-, \mathfrak{F} : \sigma^-$ and the family $\{\mathfrak{E}, \mathfrak{F}\}$ is orthogonal to $\mathfrak{D}$ if $[\![\mathfrak{D}, \mathfrak{E}, \mathfrak{F}]\!] = \dagger$). Details on this generalization are given in [12].*

Definition IV.8. *A* **behaviour** *(or* **interactive type***) on the interface Γ, is a set $\mathbf{G}$ of strategies $\mathfrak{D} : \Gamma$ such that $\mathbf{G}^{\perp\perp} = \mathbf{G}$ (it is closed by bi-orthogonal).*

We say that a behaviour $\mathbf{G}$ is positive or negative according to its interface.

We now give constructions on behaviours which will interpret linear formulas.

If $\mathfrak{D}_1 : \xi 1^-, \ldots, \mathfrak{D}_n : \xi n^-$ $(i \in I)$ are negative strategies, we define a new positive strategy on ξ^+:

$$\mathfrak{D}_1 \bullet \cdots \bullet \mathfrak{D}_n := (\xi, I).\{\mathfrak{D}_1, \ldots, \mathfrak{D}_n\}.$$

It is immediate to generalize the construction to strategies $\mathfrak{D}_i : \xi i^-, \Gamma$, to obtain $\mathfrak{D} : \xi^+, \Gamma$.

Conversely, given any strategy $\mathfrak{D} : \xi^+$, such that the root is linear (i.e. the action which labels the root occurs only once in $\mathfrak{D}$), we can write it as $\mathfrak{D} = x.\{\mathfrak{D}_1, \ldots \mathfrak{D}_n\}$. By the View condition, we have that each subtree $\mathfrak{D}_i$ is a strategy on ξi. Given a strategy $\mathfrak{D}$ as just described, we will write $\mathfrak{D}\!\upharpoonright_i$ for the operation which returns us $\mathfrak{D}_i$.

Let $\mathbf{A}_1, \mathbf{A}_2$ be *negative* behaviours, respectively on $\xi 1^-$ and $\xi 2^-$. We denote by $\mathbf{A}_1 \bullet \mathbf{A}_2$ the set

$$\{\mathfrak{D}_1 \bullet \mathfrak{D}_2, \mathfrak{D}_i \in \mathbf{A}_i\} \cup \{\dagger\}.$$

We define:
$\mathbf{A}_1 \otimes \mathbf{A}_2 := (\mathbf{A}_1 \bullet \mathbf{A}_2)^{\perp\perp}$, positive behaviour on ξ^+;
$\mathbf{A}_1{}^\perp \parr \mathbf{A}_2{}^\perp := (\mathbf{A}_1 \bullet \mathbf{A}_2)^\perp$, negative behaviour on ξ^-.

In the sequel, $\mathbf{P}$ will always denote a positive behaviour, $\mathbf{N}$ a negative one.

The interpretation $\mathbf{G}$ of a formula G will be a behaviour, i.e. a set of strategies closed by biorthogonal: $\mathfrak{D} \in \mathbf{G}$ *if and only if* $\mathfrak{D} \perp \mathfrak{E}$, *for each* $\mathfrak{E} \in \mathbf{G}^\perp$. The interpretation of a sequent $\vdash G_1, \ldots, G_n$ naturally extends this definition:

Definition IV.9. *A* **sequent of behaviours** *is a sequence of behaviours, noted by $\vdash \mathbf{G}_1, \ldots, \mathbf{G}_n$, that satisfies: $\mathfrak{D} \in \vdash \mathbf{G}_1, \ldots, \mathbf{G}_n$ if and only if $[\![\mathfrak{D}, \mathfrak{E}_1, \ldots, \mathfrak{E}_n]\!] = \dagger$, for each $\mathfrak{E}_1 \in \mathbf{G}_1{}^\perp, \ldots, \mathfrak{E}_n \in \mathbf{G}_n{}^\perp$.*

It is clear that a sequent of behaviours is itself a behaviour, i.e, a set of strategies closed by orthogonal.

V. LINEAR SETTING: INTERNAL AND FULL COMPLETENESS

In this section we restrict our attention to linear strategies. We introduce internal completeness, as well as full completeness. All these results can be proven without relying on separation.

In [12], the set of strategies which interpret $\mathbf{MALL}$ formulas satisfies a remarkable closure property, called *internal completeness*: the set $\mathbf{S}$ of strategies produced by the construction is (essentially) equal to its biorthogonal ($\mathbf{S} = \mathbf{S}^{\perp\perp}$). This means that we have a complete description of all strategies in the behaviour.

The best example is the interpretation $\mathbf{A}_1 \otimes \mathbf{A}_2 := (\mathbf{A}_1 \bullet \mathbf{A}_2)^{\perp\perp}$ of a Tensor formula. One proves that $(\mathbf{A}_1 \bullet \mathbf{A}_2) = (\mathbf{A}_1 \bullet \mathbf{A}_2)^{\perp\perp}$, i.e. we do not add new objects when closing by biorthogonal.

From this, full completeness follows. In fact, because of internal completeness, if $\mathfrak{D} \in \mathbf{A}_1 \otimes \mathbf{A}_2$ we know we can decompose it as $\mathfrak{D}_1 \bullet \mathfrak{D}_2$, with $\mathfrak{D}_1 \in \mathbf{A}_1$ and $\mathfrak{D}_2 \in \mathbf{A}_2$. This corresponds to writing the rule:

$$\dfrac{\dfrac{\vdots}{\vdash A_1} \qquad \dfrac{\vdots}{\vdash A_2}}{\vdash A_1 \otimes A_2}\, {\scriptstyle A_1 \otimes A_2}$$

i.e., if each $\mathfrak{D}_i$ corresponds to a proof of A_i, $\mathfrak{D}$ corresponds to a proof of $A_1 \otimes A_2$.

For the rest of this section, we assume that $\mathbf{A}, \mathbf{B}$ are negative behaviours respectively on $\xi 1^-$ and $\xi 2^-$.

Let us consider $\mathbf{A} \bullet \mathbf{B}$. By construction, each strategy has $x^+ = (\xi, \{1, 2\})^+$ as root. By definition of linear normalization, each strategy in $(\mathbf{A} \bullet \mathbf{B})^\perp$ has as root the action x^- (otherwise, normalization would fail immediately). All strategies in $(\mathbf{A} \bullet \mathbf{B})^{\perp\perp}$ have a positive root, which, to normalize against $(\mathbf{A} \bullet \mathbf{B})^\perp$, must be either the action $\dagger$, or x^+. Hence, we know that a strategy $\mathfrak{D} \in \mathbf{A} \otimes \mathbf{B}$ has the form $x^+.\{\mathfrak{D}_1 : \xi 1, \mathfrak{D}_2 : \xi 2\}$. We now prove that if $\mathfrak{D} \in (\mathbf{A} \bullet \mathbf{B})^{\perp\perp}$ then $\mathfrak{D}_1 \in \mathbf{A}$ and $\mathfrak{D}_2 \in \mathbf{B}$.

Proposition V.1 (Internal completeness of Tensor)**.** *Let $\mathbf{A}, \mathbf{B}$ be negative behaviours, respectively on $\xi 1^-$ and $\xi 2^-$. We have that $\mathbf{A} \otimes \mathbf{B} = \mathbf{A} \bullet \mathbf{B}$.*

Proof: Given any $\mathfrak{F} : \xi 1^+ \in \mathbf{A}^\perp$, we obtain the strategy $\mathfrak{F}' : \xi^- = x^-.\mathfrak{F}$ by adding the root x^-. For any $\mathfrak{D} : \xi^+ = x^+.\{\mathfrak{D}_1, \mathfrak{D}_2\}$, we have the following equation: $[\![x^-.\mathfrak{F}, \mathfrak{D}]\!] = [\![\mathfrak{F}, \mathfrak{D}_1]\!]$, by definition of normalization, and by the fact that since in $\mathfrak{F}$ there are only names generated by $\xi 1$, $\mathfrak{F} : \xi 1^+$ only interact with the subtree $\mathfrak{D}_1 : \xi 1^-$.
$\mathfrak{F}' \in (\mathbf{A} \bullet \mathbf{B})^\perp$, because by using the equation we deduce that $\mathfrak{F}' \perp \mathfrak{D}$, for any $\mathfrak{D} \in (\mathbf{A} \bullet \mathbf{B})$.
Given $\mathfrak{D} \in \mathbf{A} \otimes \mathbf{B}$, it must be $\mathfrak{D} \perp \mathfrak{E}$ for each $\mathfrak{E} \in (\mathbf{A} \bullet \mathbf{B})^\perp$. Hence in particular, for each $\mathfrak{F} \in \mathbf{A}^\perp$, we have $\mathfrak{D} \perp \mathfrak{F}'$ ($\mathfrak{F}'$ defined as above), and hence, again because of that equation, $\mathfrak{D}_1 \perp \mathfrak{F}$, i.e., $\mathfrak{D}_1 \in (\mathbf{A}^\perp)^\perp = \mathbf{A}$. ■

Remark V.2 (Important)**.** *Observe that here we only use two properties of the strategies: the dynamics (normalization), and the fact that* the root is linear, i.e. it is the only action on the name ξ *(to say that occurrences of $\xi 1$ only appear inside $\mathfrak{D}_1$).*

Internal completeness for the connective par $\parr$ is immediate, just spelling out Definition IV.9.

Full completeness for Multiplicative Linear Logic **MLL** follows from what we have seen in this section, by using the proof of internal completeness of Tensor and Par, and Corollary V.3.

Corollary V.3. *If* $\mathfrak{D} \in \vdash \mathbf{\Gamma}, \mathbf{P}$ *if and only if for each* $\mathfrak{E} \in \mathbf{P}^{\perp}$, $[\![\mathfrak{D}, \mathfrak{E}]\!] \in \vdash \mathbf{\Gamma}$.

VI. Ludics with repetitions: what, how, why

From this section on, we abandon the hypothesis of linearity. Here we discuss the difficulties in extending the approach of Ludics to this setting, and introduce our solution, which will be technically developed in Section VII. First, let us introduce some operations which we will use to deal with repeated actions, and describe composition.

Renaming: Given a strategy $\mathfrak{E} : \xi$, let us indicate by $\sigma(\mathfrak{E})$ the strategy obtained from $\mathfrak{E}$ by renaming, in all occurrences of action, the prefix ξ into σ. I.e., each name $\xi' = \xi.\tau$ becomes $\sigma.\tau$. Obviously, if $\mathfrak{E} : \xi$, then $\sigma(\mathfrak{E}) : \sigma$.

Renaming of the root: Given a positive strategy $\mathfrak{D} : \xi^{+}$, let us indicate by $\underline{\sigma}(\mathfrak{D})$ the strategy obtained by renaming the prefix ξ into σ in the root, and in all actions which are hereditarily justified by the root. If $\mathfrak{D} : \xi^{+}$, we obtain a new strategy $\underline{\sigma}(\mathfrak{D}) : \sigma^{+}, \xi^{+}$.

We picture this in Figure 1, where we indicate an action on ξ simply with the name ξ.

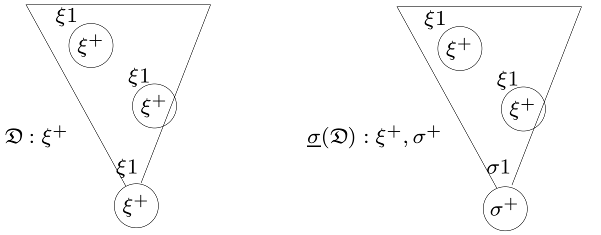


Figure 1. Renaming of the root

Copies of a behaviour: To emphasize that $\mathbf{A}$ is a set of strategies on ξ, we annotate the name ξ as a subscript: $\mathbf{A}_{\xi}$. If $\mathbf{A}_{\xi}$ is a set of strategies on the name ξ, we write $\mathbf{A}_{\sigma}$ for $\{\sigma(\mathfrak{D}), \mathfrak{D} \in A_{\xi}\}$. $\mathbf{A}_{\sigma}$ is a copy of $\mathbf{A}_{\xi}$: they are equal up to renaming.

A. Composition (normalization)

In strategies, actions can be repeated. Composition of strategies as sets of views can be described via the VAM abstract machines introduced in [6]. In this paper, what we use is that composition has a fundamental property, expressed by Proposition VI.1, where we use the operations of renaming and renaming of the root described above:

Proposition VI.1 (Copies)**.** $[\![\mathfrak{D}, \mathfrak{E}]\!] = [\![\underline{\sigma}(\mathfrak{D}), \mathfrak{E}, \sigma(\mathfrak{E})]\!]$.

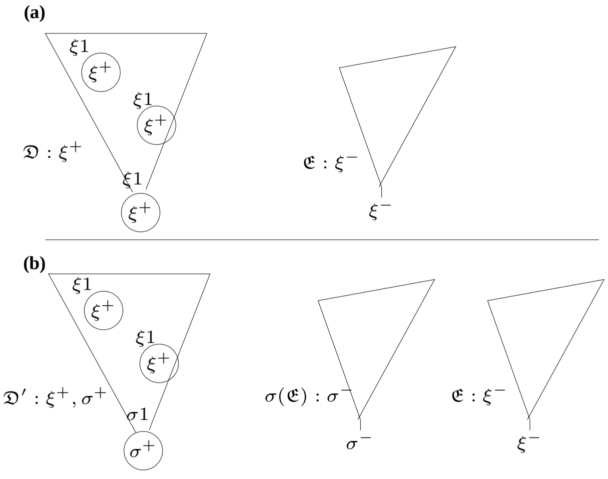


Figure 2. Composition (with repeated actions)

Let us motivate this property, which actually gives a description of the composition.

Let $\mathfrak{D} : \xi^{+}$ and $\mathfrak{E} : \xi^{-}$ be two strategies, which we represent in Figure 2 (a) (again, we indicate an action x on ξ simply with the name ξ). The idea behind the abstract machine in [6] is that, when the two strategies $\mathfrak{D}$ and $\mathfrak{E}$ interact, every time $\mathfrak{D}$ plays an action x on ξ, a copy of $\mathfrak{E}$ is created; i.e., composition works as if we had a copy of $\mathfrak{E}$ for each occurrence of x in $\mathfrak{D}$. It is rather intuitive that the result of normalization is the same if we make this explicit, by renaming one occurrence of x (namely the root), and making an explicit copy of $\mathfrak{E}$, as illustrated in Figure 2 (b).

B. What are the difficulties

We are ready to discuss which are the difficulties in extending the approach of Ludics to a setting where strategies are non linear.

Problem 1: Separation: The first problem is the failure of separation (we discuss an example of this in [4]). A main reason why previous attempts at the extension of Ludics with exponentials blocked on that, is because all proofs in [12] make essential use of a property built on it. Our key observation is that, even if separation is an important property, its failure is a relative problem, in the sense that we can still have interactive types and internal completeness.

Problem 2: Enough tests (counter-strategies): The second problem has to do with having enough tests, i.e. enough counter-strategies. Let us explain this.
As in [12], we define an interactive type to be any set of strategies closed by biorthogonal. Assume we have defined how to interpret formulas, and in particular $?A$ and $!A^{\perp}$. We would like to associate to each "good" strategy in the interpretation of a formula, for example $?\mathbf{A}$, a syntactic proof of $?A$ (full completeness).

If $\mathfrak{D} : \xi^{+} \in ?\mathbf{A}_{\xi}$, we would like to transform it into a strategy $\mathfrak{D}' \in \vdash ?\mathbf{A}_{\xi}, ?\mathbf{A}_{\sigma}$ (where distinct names indicate distinct copies). This corresponds to a contraction rule (in its upwards reading).

A natural idea is to rename the root, and all the actions which are hereditarily justified by it. We have already illustrated this operation Figure 1. From $\mathfrak{D} : \xi^+$, we obtain a new strategy $\mathfrak{D}' : \xi^+, \sigma^+$, where $\mathfrak{D}' = \underline{\sigma}(\mathfrak{D})$.
We would like to prove that:

$$(*)\ \mathfrak{D} \in \vdash ?\mathbf{A}_\xi \quad \Rightarrow \quad (**)\ \underline{\sigma}(\mathfrak{D}) \in \vdash ?\mathbf{A}_\xi, ?\mathbf{A}_\sigma$$

To have $(**)$, we need (see Definition IV.9) to know that $[\![\underline{\sigma}(\mathfrak{D}), \mathfrak{E}, \mathfrak{F}]\!] = \dagger$ for each $\mathfrak{E} \in (?\mathbf{A}_\xi)^\perp$ and each $\mathfrak{F} \in (?\mathbf{A}_\sigma)^\perp$. Since $(?\mathbf{A}_\sigma)^\perp$ is a copy (renamed in σ) of $(?\mathbf{A}_\xi)^\perp$, we can also write this condition as: $[\![\underline{\sigma}(\mathfrak{D}), \mathfrak{E}, \sigma(\mathfrak{F})]\!] = \dagger$, where both $\mathfrak{F}$ and $\mathfrak{E}$ vary in $(?\mathbf{A}_\xi)^\perp$.

Unfortunately Proposition VI.1 only gives us that $[\![\underline{\sigma}(\mathfrak{D}), \mathfrak{E}, \sigma(\mathfrak{E})]\!] = \dagger$, where we have two copies of the *same* (up to renaming) strategy $\mathfrak{E}$. This correspond to the fact that in the HO-style setting, strategies in $!\mathbf{C}$ are *uniform:* every time we find a repeated action of ”type” $?\mathbf{C}^\perp$, Opponent ($!\mathbf{C}$) reacts in the same way.

C. A solution: non-uniform tests

The need of having enough tests is similar to the one which has led Girard to the introduction of the *daimon rule*. In our case, this need leads us to enlarge the universe of tests by introducing *non-uniform counter-strategies*. This is extremely natural to realize in an AJM-style setting [1], [3], where a strategy of type $!\mathbf{C}$ is a sort of infinite tensor of strategies on $\mathbf{C}$, each one with its *index of copy*. To have HO-style non-uniform counter-strategies, we introduce a non-deterministic sum of strategies. Let us illustrate the idea, which we will formalize in the next section.

Non-uniform counter-strategies: The idea is to allow a *“non-deterministic sum”* of *negative* strategies. Let us, for now, *informally* write the sum of $\mathfrak{E}$ and $\mathfrak{F}$ this way:

$$\tau.\mathfrak{E} + \tau.\mathfrak{F}$$

Normalization may have to use several times this strategy, hence entering the strategy several times. Every time it is presented with this choice, normalization will non-deterministically choose one of the two possible continuations. The choice can be different at each repetition.

Let us define **orthogonality**, by setting $\mathfrak{D}\perp(\tau.\mathfrak{E} + \tau.\mathfrak{F})$ iff $[\![\mathfrak{D}, \tau.\mathfrak{E} + \tau.\mathfrak{F}]\!] = \dagger$ for each possible choice.
It is immediate that

$$(\text{***})\ \mathfrak{D}\perp(\tau.\mathfrak{E} + \tau.\mathfrak{F}) \Rightarrow \mathfrak{D}\perp\mathfrak{E} \textit{ and } \mathfrak{D}\perp\mathfrak{F}.$$

As we will see, if $\mathfrak{E} \in \mathbf{G}$ and $\mathfrak{F} \in \mathbf{G}$ for $\mathbf{G}$ interpretation of a formula, we have that $(\tau.\mathfrak{E} + \tau.\mathfrak{F}) \in \mathbf{G}$, and vice-versa. Hence, if $\mathfrak{D} \in ?\mathbf{A}$, for each $\mathfrak{E}, \mathfrak{F} \in (?\mathbf{A})^\perp$ we have $[\![\mathfrak{D}, \tau.\mathfrak{E} + \tau.\mathfrak{F}]\!] = \dagger$. Using Proposition VI.1 we have that $[\![\underline{\sigma}(\mathfrak{D}), (\tau.\mathfrak{E} + \tau.\mathfrak{F}), \sigma(\tau.\mathfrak{E} + \tau.\mathfrak{F})]\!] = \dagger$. Using (***), we deduce that $[\![\underline{\sigma}(\mathfrak{D}), \mathfrak{E}, \sigma(\mathfrak{F})]\!] = \dagger$, as we wanted.

D. Linearity of the root

Observe that, by construction, in $\underline{\sigma}(\mathfrak{D})$ the action at the root is positive and it is the only action on the name σ. We can hence apply the argument we have already given in Section V.1 for the internal completeness of Tensor.

As a consequence, if $A = A_1 \otimes A_2$, given $\mathfrak{D} \in \vdash ?\mathbf{A}$, we have that $\underline{\sigma}(\mathfrak{D})$ actually belongs to $\vdash \mathbf{A}, ?\mathbf{A}$, and can be decomposed in strategies $\underline{\sigma}(\mathfrak{D})_i \in \vdash \mathbf{A}_i, ?\mathbf{A}$.

This allows us to associate to $\mathfrak{D} \in \vdash ?\mathbf{A}, \mathbf{\Gamma}$ a proof which essentially has this form:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash A_1, ?(A_1 \otimes A_2), \Gamma} \qquad \dfrac{\vdots}{\vdash A_2, ?(A_1 \otimes A_2), \Gamma}}{\vdash A_1 \otimes A_2, ?(A_1 \otimes A_2), \Gamma}\ {\scriptstyle \otimes + contr}}{\vdash ?(A_1 \otimes A_2), \Gamma}\ {\scriptstyle dereliction + contr}$$

VII. Ludics with repetitions: Non-Uniform strategies

In this Section, we implement technically the ideas presented in the Section VI-C. In particular, we revise the definition of arena and strategy so to accommodate neutral actions, which correspond to the τ action we have just seen.

We extend the set of the **polarities** with a *neutral polarity*, hence we have now three possibilities: positive, negative and neutral.
We extend the set of **actions** with a set $T = \{\tau_i, i \in \mathbb{N}\}$ of *indexed tau actions*, whose polarity is defined to be neutral. We denote by T also the neutral arena, where the set of moves is T, the enabling relation is empty, and the polarity is neutral. We revise strategies (Definition III.2) giving the following definitions.

Definition VII.1 (N.U. Strategies)**.** *Let A be a (positive or negative) arena.*

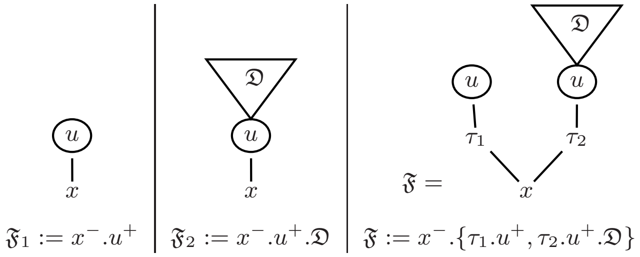
- *A* **Non-Uniform justified sequence** *s on A is a justified sequence (in the sense of Definition III.2) on $A \cup T$, which moreover satisfies the following property:*
  - Neutral actions. *If s_i is a tau action, then s_{i-1} is negative and s_{i+1} is positive.*
- *A* **Non-Uniform view** *on A is a view on $A \cup T$.*
- *A* ***Non-Uniform strategy*** *(or N.U. strategy) on A is a strategy on $A \cup T$, with the following modified coherence condition:*
  - Coherence. *If $s.m, s.n \in \mathfrak{D}$ and $s.m \neq s.n$ then m, n are either both negative or both neutral actions.*
- ***A Non-Uniform strategy*** $\mathfrak{D}$ ***on*** Γ*, written $\mathfrak{D} : \Gamma$, is a Non-Uniform strategy on the arena $U(\Gamma) \cup T$.*

From now on, we only consider N.U. strategies (the usual ones being a special case). Figure 3 below shows an example of N.U. strategy.

A. Sum of strategies

As anticipated in Section VI-C, our N.U. strategies can be seen as a non deterministic sum of standard strategies. We use N.U. strategies to capture the idea of "non uniform" tests. Let us make precise what does it mean for a strategy to be uniform or not.

Definition VII.2 (Uniform actions)**.** *Given an N.U. strategy $\mathfrak{D}$, we say that a* negative *occurrence of action x^- is* **uniform** *if x^- is immediately followed by a positive occurrence of action (and not by tau actions). If $\mathbf{N}$ is a set of negative strategies, we define:* $\mathsf{Unif}\ \mathbf{N} := \{\mathfrak{D} \in \mathbf{N} :$ *the root x^- of $\mathfrak{D}$ is uniform* $\}$*.*



u
x
$\mathfrak{F}_1 := x^-.u^+$
$\mathfrak{D}$
u
x
$\mathfrak{F}_2 := x^-.u^+.\mathfrak{D}$
$\mathfrak{D}$
u
u
τ_1
τ_2
$\mathfrak{F} =$
x
$\mathfrak{F} := x^-.\{\tau_1.u^+, \tau_2.u^+.\mathfrak{D}\}$

Figure 3. N.U. strategies

In Figure 3, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ have uniform root, while the root of $\mathfrak{F}$ is not. $\mathfrak{F}$ can be seen as a non-deterministic sum of $\mathfrak{F}_1$ and $\mathfrak{F}_2$.

It is immediate that a strategy whose root is non uniform can always be written as a sum of strategies whose root is uniform. We formalize this in the following:

Definition VII.3 (τ-sum)**.** *Let $\{\mathfrak{D}_i : \Gamma\}_{i \in S}$ be a family of negative N.U. strategies such that all $\mathfrak{D}_i = x^-.\mathfrak{E}_i$ have the same uniform root. We define their* ***sum****: $\sum_{i \in S}^{\tau} \mathfrak{D}_i := \bigcup_{i \in S} x^-.\{\tau_i.\mathfrak{E}_i\}$. When S is a finite set, say $\{1, \ldots, k\}$, we write $\mathfrak{D}_1 +^{\tau} \ldots +^{\tau} \mathfrak{D}_k$.*

A τ-sum of strategies can be seen as a *superposition* of negative strategies in a such way that they do not overlap (except for the first negative action). The following proposition is an immediate consequence of the definitions:

Proposition VII.4. *If $x^-.\mathfrak{F}$ is a negative N.U. strategy, either its root is uniform or $x^-.\mathfrak{F} = \sum^{\tau} x^-.\mathfrak{F}_i$, where the root of each $x^-.\mathfrak{F}_i$ is uniform.*

B. Orthogonality

We have sketched the definition of *composition* for N.U. strategies in Section VI-B[1].The reader here does not need the details of composition. All results use Corollary VII.6 and the fundamental property expressed by Proposition VIII.1.

The result of normalizing N.U. strategies will depend on the choice performed at the τ-sums, in other words there are several normal forms: there is a result for each possible choice. We say that two strategies are orthogonal if, for each possible choice, the result is always $\dagger$.

Definition VII.5 (Orthogonality)**.** *Let $\mathfrak{F} : \xi^-, \mathfrak{E} : \xi^+$ be N.U. strategies. $\mathfrak{E} \perp \mathfrak{F}$ if and only if $[\![\mathfrak{E}, \mathfrak{F}]\!] = \dagger$, for each possible result.*

The following is an immediate consequence:

Corollary VII.6. *Let $\mathfrak{F}$ be a negative strategy such that $\mathfrak{F} = \sum^{\tau} \mathfrak{F}_i$. We have that: $\mathfrak{E} \perp \mathfrak{F} \Rightarrow \forall i, \mathfrak{E} \perp \mathfrak{F}_i$.*

VIII. Ludics with repetitions: internal (and full) completeness

We give constructions on behaviours, and prove that they enjoy internal completeness. Since these constructions will be used to interpret formulas, full completeness will be a consequence.

Constant types: We define the positive (resp. negative) constant behaviour on ξ as follows:

$$?\mathbf{0} := \{\dagger\}^{\perp\perp} \qquad !\top := \{\dagger\}^{\perp}$$

It is immediate that $\{\dagger\}^{\perp\perp} = \{\dagger\}$ and that $!\top$ is the set of all negative strategies on ξ.

Compound types: In this section, we use the same constructions on strategies and operations on sets of strategies as in Section IV-C (observe that since strategies have repetitions, even when starting from the same set, the closure by bi-orthogonal introduces many more strategies than in the linear case).

Let $\mathbf{N}_1, \ldots, \mathbf{N}_n$ be negative behaviours respectively on $\xi 1, \ldots, \xi n$. We define a new positive (resp. negative) behaviour on ξ as follows:

$$\begin{array}{rcl} \mathbf{F}^+(\mathbf{N}_1, \ldots, \mathbf{N}_n) & := & (\mathbf{N}_1 \bullet \cdots \bullet \mathbf{N}_n)^{\perp\perp} \\ \mathbf{F}^-(\mathbf{N}_1{}^{\perp}, \ldots, \mathbf{N}_n{}^{\perp}) & := & (\mathbf{N}_1 \bullet \cdots \bullet \mathbf{N}_n)^{\perp} \end{array}$$

From now on, we write $\mathbf{N}$ for negative and $\mathbf{P}$ for positive behaviours given by the constructions above.

A. Internal completeness

We have the following property which characterizes the relation of orthogonality for τ-sums of strategies that belong to negative behaviours.

Proposition VIII.1. *Let $\mathbf{N} = (\mathbf{N}_1 \bullet \cdots \bullet \mathbf{N}_n)^{\perp}$ be a negative behaviour. If $\{\mathfrak{E}_i\} \subseteq \mathsf{Unif}\ \mathbf{N}$ is a non empty denumerable set of negative strategies of the form $\mathfrak{E}_i = x^-.\mathfrak{F}_i$, we have that*

$$\textstyle\sum^{\tau} x^-.\mathfrak{F}_i \in \mathbf{N}.$$

Together with Corollary VII.6, this gives the following.

Lemma VIII.2. $(\mathsf{Unif}\ \mathbf{N})^{\perp} = \mathbf{N}^{\perp}$. *Hence* $\mathbf{N} = \mathsf{Unif}\ \mathbf{N})^{\perp\perp}$.

[1] The details are given in the full paper [4].

This lemma expresses the fact that the study of a negative behaviour $\mathbf{N}$ can be reduced to the study of $\mathsf{Unif}\ \mathbf{N}$. We will exploit this property both in internal and full completeness.

Proposition VIII.3 (Internal completeness of $\mathbf{F}^-$)**.** *If* $x^-.\mathfrak{F} \in \mathsf{Unif}\ \mathbf{F}^-(\mathbf{P}_1, \dots, \mathbf{P}_n)$ *then* $\mathfrak{F} \in\ \vdash \mathbf{P}_1, \dots, \mathbf{P}_n$*.*

Proof: The proof is as in the linear case. ■

Lemma VIII.4. *If* $\mathfrak{D} \in \mathbf{P}_\xi$ *then* $\underline{\sigma}(\mathfrak{D}) \in\ \vdash \mathbf{P}_\xi, \mathbf{P}_\sigma$*. Moreover, the only occurrence of action on* σ *is the root.*

Proof: If $\mathfrak{D} \in \mathbf{P}_\xi$, then for all pairs $\mathfrak{E}, \mathfrak{F} \in \mathbf{P}_\xi{}^\perp$, $\mathfrak{D} \perp \mathfrak{E}$ and $\mathfrak{D} \perp \mathfrak{F}$. By Proposition VIII.1, we have that $\mathfrak{D} \perp \mathfrak{E} +^\tau \mathfrak{F}$, that is $\llbracket \mathfrak{D}, \mathfrak{E} +^\tau \mathfrak{F} \rrbracket = \dagger$. Using Proposition VI.1, we have that $\llbracket \underline{\sigma}(\mathfrak{D}), \mathfrak{E} +^\tau \mathfrak{F}, \sigma(\mathfrak{E} +^\tau \mathfrak{F}) \rrbracket = \dagger$ and by definition of non-deterministic normalization, we have that $\llbracket \underline{\sigma}(\mathfrak{D}), \mathfrak{E}, \sigma(\mathfrak{F}) \rrbracket = \dagger$, that is $\underline{\sigma}(\mathfrak{D}) \in\ \vdash \mathbf{P}_\xi, \mathbf{P}_\sigma$. ■

Remark VIII.5. *Non-uniform strategies allow us to* superpose *two negative strategies (with the same root). The capability of defining such a kind of superposition is the* core *of our solution to interpret contraction rule.*

Theorem VIII.6 (Internal completeness of $\mathbf{F}^+$)**.** *If* $\mathfrak{D} \in \mathbf{F}^+(\mathbf{N}_1, \dots, \mathbf{N}_n)$*, then: either* $\mathfrak{D} = \dagger$ *or* $\underline{\sigma}(\mathfrak{D}) = \mathfrak{D}'_1 \bullet \dots \bullet \mathfrak{D}'_n$ *where* $\mathfrak{D}'_i \in\ \vdash \mathbf{N}_i, \mathbf{F}^+$*.*

Proof: Let $\mathbf{F}^+ = \mathbf{F}^+(\mathbf{N}_1, \dots, \mathbf{N}_n)$. By Lemma VIII.4, we have that if $\mathfrak{D} \in \mathbf{F}^+_\xi$, then $\underline{\sigma}(\mathfrak{D}) \in\ \vdash \mathbf{F}^+_\xi, \mathbf{F}^+_\sigma$. Moreover, the root is an action on the name σ, and it is the only occurrence of action on σ. By using the same argument as in Proposition V.1, we have that $\underline{\sigma}(\mathfrak{D}) = \mathfrak{D}'_1 \bullet \dots \bullet \mathfrak{D}'_n$ and $\mathfrak{D}'_i \in\ \vdash \mathbf{N}_i, \mathbf{F}^+$, i.e. $\underline{\sigma}(\mathfrak{D})\restriction_i\ \in\ \vdash \mathbf{N}_i, \mathbf{F}^+(\mathbf{N}_1, \dots, \mathbf{N}_n)$. ■

B. Full completeness

In this paper we have chosen to give enough details on internal completeness, because it is the more peculiar, and hence the more interesting, property. Once established internal completeness, one can also obtain full completeness. Our model is fully complete with respect to **MELLS** (section II). The details are given in [4]

The interpretation $\parallel F \parallel_\xi$ of a formula F is given the corresponding behaviour $\mathbf{F}$, once chosen a name ξ:

$$\begin{array}{rclrcl} \parallel !\top \parallel_\xi & := & !\top_\xi & \parallel ?\mathbf{0} \parallel_\xi & := & ?\mathbf{0}_\xi \\ \parallel F^+(N_1, \dots, N_n) \parallel_\xi & := & \mathbf{F}^+(\parallel N_1 \parallel_{\xi 1}, \dots, \parallel N_n \parallel_{\xi n})_\xi \\ \parallel F^-(P_1, \dots, P_n) \parallel_\xi & := & \mathbf{F}^-(\parallel P_1 \parallel_{\xi 1}, \dots, \parallel P_n \parallel_{\xi n})_\xi \end{array}$$

A sequent $\vdash F_1, \dots, F_n$ is interpreted by the sequent of behaviours $\vdash \mathbf{F}_1, \dots, \mathbf{F}_n$.

The interpretation of a proof is a strategy which is *uniform* and winning according to the definition given in [12] (i.e. daimon-free, finite and *material* in its behaviour). We have the following results, whose proof can be found in [4].

Theorem VIII.7 (Interpretation)**.** *Let* π *be a proof of a sequent* $\vdash \Gamma$ *in* **MELLS***. There exists a winning strategy* $\mathfrak{D} \in\ \vdash \mathbf{\Gamma}$ *such that* $\mathfrak{D}$ *is interpretation of* π*.*

Theorem VIII.8 (Correctness of the interpretation)**.** *If* π *is a proof of* $\vdash \Gamma$ *in* **MELLS** *which reduces to* π'*, then if* $\mathfrak{D}$ *is the interpretation of* π *and* $\mathfrak{D}'$ *is the interpretation of* π'*, we have that* $\mathfrak{D} = \mathfrak{D}'$*.*

Theorem VIII.9 (Full Completeness)**.** *If* $\mathfrak{D}$ *is a winning strategy in a sequent of behaviours* $\vdash \mathbf{\Gamma}$ *then* $\mathfrak{D}$ *is the interpretation of a cut-free proof* π *of the sequent* $\vdash \Gamma$ *in* **MELLS**.

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