Partial orders, event structures and linear strategies

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Abstract. We introduce a Game Semantics where strategies are partial orders, and composition is a generalization of the merging of orders. Building on this, to bridge between Game Semantics and Concurrency, we explore the relation between Event Structures and Linear Strategies. The former are a true concurrency model introduced by Nielsen, Plotkin, Winskel, the latter a family of linear innocent strategies developed starting from Girard's work in the setting of Ludics.
We extend our construction on partial orders to classes of event structures, showing how to reduce composition of event structures to the simple definition of merging of orders. Finally, we introduce a compact closed category of event structures which embeds Linear Strategies.

1 Introduction and background

Game Semantics has been successful in providing accurate (fully abstract) models to programming languages and logical systems; the key feature of such a semantics is to be *interactive*. Computation is interpreted as a *play* (an interaction) between two players, where `Player` (`P`) represents the program/proof, and `Opponent` (`O`) represents the environment, the context. The set of the possible plays represents the operational behavior of a term, and is called a *strategy*. The play should respect some "rules of the game", expressed by an *arena,* which denotes a type.

Since interaction is the main feature also of a *concurrent system* and of *process calculi*, it appears natural to search for an extension of Game Semantics able to model parallel and concurrent computation. This is indeed an active line of research, even though still at its early stage. A way to allow parallelism is to relax sequentiality, and have plays (i.e. traces of computation) which are "partial orders" instead of totally ordered sequences of moves. The intent of this approach is actually two-folded: to allow for parallelism, and to provide partial order models of sequential computation, i.e. models where the scheduling in which the actions should be performed is not completely specified, while it is still possible to express constraints: certain tasks may have to be performed before other tasks; other actions can be performed in parallel, or scheduled in any order.

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A Game Semantics where strategies would be partial orders has been first propounded by Hyland in a seminal talk in Lyon. Our paper is an effort to pursue this direction. We introduce a Game Semantics where strategies are partial orders, and composition is the merging of the orders; on this basis, we are led to investigate how a class of strategies, here called *linear strategies*, fits inside a larger picture where we have *true concurrency* models.

Linear strategies. In this paper, we use the name *linear strategies* to designate a family of strategies originated by the work of Girard in Ludics [10]. Ludics can be seen as a Game Semantics where the foundational role of interaction is taken even further; moreover, many features (actions, names, a built-in observational equivalence) make it close to process calculi. We mention that, as established in [7], Ludics has a close relation with the Linear Pi-calculus [26], i.e. a process calculus which is *asynchronous* and internal (see [20]). The strategies defined in Ludics (called *designs* in the original paper) can be seen as a (linear) variant of Hyland-Ong innocent strategies. In [5] was then proposed a "more parallel" version of Ludics, leading to the introduction of graph strategies, called L-nets; these are in many ways close to proof-nets. Exploring this and moving through several degree of sequentiality is the object of [4].

True concurrency. In the literature, there are two main approaches in the study of models for parallel and concurrent programming languages. The first one is represented by *interleaving models*, that describe a concurrent system by means of possible scheduling of concurrent actions (all traces). The second one is represented by *causal models*, (also called *true-concurrent models*) in which concurrency, dependency and conflict relations among actions are directly expressed. A fundamental instance of true concurrent models are *event structures*, introduced by Nielsen, Plotkin and Winskel [19, 23, 24] as a theory combining Petri Nets and Domain Theory. An event structure describes a concurrent system in terms of a *partial order*, which specifies the causality relation between actions, and a *conflict relation*, which specifies what actions are mutually exclusive.

In a previous work [6], we have proposed event structures as a mathematical framework unifying proof-nets and linear strategies. Building on previous work by Varacca and Yoshida [22], there we have given a general definition of composition, based on an abstract machine, which realizes both event structures and game semantics composition. Moreover, we have shown that linear strategies (and proof-nets, in a sense) correspond to a particular subclass of event structure called *confusion free event structures*; these latter model a kind of well-behaving non-determinism, in which the choice is localized in "cells".

Contributions of this paper. In this paper, we introduce a Game Semantics, where strategies are partial orders (called *po strategies*) and composition is a generalization of *merging of order* (Section 2). The *merging of two orders* has been defined by Girard in [10] as the transitive closure of their set-theoretical union. Under certain acyclicity conditions, the result is a partial order. This is the base of composition of linear strategies, and also of our po strategies. More precisely, a central contribution of this paper is to generalize this operation, without requiring any acyclicity condition. The main advantage of this definition is that it appears mathematically clean and attractive, and this translates into direct and -we believe- clearer proofs.

The Game Semantics we defined is rather general, since we do not require our strategies *neither to be* `Player`/`Opponent` *alternating nor to be sequential*. Moreover we admit an additional neutral polarity; intuitively, neutral actions correspond to internal τ-actions of process calculus.

Following an idea proposed by Hyland, we show how to extend the notion of *innocence* to partial order strategies, i.e. to a setting which is parallel and non-alternating. In the Hyland-Ong game model [12], innocence is an important property on strategies, in order to obtain a well-defined Cartesian closed category which provides a model for PCF and whose effective part consists of morphisms (strategies) that are PCF-definable. In this paper we will see how to extend innocence to a parallel and non alternating setting showing that such a condition allow us to obtain a well defined category where arenas are objects and po strategies are morphisms.

In Section 3 we show how to generalize this construction to *event structures*, in order to define a general setting embedding linear strategies. Arenas are generalized to *ES-arenas* and po-strategies become *typed event structures*. All these structures can be described in terms of special families of po-strategies. Using this characterization, we show how to define a simple notion of composition of typed event structure relying on the definition of merging of partial orders. This definition of composition corresponds to the one given in [22]. We conclude then by generalizing the category of innocent po-strategies to a compact closed category of *innocent event structures*. Finally, starting from this category, we will retrieve linear strategies as a sub-class.

Related work. The exploration of concurrent Game Semantics has been initiated by Abramsky and Melliès with the introduction of Concurrent Games [2], which give a fully complete model of Multiplicative-Additive Linear Logic; strategies are here closure operators. Melliès and Mimram have then developed the fertile line of asynchronous games [17, 18], where plays are seen as Mazurkiewicz traces, and innocence receives a diagrammatic formulation. Since the purpose of this line of work is a better understanding and a generalization of innocence, there are several connections between our work and in particular [18], where strategies also do not require neither alternation nor sequentiality. A main difference is the fact to work with traces or with partial orders, which is the choice which allows us to rely on the merging of orders for composition.

Graph strategies have been introduced by Hyland, Schalk [14, 21]. Partial order models have then be proposed by McCusker [16], where partial order models where used to study programming languages. The work of Curien, Faggian and Maurel [4, 5] also fit in this line.

Game semantics for languages equipped with concurrency features have been developed by Ghica, Laird and Murawsky [15, 9]. Strategies are here described as set of traces.

2 Partial orders as strategies

In this section, we introduce a notion of arena and a notion of strategy on an arena, where a strategy is a partial order (*po strategies*). We follow a minimalist approach, in the spirit of [11,3]. We first introduce a rather general setting in which we have po strategies and a notion of *parallel composition* (which is well defined and associative); we then gradually add properties (innocence, alternation, arborescence), and show that these properties are all preserved by composition. Furthermore, by introducing a notion of innocence, we eventually obtain (Section 2.4) a category, where the objects are arenas, the arrows are innocent po strategies, and their *composition* factorizes as parallel composition plus hiding. However, since both parallel composition and hiding have an independent interest, we discuss them separately. At the end of Section 2.3, after giving the definitions, we will discussion the connection between the notions we introduce and the standard notion of innocent strategy.

We first recall some preliminary notions on partial orders. A **strict partial order** (**spo** for short) is a pair $\langle X, <_X \rangle$ where X is a set, and $<_X$ is binary, transitive and irreflexive relation and it is often denoted simply as X. We will use $X, Y, Z, W, \ldots$ to range over spos. We will move freely between a strict partial order $\langle X, <_X \rangle$ and the **partially ordered set** or **poset** $\langle X, \leq_X \rangle$, where $\leq_X$ is the reflexive closure of $<_X$. Given an element $x \in X$, the set of its **enabling elements** is $[x) = \{x' \in X \| x' <_X x\}$. An spo is **well founded** if the set $[x)$ is finite for all $x \in X$. It is **arborescent** if it is well founded and each $x \in X$ has at most one immediate predecessor; we call **forest** a strict partial order which is arborescent (we will talk of roots and children in the obvious way). A subset $S \subseteq X$ is said to be **downward-closed** if for all $x \in S$, if $y <_X x$ then $y \in S$.

2.1 Arenas and po strategies

A **polarity** ϵ is an element of the set $\{+, -, \pm\}$: The *positive polarity* $+$ corresponds to `Player`, while the *negative polarity* $-$ corresponds to `Opponent`. The *neutral polarity* $\pm$ plays a role similar to that of a τ action in a process calculus. Given a polarity ϵ, its *dual* $\epsilon^\perp$ is defined as $+^\perp = -$ and $-^\perp = +$, while $\pm^\perp$ is undefined.

An arena is a set of elements, here called actions, together with a polarity for each action, and possibly some structure of order on the actions (which may express causality or dependency). Our choice here is that the order to is a forest (but this is not necessary).

Definition 1. *An* arena *is a set Γ, whose elements are called* actions*, together with a strict partial order relation $<_\Gamma$ and a* polarization function $\pi_\Gamma : \Gamma \to \{+, -, \pm\}$, *which satisfies: (1.) $\langle \Gamma, <_\Gamma \rangle$ is a forest. (2.) For each $a \in \Gamma$, if $\pi(a) = \pm$, then $\pi(c) = \pm$, for all $c \in \Gamma$ which are comparable with a.*

With a slight abuse of notation, we will denote in the same way the arena and the set of actions. We use Greek capital letters to range over arenas. Given an arena Γ, we use $a, b, c, \ldots$ to range over its actions. A action a is **initial** if it is a root of $\langle \Gamma, <_\Gamma \rangle$. If b is immediate predecessor of c, we say that b **justifies** c, and write $b \vdash_\Gamma c$. An arena Γ is called **+/- polarized** if $\pi_\Gamma(\Gamma) \subseteq \{+, -\}$, it is **neutral** if $\pi_\Gamma(\Gamma) = \{\pm\}$. An arena Γ is **alternating** if $a_1 \vdash_\Gamma a_2 \Rightarrow \pi(a_1) = \pi(a_2)^\perp$.

Observe that our definition of arena is rather general, and does *not require alternation.* By definition, if $b \vdash_\Gamma c$, either none of them is neutral, or both are neutral.

Constructions on arenas. Given two arena Γ_1, Γ_2 which are **disjoint**, i.e. $\Gamma_1 \cap \Gamma_2 = \emptyset$, we write Γ, Δ or, for better readability, $\Gamma \odot \Delta$ for $\Gamma_1 \cup \Gamma_2$ with the order and polarization which are induced by the Γ_i's.

Given an arena Γ, the **neutral** arena $\Gamma^\pm$ is the arena having the same actions and order relation as Γ, but neutral polarity for all its actions: $\pi_{\Gamma^\pm}(a) = \pm$, for each $a \in \Gamma$.

Given a +/- polarized arena Γ, its **dual** $\Gamma^\perp$ is the arena having the same actions and order relation as Γ, but inverting their polarity: $\pi_{\Gamma^\perp}(a) = \pi_\Gamma(a)^\perp$.

Remark 1. Observe that $\Gamma^\perp$ is not defined on neutral arenas. By writing $\Gamma^\perp$ we implicitly assume that Γ is +/- polarized.

Po strategies. We are interested in spos X whose elements are taken on an arena Γ. We require that, for the elements, $X \subseteq \Gamma$ and, for the order, $<_X$ respects and *refines* $<_\Gamma$ (see Condition (1.) below). We will call such spos *po strategies*.

Definition 2. *Let Γ be an arena. A spo X on Γ is a well founded spo such that $X \subseteq \Gamma$ (as sets of actions). X is a* a po strategy *on the arena Γ, written $X : \Gamma$ when it satisfies: (1.) For each $b \in X$ and $a \in \Gamma$, if $a \vdash_\Gamma b$ then $a \in X$ and $a <_X b$. (2.) For each c which is maximal in X, $\pi_\Gamma(c) = +$.*

Both conditions reformulate standard conditions in Game Semantics; the first condition is usually called **justification**. The second one means that the strategy always has an answer to an opponent move.

Observe that each element in $X : \Gamma$ has a polarity, the one given by π_Γ. Given $a, b \in X$, we write $a \leftarrow_X b$ if a is an immediate predecessor of b according to $<_X$. We remind that we write $a \vdash_\Gamma b$ for the same relation in the arena.

Let X and Y be spos. If $X : \Gamma$, is a po strategy, we write $Y \sqsubseteq X$ if $Y \subseteq X$ is downward closed and for all $a, b \in Y$ $a <_Y b$ iff $a <_X b$. We write $Y \sqsubseteq^+ X$ if moreover the maximal elements of Y have positive polarity. It is immediate that if $Y \sqsubseteq^+ X$, then $Y : \Gamma$.

2.2 Parallel composition

If X is a po strategy on Γ, and $\Gamma = \Gamma' \odot \Gamma''$ we define $X \upharpoonright_{\Gamma'} = \{c \in X | c \in \Gamma'\}$ to be the **restriction** of X to Γ'. We compose two po strategies X_1, X_2 when they are **compatible**, that is $X_1 : \Gamma_1 \odot \Lambda^\perp$ and $X_2 : \Lambda \odot \Gamma_2$ ($\Gamma_1, \Gamma_2, \Lambda$ disjoint).

Definition 3 (Parallel composition). *Given two orders, Z_1 and Z_2, we write $Z_1 \sqcup Z_2$ for the transitive closure of $Z_1 \cup Z_2$ (which does not need to be an order).*

Let $X_1 : \Gamma_1 \odot \Lambda^\perp$ and $X_2 : \Lambda \odot \Gamma_2$ ($\Gamma_1, \Gamma_2, \Lambda$ disjoint). The actions on the common arena Λ are called **private**. *We define*

$$Int(X_1, X_2) = \left\{ Y \text{ spo on } \Gamma_1 \odot \Gamma_2 \odot \Lambda^\pm \mid \exists Y_1, Y_2 \text{ s.t. } \begin{array}{l} Y = Y_1 \sqcup Y_2, \\ Y \restriction_{\Gamma_1 \odot \Lambda^\perp} = Y_1 \sqsubseteq^+ X_1 \\ Y \restriction_{\Lambda \odot \Gamma_2} = Y_2 \sqsubseteq^+ X_2 \end{array} \right\}$$

We define the parallel composition *of X_1, X_2 as $X_1 \| X_2 = \max Int(X_1, X_2)$.*

We need to prove that $Int(X_1, X_2)$ has a unique maximal element. The proof makes essential use of the following immediate Lemma.

Lemma 1. *Let X_1, X_2 and Y be as in Definition 3. If $Y_1 = Y \restriction_{\Gamma_1, \Lambda^\perp} \sqsubseteq X_1$ and $Y_2 \restriction_{\Gamma_2, \Lambda} \sqsubseteq X_2$, then for each private action c, $c \in Y_1$ iff $c \in Y_2$.*

Proposition 1. *The set $Int(X_1, X_2)$ given in Definition 3 has a unique maximal element (hence, parallel composition is well defined).*

Proof. Let $U = U_1 \sqcup U_2$ and $V = V_1 \sqcup V_2$ both belong to $Int(X_1, X_2)$, where for each $i \in \{1, 2\}$, $U_i = U \restriction_{\Gamma_i \odot \Lambda}$ and $V_i = V \restriction_{\Gamma_i \odot \Lambda}$ are downward closed subsets of X_i given by Definition 3. Let us set $Y_1 = U_1 \cup V_1$ and observe that $Y_1 \sqsubseteq^+ X_1$ (and similarly for Y_2). We prove that $Y = Y_1 \sqcup Y_2$ is an spo. Let $Y = (Y, <_Y)$; we have to prove that $<_Y$ is irreflexive. Assume $c \in Y$, and $c <_Y c$. Since Y is obtained by transitive closure, we have $c = c_0 <_{Z_1} c_1,\ c_1 <_{z_2} c_2,\ \ldots,\ c_{n-1} <_{Z_n} c_n = c$ where each $Z_i \in \{Y_1, Y_2\}$, and $Z_i \neq Z_{i+1}$. Let assume $c \in U_1$.By downward-closure, if $c = c_n \in U_1$, then $c_{n-1} \in U_1$. By Lemma 1, we have that $c_{n-1} \in U_2$, and we conclude with a straightforward induction, that $c <_U c$, against hypothesis. □

We now check that $X_1 \| X_2$ is a po strategy, and that the parallel composition is associative.

Theorem 1. *Let X_1, X_2 and Y be as in Definition 3. Then $X_1 \| X_2$ is a po strategy on the arena $\Gamma_1, \Gamma_2, \Lambda^\pm$.*

Proof. We check both conditions of Definition 2. Condition (1.), the one corresponding to Justification, is preserved, as a consequence of the fact that if $c \in Y = Y_1 \sqcup Y_2$, and $d <_{Y_i} c$, then $d <_Y c$. Concerning Condition (2.), if c is maximal in Y, it has to be maximal in both Y_1 and Y_2. As a consequence, $c \notin \Lambda$ (otherwise either Y_1 or Y_2 would have a negative leaf) and c is positive. □

Lemma 2 (Monotonicity). *Let $Y_1 \sqsubseteq X_1$ and $Y_2 \sqsubseteq X_2$. Then $Y_1 \| Y_2 \sqsubseteq X_1 \| X_2$*

Proof. Immediate by the definition. □

Theorem 2 (Associativity). *If X_1, X_2, X_3 are po strategies which are pairwise compatible, we have that $(X_1 \| X_2) \| X_3 = X_1 \| (X_2 \| X_3)$.*

Proof. Let $(X_1\|X_2)\|X_3 = \langle Z, <_Z\rangle$. By using twice Definition 3, we observe that $\langle Z, <_Z\rangle = (Y_1 \sqcup Y_2) \sqcup Y_3$, with each $Y_i \sqsubseteq X_i$. We observe that the transitive closure of the union is associative, and by monotonicity, we conclude that $\langle Z, <_Z\rangle = Y_1 \sqcup (Y_2 \sqcup Y_3) \sqsubseteq X_1\|(X_2\|X_3)$. The other inclusion is similar. □

In the next section, we show that composition preserves several interesting properties. We will need the following easy observations.

Proposition 2. *Let $a \in X_1\|X_2 : \Gamma_1 \odot \Gamma_2 \odot \Lambda^{\pm}$, with $X_1 : \Gamma_1 \odot \Lambda^{\perp}$ and $X_2 : \Gamma_2 \odot \Lambda$. Then (1.) If a is private ($a \in \Lambda$), then $a \in X_1$ and $a \in X_2$. Its polarity is neutral. (2.) Otherwise, either $a \in X_1$ (and $a \notin X_2$) or $a \in X_2$(and $a \notin X_1$). Moreover, assuming $a \in X_1$, then $c \leftarrow a$ (resp. $a \leftarrow c$) in $X_1\|X_2$ iff $c \leftarrow_{X_1} a$ (resp. $a \leftarrow_{X_1} c$); similarly if $a \in X_2$.*

2.3 Innocence (and alternation, and arborescence)

Composition of po strategies is associative, but -as we will see- it is not possible to define a strategy which behaves as the identity for the composition. This motivates the restriction to a class of po strategies, which we call innocent, for reasons we discuss at the end of this section. Composition preserves innocence. Moreover, more standard notions of strategies, such as arborescent strategies and alternating strategies, will be a subclass of innocent po strategies.

Definition 4. *We say that a po strategy $X : \Gamma$ is* innocent *when for all $b, c \in X$, if $b \leftarrow_X c$ and ($\pi_\Gamma(c) = -$ or $\pi_\Gamma(b) = +$) then $b \vdash_\Gamma c$*

Observe that up to now, we have been rather general. In particular, in our definitions we *do not require alternation in the polarity of the actions.*

Remark 2. If the arena Γ is alternating and +/- polarized, then our innocence condition implies +/- alternation in $X : \Gamma$.

Remark 3. Innocent po strategies allow neutral actions. The innocence condition implies that immediate predecessors (resp. successors) of neutral actions are either negative or neutral (resp. either positive, or neutral).

Proposition 3. *If $X_1 : \Gamma_1 \odot \Lambda$ and $X_2 : \Gamma_2 \odot \Lambda^{\perp}$ are innocent, then (1.) $X_1\|X_2 : \Gamma_1 \odot \Gamma_2 \odot \Lambda^{\pm}$ is innocent. (2.) Moreover if X_1, X_2 are forests, then $X_1\|X_2$ is a forest.*

Proof. (1.) Assume $c \leftarrow a$ in $X_1\|X_2 = Y_1 \sqcup Y_2$, and $\pi(a) = -$ or $\pi(c) = +$. By Remark 2, for one of the two Y_i ($i \in \{1, 2\}$), we have $c \leftarrow_{Y_i} a$. Hence, by innocence of Y_i, $c \vdash_{\Gamma_i} a$. (2.) Let $Y = X_1\|X_2$, with $Y = Y_1 \sqcup Y_2$, defined as in 3. Let T be a maximal spo such that T is a forest and $T \sqsubseteq Y$. We consider a minimal $a \in Y$ s.t. $a \notin T$. We observe that (i) each immediate predecessor of a in Y is an immediate predecessor of a in either Y_1 or Y_2; (ii) if $a \in Y_1$ (resp. $a \in Y_2$), a has at most one predecessor. We now prove that a has a unique predecessor in Y, hence the restriction of Y to $T \cup \{a\}$ is still a forest. If a is not private, the result is immediate by Proposition 2 and by (ii). The same is true if a is private, and root in one of the Y_i. If a is private, let us assume $\pi_\Lambda(a) = -$. Let $z \leftarrow_{Y_2} a$ and $b \leftarrow_{Y_1} a$. By Innocence of Y_1, we have that $b \vdash_\Lambda a$. By Justification, we have that $b \leq_{Y_2} z$. Hence z is the only possible immediate predecessor of a also in Y. □

Discussion. *Linearity and pointers.* The strategies we have defined are linear, in the sense that there are no repetitions of actions. For this reason, pointers are not required. In fact, for each action c in the strategy there is a unique action b which justifies c. One can also say that pointers are implicit: c *points* to b.
Innocence. Intuitively, innocence captures the idea that `Player` is not able to see `Opponent`'s internal calculations; a Player strategy in the game is completely determined by the piece of the play it can see (called *view*). Since an innocent strategy is completely determined by the views, while the most standard presentation of a strategy is as the set of all its possible plays, an innocent strategy can equivalently be described as a *set of views* (see [11, 3]). This is the approach we follow here, in a sense we are going to explain.

It is immediate to associate to a po strategy X a set of views: following the construction developed in [4], we can associate to each $x \in X$ the restriction of $(X, <)$ to the set $\{x' \leq x\}$ (in a parallel setting, a view is not a sequence of action, but a partial order.) One should now see that Definition 4 generalizes the definition of what is a views to a setting which is parallel and non-alternating. In the standard approach, one call view a play where every `O` move is justified by the immediately preceding `P` move. This captures the idea that the only information that Player has on `O` moves is the dependency in the arena. Our condition exactly says that the strategy cannot refine the order given by the Arena on `O` actions.

2.4 Arenas and innocent po strategies as a category

In this section, we show that we can organize what we have seen into a category, where the objects are +/- polarized arenas, and the arrows are innocent po strategies. We define composition as standard in Game Semantics: composition = parallel composition + hiding. To complete the construction, we then verify that we have an identity arrow for each object.

Hiding consists in removing the private actions, which correspond to "internal communication", i.e. the actions which are used in the parallel composition to make the two structures communicate. The following is immediate (using Remark 3)

Proposition 4 (Hiding). *Let $X : \Gamma \odot \Lambda^{\pm}$ be innocent and $X' := X \upharpoonright_{\Gamma}$. We have that $X' : \Gamma$ is a po strategy. Moreover (i) if X is **innocent** then X' is innocent; (ii) if X is **arborescent** then X' is arborescent.*

Composition preserves innocence, and in that case also arborescence and alternation, as these properties are preserved by both parallel composition and hiding. Putting all pieces together, we have the following

Definition 5. *Let $X_1 : \Gamma_1 \odot \Lambda$ and $X_2 : \Gamma_2 \odot \Lambda^{\perp}$ ($\Gamma_1, \Gamma_2, \Lambda$ disjoint). We define their composition as $X_1; X_2 = (X_1 \| X_2) \upharpoonright_{\Gamma_1, \Gamma_2}$*

Theorem 3. *Let $X_1 : \Gamma_1 \odot \Lambda$ and $X_2 : \Lambda^{\perp} \odot \Gamma_2$. Then $X_1; X_2 : \Gamma_1 \odot \Gamma_2$. Moreover, if X_1, X_2 are* **innocent***, we have the following: (1.) $X_1; X_2$ is* **innocent***. (2.) If X_1, X_2 are po strategies on* **alternating** *arenas, then $X_1; X_2$ is alternating. (3.) If X_1, X_2 are **arborescent**, then $X_1; X_2$ is arborescent.*

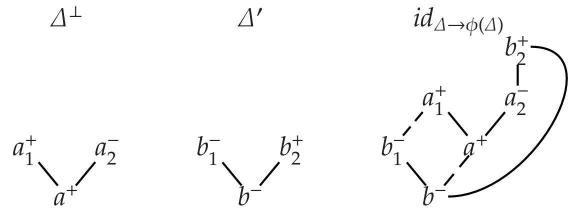
Identity (copycat). In this section we introduce a *Copycat strategy*, which generalizes what is called *fax* in [10] and copycat in Game Semantics. Our approach closely corresponds to that proposed by Hyland in [13].

All along this section we fix two *+/-polarized arenas* Δ and Δ', which are *disjoint* and *isomorphic* i.e. there is an order isomorphism $\phi : \Delta \to \Delta'$ such that $\pi_\Delta(a) = \pi_{\Delta'}(\phi(a))$. We say that ϕ is a renaming function, and say that Δ and Δ' are **equal up to renaming.** We extend this notion to po strategies too. Given two po strategies $X : \Gamma \odot \Delta$, $X' : \Gamma \odot \Delta'$, we say that they are **equal up to renaming** if X' is obtained from X by substituting each occurrence of $a \in \Delta$ with $\phi(a) \in \phi(\Delta)$.

The copycat is a strategy $id_{\Delta\to\phi(\Delta)}$ which copies any action from Δ into the corresponding action in $\phi(\Delta)$.

Definition 6 (Copycat). *Let Δ and $\Delta' = \phi(\Delta)$ be two arenas which are equal up to renaming. We define $id_{\Delta\to\Delta'} : \Delta^\perp \odot \Delta'$ as the spo X, where the order is that induced by $<_\Delta$ and $<_{\Delta'}$ with the addition of all the pairs $\{c \leftarrow_X \phi(c) \mid c \in \Delta, \pi_\Delta(c) = -\}$ and $\{\phi(c) \leftarrow_X c \mid c \in \Delta, \pi_\Delta(c) = +\}$.*

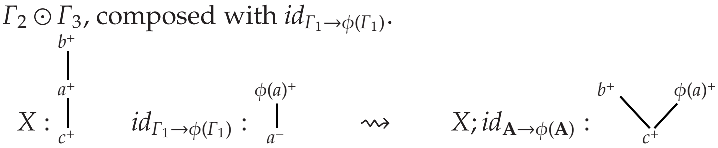
Example 1. Let $\Delta^\perp$ and Δ' be the isomorphic arenas represented below. We illustrate $id_{\Delta\to\phi(\Delta)}$ in the following picture, where we indicate with a dashed line the order which is added w.r.t. the arenas.



Remark 4. If the arena is alternating, the definition above gives us the standard copycat strategy.

Proposition 5 (Identity). *(1.) $id_{\Delta\to\phi(\Delta)}$ defined in 6 is a innocent po strategy. (2.) $id_{\Delta\to\phi(\Delta)} \upharpoonright_{\phi(\Delta)} = \phi(id_{\Delta\to\phi(\Delta)} \upharpoonright_\Delta)$. (3.) Let $X : \Gamma \odot \Delta$ be a innocent po strategy. Then $X; id_{\Delta\to\phi(\Delta)} : \Gamma\odot\phi(\Delta)$ is a innocent po strategy $\phi(X)$ which is equal to X up to renaming.*

Let us conclude with an example of the fact that, without the Innocence condition, the composition with the copycat does not produce the desired effect. Let us consider three singleton arenas $\Gamma_1 = \{a\}, \Gamma_2 = \{b\}, \Gamma_3 = \{c\}$, where $\pi_{\Gamma_1}(a) = \pi_{\Gamma_2}(b) = \pi_{\Gamma_3}(c) = +$. We then consider the following po strategy $X : \Gamma_1 \odot \Gamma_2 \odot \Gamma_3$, composed with $id_{\Gamma_1\to\phi(\Gamma_1)}$.



Hiding and observational equivalence. The neutral actions which we hide are silent actions which correspond to an internal synchronization. Hiding gives a canonical representation of an event structure with respect to observational equivalence. This idea is made precise in process calculi by the notion of weak bisimilarity.

By analogy with labelled transitions in process calculi, we generate a **labelled transition system** on spos as follows: if a is minimal in X then $X \xrightarrow{a} X \setminus \{a\}$. We then define the following reductions (i) $\Longrightarrow$ is the reflexive transitive closure of $\xrightarrow{\tau}$, where τ denotes any *neutral* action; (ii) $\overset{a}{\Longrightarrow}$ is $\Longrightarrow \xrightarrow{a} \Longrightarrow$, where a is a *non-neutral* action.

We define the **weak bisimilarity** on po strategies as the greatest binary symmetric relation $\approx$ satisfying the following property: whenever $X_1 \approx X_2$ and $X_1 \xrightarrow{a} X_1'$ then there exists X_2 such that $X_2 \overset{a}{\Longrightarrow} X_2'$ and $X_1' \approx X_2'$. we have the following

Proposition 6. *Let $X : \Gamma \odot \Lambda^{\pm}$, s.t. Γ is +/- polarized. We have that $X \approx X \upharpoonright_{\Gamma}$.*

3 Event structures as strategies

In this section, we extend the construction we have seen for partial orders to event structures.

Definition 7. *An* event structure *is a triple $\mathcal{E} = \langle E, \leq, \smile \rangle$ such that (1.) $\langle E, \leq \rangle$ is a well-founded partial order. (2.) $\smile$ is an irreflexive and symmetric relation, called* **conflict relation** *which is hereditary, i.e. for every $e_1, e_2, e_3 \in E$, if $e_1 \leq e_2$ and $e_1 \smile e_3$ then $e_2 \smile e_3$.*

Given two event structures $\mathcal{E}_1 = \langle E_1, \leq_1, \smile_1 \rangle$ and $\mathcal{E}_2 = \langle E_2, \leq_2, \smile_2 \rangle$ with $E_1 \cap E_2 = \emptyset$, we define $\mathcal{E}_1 \odot \mathcal{E}_2 = \langle E_1 \cup E_2. \leq_1 \cup \leq_2, \smile_1 \cup \smile_2 \rangle$. With an abuse of notation, given an event structure $\mathcal{E}$, we will confuse $\mathcal{E}$ with the set of its events E, writing $e \in \mathcal{E}$ (resp. $x \subseteq \mathcal{E}$) for $e \in E$ (resp. $x \subseteq E$).

Causal order and conflict are mutually exclusive. Two events which are not causally related nor in conflict are said to be *concurrent*. A conflict $e_1 \smile e_2$ is said **inherited** from the conflict $e_1 \smile e_2'$ if $e_2' \leq e_2$. If the conflict $e_1 \smile e_2$ is not inherited from any conflict, we say that it is **immediate**, written $e_1 \smile_{\mu} e_2$. We denote with $\asymp$ (resp. $\asymp_{\mu}$) the reflexive closure of $\smile$ (reps. $\smile_{\mu}$).

Given an event structure $\mathcal{E}$ a **configuration** is a set $x \subseteq \mathcal{E}$ which is *downward closed* and *conflict free*, i.e. if $e, e' \in x$ then it is never the case that $e \smile e'$. For example, given $e \in \mathcal{E}$, the sets $[e)$ and $\lceil e \rceil = [e) \cup \{e\}$ are configurations.

Observe that a configuration x is implicitly a partially ordered set (and so a spo), where the partial order is the restriction of the partial order of $\mathcal{E}$ to x. This fact is key to our approach, together with the fact that important relations in an event structure can be recovered from its configurations, and in fact an event structure can be described also as a special set of configurations, as we sketch below.

Let us denote with $\mathcal{C}(\mathcal{E})$ the family of all configurations of $\mathcal{E}$. $\mathcal{C}(\mathcal{E})$ ordered by inclusion forms a *coherent, finitary prime algebraic domain* (a dI-domain satisfying an additional condition [24]) whose set prime elements is $\{\lceil e \rceil \mid e \in \mathcal{E}\}$. A converse result holds too, i.e. every coherent finitary prime algebraic domain can be described using an event structure whose events are the prime elements, the order between events is inherited from the order between prime elements and two events are in conflict when they do not admit an upper bound, as prime elements of the domain. The following result, due to Winskel [25, pp. 60-61], summarizes the previous notions.

Theorem 4 ([25]). *Coherent finitary prime algebraic domain and event structures are equivalent.*

More details can be found in [25]. This fact will allows us to rely on the results we developed in the previous section.

Event structures form the class of objects of a category, whose **morphisms** are given by any partial map $\lambda : \mathcal{E}_1 \to \mathcal{E}_2$ satisfying

$$x \in C(\mathcal{E}_1) \Rightarrow \begin{cases} \lambda(x) \in C(\mathcal{E}_2) \\ \forall e_1, e_2 \in x.\lambda(e_1), \lambda(e_2) \text{ both defined } \wedge \lambda(e_1) = \lambda(e_2) \Rightarrow e_1 = e_2 \end{cases}$$

This category admits all finite products, co-products and pull backs [25]. A morphism is said to be **total** when the underlying map is.

3.1 Typed event structures

We now introduce the notion of typed event structures. Informally, it consists in a pair of event structures, one describing a process (a strategy) and the other one denoting a type (an arena); the typing relation between the two is represented by the existence of a total morphism relating them.

Definition 8. *An* ES-arena *is a pair* $\mathbf{\Gamma} = \langle \Gamma, \smile_\Gamma \rangle$ *where* Γ *is an arena and* $\smile_\Gamma$ *is a binary relation such that (1.)* $\langle \Gamma, \leq_\Gamma, \smile_\Gamma \rangle$ *is an event structure (2.)* π_Γ *is such that (i.) if* $\pi_\Gamma(a) = \pm$ *then* $\pi_\Gamma(b) = \pm$ *for all* b *s.t.* $b \smile_\Gamma a$*; (ii.) if* $a_1 \smile_\mu a_2$ *then* $\pi_\Gamma(a_1) = \pi_\Gamma(a_2)$*.*

All along this section, *we will consider an ES-arena* $\mathbf{\Gamma}$ *both as an event structure and as an arena in the sense of Section 2* (by ignoring the conflict relation). Furthermore, it is clear that each configuration $x \in C(\mathbf{\Gamma})$ with the induced polarization function is also an arena in the sense of Section 2. Thus, we can adapt all the definitions given in Section 2 (alternation, neutrality, duality, +/- polarization ...) to ES-arenas. In particular, given two disjoint ES-arenas $\mathbf{\Gamma_1} = \langle \Gamma_1, \smile_{\Gamma_1} \rangle$ and $\mathbf{\Gamma_2} = \langle \Gamma_2, \smile_{\Gamma_2} \rangle$ we define $\mathbf{\Gamma_1} \odot \mathbf{\Gamma_2} = \langle \Gamma_1 \odot \Gamma_2, \smile_{\Gamma_1} \cup \smile_{\Gamma_2} \rangle$.

Definition 9. *Let* $\mathbf{\Gamma}$ *be an ES-arena,* $\mathcal{E}$ *and event structure. We say that* $\mathcal{E}$ is typed in $\mathbf{\Gamma}$ *(written* $\mathcal{E} : \mathbf{\Gamma}$*) if there is a total morphism* $\lambda : \mathcal{E} \to \mathbf{\Gamma}$*, called* labeling morphism, *satisfying (1.) if* $e \in \mathcal{E}$ *is maximal w.r.t. the order of* $\mathcal{E}$*, then* $\pi_\Gamma(\lambda(e)) = +$*.*

Let $\mathcal{E} : \mathbf{\Gamma}$ be a typed event structure with a labeling morphism λ, and let $x \in C(\mathcal{E})$ be a configuration. Let us consider the structure $\langle \lambda(x), < \rangle$, where the order $<$ is defined as $\lambda(e_1) < \lambda(e_2)$ when $e_1 < e_2$ ($e_1, e_2 \in \mathcal{E}$), $\langle \lambda(x), < \rangle$ is a well-defined spo (since λ is a total morphism and an injective map on configurations) and it is isomorphic to x (viewed as spo). We say that $\langle \lambda(x), < \rangle$ a **slice** of $\mathcal{E}$ when x is such that for all $e \in x$, if e is maximal then $\pi_\Gamma(\lambda(e)) = +$. We denote with *Slices*($\mathcal{E}$) the set of all slices of $\mathcal{E}$ and we use $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2, \ldots$ to range over slices.

Lemma 3. *Let $\mathcal{E} : \mathbf{\Gamma}$ be a typed event structure and let $\mathcal{S} \in Slices(\mathcal{E})$. Then $\mathcal{S} = \langle x, < \rangle$ is a po strategy on $\mathbf{\Gamma}$.*

The following definition characterizes when a set of slices corresponds to a typed event structure; in this way, we are able to present a typed event structure as a set of spos (which are po strategies).

Definition 10. *Let $\mathbf{\Gamma}$ be an ES-Arena. A family $\mathcal{F}$ of po strategies on $\mathbf{\Gamma}$ ordered by the order $\sqsubseteq$ (see Section 2) is said to be a* po strategies family *when it satisfies (1.) If $X \in \mathcal{F}$ and $Y \sqsubseteq^{+} X$ then $Y \in \mathcal{F}$. (2.) For all $\mathcal{X} \subseteq \mathcal{F}$ such that, for every $X_1, X_2 \in \mathcal{X}$ they admit an upper bound in $\mathcal{F}$, we have $\bigsqcup \mathcal{X} \in \mathcal{F}$.*

Observe that the structure of $\mathcal{F}$ is very close to the one of a coherent finitary prime algebraic domain.

Proposition 7. *(1.) Let $\mathcal{E} : \mathbf{\Gamma}$ be a typed event structure. Then $Slices(\mathcal{E})$ is a po strategies family. (2.) Let $\mathcal{F}$ be a po strategies family on $\mathbf{\Gamma}$. Then there exists a typed event structure $Ev(\mathcal{F}) : \mathbf{\Gamma}$ such that $Slices(Ev(\mathcal{F}))$ is isomorphic to $\mathcal{F}$.*

Proof. (1.) can be obtained essentially as a corollary of Theorem 4. To prove (2.), given $X \in \mathcal{F}$ we define a view to be any subset $\mathfrak{c}_X \sqsubseteq X$ having a top element, denoted $top(\mathfrak{c}_X)$. We define $Ev(\mathcal{F}) = \langle E, \leq, \smile \rangle$ as (i) $E = \bigcup_{X \in \mathcal{F}} \{\mathfrak{c}_X \sqsubseteq X | \mathfrak{c}_X \text{ view}\}$ (we denote sets as sequences of their elements without repetitions). (ii) $\leq$ is the restriction of $\sqsubseteq$ to E. (iii) $\mathfrak{c}_X \smile \mathfrak{c}_{X'}$ when there is no $Y \in \mathcal{F}$ such that $\mathfrak{c}_X \sqcup \mathfrak{c}_{X'} \sqsubseteq Y$. We can check that $Ev(\mathcal{F})$ is an event structure and we can observe by Theorem 4, that *Slices* and *Ev* are naturally isomorphic. Moreover $Ev(\mathcal{F})$ is typed on $\mathbf{\Gamma}$ by taking $\lambda(\mathfrak{c}_X) = top(\mathfrak{c}_X)$. $\square$

We now define a notion of *parallel composition* between typed event structure, using the results given in the previous section.

Definition 11. *Let $\mathcal{E}_1 : \mathbf{\Gamma} \odot \mathbf{\Lambda}^{\perp}$ and $\mathcal{E}_2 : \mathbf{\Lambda} \odot \mathbf{\Delta}$ be two typed event structures. Then we define $\mathcal{E}_1 || \mathcal{E}_2 = Ev\Big(\{\mathcal{S}_1 || \mathcal{S}_2 \mid \mathcal{S}_1 \in Slices(\mathcal{E}_1), \mathcal{S}_2 \in Slices(\mathcal{E}_2)\}\Big)$ where $\mathcal{S}_1 || \mathcal{S}_2$ is the parallel composition between spos, defined in Section 2.*

Theorem 5. *Let $\mathcal{E}_1 : \mathbf{\Gamma} \odot \mathbf{\Lambda}^{\perp}$ and $\mathcal{E}_2 : \mathbf{\Lambda} \odot \mathbf{\Delta}$. Then $\mathcal{E}_1 || \mathcal{E}_2 : \mathbf{\Gamma} \odot \mathbf{\Lambda}^{\pm} \odot \mathbf{\Delta}^{\perp}$.*

Theorem 6 (Associativity). *Let $\mathcal{E}_1 : \mathbf{\Gamma_1} \odot \mathbf{\Lambda}^{\perp}$, $\mathcal{E}_2 : \mathbf{\Lambda} \odot \mathbf{\Gamma_2} \odot \mathbf{\Delta}^{\perp}$, $\mathcal{E}_3 : \mathbf{\Delta} \odot \mathbf{\Gamma_3}$. Then $\mathcal{E}_1 || (\mathcal{E}_2 || \mathcal{E}_3) = (\mathcal{E}_1 || \mathcal{E}_2) || \mathcal{E}_3$.*

We remark that the definition of parallel composition we give is similar to the technique of *normalization by slices* used to define normalization of designs [10] or L-nets [5] in Ludics.

3.2 A category of innocent event structures

In this section we define a category having ES-arenas as objects and (a subclass of) typed event structures as morphisms. Once again, we rely on the definitions given for po strategies.

Definition 12. *Let $\mathcal{E} : \mathbf{\Gamma}$ be a typed event structure. We say that $\mathcal{E}$ is* innocent *when for all $\mathcal{S} \in Slices(\mathcal{E})$, $\mathcal{S} = \langle x, < \rangle$ is an innocent po strategy on $\mathbf{\Gamma}$.*

By definition of parallel composition and as a consequence of Proposition 3, we observe that the class of innocent event structures is closed under parallel composition. We then define composition as parallel composition + hiding.

Definition 13. *Given two innocent event structures $\mathcal{E}_1 : \mathbf{\Gamma} \odot \mathbf{\Lambda}^\perp$, $\mathcal{E}_2 : \mathbf{\Lambda} \odot \mathbf{\Delta}$, we define $\mathcal{E}_1;\mathcal{E}_2 = Ev\Big(\{\mathcal{S}_1;\mathcal{S}_2 \mid \mathcal{S}_1 \in Slices(\mathcal{E}_1), \mathcal{S}_2 \in Slices(\mathcal{E}_2)\}\Big)$, where $\mathcal{S}_1;\mathcal{S}_2$ is the composition of innocent po strategies defined in Section 2.*

Using Proposition 3, we can prove that the class of innocent event structures is closed under composition and it is associative. Moreover, we can define a copycat event structure and prove that it plays the role of an identity with respect to the composition.

Definition 14. *Let $\mathbf{\Gamma}, \mathbf{\Gamma}'$ be two disjoint +/- polarized ES-Arenas which are isomorphic through the isomorphism $\phi : \mathbf{\Gamma} \to \mathbf{\Gamma}'$ and such that for all $e \in \mathbf{\Gamma}$, $\pi_{\Gamma}(e) = \pi_{\Gamma'}(\phi(e))$. We define the* copycat event structure *as $id_{\mathbf{\Gamma}\to\mathbf{\Gamma}'} = Ev\Big(\{id_{x\to\phi(x)} \mid x \in \mathcal{C}(\mathbf{\Gamma})\}\Big)$, where $id_{x\to\phi(x)}$ is the copycat spo defined in Section 2.*

Proposition 8. *$id_{\mathbf{\Gamma}\to\mathbf{\Gamma}'} : \mathbf{\Gamma}' \odot \mathbf{\Gamma}^\perp$ is an innocent event structure and it is the identity w.r.t. composition, i.e. for all innocent $\mathcal{E} : \mathbf{\Gamma} \odot \mathbf{\Delta}$, $\mathcal{E}$ is isomorphic to $id_{\mathbf{\Gamma}\to\mathbf{\Gamma}'};\mathcal{E}$.*

This result allows us to define a *category of innocent event structures*. We first introduce some notation. Given two event structures $\mathcal{E}_1, \mathcal{E}_2$, we write $\mathcal{E}_1 \sim \mathcal{E}_2$ when they are isomorphic in the category of event structures and we denote with $iso_{\mathcal{E},\mathcal{E}'} : \mathcal{E} \to \mathcal{E}'$ the isomorphism between them. We denote with $[\mathcal{E}]_\sim = \{\mathcal{E}' \mid \mathcal{E}' \sim \mathcal{E}\}$ the class of event structures isomorphic to $\mathcal{E}$. Moreover, given an arena $\mathbf{\Gamma}$, we define $(\!|\mathbf{\Gamma}|\!) = \{\mathbf{\Gamma}' \in [\mathbf{\Gamma}]_\sim \mid \forall a \in \mathbf{\Gamma}.\pi_{\Gamma}(a) = \pi_{\Gamma'}(iso_{\mathbf{\Gamma},\mathbf{\Gamma}'}(a))\}$ i.e. the set of all arenas isomorphic and inducing the same polarity function with respect to a given arena $\mathbf{\Gamma}$.

Definition 15. *The category $\mathcal{I}\mathbf{nn}\mathcal{E}\mathbf{v}$ is the category defined as follows. (1.) The class of* objects *is $Obj(\mathcal{I}\mathbf{nn}\mathcal{E}\mathbf{v}) = \{(\!|\mathbf{\Gamma}|\!) \mid \mathbf{\Gamma}$ +/- polarized ES-arena$\}$. (2.) The set of* morphisms *between $(\!|\mathbf{\Gamma}|\!)$ and $(\!|\mathbf{\Delta}|\!)$ is $\mathcal{I}\mathbf{nn}\mathcal{E}\mathbf{v}((\!|\mathbf{\Gamma}|\!), (\!|\mathbf{\Delta}|\!)) = \{[\mathcal{E}]_\sim \mid \mathcal{E} : \mathbf{\Delta} \odot \mathbf{\Gamma}^\perp$ innocent$\}$ (3.) The* composition *of $[\mathcal{E}_1]_\sim : (\!|\mathbf{\Gamma}|\!) \to (\!|\mathbf{\Lambda}|\!)$ and $[\mathcal{E}_2]_\sim : (\!|\mathbf{\Lambda}|\!) \to (\!|\mathbf{\Delta}|\!)$ is defined as $[\mathcal{E}_2]_\sim;[\mathcal{E}_1]_\sim = [\mathcal{E}_1;\mathcal{E}_2]_\sim : (\!|\mathbf{\Gamma}|\!) \to (\!|\mathbf{\Delta}|\!)$ (4.) Let $(\!|\mathbf{\Gamma}|\!)$ be an object. and let $\mathbf{\Gamma}' \in (\!|\mathbf{\Gamma}|\!)$ be disjoint from $\mathbf{\Gamma}$. Then, the* identity *is defined as $id_{(\!|\mathbf{\Gamma}|\!)} = [id_{\mathbf{\Gamma}\to\mathbf{\Gamma}'}]_\sim : (\!|\mathbf{\Gamma}|\!) \to (\!|\mathbf{\Gamma}|\!)$.*

Theorem 7. *$\mathcal{I}\mathbf{nn}\mathcal{E}\mathbf{v}$ is a compact closed category.*

Proof. We can define the *tensor product* $\odot$ in the following way (1.) $(\!|\mathbf{\Gamma}|\!) \odot (\!|\mathbf{\Delta}|\!) = (\!|\mathbf{\Gamma} \odot \mathbf{\Delta}|\!)$; (2.) given two morphisms $[\mathcal{E}_1]_\sim : (\!|\mathbf{\Gamma}|\!)_1 \to (\!|\mathbf{\Delta}|\!)_1$ and $[\mathcal{E}_2]_\sim : (\!|\mathbf{\Gamma}|\!)_2 \to (\!|\mathbf{\Delta}|\!)_2$, we have $[\mathcal{E}_1]_\sim \odot [\mathcal{E}_2]_\sim = [\mathcal{E}_1 \odot \mathcal{E}_2]_\sim$. It is naturally commutative, associative and it has $(\!|\emptyset|\!)$ as neutral element. Observe also that in this category every object $(\!|\mathbf{\Gamma}|\!)$ has a dual $(\!|\mathbf{\Gamma}|\!)^\perp = (\!|\mathbf{\Gamma}^\perp|\!)$: it induces a contra-variant functor $(-)^\perp$ defined as above for objects and given a morphism $[\mathcal{E}]_\sim : (\!|\mathbf{\Gamma}|\!) \to (\!|\mathbf{\Delta}|\!)$ we have that $[\mathcal{E}]_\sim^\perp = [\mathcal{E}]_\sim : (\!|\mathbf{\Delta}|\!)^\perp \to (\!|\mathbf{\Gamma}|\!)^\perp$. We can use it to define the bifunctor $\multimap$ as $(\!|\mathbf{\Gamma}|\!) \multimap (\!|\mathbf{\Delta}|\!) = (\!|\mathbf{\Gamma}|\!) \odot (\!|\mathbf{\Delta}|\!)^\perp$ and we can prove the required adjunction property. $\square$

3.3 Retrieving linear strategies: confusion freeness

In this section we sketch how linear strategies fit into the picture we have been developing, and in fact appear as a subclass of the category $\mathcal{I}$**nn**$\mathcal{E}$**v**.

In [6], we have shown that a feature of event structures representing linear strategies is that they are *confusion free*. Confusion free event structures describe a form of localized non-determinism, where the non-deterministic choice is localized in *cells*.

Given an event structure $\mathcal{E}$, a **cell** $C \subseteq \mathcal{E}$ is a maximal set of events which are pairwise in *immediate* conflict, and have the same enabling set: $\forall e, e' \in C. e \asymp_\mu e' \wedge [e) = [e')$. An event structures is said **confusion free** when cells are closed under immediate conflict.

The relation of conflict models a choice: two events which are in conflict live in two different evolutions of the system. Since conflict is inherited, the point where a choice is made corresponds to events in immediate conflict, i.e. a cell. The construct in process calculus which corresponds to a cell is a guarded sum: each events which is a cell can be seen as a guard on that choice. According to the polarity of the elements of the cell, we would hence have a sum which is guarded by output, input, or τ actions (resp. +, -, or ±). In [6], we showed a correspondence between (negative) cells and additives in Linear Logic.

We are now going to define a category where the objects are confusion free ES-arenas and morphisms are confusion free event structures which are innocent. Such a category is derived from $\mathcal{I}$**nn**$\mathcal{E}$**v**, but we first need to strengthen the conditions imposed to the labeling morphism. This is because the class of innocent confusion free event structures on a confusion free ES-arena (in the sense of Definition 9) is otherwise not closed neither composition.

Definition 16. *Let $\mathcal{E} : \mathbf{\Gamma}$ be a innocent event structure. We say that it is a* conf.-free innocent event structure *when (1.) $\mathcal{E}$ and $\mathbf{\Gamma}$ are confusion free and (2.) if $e_1 \smile_\mu e_2$ then (i.) $\lambda(e_1) \smile_\mu \lambda(e_2)$ and (ii.) $\pi_\Gamma(\lambda(e_1), \pi_\Gamma(\lambda(e_2)) \neq +$.*

Condition (1.) requires that both the event structure $\mathcal{E}$ and the arena $\mathbf{\Gamma}$ are confusion free. Now $\mathbf{\Gamma}$ has really the shape of a MALL formula tree, where immediate conflict codes the additive connectives. Condition (2.) requires that two events which are in immediate conflict in $\mathcal{E}$ are in immediate conflict also in $\mathbf{\Gamma}$ and that they are never positive. Observe that this, together with Condition (2.ii) of Definition 8 tells us that events belonging to the same cell of $\mathcal{E}$ are labeled with actions in immediate conflict in $\mathbf{\Gamma}$. Moreover, those actions have the same polarity and such a polarity can only be either negative or neutral. The correspondence innocence/asynchrony (see [7, 8]) supports this constraint, which is consistent with the fact in an asynchronous calculus only input-prefixed (external choice) and τ-prefixed (internal choice) terms can be summands in a guarded sum.

The following result allows us to define a subcategory of $\mathcal{I}$**nn**$\mathcal{E}$**v** whose objects are given by equivalence classes of confusion free ES-arenas and whose morphisms are given by equivalence classes of conf.-free innocent event structures.

Theorem 8. *Let* $\mathcal{E}_1 : \mathbf{\Gamma} \odot \mathbf{\Lambda}^{\perp}$ *and* $\mathcal{E}_2 : \mathbf{\Lambda} \odot \mathbf{\Delta}$ *be two conf.-free innocent event structures. Then (1.)* $\mathcal{E}_1 \| \mathcal{E}_2 : \mathbf{\Gamma} \odot \mathbf{\Lambda}^{\pm} \odot \mathbf{\Delta}$ *is a conf.-free innocent event structure. (2.)* $\mathcal{E}_1 ; \mathcal{E}_2 : \mathbf{\Gamma} \odot \mathbf{\Delta}$ *is a conf.-free innocent event structure.*

In the class of conf.-free innocent event structures we are now able to retrieve the *family of linear strategies*, by imposing opportune constraints. The fundamental fact is that linear strategies can be described as partial orders with a conflict relation (as detailed in [7]). We consider the following constraints on conf.-free innocent event structures: *arborescence, sequentiality, acyclicity*. An event structure $\mathcal{E} = \langle E, \leq, \smile \rangle$ is *arborescent* when $\langle E, \leq \rangle$ is. A conf.-free innocent event structure $\mathcal{E} : \mathbf{\Gamma}$ is *sequential* when it is arborescent and for all $\mathcal{S}_1, \mathcal{S}_2 \in \mathit{Slices}(\mathcal{E})$ $\mathcal{S}_1 \cap \mathcal{S}_2 \in \mathit{Slices}(\mathcal{E})$ (where the intersection of two slices is the set-theoretical intersection of the underlying set of actions with the induced order). Observe that, if the arena is alternating, this condition tells us that given $e, e_1, e_2 \in \mathcal{E}$ if $e \leftarrow_{\mathcal{E}} e_1, e \leftarrow_{\mathcal{E}} e_2$ and $\pi(e) = -$ then $\pi(e_1) = \pi(e_2) = +$ and $e_1 = e_2$.Intuitively, this condition would correspond to the constraint that in a process there is an unique output which is active at any time. Finally, we say that $\mathcal{E} : \mathbf{\Gamma}$ is *acyclic* if it satisfies the analogous of the acyclicity constraints given in [5], which we do not detail here. Intuitively, the condition guarantees the absence of deadlocks during parallel composition.

When restricting to +/- alternating arenas (and the polarity constraints given in [10]), we retrieve the family of linear strategies

Theorem 9. *Let* $\mathcal{E} : \mathbf{\Gamma}$ *be a conf.-free event structure on a* **+/- alternating** *arena. (1.) If* $\mathcal{E} : \mathbf{\Gamma}$ *is* **sequential**, *then* $\mathcal{E}$ *corresponds to a linear strategy as defined by Girard in [10] (these strategies are there called designs). (2.) If* $\mathcal{E} : \mathbf{\Gamma}$ *is* **arborescent**, *then* $\mathcal{E}$ *corresponds to a linear strategy extended with mix, as defined in [4] (and there called L-forests). (3.)If* $\mathcal{E} : \mathbf{\Gamma}$ *is* **acyclic**, *then* $\mathcal{E}$ *corresponds to an L-net, as defined in [5].*

Moreover, all the above sub-classes of conf.-free innocent event structures are closed under composition and all the induced category are all subcategories of $\mathcal{I}$**nn**$\mathcal{E}$**v**.

If we do not insist for the arena to be alternating, we would have also neutral cells, which correspond (in process calculus) to a sum guarded by τ actions. This leaves the space for a possible extension of our work to model internal choices. In future work we want to investigate this direction as a possible approach to non-deterministic Game Semantics.

A key element in this paper is linearity, which allows for the definition of composition based on the merging of orders. We believe this is not a limitation to model an expressive calculus. In ongoing work [8] we extend [7], to show that Ludics is in fact able to model a variant of the Linear π-calculus extended with recursion. Even with recursion, the game model is linear.

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