

Notes and Homework 1 (Abstract Rewriting)

Homework 2. EX2.

Let $(A, \rightarrow)$ be an ARS.

1. Prove that if (A.) holds then (B.) holds
 - A. $\forall t \in A: (t_1 \leftarrow t \rightarrow t_2)$ implies $(t_1 = t_2 \text{ or } \exists u. t_1 \rightarrow u \leftarrow t_2)$
 - B. $\forall t \in A$: if t has a normal form u (ie, $t \rightarrow^k u$, for some k), then all maximal reduction sequences from t have the same length, and all end in the same normal form u .

NOTES



1 Abstract rewriting system (ARS): the basics

We recall basic definitions.

Basics. An *abstract rewriting system (ARS)* is a pair $\mathcal{A} = (\mathcal{A}, \rightarrow)$ consisting of a set $\mathcal{A}$ and a binary relation $\rightarrow$ on $\mathcal{A}$ whose pairs are written $t \rightarrow s$ and called *steps*.

- We denote $\rightarrow^*$ (resp. $\rightarrow^=$) the transitive-reflexive (resp. reflexive) closure of $\rightarrow$. We write $t \leftarrow u$ if $u \rightarrow t$ (the reverse relation).
- If $\rightarrow_1, \rightarrow_2$ are binary relations on $\mathcal{A}$ then $\rightarrow_1 \cdot \rightarrow_2$ denotes their composition, *i.e.* $t \rightarrow_1 \cdot \rightarrow_2 s$ if there exists $u \in \mathcal{A}$ such that $t \rightarrow_1 u \rightarrow_2 s$.
- We write $(\mathcal{A}, \{\rightarrow_1, \rightarrow_2\})$ to denote the ARS $(\mathcal{A}, \rightarrow)$ where $\rightarrow = \rightarrow_1 \cup \rightarrow_2$.
- An element $u \in \mathcal{A}$ is **$\rightarrow$ -normal**, or a *$\rightarrow$ -normal form* if there is no t such that $u \rightarrow t$ (we also write $u \nrightarrow$).
- A *$\rightarrow$ -sequence* (or **reduction sequence**) from t is a (possibly infinite) sequence $t, t_1, t_2, \dots$ such that $t_i \rightarrow t_{i+1}$. $t \rightarrow^* s$ indicates that there is a finite sequence from t to s .
A $\rightarrow$ -sequence from t is *maximal* if it is either infinite or ends in a $\rightarrow$ -normal form.

We freely use the fact that the transitive-reflexive closure of a relation is a closure operator, *i.e.* satisfies

$$\rightarrow \subseteq \rightarrow^*, \quad (\rightarrow^*)^* = \rightarrow^*, \quad \rightarrow_1 \subseteq \rightarrow_2 \text{ implies } \rightarrow_1^* \subseteq \rightarrow_2^*. \quad (\text{Closure})$$

The following property is an immediate consequence:

$$(\rightarrow_1 \cup \rightarrow_2)^* = (\rightarrow_1^* \cup \rightarrow_2^*)^*. \quad (\text{TR})$$

Local vs Global Properties. An important distinction in rewriting theory is between local and global properties. A property of term t is *local* if it is quantified over only *one-step reductions* from t ; it is *global* if it is quantified over all *rewrite sequences* from t . Local properties are easier to test, because the analysis (usually) involves a finite number of cases.

Commutation and Confluence Two relations $\rightarrow_1$ and $\rightarrow_2$ on $\mathcal{A}$ *commute* if

$$\leftarrow_1^* \cdot \rightarrow_2^* \subseteq \rightarrow_2^* \cdot \leftarrow_1^*.$$

A relation $\rightarrow$ on $\mathcal{A}$ is *confluent* if it commutes with itself.

A classic tool to modularize the proof of confluence is Hindley-Rosen lemma.

Confluence of two relations $\rightarrow_1$ and $\rightarrow_2$ does not imply confluence of $\rightarrow_1 \cup \rightarrow_2$, however it does if they commute.

► **Lemma** (Hindley-Rosen). *Let $\rightarrow_1$ and $\rightarrow_2$ be relations on the set A . If $\rightarrow_1$ and $\rightarrow_2$ are **confluent** and **commute with each other**, then*

$$\rightarrow_1 \cup \rightarrow_2 \text{ is confluent.}$$

Local conditions. Commutation is a global condition, which is difficult to test. There are however *easy-to-check* sufficient conditions. One of the most useful such conditions is Hindley's strong commutation :

$$\leftarrow_1 \cdot \rightarrow_2 \subseteq \rightarrow_2^* \cdot \leftarrow_1^* \quad (\text{Strong Commutation})$$

► **Lemma** (Local test). *Strong commutation implies commutation.*

2 Factorization.

Both *confluence* and *factorization* are forms of commutation.

Let $\mathcal{A} = (A, \{\rightarrow_e, \rightarrow_i\})$ be an ARS.

■ The relation $\rightarrow = \rightarrow_e \cup \rightarrow_i$ satisfies **e-factorization**, written $\text{Fact}(\rightarrow_e, \rightarrow_i)$, if

$$\text{Fact}(\rightarrow_e, \rightarrow_i) : \rightarrow^* \subseteq \rightarrow_e^* \cdot \rightarrow_i^* \quad (\text{Factorization})$$

■ The relation $\rightarrow_i$ **postpones** after $\rightarrow_e$, written $\text{PP}(\rightarrow_e, \rightarrow_i)$, if

$$\text{PP}(\rightarrow_e, \rightarrow_i) : \rightarrow_i^* \cdot \rightarrow_e^* \subseteq \rightarrow_e^* \cdot \rightarrow_i^*. \quad (\text{Postponement})$$

Postponement can be formulated in terms of commutation, and viceversa, since clearly $(\rightarrow_i \text{ postpones after } \rightarrow_e)$ if and only if $(\leftarrow_i \text{ commutes with } \rightarrow_e)$. Note that reversing $\rightarrow_i$ introduce an asymmetry between the two relations. It is an easy result that e-factorization is equivalent to postponement, which is a more convenient way to express it.

► **Lemma 1.** *For any two relations $\rightarrow_e, \rightarrow_i$ the following are equivalent:*

1. $\rightarrow_i^* \cdot \rightarrow_e \subseteq \rightarrow_e^* \cdot \rightarrow_i^*$
2. $\rightarrow_i \cdot \rightarrow_e^* \subseteq \rightarrow_e^* \cdot \rightarrow_i^*$
3. Postponement: $\rightarrow_i^* \cdot \rightarrow_e^* \subseteq \rightarrow_e^* \cdot \rightarrow_i^*$
4. Factorization: $(\rightarrow_e \cup \rightarrow_i)^* \subseteq \rightarrow_e^* \cdot \rightarrow_i^*$

A local test. Hindley first noted that a local property implies postponement, hence factorization

We say that $\rightarrow_i$ **strongly postpones** after $\rightarrow_e$, if

$$\text{SP}(\rightarrow_e, \rightarrow_i) : \rightarrow_i \cdot \rightarrow_e \subseteq \rightarrow_e^* \cdot \rightarrow_i^* \quad (\text{Strong Postponement})$$

► **Lemma 2** (Local test for postponement). *Strong postponement implies postponement:*

$$\text{SP}(\overrightarrow{e}, \overrightarrow{i}) \text{ implies } \text{PP}(\overrightarrow{e}, \overrightarrow{i}), \text{ and so } \text{Fact}(\overrightarrow{e}, \overrightarrow{i}).$$

It is immediate to recognize that the property is exactly the postponement analog of strong commutation; indeed it is the same expression, with $\overrightarrow{i} := \leftarrow_1$ and $\overrightarrow{e} := \rightarrow_2$.

A characterization. Another property that we shall use freely is the following, which is immediate by the definition of postponement and property **TR**

► **Property.** *Given a relation φ_i such that $\varphi_i^* = \overrightarrow{i}^*$,*

$$\text{PP}(\overrightarrow{e}, \overrightarrow{i}) \text{ if and only if } \text{PP}(\overrightarrow{e}, \varphi_i).$$

A well-known use of the above is to instantiate φ_i with a notion of parallel reduction (as in [Takahashi])