

Proof Nets

A graph syntax for proofs

Reference:

Notes on proof-nets by Olivier Laurent

Credits: most figures and material come from the notes of O. Laurent

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MLL

$$\frac{}{\vdash A^\perp, A} ax \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} cut$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \wp$$

Recall that linear negation is defined :

$$\begin{aligned} (X^\perp)^\perp &= X \\ (A \otimes B)^\perp &= A^\perp \wp B^\perp \\ (A \wp B)^\perp &= A^\perp \otimes B^\perp \end{aligned}$$

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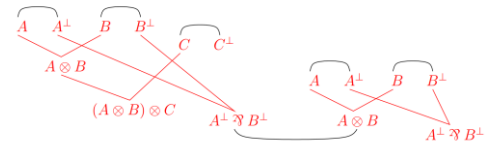
Forgetting Sequential Structure

$$\frac{\frac{\frac{}{\vdash A, A^\perp} (ax) \quad \frac{}{\vdash B, B^\perp} (ax)}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes) \quad \frac{}{\vdash C, C^\perp} (ax)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} (\otimes) \quad \frac{}{\vdash A, A^\perp} (ax) \quad \frac{}{\vdash B, B^\perp} (ax)}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes) \quad \frac{}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} (\wp) \quad \frac{}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} (cut) \quad \frac{}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} (\wp)$$

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Forgetting Sequential Structure

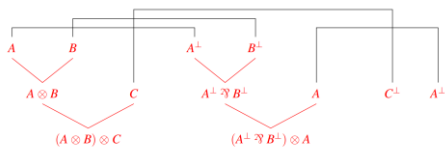
$$\frac{\frac{\frac{}{\vdash A, A^\perp} (ax) \quad \frac{}{\vdash B, B^\perp} (ax)}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes) \quad \frac{}{\vdash C, C^\perp} (ax)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} (\otimes) \quad \frac{}{\vdash A, A^\perp} (ax) \quad \frac{}{\vdash B, B^\perp} (ax)}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes) \quad \frac{}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} (\wp) \quad \frac{}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} (cut) \quad \frac{}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} (\wp)$$



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Forgetting Sequential Structure

$$\begin{aligned} (1) & \frac{}{\vdash A, A^\perp} ax \quad (2) \frac{}{\vdash B, B^\perp} ax \\ (3) & \frac{}{\vdash A \otimes B, A^\perp, B^\perp} \otimes \quad (4) \frac{}{\vdash C, C^\perp} ax \\ (5) & \frac{}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} \otimes \\ (6) & \frac{}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp} \wp \\ (7) & \frac{}{\vdash A, A^\perp} ax \\ (8) & \frac{}{\vdash (A \otimes B) \otimes C, (A^\perp \wp B^\perp) \otimes A, C^\perp, A^\perp} \otimes \end{aligned}$$



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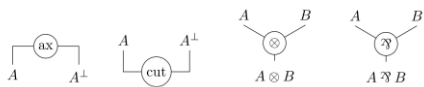
Proof Nets

A graph syntax for proofs

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Proof structures

A **proof structure** $\mathcal{M}$ is a **labelled directed acyclic graph (DAG)** with possibly pending edges (i.e. some edges may have no source and/or no target) built over the alphabet of nodes which is represented below.
(Note: in figures, the edges orientation is always top-bottom.)



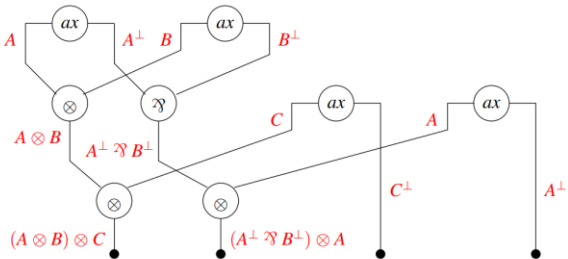
- The **nodes** are labelled by ax, cut, $\otimes$, $\wp$
- The **edges** are labelled by MLL formulas.

For each node/link: **premisses** = entering edges, **conclusions** = exiting edges

The **conclusions** of $\mathcal{M}$ is the set of pending edges of $\mathcal{M}$.

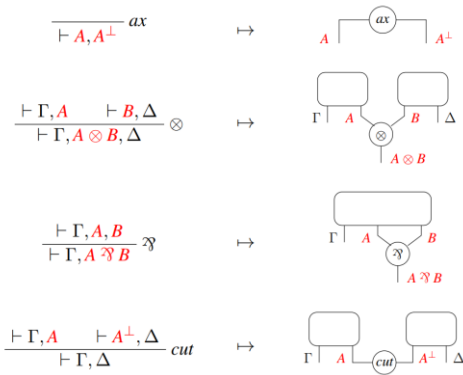
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In the graphical representation, we do not mention explicitly the direction of edges, but we draw them so that the direction is always top-down:



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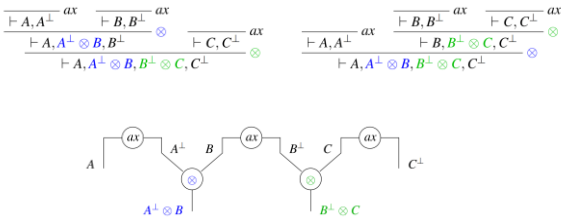
From proofs to proofs structures



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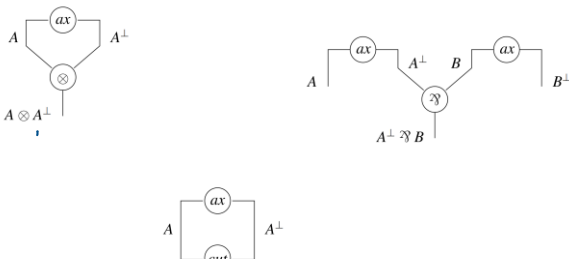
example

Translate each of these sequent calculus proofs. Start from axioms....



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Is every structure the image of an MLL proof?



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Proof Nets

A **PROOF NET** is a proof structure which is the image of an MLL proof

Internal condition!

Purely geometrical conditions (correction) characterize the proof structures which are proof nets

Theorem 1. A proof structure is *correct* iff it is a *proof net*.

Correctness criteria:

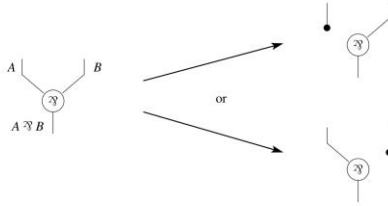
- (LT) Long trip [Girard]
- (AC) Connected-Acyclic [Danos-Regnier]
- ...

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Danos-Regnier Criterion

Correctness Criterion

Switching Graphs



Correctness

- Switching graphs are **acyclic**.
- Switching graphs are **connected**.

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Danos-Regnier Criterion (formally)

Given a proof structure $\mathcal{S}$, let $\mathcal{N}_{\gamma}(\mathcal{S})$ be the set of its γ -nodes. A **switching** of $\mathcal{S}$ is a function φ defined on $\mathcal{N}_{\gamma}(\mathcal{S})$ and such that, for each γ -node P , $\varphi(P)$ is one of its premises. The **switching graph** $\mathcal{S}^{\varphi}$ associated with φ is the graph obtained from $\mathcal{G}(\mathcal{S})$ by keeping only the premise $\varphi(P)$ for each γ -node

Danos-Regnier correctness. A proof structure $\mathcal{S}$ is *DR-correct* if for each correctness graph $\mathcal{S}^{\varphi}$, its undirected underlying graph is acyclic and connected.

A proof structure with p γ nodes induces 2^p switchings and thus 2^p switching graphs. A switching graph is not a proof structure in general since its γ nodes have only one premise.

A **connected component** of a switching graph is a connected component of its underlying (undirected) multigraph.

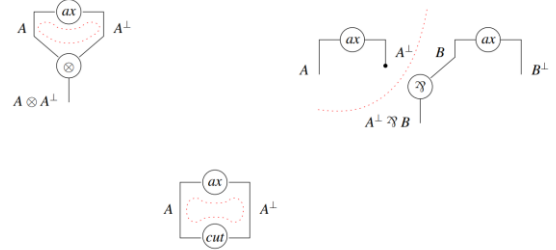
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Acyclic-only correctness

Acyclicity. A multiplicative proof structure is *acyclic* if its switching graphs do not contain any undirected cycle.

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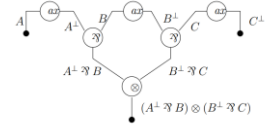
Is this correct?



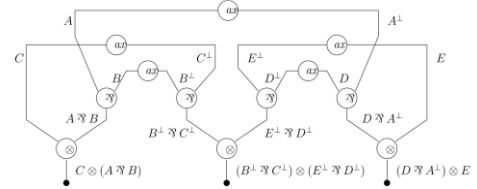
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Is this correct?

PN1:



PN2:



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Correctness guarantees:

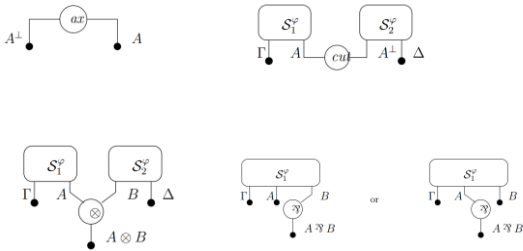
- ✓ Graph is image of a proof (sequentialization)
- ✓ Normalization progresses (no deadlocks)
- ✓ Normalization terminates (no infinite cycles)

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Sequentialization

Soundness

Proposition 4.1.1 (Soundness of Correctness). *The translation of a sequent calculus proof of MLL is a connected multiplicative proof net.*



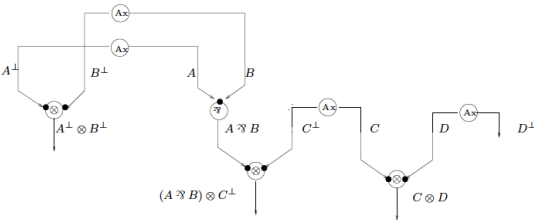
Theorem 4.1.1 (Sequentialization). *Any connected multiplicative proof net is the translation of a sequent calculus proof of MLL.*

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Sequentialization answers the question:

We have a proof net. The problem: it is the image of a sequent calculus proof? And which?



The beauty of proof nets is normalization

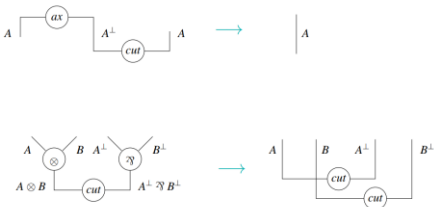
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Normalization (local graph reductions!)

Properties of normalization

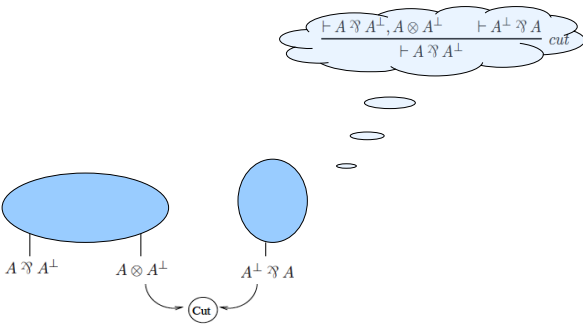
Lemma (preservation of correctness)
If the proof structure R is correct and reduces to R' , then R' is correct.



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Let us try out!



Write a proof net with this conclusion... and normalize it

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How we write a proof net of these conclusions?

$A \wp A^\perp$ must type an edge conclusion of a par link, with premisses

$A \otimes A^\perp$ must type an edge conclusion of a tensor link, with premisses

Then we have to **choose** the axiom links!

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Let us try one more. First, write a proof net with this conclusion...

$$(X \otimes X) \multimap (X \otimes X) =$$
$$(X \otimes X)^\perp \wp (X \otimes X) = (X^\perp \wp X^\perp) \wp (X \otimes X)$$

TIP: How we write a proof net? As before, all proof nets with the same conclusion, start with the same nodes (the formula tree!) What distinguishes different proofs are the axiom links

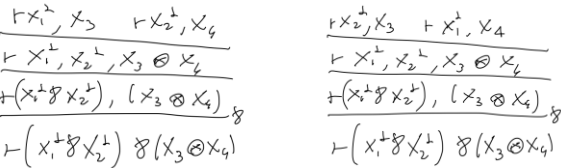
To distinguish the different occurrences of atoms, let us write indices:

$$(X_1^\perp \wp X_2^\perp) \wp (X_3 \otimes X_4)$$

In this case, we have two possible proofs, corresponding to two possible way to write axioms:
1,3 and 2,4
OR
1,4 and 2,3

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In sequent calculus, they correspond to these two proofs (one uses exchange, one no)

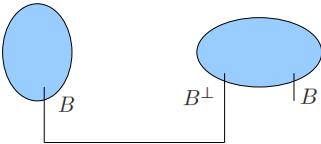


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When we have a formula whose normal proofs are exactly two, we have a good candidate to code BOOLEANS :-)

Let us indicate the formula $(X^\perp \wp X^\perp) \wp (X \otimes X)$ with B (for boolean). We call one proof **true**, and the other **false**...

We can feed one of our two values to a proof which takes a boolean, and return a boolean.



We know that the normal form (i.e the result of computation) will be of type B... Hence one of our two values.

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Try to normalize one of the proofs of $(X_1^\perp \wp X_2^\perp) \wp (X_3 \otimes X_4)$

with the proof net which has conclusions

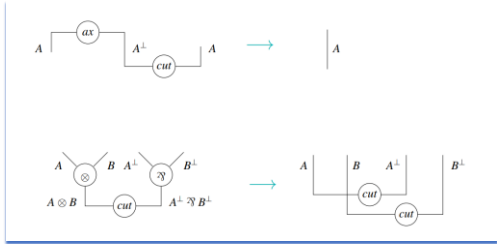
$$(X_1 \otimes X_2) \otimes (X_3^\perp \wp X_4^\perp) \quad (X_5^\perp \wp X_6^\perp) \wp (X_7 \otimes X_8)$$

and axiom links: (1,6) (2,5) (3,7) (4,8)

What is the function coded by this proof net?

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MLL normalization
(local graph reductions)



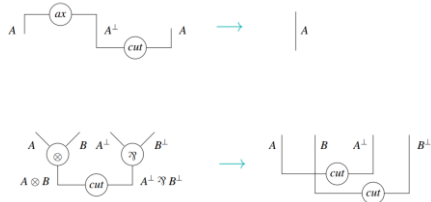
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Properties of MLL normalization

- 1. Confluence?
- 2. Is normalization weakly/strongly normalizing?
- 3. Would you be able to define a normalizing strategy?
- 4. Would you be able to define a normalizing strategy which reaches normal form in a minimal number of steps?

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Properties of MLL normalization



- Strongly normalizing
- Confluent
- Cut-elimination:
a proof-net in normal form contains no cuts

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Correctness criterion,
simplified

Correctness: if we focus on acyclicity, Danos-Regnier criterion can be reformulated (in equivalent way)

Let R be a proof structure; a *switching path* of R is a path which does not use any two edges entering on the same γ link (such edges are called *switching edges*); a *switching cycle* is a switching path which is a cycle.

Definition 3 (Correctness criterion) . A proof structure is correct if it does not contain any switching cycle.

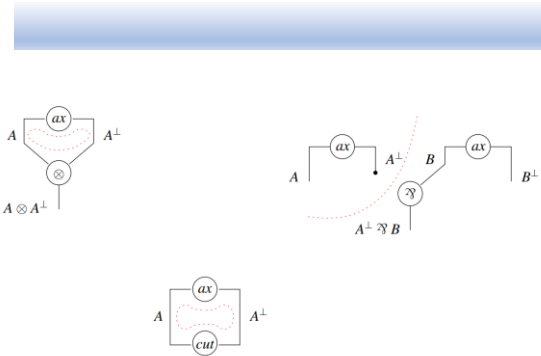
$$\frac{}{\vdash A, A^\perp} (Ax) \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} (Cut)$$
$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, A \otimes B, \Delta} (\otimes) \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} (\wp)$$

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} (Mix)$$

we can throw away MIX later
By requiring **connectness**

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Correctness: if we focus on acyclicity, Danos-Regnier criterion can be reformulated (in equivalent way)

A proof structure is **correct** if every (undirected) cycle contains a segment

→ ⌘ ←

$$\frac{}{\vdash A, A^\perp} (Ax) \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} (Cut)$$
$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, A \otimes B, \Delta} (\otimes) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} (\wp)$$

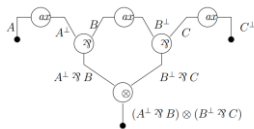
$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} (Mix)$$

we can throw away MIX later
By requiring **connectness**

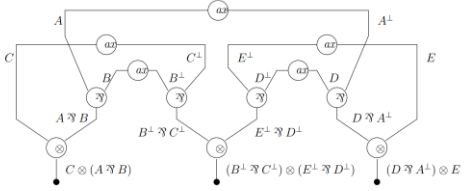
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Is this correct?

PN1:



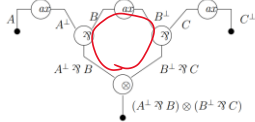
PN2:



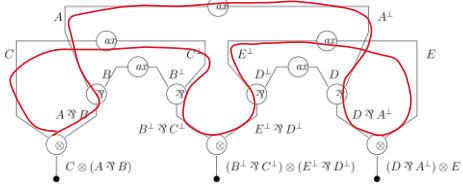
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Is this correct?

PN1:



PN2:



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A challenge (challenging bonus)

Bonus
valid till 21/10

A fully positive formula is P is defined as
 $P ::= X \mid P \otimes P$
A fully negative formula N is the dual notion, $N ::= (P)^\perp$

A proof structure is **polarized** if every edge is labelled by a fully positive or a fully negative formula.

• **Exercise**
Let M be a MLL **polarized** proof structure. We denote by $Pol(M)$ the graph which has the same nodes and edges as M , but where the edges are directed downward if the label is positive, upwards if the label negative.

Q1: Assume that all formulas in M are either atoms, or built using the connective par. Assume also that we are happy with the mix rule, so we do not care about the "connected" part of correctness criteria
Do you see any simple way to show that the following correctness criteria are equivalent ?
(1.) M is acyclic correct, (2.) $Pol(M)$ is a DAG.

Q2: What if we now assume that the polarized formulas in M use both tensor and par ?

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More expressivity: Exponentials

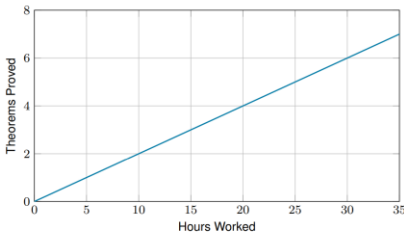
Linear vs **non-linear**
 $A \multimap B$ vs $A \rightarrow B$

Work $\multimap$ Theorem

! Work $\multimap$ Theorem

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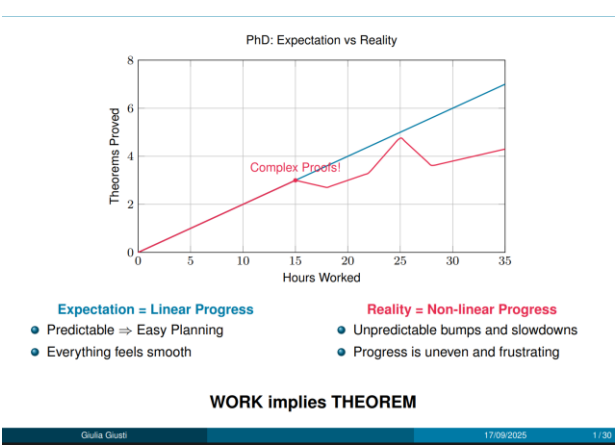
PhD: Expectation vs Reality



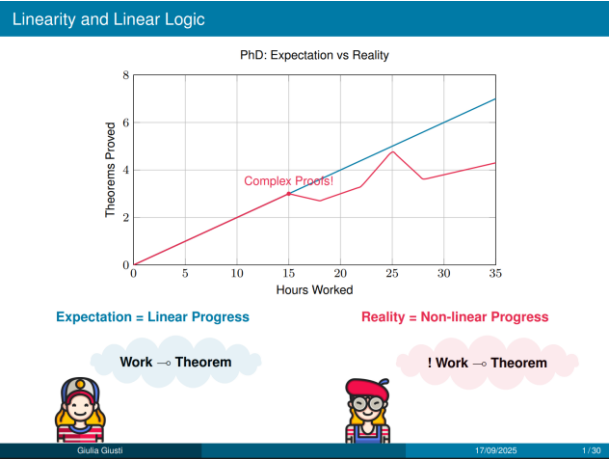
Expectation = Linear Progress

- Predictable $\Rightarrow$ Easy Planning
- Everything feels smooth

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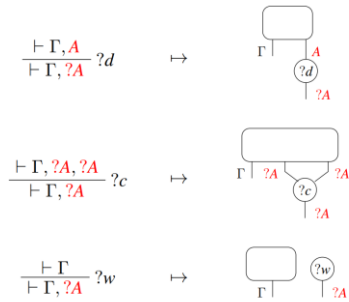


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From proofs to proofs structures

Exponentials:
MELL proof-nets

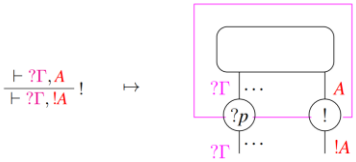
4 new nodes



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From proofs to proofs structures



- Boxes never overlap.
- Boxes are sequential (as rules in sequent calculus).
- Correctness: box by box.
- Boxes permit duplication and erasure.

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MELL proof-nets: correctness

Correctness is given by acyclicity of the switching paths, similarly to the "simplified" criterion for MLL.
The $?c$ node behave the same as the $?w$ node.

A **switching path** in the proof-structure R is an undirected path on R (regarded as an undirected graph) which uses **at most one of the two premisses** for each $?w$ node and $?c$ node.

- A **proof-structure** is **correct** if :
- none of its switching paths is cyclic, and
 - the content of each of its box is itself correct.

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What is the associated MELL proof-net?

$$\frac{\frac{\frac{}{\vdash A, A^\perp} ax}{\vdash A \wp A^\perp} \wp}{\vdash A \wp A^\perp, ?B} ?w \quad \mapsto$$

For each of the following two sequents, write a proof-net with the same conclusions (you can start from the sequent calculus proof, or just write a proof-net directly)
Please always write the proofs so that **all axioms are atomic** (assuming **A atomic formula**)

$$\vdash ?A^\perp \wp ?A^\perp, !A \quad \vdash !A \otimes !A, ?A^\perp$$

Reduction steps

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MEMO: Cut-elimination steps

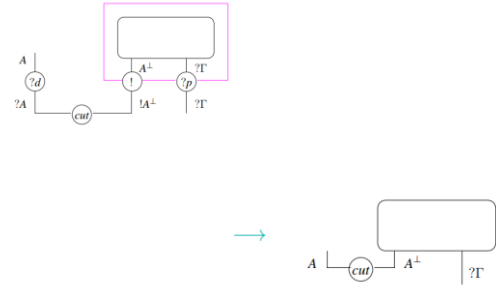
$$\frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash \Delta, A^\perp}{\vdash \Delta, ?A^\perp} ?d}{\vdash ?\Gamma, \Delta} cut \quad \rightsquigarrow \quad \frac{\vdash ?\Gamma, A \quad \vdash \Delta, A^\perp}{\vdash ?\Gamma, \Delta} cut$$

$$\frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash \Delta, ?A^\perp, ?A^\perp}{\vdash \Delta, ?A^\perp} ?c}{\vdash ?\Gamma, \Delta} cut \quad \rightsquigarrow \quad \frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash \Delta, ?A^\perp, ?A^\perp}{\vdash ?\Gamma, \Delta, ?A^\perp} cut}{\vdash ?\Gamma, \Delta, ?A^\perp} cut \quad \rightsquigarrow \quad \frac{\vdash ?\Gamma, \Delta, ?A^\perp}{\vdash ?\Gamma, \Delta} ?c$$

$$\frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash \Delta}{\vdash \Delta, ?A^\perp} ?w}{\vdash ?\Gamma, \Delta} cut \quad \rightsquigarrow \quad \frac{\vdash \Delta}{\vdash ?\Gamma, \Delta} ?w$$

$$\frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash ?\Delta, ?A^\perp, B}{\vdash ?\Delta, ?A^\perp, !B} !}{\vdash ?\Gamma, ?\Delta, !B} cut \quad \rightsquigarrow \quad \frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \vdash ?\Delta, ?A^\perp, B}{\vdash ?\Gamma, ?\Delta, B} cut \quad \rightsquigarrow \quad \frac{\vdash ?\Gamma, ?\Delta, B}{\vdash ?\Gamma, ?\Delta, !B} !$$

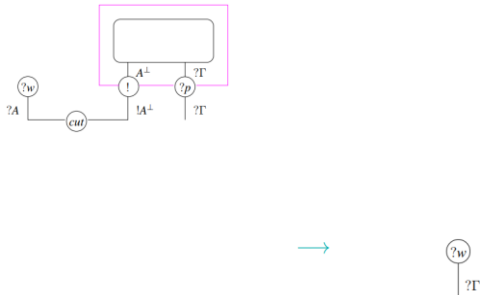
Reduction Steps: ?d



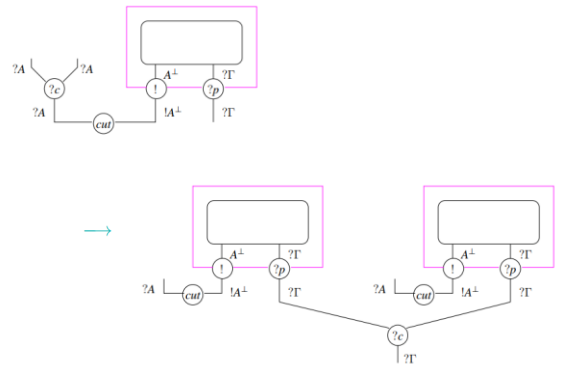
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Reduction Steps: ?w



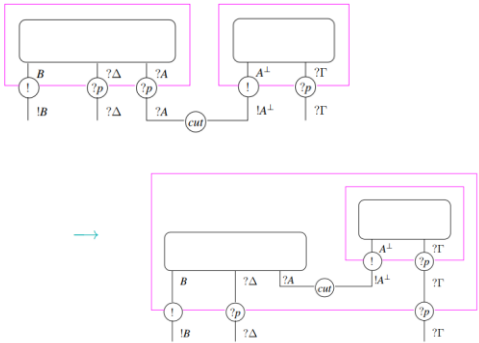
Reduction Steps: ?c



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Reduction Steps: ?p



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How many cut-free proof-net

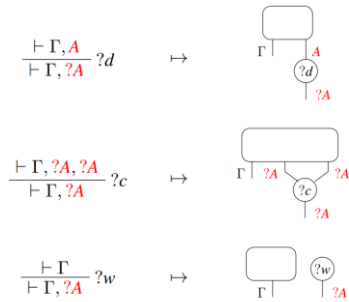
There exist exactly two cut-free proof-nets with conclusion $(X \otimes X) \multimap (X \otimes X)$
Prove that exist (essentially) two cut-free proof-nets with conclusion $!(?X^\perp \wp (?X^\perp \wp X))$

Q1: ? how many cut-free proof-nets have conclusion
 $!(X \multimap X) \multimap (X \multimap X)$

Q2: ? how many cut-free proof-nets have conclusion
 $!(X \multimap X) \multimap !(X \multimap X)$

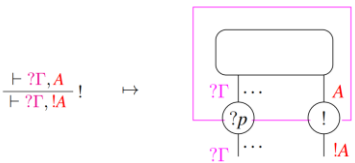
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From proofs to proofs structures



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From proofs to proofs structures



- Boxes never overlap.
- Boxes are sequential (as rules in sequent calculus).
- Correctness: box by box.
- Boxes permit duplication and erasure.

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Confluence?
Weak Normalization?
Strong Normalization?

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Properties of MELL reduction

- 1. Is confluent?
- 2. Is weakly normalizing?

Tip for WN.

Given a proof-net R, try to make decrease a size S(R).

For example:

- **Size of a cut:** pair (s,t) where
s is the size of the cut formula and,
t is the size of the ?-tree above the ? premisses of the cut if any, or 0

- **Size S(R)** of the **proof-net R**:
the multiset of the sizes of all its cuts

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Memo: multiset ordering

Notation: we write $M = [a, c, c, c, d, d, d]$ for the finite multiset

$$M = a \mapsto 1, \quad b \mapsto 0, \quad c \mapsto 3, \quad d \mapsto 4$$

on the set $S = \{a, b, c, d\}$. We also write $s \in M$ when $M(s) \neq 0$ or equivalently, when $s \in \text{dom}(M)$. Given two multisets M, N , we define the multiset $M \uplus N$ as follows:

$$M \uplus N = s \mapsto M(s) + N(s).$$

We write $M \subseteq N$ when $\forall s \in S, M(s) \leq N(s)$ and define in that case the multiset $M \setminus N$ as follows:

$$M \setminus N = s \mapsto M(s) - N(s).$$

$$M \succ_{\text{mset}} N$$

for two finite multisets M, N on S , when there exists two finite multisets X, Y such that

$$N = (M \setminus X) \uplus Y \quad \text{where} \quad \forall y \in Y, \exists x \in X, x \succ y.$$

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Memo: multiset ordering

Notation: we write $M = [a, c, c, c, d, d, d]$ for the finite multiset

$$M = a \mapsto 1, \quad b \mapsto 0, \quad c \mapsto 3, \quad d \mapsto 4$$

on the set $S = \{a, b, c, d\}$. We also write $s \in M$ when $M(s) \neq 0$ or equivalently, when $s \in \text{dom}(M)$. Given two multisets M, N , we define the multiset $M \uplus N$ as follows:

$$M \uplus N = s \mapsto M(s) + N(s).$$

We write $M \subseteq N$ when $\forall s \in S, M(s) \leq N(s)$ and define in that case the multiset $M \setminus N$ as follows:

$$M \setminus N = s \mapsto M(s) - N(s).$$

Let us look just one element of M: replace x with Y

§1. Given two finite multisets M, N , we write $M \succ_{\text{mset}} N$ when

$$N = (M \setminus [x]) \uplus Y$$

for an element $x \in M$ and a finite multiset Y such that $\forall y \in Y, x \succ y$.

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Are comparable? One is smaller?

Suppose the underlying set is $S = \{a, b, 1, 2\}$ and we define an ordering $a < b, 1 < 2$. Consider two multisets:

- $M_1 = \{2, 2, b\}$
- $M_2 = \{a, a, 1, 1\}$

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Properties of MELL reduction

- 1. Is confluent?
- 2. Is weakly normalizing?

Tip for WN.

Given a proof-net R, try to make decrease a size S(R).

For example:

- **Size of a cut:** pair (s,t) where
s is the size of the cut formula and,
t is the size of the ?-tree above the ? premisses of the cut if any, or 0

- **Size S(R)** of the **proof-net R**:
the multiset of the sizes of all its cuts

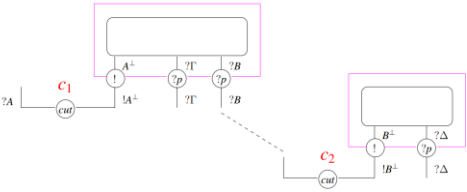
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Weak Normalization

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Weak Normalization

- cut relation: $c_1 \prec c_2$ (exponential cuts only)



- correctness $\Rightarrow$ no cycle in $\prec^*$ $\Rightarrow$ maximal cuts
- reduction of a maximal cut $\Rightarrow$ $[(|formula|, |?-tree|)]$ decreases

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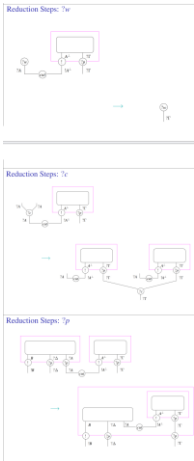
Weak Normalization

Size of a cut: pair (s,t) where s is the size of the cut and, t is the size of the $?$ -tree above the $?$ premiss of the cut if any, or 0

Size $S(R)$ of the proof-net R : the multiset of the sizes of all its cuts

- Non-bouncing path:** a path which “descends”, turning at cuts, ax and boxes:
- if the path reaches a node by a conclusion continues by a premiss and viceversa
 - cut and ax: enters by one edge, continues on the other
 - exponential boxes:
 - From “outside”: if the path reaches a box by the conclusion of the $!$ -door, it leaves by a $?$ -conclusion and vice versa.
 - From “inside”: if the path arrives to the premiss of the $!$ -door it stops. If it arrives by a premiss of an auxiliary door, it continues by descending on the $?$ -conclusion
- What are the extremities?
- Is a cycle possible?

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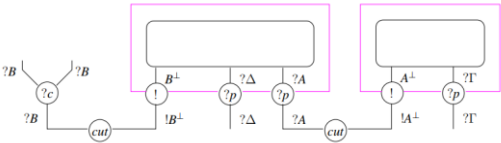


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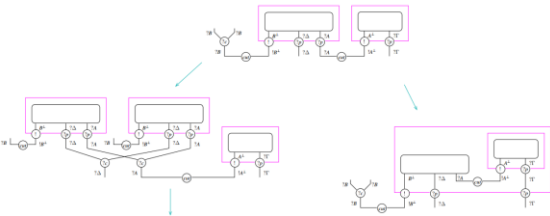
Confluence

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Is the reduction confluent?

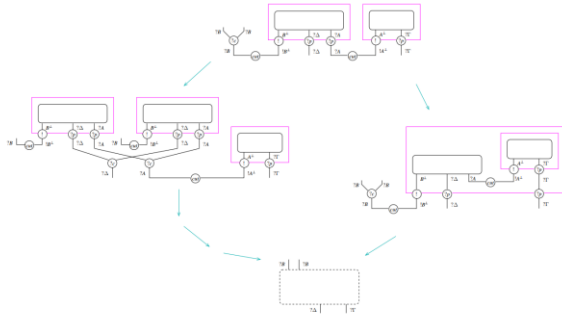


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Local Confluence ...



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- a/a (shared ax)

$$\begin{array}{c} n \\ \swarrow \searrow \\ a \quad a \\ \downarrow \quad \downarrow \\ n+1 \end{array}$$

- a/a (shared cut)

$$\begin{array}{c} n \\ \swarrow \searrow \\ a \quad a \\ \downarrow \quad \downarrow \\ n+1 \end{array}$$

- d/in

$$\begin{array}{c} n, m \\ \swarrow \searrow \\ n+m+1 \quad n+m' \\ \swarrow \searrow \quad \swarrow \searrow \\ in \quad d \quad in \quad d \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ n+m'+1 \end{array}$$

- c/in

$$\begin{array}{c} n, m \\ \swarrow \searrow \\ n, m, m' \quad n, m' \\ \swarrow \searrow \quad \swarrow \searrow \\ in \quad c \quad in \quad c \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ n, m', m \quad n, m', m' \end{array}$$

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- d/p

$$\begin{array}{c} n, m, k \\ \swarrow \searrow \\ n+m+1, k \end{array}$$

- w/p

$$\begin{array}{c} n, m, k \\ \swarrow \searrow \\ n, m, k \end{array}$$

- c/p

$$\begin{array}{c} n, m, k \\ \swarrow \searrow \\ n, m, m, k \quad n, m, k \\ \swarrow \searrow \quad \swarrow \searrow \\ n, m, m, k, k \quad n, m, k \\ \swarrow \searrow \quad \swarrow \searrow \\ n, m, m, k, k \quad n, m, m, k, k \\ \swarrow \searrow \quad \swarrow \searrow \\ n, m, m, k, k \quad n, m, m, k, k \end{array}$$

- p/p

$$\begin{array}{c} n, m, k, l \\ \swarrow \searrow \\ n, m, k, l \quad n, m, k, l \\ \swarrow \searrow \quad \swarrow \searrow \\ n, m, k, l \quad n, m, k, l \\ \swarrow \searrow \quad \swarrow \searrow \\ n, m, k, l \quad n, m, k, l \end{array}$$

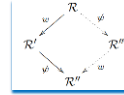
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From weak to strong normalization

Proposition B.5 (Increasing Normalization)

For any ARS, local confluence $\wedge \mu$ -increasing $\wedge$ weak normalization $\implies$ strong normalization.

Strong normalization of proof-nets just a matter of some technical steps



Lemma 2.7 (Weak non w Normalization)

The $\rightarrow_{\mu}$ reduction of numbered proof nets is weakly normalizing.

Lemma 2.8 (Increasing non w Reduction)

The reduction $\rightarrow_{\mu}$ on numbered proof nets is μ -increasing where, for a numbered proof net $\mathcal{R}$, $\mu(\mathcal{R}) = l^2 + p$ with:

- l is the sum of all the labels of $\mathcal{R}$,
- p is the sum of the depths of the boxes of $\mathcal{R}$.

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- w/in

$$\begin{array}{c} n, m \\ \swarrow \searrow \\ w \quad in \\ \downarrow \quad \downarrow \\ n \quad n, m' \end{array}$$

- p/in (left side)

$$\begin{array}{c} n, m, k \\ \swarrow \searrow \\ n, m, k \quad n, m', k \\ \swarrow \searrow \quad \swarrow \searrow \\ in \quad p \quad in \quad p \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ n, m', k \end{array}$$

- p/in (right side)

$$\begin{array}{c} n, m, k \\ \swarrow \searrow \\ n, m, k \quad n, m, k' \\ \swarrow \searrow \quad \swarrow \searrow \\ in \quad p \quad in \quad p \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ n, m, k' \end{array}$$

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How many cut-free proof-net

Homework

There exist exactly two cut-free proof-nets with conclusion

$$(X \otimes X) \multimap (X \otimes X)$$

Q1. Prove that exist (essentially) two cut-free proof-nets with conclusion

$$!(?X^\perp \wp (?X^\perp \wp X))$$

Q2. Taking the above as exponential encoding for Booleans, propose an encoding for the negation.

Q3: ? how many cut-free proof-nets have conclusion

$$!(X \multimap X) \multimap (X \multimap X)$$

Q4: ? how many cut-free proof-nets have conclusion

$$!(X \multimap X) \multimap !(X \multimap X)$$

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Additive encoding of the additives Homework

Let us work in LL **sequent calculus**.

Another way to encode the Booleans in LL is with the additive formula $\mathbf{B} := \mathbf{1} \oplus \mathbf{1}$.

There are two cut-free derivations of $\mathbf{1} \oplus \mathbf{1}$ (which ones?)
Let us call one True and the other False

Please provide an encoding (as sequent calculus derivation) of

1. True and False
2. the **not** function
3. The **if-then-else** function

Please justify you answer.

Translating lambda-calculus into LL

CbN and CbV

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Simply typed lambda calculus

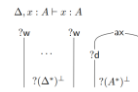
$$\frac{}{\Gamma, x : \tau \vdash x : \tau} (var) \quad \frac{\Gamma, x : \tau \vdash t : \sigma}{\Gamma \vdash \lambda x. t : \tau \rightarrow \sigma} (abs) \quad \frac{\Gamma \vdash t : \tau \rightarrow \sigma \quad \Gamma \vdash u : \tau}{\Gamma \vdash t u : \sigma} (app)$$

$$\begin{aligned} X^* &= X \\ (A \rightarrow B)^* &= !A^* \multimap B^* \\ (\Gamma \vdash A)^* &= !\Gamma^* \vdash A^* \end{aligned}$$

$$\begin{aligned} X^* &= !X \\ (A \rightarrow B)^* &= !(A^* \multimap B^*) \\ (\Gamma \vdash A)^* &= \Gamma^* \vdash A^* \end{aligned}$$

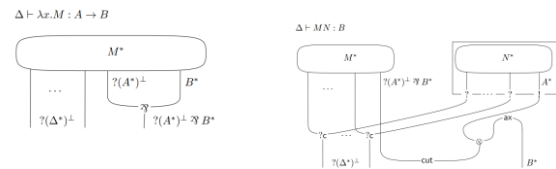
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CbN translation



$$\begin{aligned} X^* &= X \\ (A \rightarrow B)^* &= !A^* \multimap B^* \\ (\Gamma \vdash A)^* &= !\Gamma^* \vdash A^* \end{aligned}$$

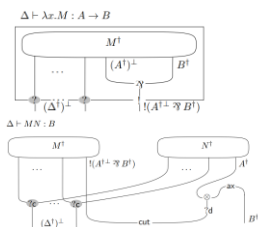
$$\frac{\Gamma, x : \tau \vdash t : \sigma}{\Gamma \vdash \lambda x. t : \tau \rightarrow \sigma} (abs) \quad \frac{\Gamma \vdash t : \tau \rightarrow \sigma \quad \Gamma \vdash u : \tau}{\Gamma \vdash t u : \sigma} (app)$$



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CbV translation

$$\begin{aligned} X^* &= !X \\ (A \rightarrow B)^* &= !(A^* \multimap B^*) \\ (\Gamma \vdash A)^* &= \Gamma^* \vdash A^* \end{aligned}$$

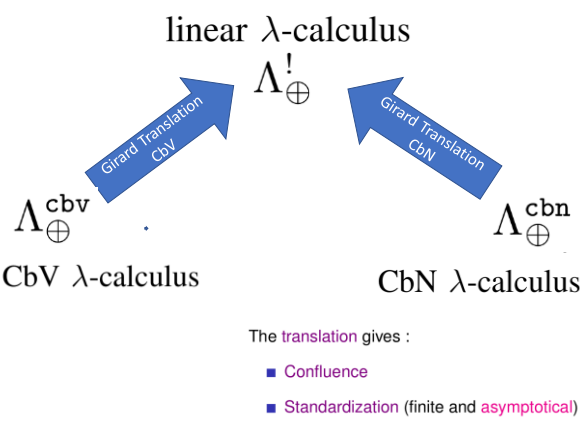


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CbV



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Three lambda calculi... Or indeed just one.

The [two Girard's translations](#) (intuitionistic into linear logic) are well-known to correspond to the **CbN** and **CbV**.

Taking this point of view provides a modular approach.

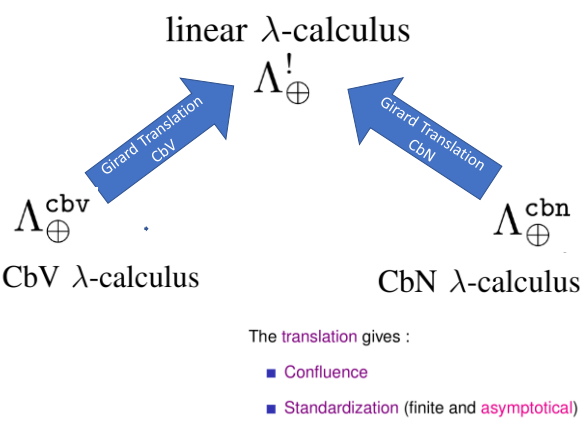
In $\Lambda_{\oplus}^!$ the natural constraint is:

■ no reduction in the scope of a !-box

Girard's translations transform this policy respectively in:

- $\Lambda_{\oplus}^{cbv}$: no reduction under λ -abstraction
- $\Lambda_{\oplus}^{cbn}$ no no reduction argument position.

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