

Linear Logic

[J.-Y. Girard, '86]

Recommended reference:

Lecture Notes by Olivier Laurent (in French)

Online Handbook of Linear Logic (on-going, collective project)

Credits:

In my LL slides, most figures are taken from Laurent's notes, or from the Handbook of Linear Logic.

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Let us start from **syntax**: recall LK ?

Groupe logique multiplicatif.

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \wedge^{\text{mul}}R \quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge^{\text{mul}}L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee^{\text{mul}}R \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \vee^{\text{mul}}L$$

Groupe logique additif.

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge^{\text{add}}R \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge^{\text{add}}L_1 \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge^{\text{add}}L_2$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee^{\text{add}}R_1 \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee^{\text{add}}R_2 \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee^{\text{add}}L$$

Groupe structurel.

$$\frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \pi(\Delta)} \text{sch}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ctrL} \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, A, \Delta} \text{ctrR}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{wL} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{wR}$$

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LL Sequent Calculus

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• **Classical** logic is about **truth**

• **Constructive** logic is about **proofs** (and programs as functions)

• **Linear** Logic is about **resources** and the dynamics of computation : **how to compute?**

Menu a 75 Frs
Entree
quiche lorraine
ou
saumon fume
et
Plat
pot-au-feu
ou
filet de canard
et
Fruit
selon saison (banane ou raisin ou oranges ou ananas)
ou
Dessert au choix (mistere, glace, tarte aux pommes)

Mousse triceps

Ingredients: chocolate, eggs

- 1 Get the eggs
- 2 Get the chocolate
- 3 Warm the chocolate
- 4 Separate the eggs
- 5 Beat the whites
- 6 Add the yolks in the chocolate
- 7 Add the whites in the mixture



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What's in a conjunction ?

euro $\vdash$ coffee euro $\vdash$ coffee
euro, euro $\vdash$ coffee $\otimes$ coffee euro $\vdash$ croissant
euro, euro, euro $\vdash$ (coffee $\otimes$ coffee) $\otimes$ croissant

spring $\vdash$ birds spring $\vdash$ blooming
spring $\vdash$ birds & blooming

- Resources
- Time of computation
- Memory space
- Number of qubits
-

- Shared resources
- Mathematical lemmas
- Choice (only one of several)
- Shared dependency/causality
- Random variables
-

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LL Formulas:

$$A ::= X \mid A \wp A \mid A \& A \mid ?A \mid X^\perp \mid A \otimes A \mid A \oplus A \mid !A$$

SEQUENT CALCULUS RULES (*Sequents* are multi-sets of formulas)

Multiplicative group:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes R \quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \wp B, \Delta} \wp R \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \wp B \vdash \Delta, \Delta'} \wp L$$

Additive group:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \& R \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \& L_1 \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \& L_2$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus R_1 \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus R_2 \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus L$$

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$$\frac{}{A \vdash A} (ax) \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} (cut) \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} (n_L) \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} (n_R)$$

Identity and negation group

Exponential group:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta} ?R \quad \frac{! \Gamma, A \vdash ? \Delta}{! \Gamma, ?A \vdash ? \Delta} ?L$$

$$\frac{! \Gamma \vdash A, ? \Delta}{! \Gamma \vdash !A, ? \Delta} !R \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} !L$$

$$\frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta} ctrR \quad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} ctrL$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta} wkR \quad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} wkL$$

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Taxonomy (and how to pronounce it)

	Positive	Negative	Class
α	atom	$A^\perp$	negation
$A \otimes B$	tensor	$A \wp B$	par
1	one	$\perp$	bottom
$A \oplus B$	plus	$A \& B$	with
0	zero	$\top$	top
$!A$	of course	$?A$	why not
$\exists \xi. A$	there exists	$\forall \xi. A$	for all
			quantifiers

Table 1.1: Formulas of Linear Logic.

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MALL fragment

$$\frac{}{A \vdash A} ax \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} cut$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes R \quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \wp B, \Delta} \wp R \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \wp B \vdash \Delta, \Delta'} \wp L$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \& R \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \& L_1 \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \& L_2$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus R_1 \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus R_2 \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus L$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} (\cdot)^\perp R \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} (\cdot)^\perp L$$

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Lafont's
menu

Menu a 75 Frs
Entree
quiche lorraine
ou
saumon fume
et
Plat
pot-au-feu
ou
filet de canard
et
Fruit
selon saison (banane ou raisin ou oranges ou ananas)
ou
Dessert au choix (mistere, glace, tarte aux pommes)

$$75FF^\perp \wp (Q \& S) \otimes (P \& F) \otimes ((B \oplus R \oplus O \oplus A) \& (M \& G \& T))$$

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Define negation:

$$A ::= X \mid A \wp A \mid A \& A \mid X^\perp \mid A \otimes A \mid A \oplus A$$

$$(X^\perp)^\perp = X$$

$$(A \otimes B)^\perp = A^\perp \wp B^\perp$$

$$(A \wp B)^\perp = A^\perp \otimes B^\perp$$

$$(A \& B)^\perp = A^\perp \oplus B^\perp$$

$$(A \oplus B)^\perp = A^\perp \& B^\perp$$

ce qui donne $A^{\perp\perp} = A$.

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One-sided sequent calculus

$$(X^\perp)^\perp = X$$

$$(A \otimes B)^\perp = A^\perp \wp B^\perp$$

$$(A \wp B)^\perp = A^\perp \otimes B^\perp$$

$$(A \& B)^\perp = A^\perp \oplus B^\perp$$

$$(A \oplus B)^\perp = A^\perp \& B^\perp$$

$$\frac{}{\vdash A^\perp, A} ax \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} cut$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \wp$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \& \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \oplus_1 \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \oplus_2$$

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To familiarize...

EX.

prove the following **distributivity laws** :

$$A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$$

$$A \wp (B \& C) \dashv\vdash (A \wp B) \& (A \wp C)$$

EX.

The axiom can be restricted to atomic formulas:
the axiom rule is admissible for every formula

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$$A \multimap B = A^\perp \wp B$$

The standard rules are hence derivable:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta} \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \multimap B \vdash \Delta, \Delta'}$$

The same, single-sided:

$$\frac{\vdash \Gamma, A^\perp, B}{\vdash \Gamma, A \multimap B} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B^\perp}{\vdash \Gamma, \Delta, (A \multimap B)^\perp}$$

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Homework

EX.

How do we define the non-logical steps?

Give the following two examples:

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes \quad \text{Cut with a formula } C \text{ inside } \Gamma = \Gamma', C. \text{ How do we reduce this cut?}$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \& \quad \text{Cut with a formula } C \text{ inside } \Gamma = \Gamma', C. \text{ How do we reduce this cut?}$$

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Cut elimination ("the engine of sequent calculus")

Logical-step (the key cases):

$$\begin{aligned} & \frac{\vdash \Gamma, A \quad \vdash A^\perp, A}{\vdash \Gamma, A} \text{cut} \rightsquigarrow \vdash \Gamma, A \\ & \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes \quad \frac{\vdash \Xi, A^\perp, B^\perp}{\vdash \Xi, A^\perp \wp B^\perp} \wp \rightsquigarrow \frac{\vdash \Gamma, A \quad \vdash \Delta, B \quad \vdash \Xi, A^\perp, B^\perp}{\vdash \Gamma, \Delta, \Xi} \text{cut} \\ & \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \& \quad \frac{\vdash \Delta, A^\perp}{\vdash \Delta, A^\perp \oplus B^\perp} \oplus_1 \rightsquigarrow \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} \text{cut} \\ & \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \& \quad \frac{\vdash \Delta, B^\perp}{\vdash \Delta, A^\perp \oplus B^\perp} \oplus_2 \rightsquigarrow \frac{\vdash \Gamma, B \quad \vdash \Delta, B^\perp}{\vdash \Gamma, \Delta} \text{cut} \end{aligned}$$

Non-logical steps are the obvious ones.

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Re-introducing the
Structural Rules

In a controlled way

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Exponentials

$$A ::= \begin{array}{l} X \mid A \wp A \mid A \& A \\ X^\perp \mid A \otimes A \mid A \oplus A \end{array} \quad \boxed{\begin{array}{l} ?A \\ !A \end{array}}$$

$$(X^\perp)^\perp = X$$

$$(A \otimes B)^\perp = A^\perp \wp B^\perp$$

$$(A \wp B)^\perp = A^\perp \otimes B^\perp$$

$$(A \& B)^\perp = A^\perp \oplus B^\perp$$

$$(A \oplus B)^\perp = A^\perp \& B^\perp$$

$$\boxed{(!A)^\perp := ?(A)^\perp \quad (?A)^\perp := !(A)^\perp}$$

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Exponentials rules

$$A ::= \begin{array}{l} X \mid A \wp A \mid A \& A \\ X^\perp \mid A \otimes A \mid A \oplus A \end{array} \quad \boxed{\begin{array}{l} ?A \\ !A \end{array}}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} ?d$$

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} ?c$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} ?w$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} !$$

The ?d rule ("dereliction") allows any formula to become subject to the structural rules. The rules ?w ("weakening") and ?c ("contraction") are the usual structural rules, but they apply only to formulas whose principal connective is ?.

The ! rule is called "promotion"/

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! rule

The ! Rule is the most subtle one.

It makes a formula (and above all the corresponding proof) duplicable (or erasable), but this requires a suitable context.

One can understand !A as "A as many times as one wants". The rule, written in the form:

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A} !$$

says that if A is obtained from hypotheses that can be used as many times as one wants, then A itself can be used as many times as one wants.

A good way to understand the context constraint of this rule is to look at cut elimination.

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Cut-elimination steps for exponentials

$$\frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash \Delta, A^\perp}{\vdash \Delta, ?A^\perp} ?d}{\vdash ?\Gamma, \Delta} cut \rightsquigarrow \frac{\vdash ?\Gamma, A \quad \vdash \Delta, A^\perp}{\vdash ?\Gamma, \Delta} cut$$

$$\frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash \Delta, ?A^\perp, ?A^\perp}{\vdash \Delta, ?A^\perp} ?c}{\vdash ?\Gamma, \Delta} cut \rightsquigarrow \frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash ?\Gamma, !A \quad \vdash \Delta, ?A^\perp, ?A^\perp}{\vdash ?\Gamma, \Delta, ?A^\perp} cut}{\vdash ?\Gamma, \Delta} ?c$$

$$\frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash \Delta}{\vdash \Delta, ?A^\perp} ?w}{\vdash ?\Gamma, \Delta} cut \rightsquigarrow \frac{\vdash \Delta}{\vdash ?\Gamma, \Delta} ?w$$

$$\frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \frac{\vdash ?\Delta, ?A^\perp, B}{\vdash ?\Delta, ?A^\perp, !B} !}{\vdash ?\Gamma, ?\Delta, !B} cut \rightsquigarrow \frac{\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} ! \quad \vdash ?\Delta, ?A^\perp, B}{\vdash ?\Gamma, ?\Delta, B} cut \quad \frac{\vdash ?\Gamma, ?\Delta, B}{\vdash ?\Gamma, ?\Delta, !B} !$$

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Ex (Homework)

Two formulas A and B are (linearly) equivalent, written $A \dashv\vdash B$, if both implications $A \multimap B$ and $B \multimap A$ are provable. Equivalently, $A \dashv\vdash B$ if both $A \vdash B$ and $B \vdash A$ are provable.

Ex. Pick and prove one equivalence in each of the 3 groups

1. $A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$
 $A \wp (B \& C) \dashv\vdash (A \wp B) \& (A \wp C)$
2. $!(A \& B) \dashv\vdash !A \otimes !B$
 $?(A \oplus B) \dashv\vdash ?A \wp ?B$
3. $!A \otimes !A \dashv\vdash !A$
 $?A \wp ?A \dashv\vdash ?A$



Homework: points 3 (only).
 Please use **one-sided** sequents, and **atomic axioms**: assume A is an **atom**.

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Neutral Elements

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On définit quatre éléments neutres pour les quatre connecteurs multiplicatifs et additifs : 1 ("un") pour $\otimes$, $\perp$ ("bottom") pour $\wp$, $\top$ ("top") pour $\&$ et 0 ("zéro") pour $\oplus$:

$$A ::= \begin{array}{c|c|c|c|c|c|c} X & A \wp A & A \& A & \perp & \top & ?A \\ \hline X^\perp & A \otimes A & A \oplus A & 1 & 0 & !A \end{array}$$

Les règles sont obtenues comme les cas 0-aires des règles de connecteurs binaires correspondants (en particulier deux règles pour $\oplus$ donc aucune pour 0) :

$$\frac{}{\vdash 1} 1 \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp} \perp \quad \frac{}{\vdash \Gamma, \top} \top$$

Les étapes clés d'élimination des coupures sont :

$$\frac{\frac{}{\vdash 1} 1 \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp} \perp}{\vdash \Gamma} cut \rightsquigarrow \vdash \Gamma$$

puisqu'il n'y a pas de règle pour 0.

Consequences of Cut-elimination

Theorem 3.4.1 (cut elimination). *For every sequent $\Gamma \vdash \Delta$, there is a proof of $\Gamma \vdash \Delta$ if and only if there is a proof of $\Gamma \vdash \Delta$ that does not use the cut rule.*

Theorem 3.4.3 (subformula property). *A sequent $\Gamma \vdash \Delta$ is provable if and only if it is the conclusion of a proof in which each intermediate conclusion is made of subformulas of the formulas of Γ and Δ .*

Theorem 3.4.4 (consistency). *The empty sequent $\vdash$ is not provable. Subsequently, it is impossible to prove both a formula A and its negation $A^\perp$; it is impossible to prove 0 or $\perp$.*

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On Proof Search

Reversing the Reversible Rules

Definition 1.6.1 (Reversibility). A connective c is called *reversible* if for every proof $\pi : \vdash \Gamma, c(A_1, \dots, A_n)$, there is a proof π' with the same conclusion in which $c(A_1, \dots, A_n)$ is introduced by the last rule (i.e. principal).

$$\begin{array}{c} \frac{\vdash A \wp B, \Gamma}{\vdash A, B, \Gamma} (\wp^{rev}) \quad \frac{\vdash \perp, \Gamma}{\vdash \Gamma} (\perp^{rev}) \quad \frac{\vdash A_1 \& A_2, \Gamma}{\vdash A_1, \Gamma} (\&^{rev}) \\ \\ \frac{\frac{}{\vdash A, A^\perp} (ax) \quad \frac{}{\vdash B, B^\perp} (ax)}{\vdash A, B, A^\perp \otimes B^\perp} (\otimes) \quad \frac{}{\vdash A \wp B, \Gamma} (cut) \\ \vdash A, B, \Gamma \\ \\ \frac{\frac{}{\vdash A, A^\perp} (ax)}{\vdash A, A^\perp \oplus B^\perp} (\oplus_1) \quad \frac{}{\vdash A \& B, \Gamma} (\&) \\ \vdash A, \Gamma \end{array}$$

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Reversible Rules

$$\vdash \Gamma, A \wp B \rightsquigarrow \frac{\frac{\frac{}{\vdash A^\perp, A} ax \quad \frac{}{\vdash B^\perp, B} ax}{\vdash A^\perp \otimes B^\perp, A, B} \otimes}{\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \wp} cut$$

$$\vdash \Gamma, A \& B \rightsquigarrow \frac{\frac{\frac{}{\vdash A^\perp, A} ax}{\vdash A^\perp \oplus B^\perp, A} \oplus_1}{\vdash \Gamma, A} \oplus_1 \quad \frac{\frac{\frac{}{\vdash B^\perp, B} ax}{\vdash A^\perp \oplus B^\perp, B} \oplus_2}{\vdash \Gamma, B} \oplus_2 \quad \frac{\vdash \Gamma, A \& B \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \&$$

Reversible Rules

$$\vdash \Gamma, \perp \rightsquigarrow \frac{\frac{\vdash \Gamma, \perp}{\vdash \Gamma, \perp} \perp \quad \frac{}{\vdash 1} 1}{\vdash \Gamma} cut$$

$$\vdash \Gamma, \top \rightsquigarrow \frac{}{\vdash \Gamma, \top} \top$$

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Positive/negative connective

	Positive	Negative	Class
α	atom	$A^\perp$	negation
$A \otimes B$	tensor	$A \wp B$	par
$\mathbf{1}$	one	$\perp$	bottom
$A \oplus B$	plus	$A \& B$	with
$\mathbf{0}$	zero	$\top$	top
$!A$	of course	$?A$	why not
			exponentials

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Focalization

A consequence of reversibility is that, **when searching for a proof** of a sequent Γ , we can always start by decomposing negative formulas in Γ without losing provability.

For instance:

- $\vdash \Gamma, (A \wp B) \wp (B \& C)$ is provable
- iff $\vdash \Gamma, A \wp B, B \& C$ is provable
- iff $\vdash \Gamma, A \wp B, B$ and $\vdash \Gamma, A \wp B, C$ are provable
- iff $\vdash \Gamma, A, B, B$ and $\vdash \Gamma, A, B, C$ are provable

So without loss of generality, we can assume that any proof of $\vdash \Gamma, (A \wp B) \wp (B \& C)$ ends like

$$\frac{\frac{\vdash \Gamma, A, B, B \quad (\wp)}{\vdash \Gamma, A \wp B, B} \quad (\wp) \quad \frac{\vdash \Gamma, A, B, C}{\vdash \Gamma, A \wp B, C} (\wp)}{\vdash \Gamma, A \wp B, B \& C} (\&) \quad (\wp)$$

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Focalization

Definition 1.6.9. A formula is *positive* if it has a principal connective among $\otimes, \oplus, \mathbf{1}, \mathbf{0}$. It is called *negative* if it has a principal connective among $\wp, \&, \perp, \top$. It is called *neutral* if it is neither positive nor negative.

Prop. (positive focus)

Let $\vdash \Gamma$ be a sequent **containing only positive formulas** (and possibly atoms). If $\vdash \Gamma$ is provable then there **exists a formula P** in Γ such that $\vdash \Gamma$ is provable by a derivation that ends with the rules for introducing the **positive layer of P** (the proof is then said to be focused).

Theorem 1.6.11. A sequent $\vdash \Gamma$ is ☐ provable if and only if it is provable by a cut-free proof that is positively focused.

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Focalization

Definition 1.6.9. A formula is *positive* if it has a principal connective among $\otimes, \oplus, \mathbf{1}, \mathbf{0}$. It is called *negative* if it has a principal connective among $\wp, \&, \perp, \top$. It is called *neutral* if it is neither positive nor negative.

Definition 1.6.10. A proof $\pi : \vdash \Gamma, A$ is said to be *positively focused* on A if it has the shape

$$\frac{\pi_1 : \vdash \Gamma_1, A_{i_1} \quad \dots \quad \pi_\ell : \vdash \Gamma_\ell, A_{i_\ell}}{\vdash \Gamma_1, \dots, \Gamma_\ell, P[A_{i_1}, \dots, A_{i_\ell}]} (P_{(i_1, \dots, i_\ell)})$$

where P is a positive generalized connective, the A_i are non-positive and $A = P[A_{i_1}, \dots, A_{i_\ell}]$. The formula A is called the *focus* of the proof π .

In other words, a proof is positively focused on a conclusion A if its last rules build A from some of its non-positive subformulas in one cluster of inferences.

Theorem 1.6.11. A sequent $\vdash \Gamma$ is ☐ provable if and only if it is provable by a cut-free proof that is positively focused.

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LL formulas and Linear negation

Grammar of LL formulas:

$$F, G ::= a \mid F \otimes G \mid \mathbf{1} \mid F \oplus G \mid \mathbf{0} \mid !F \mid a^\perp \mid F \wp G \mid \perp \mid F \& G \mid \top \mid ?F$$

Summing-Up: Inference rules for Linear Logic Sequent Calculus

$$\begin{array}{ll} (\alpha)^\perp := \alpha^\perp & (\alpha^\perp)^\perp := \alpha \\ (\mathbf{1})^\perp := \perp & (\perp)^\perp := \mathbf{1} \\ (A \otimes B)^\perp := (A)^\perp \wp (B)^\perp & (A \wp B)^\perp := (A)^\perp \otimes (B)^\perp \\ (\top)^\perp := \mathbf{0} & (\mathbf{0})^\perp := \top \\ (A \oplus B)^\perp := (A)^\perp \& (B)^\perp & (A \& B)^\perp := (A)^\perp \oplus (B)^\perp \\ (!A)^\perp := ?(A)^\perp & (?A)^\perp := !(A)^\perp \end{array}$$

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Identity group $\frac{}{\vdash F^\perp, F} (ax) \quad \frac{\vdash F, \Gamma \quad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta} (cut)$	
Multiplicative group $\frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta} (\otimes) \quad \frac{}{\vdash \mathbf{1}} (1) \quad \frac{\vdash F, G, \Gamma}{\vdash F \wp G, \Gamma} (\wp) \quad \frac{\vdash \Gamma}{\vdash \perp, \Gamma} (\perp)$	
Additive group $\frac{\vdash F_i, \Gamma}{\vdash F_1 \oplus F_2, \Gamma} (\oplus_i) \quad \frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \& G, \Gamma} (\&) \quad \frac{}{\vdash \top, \Gamma} (\top)$	
Exponential group $\frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma} (!) \quad \frac{\vdash F, \Gamma}{\vdash ?F, \Gamma} (?)$	
Structural group $\frac{\vdash \Gamma, F, G, \Delta}{\vdash \Gamma, G, F, \Delta} (ex) \quad \frac{\vdash \Gamma}{\vdash ?F, \Gamma} (w) \quad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma} (c)$	