TD1: CbN and CbV Lambda Calculus

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- RECAP:

CbN and CbV Calculi.

- The (pure) **Call-by-Name** calculus $\Lambda^{\mathsf{cbn}} = (\Lambda, \to_\beta)$ is the set of terms equipped with the contextual closure of the β-rule.

$$(\lambda x.M)N \mapsto_\beta M\{N/x\}$$

- The (pure) **Call-by-Value** calculus $\Lambda^{\mathsf{cbv}} = (\Lambda, \to_{\beta_v})$ is the same set equipped with the contextual closure of the β_v-rule.

$$(\lambda x.M)V \mapsto_{\beta_v} M\{V/x\} \text{ where } V \in \mathcal{V}$$

Head reduction in CbN

Head reduction is the closure of β under head context

$\lambda x_1 ... x_n . (\!|\ |\!) M_1 ... M_k$

Head normal forms (hnf), whose set is denoted by $\mathcal{H}$, are its normal forms.

- Given a rule ρ, we write $\underset{\mathsf{h}}{\to}_\rho$ for its closure under head context.
- A step $\to_\rho$ is non-head, written $\underset{\neg\mathsf{h}}{\to}_\rho$ if it is not head.

Weak reductions in CbV

The result of interest are **values** (*i.e.* functions).
In languages, in general the reduction is *weak*, that is, it does not reduce in the body of a function.

There are three main weak schemes: left, right and in arbitrary order.
Left contexts $\mathbf{L}$, *right* contexts $\mathbf{R}$, and (arbitrary order) *weak* contexts $\mathbf{W}$ are defined by

$\mathbf{L} ::= (\!|\ |\!) \mid \mathbf{L}M \mid V\mathbf{L}$
$\mathbf{R} ::= (\!|\ |\!) \mid M\mathbf{R} \mid \mathbf{R}V$
$\mathbf{W} ::= (\!|\ |\!) \mid \mathbf{W}M \mid M\mathbf{W}$

Given a rule $\mapsto$ on Λ, *weak reduction* $\underset{\mathsf{w}}{\to}$ is the closure of $\mapsto$ under context $\mathbf{W}$.
A step $T \to S$ is non-weak, written $T \underset{\neg\mathsf{w}}{\to} S$ if it is not weak. Similarly for left ($\underset{\mathsf{l}}{\to}$ and $\underset{\neg\mathsf{l}}{\to}$), and right ($\underset{\mathsf{r}}{\to}$ and $\underset{\neg\mathsf{r}}{\to}$).

▶ **Fact 3** (Weak normal forms). *Given M a closed term, M is $\underset{\mathsf{s}}{\to}$-normal* iff *$M$ is a value.*

- TD 1. We work on the properties and exercises which are highlighted

☐ BASIC PROPERTIES OF THE CONTEXTUAL CLOSURE

If a step $T \to_\gamma T'$ is obtained by closure under *non-empty context* of a rule $\mapsto_\gamma$, then T and T' have *the same shape*, i.e. *both terms are an application (resp. an abstraction, a variable).*

▶ **Fact 5** (Shape preservation).

- *Assume $T = \mathbf{C}(\!|R|\!) \to \mathbf{C}(\!|R'|\!) = T'$ and that the context $\mathbf{C}$ is* non-empty. *Then T and T' have the same shape.*
- *Hence, for any internal step $M \underset{\neg\mathsf{s}}{\to} M'$ ($\mathsf{s} \in \{\mathsf{h}, \mathsf{w}, \mathsf{l}, \mathsf{r}, ...\}$) M and M have the same shape.*

The following is an easy to verify consequence.

▶ **Lemma 6** (Redexes preservation).
1. *CbN: Assume* $T \underset{\neg h}{\rightarrow}_\beta S$. *T is a* β*-redex* iff *so is* S.

2. *CbV. Assume* $T \underset{\neg w}{\rightarrow}_{\beta_v} S$. *T is a* β_v*-redex* iff *so is* S.

Fixed a set of redexes $\mathcal{R}$, M is w-normal (resp. h-normal) if there is no redex $R \in \mathcal{R}$ such that $M = \mathbf{W}(\!|R|\!)$ (resp. $M = \mathbf{H}(\!|R|\!)$)

▶ **Lemma 7** (Surface normal forms). **1.** *CbN. Let* $\mathcal{R}$ *be the set of* β_v*-redexes.*
Assume $M \underset{\neg h}{\rightarrow}_\beta M'$. *M is* h*-normal* $\Leftrightarrow$ M' *is* h*-normal.*
2. *CbV. Let* $\mathcal{R}$ *be the set of* β_v*-redexes.*
Assume $M \underset{\neg w}{\rightarrow}_{\beta_v} M'$. *M is* w*-normal* $\Leftrightarrow$ M' *is* w*-normal.*

Using FACTORIZATION

CbN:
- *Head Factorization:* $\rightarrow_\beta{}^* \subseteq \underset{h}{\rightarrow}_\beta{}^* \cdot \underset{\neg h}{\rightarrow}_\beta{}^*$.
- EX. A Prove that M has *hnf* if and only if head reduction from M terminates .

CbV: *Left* contexts **L**, *right* contexts **R**, and (arbitrary order) *weak* contexts **W** are def by

$$\mathbf{L} ::= (\!|\ |\!) \mid \mathbf{L}M \mid V\mathbf{L}$$
$$\mathbf{R} ::= (\!|\ |\!) \mid M\mathbf{R} \mid \mathbf{R}V$$
$$\mathbf{W} ::= (\!|\ |\!) \mid \mathbf{W}M \mid M\mathbf{W}$$

The closure under **L** (resp. **W**,**R**) context is noted $\underset{l}{\rightarrow}$ (resp $\underset{w}{\rightarrow}$, $\underset{r}{\rightarrow}$)

Let $s \in \{w,l,r\}$
- *weak factorization of* $\rightarrow_{\beta_v}$: $\rightarrow_{\beta_v}{}^* \subseteq \underset{s}{\rightarrow}_{\beta_v}{}^* \cdot \underset{\neg s}{\rightarrow}_{\beta_v}{}^*$.

▶ **Fact X** (?). *Let M be a closed term. We say that M returns a value when* $M \rightarrow^*_\beta V$ *for some V.* ***EX B*** *Prove any of the following*
1. *M returns a value, if and only if* $\underset{l}{\rightarrow}_{\beta_v}$*-reduction from M terminates.*
2. *M returns value, if and only if* $\underset{w}{\rightarrow}_{\beta_v}$*-reduction from M terminates.*

EX. C. Normalization.

1. Give an inductive definition of **leftmost reduction**, completing the following b

Consider $(\Lambda, \to)$, where $\to = \to_\beta$. The relation $\underset{\text{LO}}{\to} \subseteq \to$ is induc

- If $M \underset{\mathsf{h}}{\to} M'$ then $M \underset{\text{LO}}{\to} M'$.
- If $M \underset{\mathsf{h}}{\not\to}$ (*i.e.*, M is h-normal) then:

$$\frac{P \underset{\text{LO}}{\to} P'}{M := (\lambda x.P) \underset{\text{LO}}{\to} (\lambda x.P')} \qquad \frac{P \underset{\text{LO}}{\to} P'}{M := PQ \underset{\text{LO}}{\to} P'Q} \qquad \frac{Q \underset{\text{LO}}{\to} Q'}{M := PQ \underset{\text{LO}}{\to} PQ'}$$

2. Prove that **it is a normalizing strategy**, using head factorization.

Normalization (or, make your own Normalizing strategy)

▶ **Definition 8** (Iteration of surface reduction). *Given $(\Lambda, \to)$, where $\to$ is the context closure of a rule $\mathsf{b} \in \{\beta, \beta_v\}$, let $\underset{\mathsf{s}}{\to} \subseteq \to$ be as follows:*

$$\underset{\mathsf{s}}{\to} = \underset{\mathsf{h}}{\to} \textit{ if } \mathsf{b} = \beta \textit{ (CbN)} \qquad \underset{\mathsf{s}}{\to} \in \{\underset{\mathsf{w}}{\to}, \underset{\mathsf{l}}{\to}, \underset{\mathsf{r}}{\to}\} \textit{ if } \mathsf{b} = \beta_v \; \boxed{\text{CbV}}$$

The relation $\underset{E}{\to} \subseteq \to$ is inductively defined by

- *If $M \underset{\mathsf{s}}{\to} M'$ then $M \underset{E}{\to} M'$.*
- *If $M \underset{\mathsf{s}}{\not\to}$ (i.e., M is s-normal) then:*

$$\frac{P \underset{E}{\to} P'}{M := (\lambda x.P) \underset{E}{\to} (\lambda x.P')} \qquad \frac{P \underset{E}{\to} P'}{M := PQ \underset{E}{\to} P'Q} \qquad \frac{Q \underset{E}{\to} Q'}{M := PQ \underset{E}{\to} PQ'}$$

▶ **Theorem 10** (Normalization).
CbN: $\underset{E}{\to}_\beta$ *is a normalizing strategy for* $\to_\beta$
CbV: $\underset{E}{\to}_{\beta_v}$ *is a normalizing strategy for* $\to_{\beta_v}$.