TD0: Abstract Rewriting

Abstract rewrite system (ARS) is a set *A* equipped with binary relation →

- The transitive-reflexive closure of a relation is a closure operator, *i.e.* satisfies

$$\to \subseteq \to^*, \qquad (\to^*)^* = \to^*, \qquad \to_1 \subseteq \to_2 \text{ implies } \to_1^* \subseteq \to_2^*$$

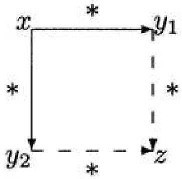
- As a consequence

$$(\to_1 \cup \to_2)^* = (\to_1^* \cup \to_2^*)^*.$$

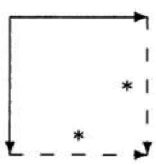
- **EX1.** Prove it!

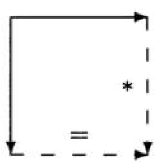
- **CONFLUENCE**

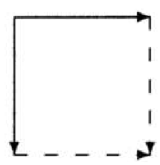
Confluent



Locally confluent **Strongly confluent** **Diamond**

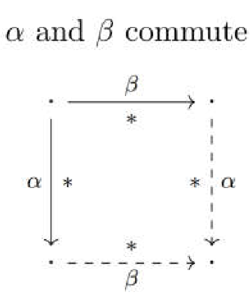






- **EX. 2**
  (a) Prove that strongly confluent implies confluent
  (b) As a preliminary step, prove: $\leftarrow^* \cdot \to \subseteq \to^* \cdot \leftarrow^*$ *implies confluence*

EX 4. Two relations commute if

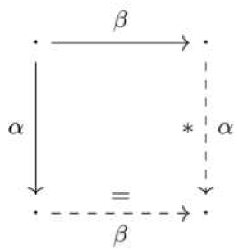


Prove that

▶ **Lemma** (Hindley-Rosen). *Let* $\rightarrow_1$ *and* $\rightarrow_2$ *be relations on the set* A. *If* $\rightarrow_1$ *and* $\rightarrow_2$ *are* ***confluent*** *and* ***commute with each other***, *then*

$$\rightarrow_1 \cup \rightarrow_2 \textit{ is confluent.}$$

EX. 5 Two relations strongly commute if



- Prove that strong commutation implies commutation

- **TERMINATION**

  - The element s is $\mathcal{R}$-weakly normalising (WN) iff s has at least one normal form
  - The element s is $\mathcal{R}$- strongly normalising (SN) iff there is no infinite sequence

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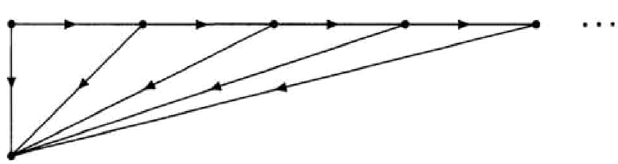
Consider



- **EX** Say which properties hold

1. Confluent
2. Locally confluent
3. Normalizing (weakly normalizing, WN)
4. Terminating (strongly normalizing, SN)

- **EX. 8**

Newman's Lemma. *Every terminating and locally confluent* ARS *is confluent.*

A second Proof.
It suffices to show that every element has unique normal forms

- suppose $B = \{ a \in A \mid \neg \mathrm{UN}(a) \} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\rightarrow$)
- $b \rightarrow^! n_1$ and $b \rightarrow^! n_2$ with $n_1 \neq n_2$

➢ **Conclude** by showing that it is impossible (**absurd**)