Proof Nets

A graph syntax for proofs

Reference:
Notes on proof-nets by Olivier Laurent

(Note: most slides are taken from the notes of Olivier Laurent)

1

MLL

$$\dfrac{}{\vdash A^\perp, A}\,ax \qquad \dfrac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut$$

$$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr$$

Recall that linear negation is defined :

$$(X^\perp)^\perp = X$$
$$(A \otimes B)^\perp = A^\perp \parr B^\perp$$
$$(A \parr B)^\perp = A^\perp \otimes B^\perp$$

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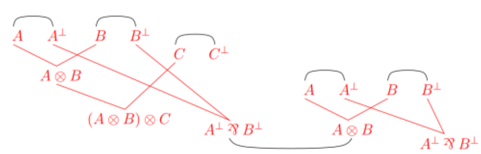
Forgetting Sequential Structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \quad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A \otimes B, A^\perp, B^\perp}(\otimes) \quad \dfrac{}{\vdash C, C^\perp}(ax)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}(\otimes)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}(\parr) \quad \dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \quad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A \otimes B, A^\perp, B^\perp}(\otimes)}{\dfrac{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}(\parr)}(cut)$$

3

Forgetting Sequential Structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \quad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A \otimes B, A^\perp, B^\perp}(\otimes) \quad \dfrac{}{\vdash C, C^\perp}(ax)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}(\otimes)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}(\parr) \quad \dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \quad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A \otimes B, A^\perp, B^\perp}(\otimes)}{\dfrac{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}(\parr)}(cut)$$



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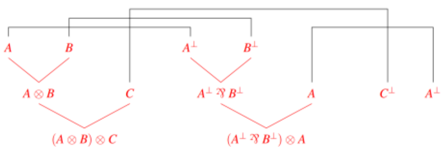
Forgetting Sequential Structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}ax\,(1) \quad \dfrac{}{\vdash B, B^\perp}ax\,(2)}{\vdash A \otimes B, A^\perp, B^\perp}\otimes\,(3) \quad \dfrac{}{\vdash C, C^\perp}ax\,(4)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}\otimes\,(5)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}\parr\,(6) \quad \dfrac{}{\vdash A, A^\perp}ax\,(7)}{\vdash (A \otimes B) \otimes C, (A^\perp \parr B^\perp) \otimes A, C^\perp, A^\perp}\otimes\,(8)$$



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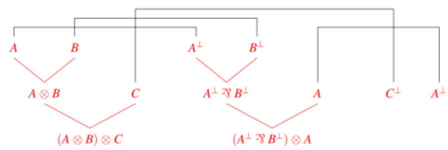
Forgetting Sequential Structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}ax\,(1) \quad \dfrac{}{\vdash B, B^\perp}ax\,(2)}{\vdash A \otimes B, A^\perp, B^\perp}\otimes\,(3) \quad \dfrac{}{\vdash C, C^\perp}ax\,(4)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}\otimes\,(5)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}\parr\,(6) \quad \dfrac{}{\vdash A, A^\perp}ax\,(7)}{\vdash (A \otimes B) \otimes C, (A^\perp \parr B^\perp) \otimes A, C^\perp, A^\perp}\otimes\,(8)$$



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Proof Nets

A graph syntax for proofs

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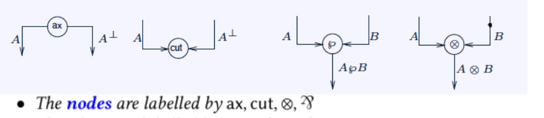
Proof structures

A ***proof structure*** $\mathcal{M}$ is ***a labelled directed acyclic graph (DAG)*** with possibly pending edges (i.e. some edges may have no source and/or no target) built over the alphabet of nodes which is represented below.
(Note: in figures, the edges orientation is always top-bottom.)



- *The* ***nodes*** *are labelled by* ax, cut, $\otimes$, $⅋$
- *The* ***edges*** *are labelled by* MLL *formulas.*

For each node/link: ***premisses*** *= entering edges,* ***conclusions*** *= exiting edges*

The ***conclusions*** *of* $\mathcal{M}$ *is the set of* pending edges *of* $\mathcal{M}$.

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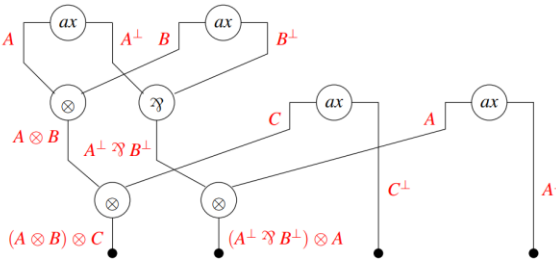
In the graphical representation of a proof structure, we do not mention explicitly the direction of edges, but we draw them in such a way that direction in represented in a top-down way:



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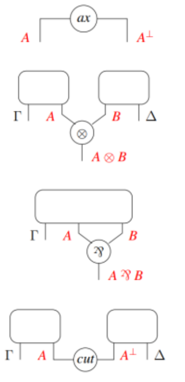
From proofs to proofs structures

$$\dfrac{}{\vdash A, A^\perp}\ ax \quad \mapsto$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ \otimes \quad \mapsto$$

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A ⅋ B}\ ⅋ \quad \mapsto$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ cut \quad \mapsto$$



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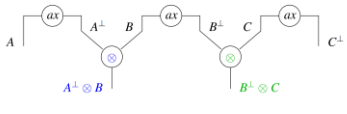
example

Translate each of these sequent calculus proofs. Start from axioms....

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\ ax \quad \dfrac{}{\vdash B, B^\perp}\ ax}{\vdash A, A^\perp \otimes B, B^\perp}\ \otimes \qquad \dfrac{}{\vdash C, C^\perp}\ ax}{\vdash A, A^\perp \otimes B, B^\perp \otimes C, C^\perp}\ \otimes \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\ ax \qquad \dfrac{\dfrac{}{\vdash B, B^\perp}\ ax \quad \dfrac{}{\vdash C, C^\perp}\ ax}{\vdash B, B^\perp \otimes C, C^\perp}\ \otimes}{\vdash A, A^\perp \otimes B, B^\perp \otimes C, C^\perp}\ \otimes$$



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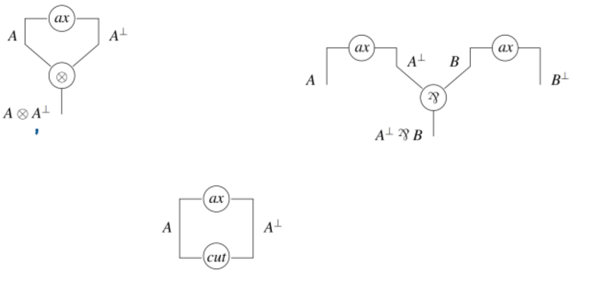
Is every structure the image of an MLL proof?



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Proof Nets

A PROOF NET is a proof structure which is the image of an MLL proof

Internal condition!

Purely geometrical conditions (correction) characterize the proof structures which are proof nets

Theorem 1. *A proof structure is* **correct** *iff it is a* **proof net**.

Correctness criteria:

- (LT) Long trip [Girard]
- (AC) Connected-Acyclic [Danos-Regnier]
- ...

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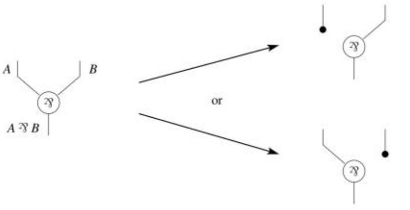
Danos-Regnier Criterion

Correctness Criterion

Switching Graphs



Correctness

- Switching graphs are acyclic.
- Switching graphs are connected.

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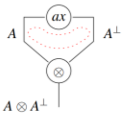

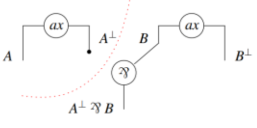

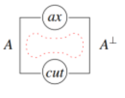

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Definition 2 (Correctness criterion AC (Danos-Regnier)). *Let R be a proof structure.*

A **switching** *s is a function on the nodes of R, which chooses, for each ⅋-link, either the left or the right premise.*

A proof structure R is **correct** *if* **for each switching***, the* **unoriented graph** *obtained by erasing for each ⅋-link of R the edges not chosen by s, is:*

connected and acyclic

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Acyclicity. A multiplicative proof structure is *acyclic* if its switching graphs do not contain any undirected cycle.

A proof structure with p ⅋ nodes induces 2^p switchings and thus 2^p switching graphs. A switching graph is not a proof structure in general since its ⅋ nodes have only one premisse.

A *connected component* of a switching graph is a connected component of its underlying (undirected) multigraph.

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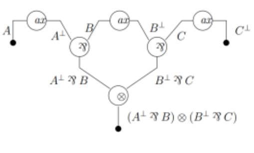
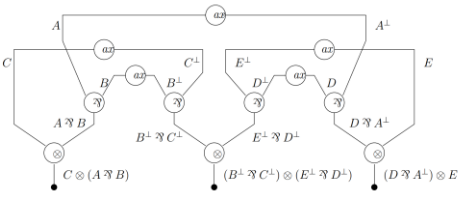
Is this correct?

PN1:



PN2:



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- Correctness guarantees:
  - ✓ Graph is image of a proof (sequentialization)
  - ✓ Normalization progresses (no deadlocks)
  - ✓ Normalization terminates (no infinite cycles)

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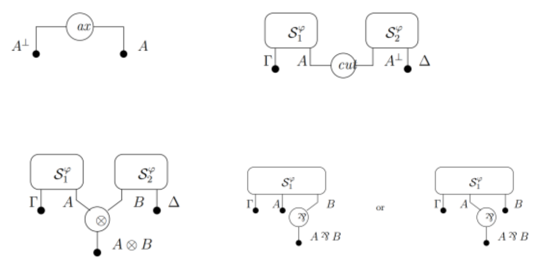
Soundness

Proposition 4.1.1 (Soundness of Correctness). *The translation of a sequent calculus proof of MLL is a connected multiplicative proof net.*



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Sequentialization

Theorem 4.1.1 (Sequentialization). *Any connected multiplicative proof net is the translation of a sequent calculus proof of MLL.*

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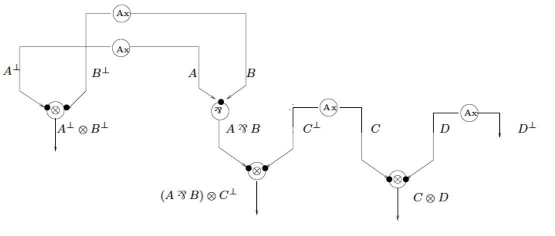
Sequentialization answers the question:

We have a proof net. The problem: it is the image of a sequent calculus proof? And which?



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The beauty of proof nets is **normalization**

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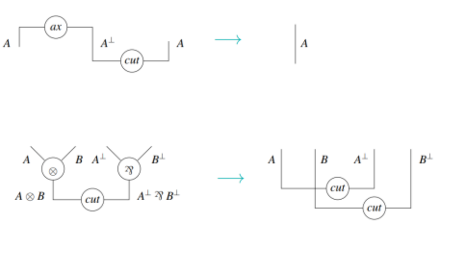
Normalization (local graph reductions!)



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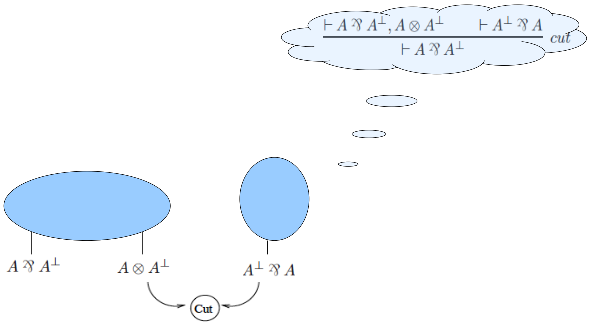
Let us try out!



Write a proof net with this conclusion... and normalize it

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How we write a proof net of these conclusions?

$A \parr A^\perp$ must type an edge conclusion of a par link, with premisses ....

$A \otimes A^\perp$ must type an edge conclusion of a tensor link, with premisses ....

Then we have to choose the axiom links!

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Homework

Let us try one more. First, write a proof net with this conclusion...

$(X \otimes X) \multimap (X \otimes X)$ =

$(X \otimes X)^\perp \parr (X \otimes X)$ = $(X^\perp \parr X^\perp) \parr (X \otimes X)$

TIP: How we write a proof net? As before, all proof nets with the same conclusion, start with the same nodes (the formula tree!)
What distinguishes different proofs are the axiom links

To distinguish the different occurrences of atoms, let us write indices:

$(X_1{}^\perp \parr X_2{}^\perp) \parr (X_3 \otimes X_4)$

In this case, we have two possible proofs, corresponding to two possible way to write axioms:
1,3 and 2,4
OR
1,4 and 2,3

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Homework

In sequent calculus, they correspond to these two proofs (one uses exchange, one no)

$$\dfrac{\dfrac{\dfrac{\vdash X_3{}^\perp, X_1 \qquad \vdash X_4{}^\perp, X_2}{\vdash X_1{}^\perp, X_2{}^\perp, X_3 \otimes X_4}\otimes}{\vdash X_1{}^\perp \parr X_2{}^\perp, X_3 \otimes X_4}\parr}{\vdash (X_1 \otimes X_3)^\perp \parr (X_2 \otimes X_4)}\parr \qquad \dfrac{\dfrac{\dfrac{\vdash X_3{}^\perp, X_2 \qquad \vdash X_4{}^\perp, X_1}{\vdash X_1{}^\perp, X_2{}^\perp, X_3 \otimes X_4}\otimes}{\vdash X_1{}^\perp \parr X_2{}^\perp, X_3 \otimes X_4}\parr}{\vdash (X_1 \otimes X_3)^\perp \parr (X_2 \otimes X_4)}\parr$$

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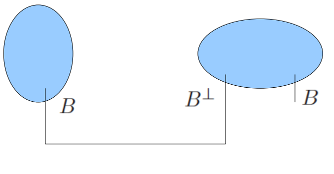
Homework

When we have a formula whose normal proofs are exactly two, we have a good candidate to code BOOLEANS :)

Let us indicate the formula $(X^\perp \parr X^\perp) \parr (X \otimes X)$ with B (for boolean).
We call one proof ***true***, and the other ***false***...

We can feed one of our two values to a proof which takes a boolean, and return a boolean.



We know that the normal form (i.e the result of computation) will be of type B... Hence one of our two values.

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Homework

Try to normalize one of the proofs of $(X_1{}^\perp \parr X_2{}^\perp) \parr (X_3 \otimes X_4)$

with the proof net which has conclusions

$(X_1 \otimes X_2) \otimes (X_3{}^\perp \parr X_4{}^\perp)$ $\quad (X_5{}^\perp \parr X_6{}^\perp) \parr (X_7 \otimes X_8)$

and axiom links: (1,6) (2,5) (3,7) (4,8)

What is the function coded by this proof net?

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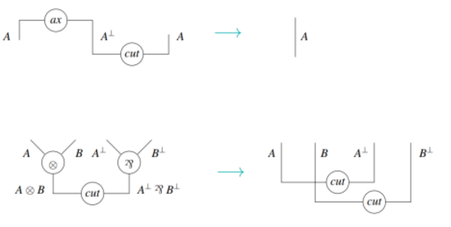
MLL normalization (local graph reductions)



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Properties of MLL normalization

1. Confluence?
2. Is normalization weakly/strongly normalizing?
3. Would you be able to define a normalizing strategy?
4. Would you be able to define a normalizing strategy which reaches normal form in a **minimal** number of steps?

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Properties of normalization

Lemma (preservation of correctness)
If the proof structure R is correct and reduces to R', then R' is correct.

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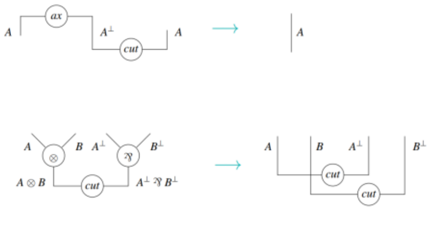
Properties of MLL normalization



- Strongly normalizing
- Confluent
- **Cut-elimination**:
  a proof-net in normal form contains no cuts

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Correctness criterion, simplified

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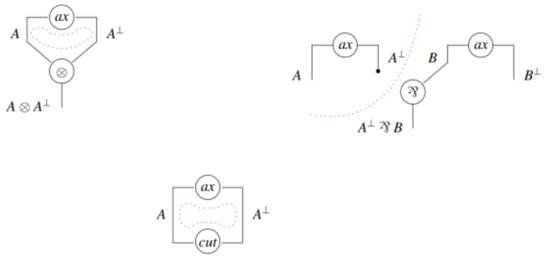

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Correctness: if we focus on acylicity, Danos-Regnier criterion can be reformulated (in equivalent way)

A proof structure is **correct** if every (undirected) cycle contains a segment

→ ⅋ ←

$$\frac{}{\vdash A,\ A^\perp}(Ax) \qquad \frac{\vdash \Gamma,\ A \qquad \vdash \Delta,\ A^\perp}{\vdash \Gamma,\ \Delta}(Cut)$$

$$\frac{\vdash \Gamma,\ A \qquad \vdash \Delta,\ B}{\vdash \Gamma,\ A \otimes B,\ \Delta}(\otimes) \qquad \frac{\vdash \Gamma,\ A,\ B}{\vdash \Gamma,\ A ⅋ B}(⅋)$$

$$\boxed{\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}(Mix)}$$

we can throw away MIX later
By requiring **connectness**

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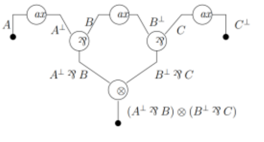
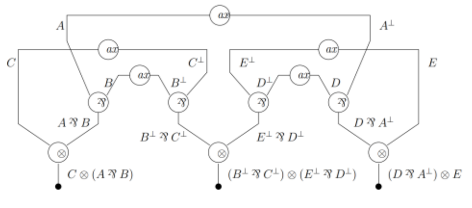
Is this correct?

PN1:



PN2:



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Exponentials

4 new nodes

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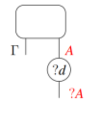
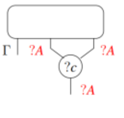
From proofs to proofs structures

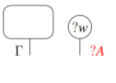
$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?d \quad \mapsto$$



$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c \quad \mapsto$$



$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w \quad \mapsto$$



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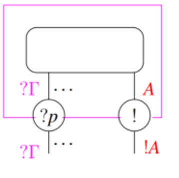
From proofs to proofs structures

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \quad \mapsto$$



- Boxes never overlap.
- Boxes are sequential (as rules in sequent calculus).
- Correctness: box by box.
- Boxes permit duplication and erasure.

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Cut-elimination steps

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta, A^{\perp}}{\vdash \Delta, ?A^{\perp}}\,?d}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\vdash ?\Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash ?\Gamma, \Delta}\,cut$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta, ?A^{\perp}, ?A^{\perp}}{\vdash \Delta, ?A^{\perp}}\,?c}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \vdash \Delta, ?A^{\perp}, ?A^{\perp}}{\vdash ?\Gamma, \Delta, ?A^{\perp}}\,cut}{\vdash ?\Gamma, ?\Gamma, \Delta}\,cut}{\vdash ?\Gamma, \Delta}\,?c$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta}{\vdash \Delta, ?A^{\perp}}\,?w}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\vdash \Delta}{\vdash ?\Gamma, \Delta}\,?w$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash ?\Delta, ?A^{\perp}, B}{\vdash ?\Delta, ?A^{\perp}, !B}\,!}{\vdash ?\Gamma, ?\Delta, !B}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \vdash ?\Delta, ?A^{\perp}, B}{\vdash ?\Gamma, ?\Delta, B}\,cut}{\vdash ?\Gamma, ?\Delta, !B}\,!$$

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What is the associated proof-net?

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,ax}{\vdash A \parr A^{\perp}}\,\parr}{\vdash A \parr A^{\perp}, ?B}\,?w \qquad \mapsto$$

$$!A \otimes !A \dashv\vdash !A$$
$$?A \parr ?A \dashv\vdash ?A$$

Can you write the proof so that all axioms are atomic (assuming A atomic formula)?

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Reduction steps

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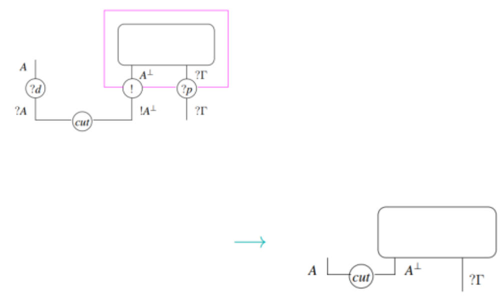
Reduction Steps: $?d$



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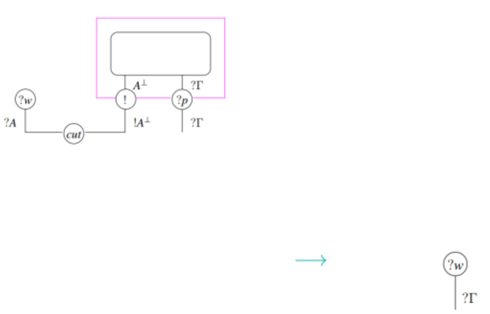
Reduction Steps: $?w$



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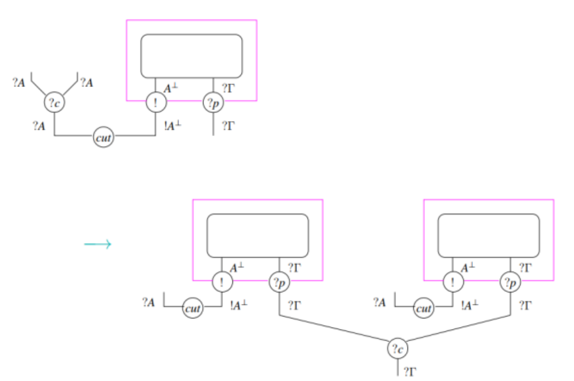
Reduction Steps: $?c$



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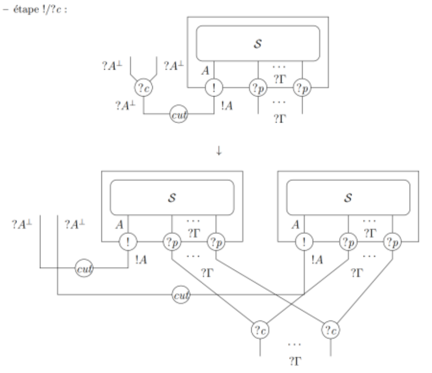
– étape !/?c :



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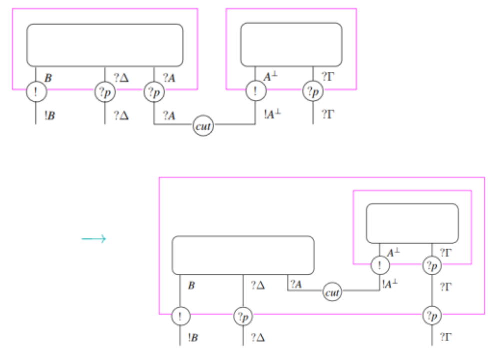
Reduction Steps: $?p$



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Properties of MELL reduction:

Confluence?
Weak Normalization?
Strong Normalization?

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Confluence

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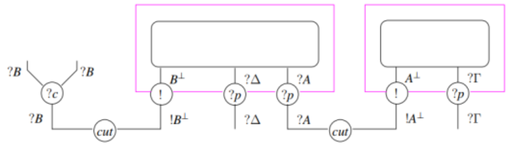
Is the reduction confluent?



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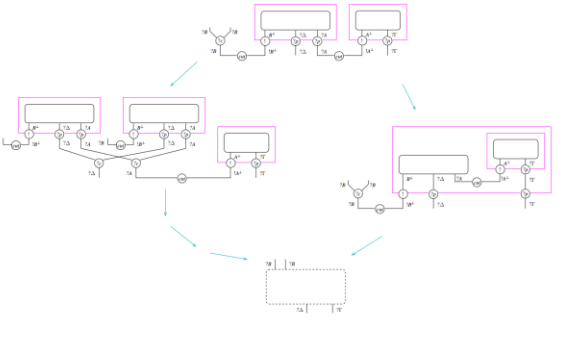
Local Confluence ...



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Properties of MELL reduction

1. Is confluent?
2. Is weakly normalizing?

Tip for WN.

Given a proof-net R, *try to make decrease a size S(R).*

For example:

- **Size of a cut**: pair (s,t) where

s is the size of the cut formula, and

t is the size of the ?-tree above the ? premisse of the cut if any, or 0

- Size S(R) of the **proof-net R**:
  the multiset of the sizes of all its cuts

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Weak Normalization

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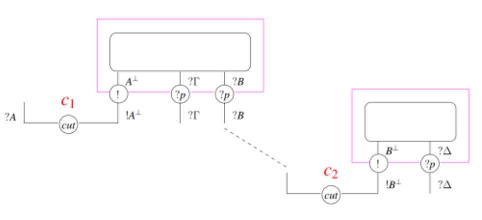
Weak Normalization

- cut relation: $c_1 \prec c_2$ *(exponential cuts only)*



- correctness $\Longrightarrow$ no cycle in $\prec^*$ $\Longrightarrow$ maximal cuts
  - reduction of a maximal cut
    $\Longrightarrow$ [(|formula| , |?-tree|)] decreases

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Weak Normalization

Size of a cut: pair (s,t) where
s is the size of the cut and,
t is the size of the ?-tree above the ? premisse of the cut if any, or 0

Size S(R) of the **proof-net R**:
the multiset of the sizes of all its cuts

The *size* of a ?-tree is its number of nodes.
descent path (bis): from a node downwards to a conclusion or to a cut or to a premisse of ! node (that is we do not continue down through an ! node)

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From weak to **strong normalization**

Proposition B.5 (Increasing Normalization)
For any ARS, *local confluence* $\wedge$ *μ-increasing* $\wedge$ *weak normalization* $\Longrightarrow$ *strong normalization.*

Strong normalization of proof-nets just a matter of some technical steps

$$\begin{tikzcd} & \mathcal{R} \arrow[dl, "w"'] \arrow[dr, dotted, "\not w"] & \\ \mathcal{R}' \arrow[dr, "\not w"'] & & \mathcal{R}''' \arrow[dl, dotted, "w"] \\ & \mathcal{R}'' & \end{tikzcd}$$

Lemma 2.7 (Weak non w Normalization)
The $\rightarrow_{\not w}$ reduction of numbered proof nets is weakly normalizing.

Lemma 2.8 (Increasing non w Reduction)
The reduction $\rightarrow_{\not w}$ on numbered proof nets is μ-increasing where, for a numbered proof net $\mathcal{R}$, $\mu(\mathcal{R}) = l^2 + p$ with:

- *l is the sum of all the labels of $\mathcal{R}$,*
- *p is the sum of the depths of the boxes of $\mathcal{R}$.*

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- a/a (shared ax)

$$\begin{tikzcd} n \arrow[d, bend right, "a"'] \arrow[d, bend left, "a"] \\ n+1 \end{tikzcd}$$

- a/a (shared cut)

$$\begin{tikzcd} n \arrow[d, bend right, "a"'] \arrow[d, bend left, "a"] \\ n+1 \end{tikzcd}$$

- d/in

$$\begin{tikzcd} & n,m \arrow[dl, "d"'] \arrow[dr, "\text{in}"] & \\ n+m+1 \arrow[dr, dotted, "\text{in}"'] & & n+m' \arrow[dl, dotted, "d"] \\ & n+m'+1 & \end{tikzcd}$$

- c/in

$$\begin{tikzcd} & n,m \arrow[dl, "c"'] \arrow[dr, "\text{in}"] & \\ n,m,m \arrow[d, dotted, "\text{in}"'] & & n,m' \arrow[ddl, dotted, "c"] \\ n,m',m \arrow[dr, dotted, "\text{in}"'] & & \\ & n,m',m' & \end{tikzcd}$$

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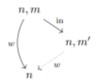
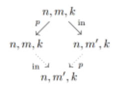
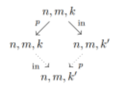
- w/in



- p/in (left side)



- p/in (right side)



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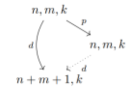
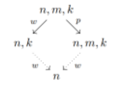
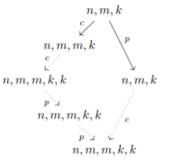
- d/p



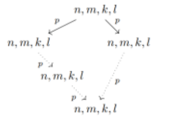
- w/p



- c/p



- p/p



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Bonus Exercise

A proof-nets is *polarized if* every edge is labelled by a positive or a negative formula

Let *M* be a **MLL** polarized proof structure. We *denote by Pol(M) the graph which has the same nodes and edges as M, but where the edges are directed downward if positive, upwards if negative.*

Do you see any *simple* way to show that the following are equivalent?

(1.) $\mathcal{M}$ is acyclic correct, (2.) Pol($\mathcal{M}$) is a DAG.

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Sequentialization

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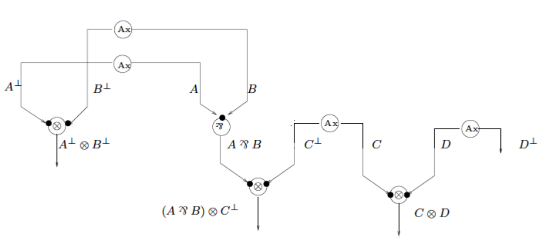
We have a proof net. The problem: it is the image of a sequent calculus proof? And which?



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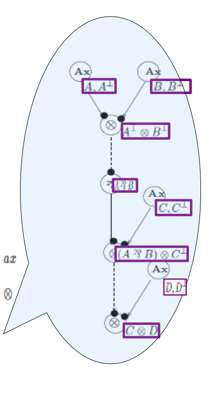
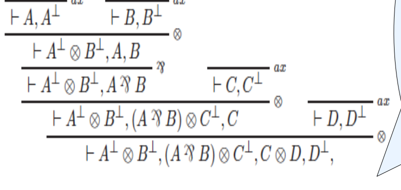
In fact, what is a sequent calculus proof?
A sequent calculus proof is a tree of rules...



$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}ax \quad \dfrac{}{\vdash B,B^\perp}ax}{\vdash A^\perp\otimes B^\perp, A, B}\otimes}{\vdash A^\perp\otimes B^\perp, A\parr B}\parr \quad \dfrac{}{\vdash C,C^\perp}ax}{\vdash A^\perp\otimes B^\perp,(A\parr B)\otimes C^\perp, C}\otimes \quad \dfrac{}{\vdash D,D^\perp}ax}{\vdash A^\perp\otimes B^\perp,(A\parr B)\otimes C^\perp, C\otimes D, D^\perp,}\otimes$$



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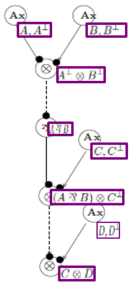
Such a tree directly corresponds to the following sequent calculus proof:



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,\mathrm{ax} \qquad \dfrac{}{\vdash B, B^{\perp}}\,\mathrm{ax}}{\vdash A^{\perp}\otimes B^{\perp}, A, B}\,\otimes}{\vdash A^{\perp}\otimes B^{\perp}, A \parr B}\,\parr \qquad \dfrac{}{\vdash C, C^{\perp}}\,\mathrm{ax}}{\vdash A^{\perp}\otimes B^{\perp}, (A\parr B)\otimes C^{\perp}, C}\,\otimes \qquad \dfrac{}{\vdash D, D^{\perp}}\,\mathrm{ax}}{\vdash A^{\perp}\otimes B^{\perp}, (A\parr B)\otimes C^{\perp}, C\otimes D, D^{\perp},}\,\otimes$$

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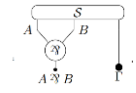
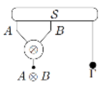
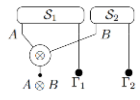
Look at the terminal nodes

- Ax-node
- Par-node

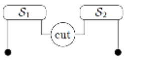


- Tensor-node





- Cut-node



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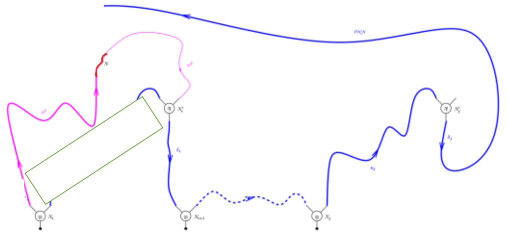

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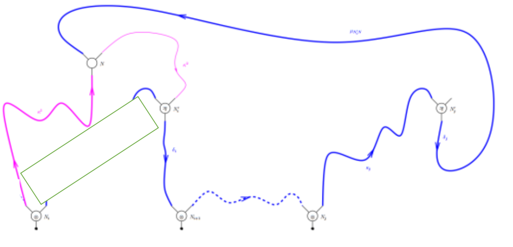

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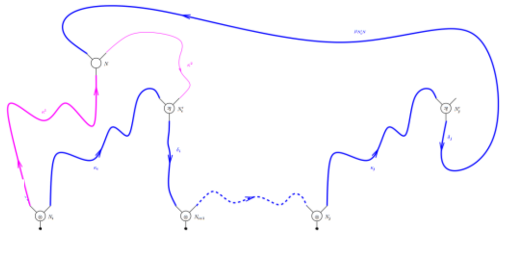

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Translating lambda-calculus into LL

CbN and CbV

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Simply typed lambda calculus

$$\frac{}{\Gamma, x:\tau \vdash x:\tau}\,(var) \qquad \frac{\Gamma, x:\tau \vdash t:\sigma}{\Gamma \vdash \lambda x.t : \tau \to \sigma}\,(abs) \qquad \frac{\Gamma \vdash t:\tau \to \sigma \quad \Gamma \vdash u:\tau}{\Gamma \vdash t\,u : \sigma}\,(app)$$

$$X^\bullet = X \qquad (A \to B)^\bullet = !A^\bullet \multimap B^\bullet \qquad (\Gamma \vdash A)^\bullet = !\Gamma^\bullet \vdash A^\bullet$$

$$X^\star = !X \qquad (A \to B)^\star = !(A^\star \multimap B^\star) \qquad (\Gamma \vdash A)^\star = \Gamma^\star \vdash A^\star$$

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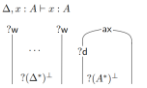
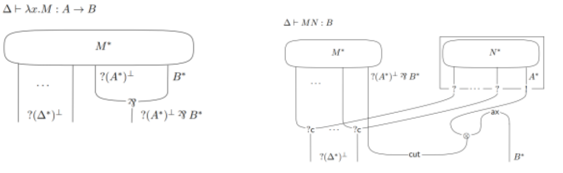
CbN translation

$$X^\bullet = X \qquad (A \to B)^\bullet = !A^\bullet \multimap B^\bullet \qquad (\Gamma \vdash A)^\bullet = !\Gamma^\bullet \vdash A^\bullet$$



$$\frac{\Gamma, x:\tau \vdash t:\sigma}{\Gamma \vdash \lambda x.t : \tau \to \sigma}\,(abs) \qquad \frac{\Gamma \vdash t:\tau \to \sigma \quad \Gamma \vdash u:\tau}{\Gamma \vdash t\,u : \sigma}\,(app)$$



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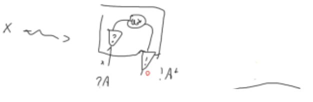
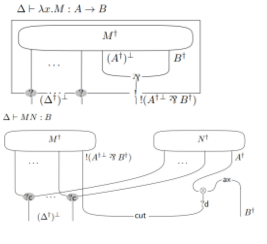
CbV translation

$$X^\star = !X \qquad (A \to B)^\star = !(A^\star \multimap B^\star) \qquad (\Gamma \vdash A)^\star = \Gamma^\star \vdash A^\star$$





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CbV

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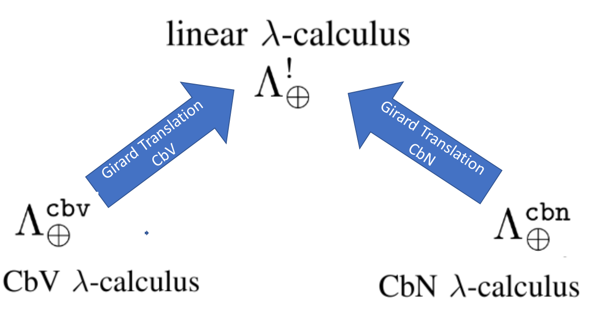

The translation gives :

- Confluence
- Standardization (finite and asymptotical)

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Three lambda calculi... Or indeed just one.

The two Girard's translations (intuitionistic into linear logic) are well-known to correspond to the CbN and CbV.

Taking this point of view provides a modular approach.

In $\Lambda^!_\oplus$ the natural constraint is:

- **no reduction** in the scope of a !-box

Girard's translations transform this policy respectively in:

- $\Lambda^{cbv}_\oplus$: **no reduction** under λ-abstraction
- $\Lambda^{cbn}_\oplus$ no **no reduction** argument position.

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