M2 LMFI

Proofs and programs: advanced topics

Linear Logic and Quantitative Semantics

Teachers:

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1

Organization

- Lectures:
  - Wednesday 14h00-16h00
  - Friday 14h00-16h00
- **Grading:**
  - **weekly homework** projects

2

Plan

A foundational study of functional programming languages,

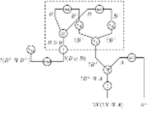
- building on:
  - *proof theory* (Types, Curry-Howard isomorphism) and
  - the theory of *lambda-calculus,*
- adopting *the **dynamic and quantitative view brought by Linear Logic.***
  - Focus first part: a **quantitative view in Operational Semantics**
  - Focus second prat: a **quantitative view in Denotational Semantics**
  - Openings towards active research topics: Bayesian learning/ probabilistic programming, ….
- Courses from **LMFI** first term we build on:
  - *Proof Theory (cut-elimination, lambda calculus,* Curry-Howard *iso)*
- Connected to the **MPRI** course: Semantics of Programming Language (which builds on the models of Linear Logic)

3

Linear Logic [Girard87] breakthroughs

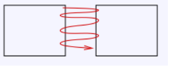
New insights into **proof theory** and
(via the **Curry-Howard correspondence** between proofs and programs)
into the **semantics of programming language**.

- **Proof Nets:** advanced formal system



- representation of proofs (λ-terms, functional programs) by graphs
- tool for the analysis of cut-elimination (= execution) as graph-rewriting process

- Dynamic view, capturing the **flow of computation**:



  - **Game Semantics**
  - **Geometry of Interaction**

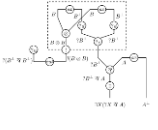
4

Linear Logic [Girard87] breakthroughs

New insights into **proof**
(via the **Curry-Howard co**
into the **semantics of pro**

especially suitable for
- Cost analysis (runtime/memory space/ other resources)
- modelling probabilistic & quantum programming

- **Proof Nets:** advanced formal system



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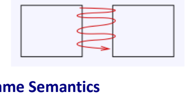

  - **Game Semantics**
  - **Geometry of Interaction**

- Account for resources
  - Quantitative Semantics
  - Quantitative Type Systems

5

Resource awareness (Quantitative Types)

$$\frac{}{\mathcal{P}, x : P \vdash x : P}\ \text{S-VAR} \qquad \frac{}{\mathcal{P} \vdash \mathtt{b} : \mathtt{B}}\ \text{S-BOOL} \qquad \frac{\mathcal{P} \vdash u : P \quad \mathcal{P}, x : P \vdash t : A}{\mathcal{P} \vdash \mathtt{let}\ x = u\ \mathtt{in}\ t : A}\ \text{S-LET}$$

$$\frac{\mathcal{P} \vdash v : L_1 \quad \mathcal{P} \vdash w : L_2}{\mathcal{P} \vdash \langle v, w \rangle : L_1 \otimes L_2}\ \text{S-PAIR} \qquad \frac{\mathcal{P} \vdash v : L_1 \otimes L_2 \quad \mathcal{P}, x : L_1, y : L_2 \vdash t : A}{\mathcal{P} \vdash \mathtt{letp}\ \langle x, y \rangle = v\ \mathtt{in}\ t : A}\ \text{S-LETP}$$

$$\frac{\mathcal{P}, x : P \vdash t : A}{\mathcal{P} \vdash \lambda x.t : P \multimap A}\ \text{S-ABS} \qquad \frac{\mathcal{P} \vdash t : P \multimap A \quad \mathcal{P} \vdash v : P}{\mathcal{P} \vdash tv : A}\ \text{S-APP}$$

$$\frac{\mathcal{P} \vdash t : A}{\mathcal{P} \vdash !t : !A}\ \text{S-BANG} \qquad \frac{\mathcal{P} \vdash v : !A}{\mathcal{P} \vdash \mathtt{der}\ v : A}\ \text{S-DER}$$

$$\frac{}{\Lambda, x : P \vdash x : P}\ \text{I-VAR} \qquad \frac{\Lambda, \Gamma_1 \vdash u : P \quad \Lambda, \Gamma_2, x : P \vdash t : A}{\Lambda, \Gamma_1 \uplus \Gamma_2 \vdash \mathtt{let}\ x = u\ \mathtt{in}\ t : A}\ \text{I-LET}$$

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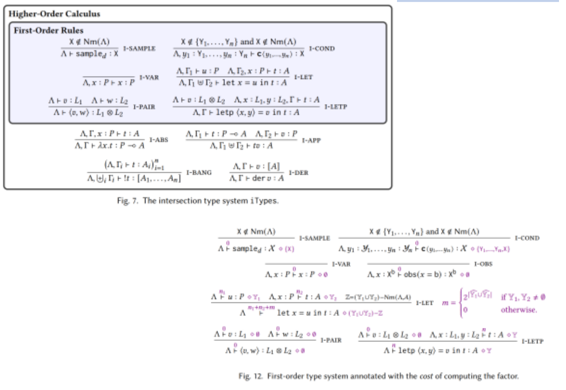
$$\frac{(\Lambda, \Gamma_i \vdash t : A_i)_{i=1}^{n}}{\Lambda, \biguplus_i \Gamma_i \vdash !t : [A_1, \ldots, A_n]}\ \text{I-BANG} \qquad \frac{\Lambda, \Gamma \vdash v : [A]}{\Lambda, \Gamma \vdash \mathtt{der}\ v : A}\ \text{I-DER}$$

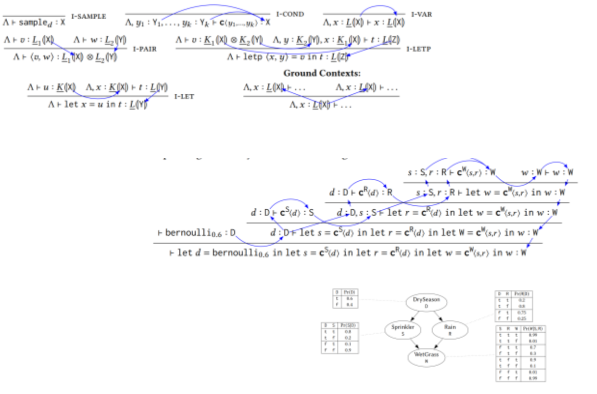
6

Higher-Order Bayesian Networks



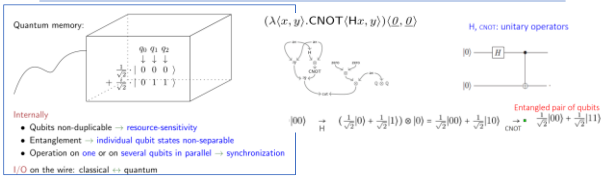
7

Higher-Order Bayesian Networks

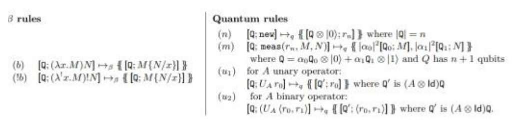


8

Quantum lambda calculus



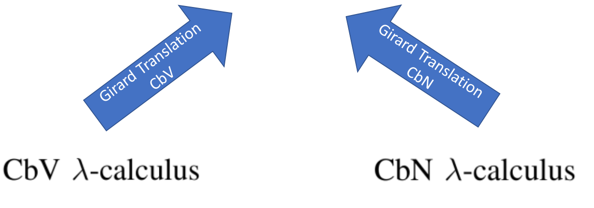
$$M, N, P ::= x \mid !M \mid \lambda x.M \mid \lambda^! x.M \mid MN \mid r_i \mid U_A \mid \mathtt{new} \mid \mathtt{meas}(P, M, N) \qquad \textbf{(terms } \Lambda_q\textbf{)}$$



9

LINEAR LOGIC

Proof-nets / ! λ-calculus



10

Plan / topics for Part 1

HANDS-ON

- Theoretical tools to study the operational properties of a system:
  - Rewrite Theory (rewriting=abstract form of program execution)
- Linear Logic and Proof-Nets.
- Bridging between lambda-calculus and functional programming:
  - Call-by-Value and Call-by Name, weak and lazy calculi.
- Beyond pure functional:
  - Probabilistic programming and Bayesian Inference: Probabilistic lambda calculi, Bayesian proof-nets

  (Internships possible on operational aspects of probabilistic and quantum computation)

11

Resources

- **Webpage** https://www.irif.fr/~faggian/LMFI2025
- **Lecture Notes** (by A. Middeldorp, O. Laurent, L. Ong)

12

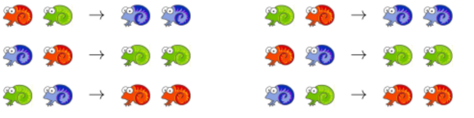
Operational semantics
of formal calculi and programming languages

Rewriting theory

- **Rewriting = abstract form of program execution**
- Paradigmatic example: **λ-calculus**
  (functional programming language, in its essence)

13

A colony of chameleons includes 20 red, 18 blue, and 16 green individuals. Whenever two chameleons of different color meet, each changes to the third color. Some time passes during which no chameleons are born or die nor do any enter or leave the colony. Is it possible that at the end of this period, all 54 chameleons are the same color?



14

Math formalizations…

Example (Group Theory)

signature	e (constant)	$^-$ (unary, postfix)	$\cdot$ (binary, infix)
equations	$e \cdot x \approx x$ $x^- \cdot x \approx e$	$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$	$\mathcal{E}$
theorems	$e^- \approx_{\mathcal{E}} e$	$(x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot x^-$	
rewrite rules	$e \cdot x \to x$ $x^- \cdot x \to e$ $(x \cdot y) \cdot z \to x \cdot (y \cdot z)$ $e^- \to e$ $x^- \cdot (x \cdot y) \to y$	$x \cdot e \to x$ $x \cdot x^- \to e$ $x^{--} \to x$ $(x \cdot y)^- \to y^- \cdot x^-$ $x \cdot (x^- \cdot y) \to y$	$\mathcal{R}$

① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same $\mathcal{R}$-normal form

② $\mathcal{R}$ admits no infinite computations

① & ② $\implies$ $\mathcal{E}$ has decidable validity problem

15

Modelling computation

Example (Lambda Calculus)

signature	λ (binds variables) $\cdot$ (application, binary, infix)
terms	$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$
α conversion	$\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$
β reduction	$(\lambda x.\, M) \cdot N \to_\beta M[x := N]$ replace free occurrences of x in M by N (and avoid variable capturing)
rewriting	$(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \to (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$
inventor	Alonzo Church (1932)

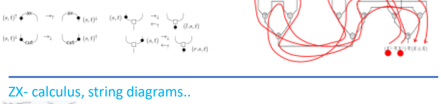
both Combinatory Logic and Lambda Calculus are Turing-complete

17

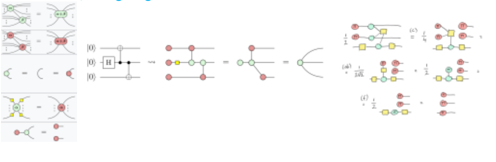
Graph Rewriting

LL proof-nets

Geometry of Interaction



ZX- calculus, string diagrams..



18

Rewriting

- Rewrite Theory provides a powerful set of tools to study computational and operational properties of a system : normalization, termination , confluence, uniqueness of normal forms
- tools to study and compare strategies:
  - Is there a strategy guaranteed to lead to normal form, if any (*normalizing strat.*)?
- Abstract Rewrite Systems (ARS) capture the common substratum of rewrite theory (independently from the particular structure of terms) - can be uses in the study of any calculus or programming language.

19

Abstract Rewriting: motivations

concrete rewrite formalisms / concrete operational semantics:

- λ-calculus
- *Quantum/ probabilistic/ non-deterministic/*……….. λ-calculus
- Proof-nets / graph rewriting
- Sequent calculus and cut-elimination
- string rewriting
- term rewriting

abstract rewriting

- **independent from structure** of objects that are rewritten
- **uniform** presentation of properties and proofs

20

Why a theory of rewriting matters?

- **Rewriting = abstract form of program execution**

Rewriting theory provides a *sound framework* for reasoning about
- **programs transformations**, such as compiler optimizations or parallel implementations,
- **program equivalence**.

21

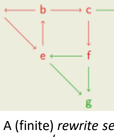
Abstract Rewriting

Basic language

22

ARS

Definition 1.1.1. An *abstract rewrite system* (ARS for short) is a pair $\mathcal{A} = \langle A, \to \rangle$ consisting of a set A and a binary relation $\to$ on A. Instead of $(a, b) \in \to$ we write $a \to b$ and we say that $a \to b$ is a *rewrite step*.



ARS $\mathcal{A} = \langle A, \to \rangle$
- $A = \{ a, b, c, d, e, f, g \}$
- $\to = \left\{ \begin{array}{l} (a,e), (b,a), (b,c), (c,d), (c,f) \\ (e,b), (e,g), (f,e), (f,g) \end{array} \right\}$

- A (finite) *rewrite sequence* is a non-empty sequence $(a_0, \ldots\ a_n)$ of elements in A such that $a_i \to a_{\{i+1\}}$

We write $a_0 \to^n a_n$ or simply $a_0 \to^* a_n$

- rewrite sequence
  - finite $a \to e \to b \to c \to f$
  - empty a
  - infinite $a \to e \to b \to a \to e \to b \to \cdots$

23

- $\leftarrow$ inverse of $\to$
- $\to^*$ transitive and reflexive closure of $\to$
- ${}^*\!\leftarrow$ inverse of $\to^*$

$s \leftrightarrow_{\mathcal{R}} t$ iff $s \to_{\mathcal{R}} t$ or $t \to_{\mathcal{R}} s$

$s \leftrightarrow^*_{\mathcal{R}} t$ iff $s = s_0 \leftrightarrow_{\mathcal{R}} s_1 \leftrightarrow_{\mathcal{R}} \ldots \leftrightarrow_{\mathcal{R}} s_n = t$ for $n \geq 0$

- $\leftrightarrow$ symmetric closure of $\to$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\to$) **
- $\to^+$ transitive closure of $\to$
- $\to^=$ reflexive closure of $\to$

$\cdot$ is relation composition: $R \cdot S = \{ (a,c) \mid a\ R\ b \text{ and } b\ S\ c \}$

$\downarrow = \to^* \cdot {}^*\!\leftarrow$

24

Composition

- If $\to_1, \to_2$ are binary relations on A then $\to_1 \cdot \to_2$ denotes their composition, *i.e.* $t \to_1 \cdot \to_2 s$ iff there exists $u \in A$ such that $t \to_1 u \to_2 s$.
- We write $(A, \{\to_1, \to_2\})$ to denote the ARS $(A, \to)$ where $\to = \to_1 \cup \to_2$.

25

Closure

The transitive-reflexive closure of a relation is a closure operator, *i.e.* satisfies

$$\to \subseteq \to^*, \qquad (\to^*)^* = \to^*, \qquad \to_1 \subseteq \to_2 \text{ implies } \to_1^* \subseteq \to_2^*$$

As a consequence

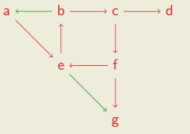
$$(\to_1 \cup \to_2)^* = (\to_1^* \cup \to_2^*)^*.$$

26

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \; {}^*\!\leftarrow y$ then z is common reduct of x and y
- if $x \leftrightarrow^* y$ then x and y are convertible

Example

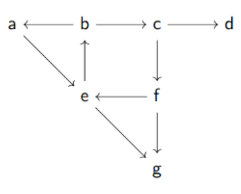


- a $\to^*$ f
- e $\downarrow$ f   f $\downarrow$ d   not g $\downarrow$ d
- g $\leftrightarrow^*$ d

27

Normal forms model **results**

Definition 1.1.11. Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS. An element $a \in A$ is *reducible* if there exists an element $b \in A$ with $a \to b$. A *normal form* is an element that is not reducible. The set of normal forms of $\mathcal{A}$ is denoted by $\mathsf{NF}(\mathcal{A})$ or $\mathsf{NF}(\to)$ when A can be inferred from the context. An element $a \in A$ *has a normal form* if $a \to^* b$ for some normal form b. In that case we write $a \to^! b$.



Element **a** has normal forms ?
How many normal forms has this ARS?

ARS $\mathcal{A} = \langle A, \to \rangle$

- d is normal form
- $\mathsf{NF}(\mathcal{A}) = \{$ d, g $\}$
- b $\to^!$ g

28

Operational properties of interest

- Termination and Confluence

Existence and uniqueness of normal forms

- How to Compute

reduction strategies with good properties:

- standardization,
- normalization

29

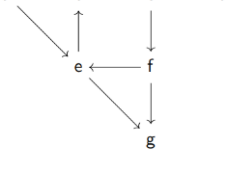
- SN strong normalization termination
  - no infinite rewrite sequences
- WN (weak) normalization
  - every element has at least one normal form
  - $\forall a \, \exists b \quad a \to^! b$
- UN unique normal forms
  - no element has more than one normal form
  - $\forall a, b, c \quad$ if $a \to^! b$ and $a \to^! c$ then $b = c$

30

Termination

Definition 1.2.1. Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS. An element $a \in A$ is called *terminating* or *strongly normalizing* (SN) if there are no infinite rewrite sequences starting at a. The ARS $\mathcal{A}$ is terminating or strongly normalizing if all its elements are terminating. An element $a \in A$ has *unique normal forms* (UN) if it does not have different normal forms ($\forall\, b, c \in A$ if $a \to^! b$ and $a \to^! c$ then $b = c$). The ARS $\mathcal{A}$ has unique normal forms if all its elements have unique normal forms.

An element *a* is *weakly normalizing* (WN) (or simply *normalizing)* if it has a normal form.



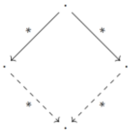
a is WN? SN?
c is WN? SN?
a or c has UN ?

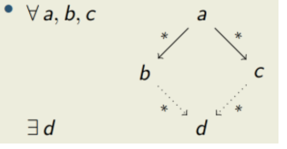
The nf are convertible?

31

Confluence

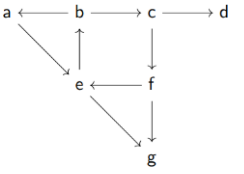
Definition 1.2.3. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS. An element $a \in A$ is *confluent* if for all elements $b, c \in A$ with $b \ {}^*\!\!\leftarrow a \rightarrow^* c$ we have $b \downarrow c$. The ARS $\mathcal{A}$ is confluent if all its elements are confluent.





Every confluent ARS *has unique normal forms.*

32



1. a is confluent?
2. f is confluent?

Bonus Point

3. Can you add a single arrow so that the resulting ARS has **unique normal forms without being confluent ?**

33

Given

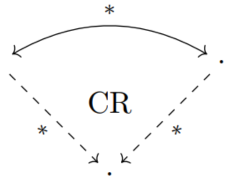
$$\mathcal{R} = \begin{cases} f(x,x) & \rightarrow & c \\ a & \rightarrow & b \\ f(x,b) & \rightarrow & d \end{cases}$$

f(a,a) has normal form?
Can you produce two different nf?

we can compute from the same term $f(a, a)$ two different normal-forms c and d
different meaning for same term!
(also: different meaning for equivalent terms)

34

Same meaning for *equivalent* terms

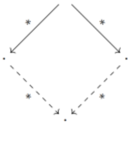


35

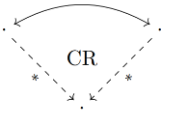
Confluence & CR

Definition 1.2.3. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS. An element $a \in A$ is *confluent* if for all elements $b, c \in A$ with $b \ {}^*\!\!\leftarrow a \rightarrow^* c$ we have $b \downarrow c$. The ARS $\mathcal{A}$ is confluent if all its elements are confluent.

Confluence



Church-Rosser



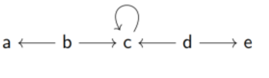
An ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ *is confluent if and only if* $\leftrightarrow^* \subseteq \downarrow$.

36

Definition 1.2.10. An ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has *unique normal forms with respect to conversion* (UNC) if different normal forms are not convertible ($\forall\, a, b \in \mathsf{NF}(\mathcal{A})$ if $a \leftrightarrow^* b$ then $a = b$).

in an ARS with the property UNC every equivalence class of convertible elements contains at most one normal form.

Q: are UN and UNC equivalent?



37

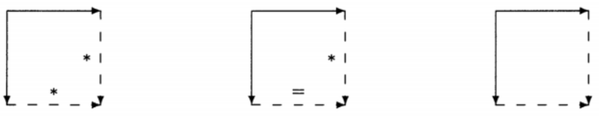
Global vs Local

38

Confluence

A property of term *t* is *local* if it is quantified over only *one-step reductions* from *t*; it is *global* if it is quantified over all *rewrite sequences* from *t*.

Locally confluent (WCR) **Strongly confluent** **Diamond**



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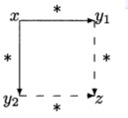
Global property:



39

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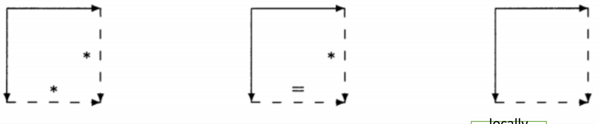
Locally confluent (WCR) **Strongly confluent** **Diamond**

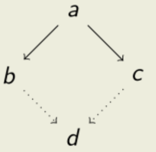


Local confluence Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS. An element $a \in A$ is locally confluent for all elements $b, c \in A$ with $b \leftarrow a \rightarrow c$ we have $b \downarrow c$. The ARS $\mathcal{A}$ is confluent if all its elements are confluent.

An ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has the *diamond property* ($\diamond$) if $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$

40

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$
  - $\forall a, b, c$



$\exists d$

- *every ARS with diamond property is confluent*

Proof by tiling

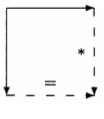
41

An ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is *strongly confluent* (SCR) if $\leftarrow \cdot \rightarrow \subseteq \rightarrow^{=} \cdot {}^*\!\leftarrow$, see Figure

a Show that every strongly confluent ARS is confluent.

b Does the converse also hold?

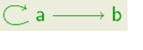
c Show that an ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if and only if ${}^*\!\leftarrow \cdot \rightarrow \subseteq \rightarrow^* \cdot {}^*\!\leftarrow$



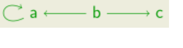
42

Which is true?

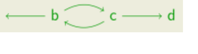
1. SN => WN
2. WN => SN



3. Confluence => UN
4. UN => Confluence



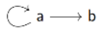
5. Confluence => Local confluence
6. Local confluence => Confluence



7. WN & UN => Confluence
8. WN & Local Conf. => Confluence
9. SN & Local Conf. => Confluence

43

WN vs SN



(ii) WN $\not\Rightarrow$ SN

$$\mathcal{R} = \begin{cases} f(a) & \to & c \\ f(x) & \to & f(a) \end{cases}$$

The system is weakly normalising but not strongly normalising:

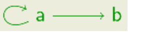
Can you find an infinite reduction sequence from f(b)?

$$f(b) \to f(a) \to c$$

$$f(b) \to f(a) \to f(a) \ldots$$

44

1. SN => WN
2. WN => SN



3. Confluence => UN
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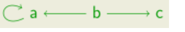


5. Confluence => Local confluence
6. Local confluence => Confluence

7. WN & UN => Confluence
8. WN & Local Conf. => Confluence
9. SN & Local Conf. => Confluence

Newman's Lemma

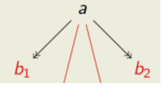
45

Lemma

WN & UN $\implies$ CR

Proof

- WN $\implies$ $\exists n_1, n_2 \colon\ b_1 \to^! n_1$ and $b_2 \to^! n_2$
- UN $\implies$ $n_1 = n_2$ $\implies$ $b_1 \downarrow b_2$



46

Newman Lemma

Newman's Lemma. *Every terminating and locally confluent ARS is confluent.*

By well-founded induction

47

Memo: Well-founded Induction

given

- property P of ARSs with $P(\mathcal{A}) \iff \forall a\colon P(a)$
- strongly normalizing ARS $\mathcal{A} = \langle A, \to \rangle$

to conclude

- $P(\mathcal{A})$

it is sufficient to prove

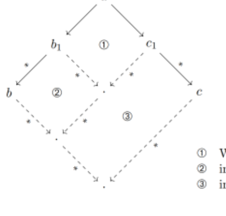
- if $\underbrace{P(b) \text{ for every } b}_{\text{induction hypothesis}}$ with $a \to b$ then $P(a)$

for arbitrary element a

48

Newman Lemma

Newman's Lemma. *Every terminating and locally confluent ARS is confluent.*



① WCR
② induction hypothesis ($a \to b_1 \implies b_1$ is CR)
③ induction hypothesis ($a \to c_1 \implies c_1$ is CR)

49

Newman Lemma

Bonus Exercise

Newman's Lemma. *Every terminating and locally confluent ARS is confluent.*

A second Proof. Let $\mathcal{A} = \langle A, \to \rangle$ *terminating and locally confluent*

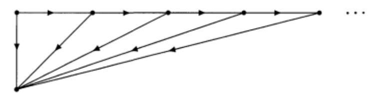
It suffices to show that every element has unique normal forms

- suppose $B = \{ a \in A \mid \neg \mathrm{UN}(a) \} \neq \varnothing$
- let $b \in B$ be minimal element (with respect to $\to$)
- $b \to^! n_1$ and $b \to^! n_2$ with $n_1 \neq n_2$

➤ **Conclude** by showing that it is impossible (**absurd**)

50

Recap Flash Ex



➤ **EX** Say which properties hold

1. Confluent
2. Locally confluent
3. Normalizing (weakly normalizing, WN)
4. Terminating (strongly normalizing, SN)

51

Recap basics

- An *abstract rewriting system (ARS)* is a pair $(\mathcal{A}, \to)$ consisting of a set $\mathcal{A}$ and a binary relation $\to$ on $\mathcal{A}$ whose pairs are written $t \to s$ and called *steps*.
- We denote $\to^*$ (resp. $\to^=$) the transitive-reflexive (resp. reflexive) closure of $\to$. We write $t \leftarrow u$ if $u \to t$.
- If $\to_1, \to_2$ are binary relations on $\mathcal{A}$ then $\to_1 \cdot \to_2$ denotes their composition, *i.e.* $t \to_1 \cdot \to_2 s$ if there exists $u \in \mathcal{A}$ such that $t \to_1 u \to_2 s$.
- We write $(\mathcal{A}, \{\to_1, \to_2\})$ to denote the *compound system* $(\mathcal{A}, \to)$ where $\to \; = \; \to_1 \cup \to_2$.

- A $\to$-*sequence* (or **reduction sequence**) from t is a (possibly infinite) sequence $t, t_1, t_2, \ldots$ such that $t_i \to t_{i+1}$.
  $t \to^* s$ indicates that there is a finite sequence from t to s.
  A $\to$-sequence from t is *maximal* if it is either infinite or ends in a $\to$-*nf*.

52

The heart of confluence is a diamond

Prop. DIAMOND implies CONFLUENCE

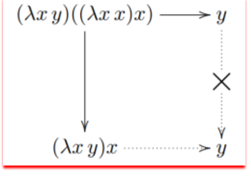
Can rarely be used ***directly****:*
Most relations of interest do not satisfy it

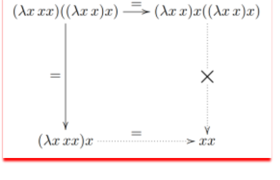
Lemma **(Characterize Confluence).** $\to$ *is confluent* if and only if *there exists a relation* $\twoheadrightarrow$ *such that*

a. $\twoheadrightarrow^* \; = \; \to^*$,
b. $\twoheadrightarrow$ *is diamond.*

53

You have already seen an example: in the notes by Joly





Definition *The* development relation *is the least reflexive relation* $\rhd$ *on* Λ *such that:*

- $t \rhd t' \implies \lambda x\, t \rhd \lambda x\, t'$
- $t \rhd t',\ u \rhd u' \implies tu \rhd t'u'$
- $t \rhd t',\ u \rhd u' \implies (\lambda x\, t)u \rhd t'[x{:=}u']$.

Lemma 1 $\to \; \subseteq \; \rhd \; \subseteq \; \twoheadrightarrow$.

54

You have already seen an example: in the notes by Joly

Lemma 3 (Characterize Confluence). $\to$ *is confluent* if and only if *there exists a relation* $\twoheadrightarrow$ *such that*

a. $\twoheadrightarrow^* \; = \; \to^*$,
b. $\twoheadrightarrow$ *is diamond.*

Definition *The* development relation *is the least reflexive relation* $\rhd$ *on* Λ *such that:*

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Lemma 1 $\to \; \subseteq \; \rhd \; \subseteq \; \twoheadrightarrow$.

55

Closure

$\twoheadrightarrow_R$ is the reflexive, transitive closure of $\rightarrow_R$:
(1) $M \rightarrow_R N \Rightarrow M \twoheadrightarrow_R N$,
(2) $M \twoheadrightarrow_R M$,
(3) $M \twoheadrightarrow_R N,\ N \twoheadrightarrow_R L \Rightarrow M \twoheadrightarrow_R L$.

The transitive-reflexive closure of a relation is a closure operator, *i.e.* satisfies

$$\rightarrow \subseteq \rightarrow^*, \qquad (\rightarrow^*)^* = \rightarrow^*, \qquad \rightarrow_1 \subseteq \rightarrow_2 \text{ implies } \rightarrow_1^* \subseteq \rightarrow_2^*$$

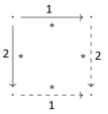
As a consequence

$$(\rightarrow_1 \cup \rightarrow_2)^* = (\rightarrow_1^* \cup \rightarrow_2^*)^*.$$

56

Commutation

Commutation. Two relations $\rightarrow_1$ and $\rightarrow_2$ on A *commute* if $\leftarrow_1{}^* \cdot \rightarrow_2{}^* \subseteq \rightarrow_2{}^* \cdot \leftarrow_1{}^*$.



Confluence. A relation $\rightarrow$ on A is confluent if it commutes with itself.

57

Proving confluence modularly

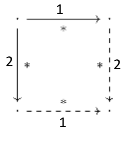
Lemma (Hindley-Rosen)
If two relations $\rightarrow_1$ and $\rightarrow_2$ are ***confluent*** *and* ***commute with each other****, then*
$\rightarrow_1 \cup \rightarrow_2$ is confluent.

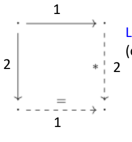
58

An effective usable technique

Lemma (Hindley-Rosen)
If two relations $\rightarrow_1$ and $\rightarrow_2$ are ***confluent*** *and* ***commute with each other****, then*
$\rightarrow_1 \cup \rightarrow_2$ is confluent.

Global condition
(all sequences)





Local condition
(one-step test)

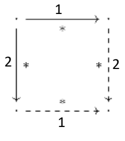
Lemma (Hindley's local test)
Strong commutation $\leftarrow_1 \cdot \rightarrow_2 \subseteq \rightarrow_2{}^* \cdot \leftarrow_1{}^=$ implies commutation.

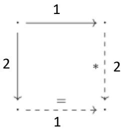
59

an effective usable technique

Lemma (Hindley-Rosen)
If two relations $\rightarrow_1$ and $\rightarrow_2$ are ***confluent*** *and* ***commute with each other****, then*
$\rightarrow_1 \cup \rightarrow_2$ is confluent.

Global condition
(all sequences)





Local condition
(one-step test)

$\leftarrow_1 \cdot \rightarrow_2 \subseteq \rightarrow_2{}^* \cdot \leftarrow_1{}^=$ **(Strong Commutation)**

▶ **Lemma** (Local test). *Strong commutation implies commutation.*

60

Strategies

61