TD Programs Equivalence and Applicative Simulation

Contextual equivalence

Definition (contextual pre-order and equivalence) We define the contextual preorder on terms as:

 $M \leq N \text{ if for all closing } C (C[M] \Downarrow \text{ implies } [N] \Downarrow)$

Contextual equivalence is derived by defining:

 $M\approx_C N$ if $M\leq_C N$ and $N\leq_C M$.

Applicative Simulation and Bi-simulation

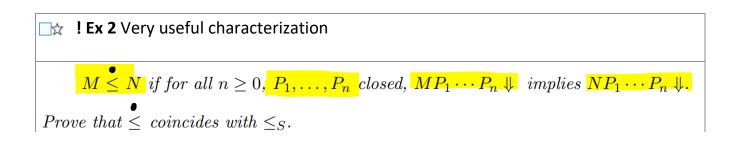
CbN **Definition** (simulation) We say that a binary relation on closed terms S is a simul simulation if whenever $(M, N) \in S$ we have: ation (1) if $M \Downarrow$ then $N \Downarrow$ and (2) for all P closed $(MP, NP) \in S$. We define \leq_S as the largest simulation. Bisimil $M =_{S} N$ if $M \leq_{S} N$ and $N \leq_{S} M$. arity CbV A (call-by-value) simulation is a binary relation S on $closed \lambda$ terms such that whenever simul $(M,N) \in S$ we have: (1) if $M \Downarrow$ then $N \Downarrow$ and (2) for all closed values $V, (MV,NV) \in S$. ation We denote with \leq_S the largest simulation.

Prop.

 \leq_S is the largest fixed point of the following function on binary relations:

$$f(S) = \{(M,N) \mid M \Downarrow \text{ implies } N \Downarrow \text{ , } \forall P \text{ closed } (MP,NP) \in S\} \text{ .}$$

EX 1 How would you prove that $(\lambda x.M)N \leq_C M\{N/x\}$?



Ex 3 Which of the following are true?
1. If M ∉ ↓ then M ≤_S N (for all N)
2. λx.λy.x ∉_S λx.λy.y
3. Let Ω_n ≡ λx₁....λx_n.Ω. Then Ω_n <_S Ω_{n+1} (strictly) and, for all M, Ω₀ ≤_S M.

Ex. 4. Is bisimilarity $=_s$ an equivalence relation on programs (ie closed terms)? Why?

Definition We extend ≤_S to open terms by defining: $M ≤_S N$ if for all closing substitutions σ ($\sigma M ≤_S \sigma N$). We also write $M =_S N$ if $M ≤_S N$ and $N ≤_S M$. The following properties of simulation hold: 1. ≤_S is a pre-order (on open terms). 2. If $M ≤_S N$ then for any substitution σ (not necessarily closed) $\sigma M ≤_S \sigma N$. 3. If $M \Downarrow V$ and $N \Downarrow V$, M, N closed, then $M =_S N$. 4. ($\lambda x.M$) $N =_S M\{N/x\}$ (M, N can be open). 5. If $M ≤_C N$ then $M ≤_S N$ (M, N can be open). 5. If $M ≤_C N$ then $M ≤_S N$ (M, N can be open). **Ex 5.** Prove properties 3, 4, and 5. In 4, simply assume that $\lambda x.M$ and N are closed terms. Similarly, in 5. simply assume M and N closed. Memo: To prove that $M ≤_S N$ (M, N closed) it suffices to find a relation S which is a simulation and such that (M, N) ∈ S Applicative bisimilarity $=_s$ is an equivalence relation on programs (closed terms). We want to say also that:

- It is a congruence (ie, it is preserved by contexts)
- It coincide with contextual equivalence

We have all the elements to prove the following:

Let x be a variable and M, N, P be terms. If $M \leq_S N$ then:

(1) $MP \leq_S NP$ and

(2) $\lambda x.M \leq_S \lambda x.N.$

EX.6.1 Prove point 2

 \Box_{x} EX 6.2 What is still missing to prove that $\leq_s is$ preserved by contexts?

<u>√☆</u> Ex 7

Assume we have defined a *relation* \leq_A on *open* terms such that

1. $\leq_S \subseteq \leq_A$

- 2. is preserved by contexts
- 3. \leq_A is a simulation
- ☆ 1. Prove that $M \leq_A N$ implies $M \leq_C N$ (where \leq_C is the contextual preorder)
- ☆ 2. Can we conclude that applicative simulation and contextual preorder coincide? How?