TD Programs Equivalence and Applicative Simulation

Contextual equivalence
Definition (contextual pre-order and equivalence) *We define the contextual* preorder *on terms as:* $$M \leq \ N \text{ if for all closing } C \ (C[M] \Downarrow \text{ implies } [N] \Downarrow)$$
Contextual equivalence *is derived by defining:* $$M \approx_C N \text{ if } M \leq_C N \text{ and } N \leq_C M \ .$$

	Applicative Simulation and Bi-simulation
CbN simulation	**Definition (simulation)** *We say that a binary relation on* closed terms S *is a* simulation *if whenever* $(M, N) \in S$ *we have:* *(1) if* $M \Downarrow$ *then* $N \Downarrow$ *and (2) for all* P *closed* $(MP, NP) \in S$. *We define* $\leq_S$ *as the largest simulation.*
Bi-similarity	$$M =_S N \text{ if } M \leq_S N \text{ and } N \leq_S M.$$
CbV simulation	*A (call-by-value) simulation is a binary relation* S *on closed* λ *terms such that whenever* $(M, N) \in S$ *we have: (1) if* $M \Downarrow$ *then* $N \Downarrow$ *and (2) for all closed values* V, $(MV, NV) \in S$. *We denote with* $\leq_S$ *the largest simulation.*

Prop.
$\leq_S$ *is the largest fixed point of the following function on binary relations:* $$f(S) \ = \{(M, N) \mid \ M \Downarrow \text{ implies } N \Downarrow \ , \ \ \forall P \text{ closed } (MP, NP) \in S\} \ .$$

EX 1 How would you prove that (λx.M)N $\leq_C$ M{N/x} ?

☐☆ **! Ex 2** Very useful characterization
$M \overset{\bullet}{\leq} N$ *if for all* $n \geq 0$, $P_1, \ldots, P_n$ *closed,* $MP_1 \cdots P_n \Downarrow$ *implies* $NP_1 \cdots P_n \Downarrow$. *Prove that* $\overset{\bullet}{\leq}$ *coincides with* $\leq_S$.

☐☆ Ex 3 Which of the following are true?

1. If M $\not\in$ $\Downarrow$ then M $\leq_S$ N (for all N)
2. $\lambda x.\lambda y.x \not\leq_S \lambda x.\lambda y.y$
3. *Let* $\Omega_n \equiv \lambda x_1.\ldots.\lambda x_n.\Omega$. *Then* $\Omega_n <_S \Omega_{n+1}$ *(strictly) and, for all* M, $\Omega_0 \leq_S M$.

☐☆ Ex. 4. Is bisimilarity $=_S$ an equivalence relation on programs (ie closed terms) ? Why?

Definition *We extend* $\leq_S$ *to* open terms *by defining:*

$$M \leq_S N \text{ if for all closing substitutions } \sigma \ (\sigma M \leq_S \sigma N)\ .$$

We also write $M =_S N$ *if* $M \leq_S N$ *and* $N \leq_S M$.

The following properties of simulation hold:

1. $\leq_S$ *is a pre-order (on open terms).*
2. *If* $M \leq_S N$ *then for any substitution* σ *(not necessarily closed)* $\sigma M \leq_S \sigma N$.
3. *If* $M \Downarrow V$ *and* $N \Downarrow V$, M, N *closed, then* $M =_S N$.
4. $(\lambda x.M)N =_S M\{N/x\}$ *(M, N can be open).*
5. *If* $M \leq_C N$ *then* $M \leq_S N$ *(M, N can be open).*

☐☆ **Ex 5.** Prove properties 3 , 4, and 5.
In 4, simply assume that $\lambda x.M$ and N are closed terms. Similarly, in 5. simply assume M and N closed.

Memo: To prove that $M \leq_S N$ (*M, N* closed) it suffices to find a relation *S* which is a simulation and such that $(M,N) \in S$

Applicative bisimilarity $=_S$ is an equivalence relation on programs (closed terms). We want to say also that: • It is a congruence (ie, it is preserved by contexts) • It coincide with contextual equivalence
We have all the elements to prove the following:
Let x be a variable and M, N, P be terms. If $M \leq_S N$ then: *(1) $MP \leq_S NP$ and* *(2) $\lambda x.M \leq_S \lambda x.N$.*
☐ EX.6.1 Prove point 2 ☐☆ **EX 6.2 What is still missing to prove that $\leq_S$ *is* *preserved by contexts*** ?

☑☆ **Ex 7**
Assume we have defined a *relation* $\leq_A$ on *open* terms such that
1. $\leq_S \subseteq \leq_A$ 2. is preserved by contexts 3. $\leq_A$ *is a simulation*
☆1. *Prove that $M \leq_A N$ implies $M \leq_C N$ (where $\leq_C$ is the contextual preorder)*
☆2. *Can we conclude that applicative simulation and contextual preorder coincide? How?*