TD1: CbN and CbV Lambda Calculus

Wednesday, February 24, 2021 9:34 PM

- **RECAP:**

CbN and CbV Calculi.

- The (pure) **Call-by-Name** calculus $\Lambda^{\mathsf{cbn}} = (\Lambda, \rightarrow_\beta)$ is the set of terms equipped with the contextual closure of the β-rule.

$$(\lambda x.M)N \;\mapsto_\beta\; M\{N/x\}$$

- The (pure) **Call-by-Value** calculus $\Lambda^{\mathsf{cbv}} = (\Lambda, \rightarrow_{\beta_v})$ is the same set equipped with the contextual closure of the β_v-rule.

$$(\lambda x.M)V \;\mapsto_{\beta_v}\; M\{V/x\} \quad \text{where } V \in \mathcal{V}$$

Head reduction in CbN

Head reduction is the closure of β under head context

$\lambda x_1 \dots x_n . (\!|\;|\!) M_1 \dots M_k$

Head normal forms (*hnf*), whose set is denoted by $\mathcal{H}$, are its normal forms.

- Given a rule ρ, we write $\underset{\mathsf{h}}{\rightarrow}_\rho$ for its closure under head context.
- A step $\rightarrow_\rho$ is non-head, written $\underset{\neg\mathsf{h}}{\rightarrow}_\rho$ if it is not head.

Weak reductions in CbV

The result of interest are **values** (*i.e.* functions).
In languages, in general the reduction is *weak*, that is, it does not reduce in the body of a function.

There are three main weak schemes: left, right and in arbitrary order.
Left contexts $\mathbf{L}$, *right* contexts $\mathbf{R}$, and (arbitrary order) *weak* contexts $\mathbf{W}$ are defined by

$\mathbf{L} ::= (\!|\;|\!) \mid \mathbf{L}M \mid V\mathbf{L}$
$\mathbf{R} ::= (\!|\;|\!) \mid M\mathbf{R} \mid \mathbf{R}V$
$\mathbf{W} ::= (\!|\;|\!) \mid \mathbf{W}M \mid M\mathbf{W}$

Given a rule $\mapsto$ on Λ, *weak reduction* $\underset{\mathsf{w}}{\rightarrow}$ is the closure of $\mapsto$ under context $\mathbf{W}$.
A step $T \rightarrow S$ is non-weak, written $T \underset{\neg\mathsf{w}}{\rightarrow} S$ if it is not weak. Similarly for left ($\underset{\mathsf{l}}{\rightarrow}$ and $\underset{\neg\mathsf{l}}{\rightarrow}$), and right ($\underset{\mathsf{r}}{\rightarrow}$ and $\underset{\neg\mathsf{r}}{\rightarrow}$).

▶ **Fact 3** (Weak normal forms). *Given M a closed term, M is $\underset{\mathsf{s}}{\rightarrow}$-normal* iff *$M$ is a value.*

- **TD 1. We work on the properties and exercises which are highlighted**

☐ BASIC PROPERTIES OF THE CONTEXTUAL CLOSURE

If a step $T \rightarrow_\gamma T'$ is obtained by closure under *non-empty context* of a rule $\mapsto_\gamma$, then T and T' have *the same shape*, i.e. *both terms are an application (resp. an abstraction, a variable).*

▶ **Fact 5** (Shape preservation).

- *Assume $T = \mathbf{C}(\!|R|\!) \rightarrow \mathbf{C}(\!|R'|\!) = T'$ and that the context $\mathbf{C}$ is* non-empty. *Then T and T' have the same shape.*
- *Hence, for any internal step $M \underset{\neg\mathsf{s}}{\rightarrow} M'$ ($\mathsf{s} \in \{\mathsf{h}, \mathsf{w}, \mathsf{l}, \mathsf{r}, \dots\}$) M and M have the same shape.*

The following is an easy to verify consequence.

▶ **Lemma 6** (Redexes preservation).
1. *CbN: Assume* $T \underset{\neg h}{\to}_\beta S$. *T is a β-redex* iff *so is S.*

2. *CbV. Assume* $T \underset{\neg w}{\to}_{\beta_v} S$. *T is a* β_v*-redex* iff *so is S.*

Fixed a set of redexes $\mathcal{R}$, M is w-normal (resp. h-normal) if there is no redex $R \in \mathcal{R}$ such that $M = \mathbf{W}(\!|R|\!)$ (resp. $M = \mathbf{H}(\!|R|\!)$)

▶ **Lemma 7** (Surface normal forms). **1.** *CbN. Let* $\mathcal{R}$ *be the set of* β_v*-redexes.*
Assume $M \underset{\neg h}{\to}_\beta M'$. *M is h-normal* $\Leftrightarrow$ *M′ is h-normal.*
2. *CbV. Let* $\mathcal{R}$ *be the set of* β_v*-redexes.*
Assume $M \underset{\neg w}{\to}_{\beta_v} M'$. *M is w-normal* $\Leftrightarrow$ *M′ is w-normal.*

Using FACTORIZATION

CbN:

- *Head Factorization:* $\to_\beta^* \subseteq \underset{h}{\to}_\beta^* \cdot \underset{\neg h}{\to}_\beta^*$.
- EX. A Prove that M has *hnf* if and only if head reduction from M terminates .

CbV: *Left* contexts **L**, *right* contexts **R**, and (arbitrary order) *weak* contexts **W** are def by

$$\mathbf{L} ::= (\!|\ |\!) \mid \mathbf{L}M \mid V\mathbf{L}$$
$$\mathbf{R} ::= (\!|\ |\!) \mid M\mathbf{R} \mid \mathbf{R}V$$
$$\mathbf{W} ::= (\!|\ |\!) \mid \mathbf{W}M \mid M\mathbf{W}$$

The closure under **L** (resp. **W**,**R**) context is noted $\underset{l}{\to}$ (resp $\underset{w}{\to}$, $\underset{r}{\to}$)

Let $s \in \{w,l,r\}$
- *weak factorization of* $\to_{\beta_v}$*:* $\to_{\beta_v}^* \subseteq \underset{s}{\to}_{\beta_v}^* \cdot \underset{\neg s}{\to}_{\beta_v}^*$.

▶ **Fact 7** (?). *Let M be a closed term. We say that M returns a value when* $M \to_\beta V$ *for some V.* ***EX B*** *Prove any of the following*
1. *M returns a value, if and only if* $\underset{l}{\to}_{\beta_v}$*-reduction from M terminates.*
2. *M returns value, if and only if* $\underset{w}{\to}_{\beta_v}$*-reduction from M terminates.*

EX. C. Normalization.

1. Give an inductive definition of **leftmost reduction**, completing the following b

Consider $(\Lambda,\rightarrow)$, where $\rightarrow=\rightarrow_\beta$. The relation $\underset{\text{LO}}{\rightarrow}\subseteq\rightarrow$ is induc

- If $M\underset{\mathsf{h}}{\rightarrow}M'$ then $M\underset{\text{LO}}{\rightarrow}M'$.
- If $M\underset{\mathsf{h}}{\not\rightarrow}$ (*i.e.*, M is h-normal) then:

$$\frac{P\underset{\text{LO}}{\rightarrow}P'}{M:=(\lambda x.P)\underset{\text{LO}}{\rightarrow}(\lambda x.P')}\qquad\frac{P\underset{\text{LO}}{\rightarrow}P'}{M:=PQ\underset{\text{LO}}{\rightarrow}P'Q}\qquad\frac{Q\underset{\text{LO}}{\rightarrow}Q'}{M:=PQ\underset{\text{LO}}{\rightarrow}PQ'}$$

2. Prove that **it is a normalizing strategy**, using head factorization.

Normalization (or, make your own Normalizing strategy)

▶ **Definition 8** (Iteration of surface reduction). *Given* $(\Lambda,\rightarrow)$, *where* $\rightarrow$ *is the context closure of a rule* $\mathsf{b}\in\{\beta,\beta_v\}$, *let* $\underset{\mathsf{s}}{\rightarrow}\subseteq\rightarrow$ *be as follows:*

$\underset{\mathsf{s}}{\rightarrow}=\underset{\mathsf{h}}{\rightarrow}$ *if* $\mathsf{b}=\beta$ *(CbN)* $\qquad\underset{\mathsf{s}}{\rightarrow}\in\{\underset{\mathsf{w}}{\rightarrow},\underset{\mathsf{l}}{\rightarrow},\underset{\mathsf{r}}{\rightarrow}\}$ *if* $\mathsf{b}=\beta_v$ CbV

The relation $\underset{E}{\rightarrow}\subseteq\rightarrow$ *is inductively defined by*

- *If* $M\underset{\mathsf{s}}{\rightarrow}M'$ *then* $M\underset{E}{\rightarrow}M'$.
- *If* $M\underset{\mathsf{s}}{\not\rightarrow}$ (i.e., M *is* s*-normal) then:*

$$\frac{P\underset{E}{\rightarrow}P'}{M:=(\lambda x.P)\underset{E}{\rightarrow}(\lambda x.P')}\qquad\frac{P\underset{E}{\rightarrow}P'}{M:=PQ\underset{E}{\rightarrow}P'Q}\qquad\frac{Q\underset{E}{\rightarrow}Q'}{M:=PQ\underset{E}{\rightarrow}PQ'}$$

▶ **Theorem 10** (Normalization).
CbN: $\underset{E}{\rightarrow}_\beta$ *is a normalizing strategy for* $\rightarrow_\beta$
CbV: $\underset{E}{\rightarrow}_{\beta_v}$ *is a normalizing strategy for* $\rightarrow_{\beta_v}$.