Lambda Calculus, Call-by-Name and Call-by-Value (summary)

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Terms and *values* are generated by the following grammars

$$\begin{array}{lcll} V & ::= & x \mid \lambda x.M & (\textit{Values}, \mathcal{V}) \\ M & ::= & x \mid c \mid \lambda x.M \mid MM & (\textit{Terms}) \end{array}$$

where x ranges over a countable set of *variables*, and c over a disjoint (possibly empty) set $\mathcal{O}$ of constants.

- If the set of constants is empty, the calulus is *pure*, and the set of terms is denoted Λ.
- Otherwise, the calculus is called *applied*, and the set of terms is often indicated as $\Lambda_{\mathcal{O}}$.

REDUCTION:

Contexts (with one hole $(\!|\,|\!)$) are generated as follows. $\mathbf{C}(\!|M|\!)$ stands for the term obtained from $\mathbf{C}$ by replacing the hole with the term M (possibly capturing free variables of M).

$\mathbf{C} ::= (\!|\,|\!) \mid\mid M\mathbf{C} \mid \mathbf{C}M \mid \lambda x.\mathbf{C}$ (*Contexts*)

- A **rule** ρ is a binary relation on $\Lambda_{\mathcal{O}}$, which we also denote $\mapsto_\rho$, writing $R \mapsto_\rho R'$. R is called a ρ-*redex*.
  The best known rule is β:

  $$(\lambda x.M)N \mapsto_\beta M\{N/x\}$$

- A **reduction step** $\to_\rho$ is the closure under context $\mathbf{C}$ of ρ.
  Explicitly, $T \to T'$ holds if $T = \mathbf{C}(\!|R|\!)$, $T' = \mathbf{C}(\!|R'|\!)$, and $R \mapsto_\rho R'$.

CbN and CbV Calculi.

- The (pure) **Call-by-Name** calculus $\Lambda^{\mathsf{cbn}} = (\Lambda, \to_\beta)$ is the set of terms equipped with the contextual closure of the β-rule.

  $$(\lambda x.M)N \mapsto_\beta M\{N/x\}$$

- The (pure) **Call-by-Value** calculus $\Lambda^{\mathsf{cbv}} = (\Lambda, \to_{\beta_v})$ is the same set equipped with the contextual closure of the β_v-rule.

  $$(\lambda x.M)V \mapsto_{\beta_v} M\{V/x\} \quad \text{where } V \in \mathcal{V}$$

Restricted reductions: head, leftmot-outermost, weak...

Head reduction in CbN

Head reduction is the closure of β under head context

$\lambda x_1 ... x_n.(\!|\ |\!) M_1 ... M_k$

Head normal forms (*hnf*), whose set is denoted by $\mathcal{H}$, are its normal forms.

- Given a rule ρ, we write $\underset{h}{\rightarrow}_\rho$ for its closure under head context.
- A step $\rightarrow_\rho$ is non-head, written $\underset{\neg h}{\rightarrow}_\rho$ if it is not head.

What about?

$$\mathbf{H} ::= (\!|\ |\!) \mid \lambda x.\mathbf{H} \mid \mathbf{H} M$$

Head Factorization

Head factorization allows for a characterization of the terms which have head normal form, that is M has *hnf* if and only if $\underset{h}{\rightarrow}$-reduction from M terminates.

▶ **Theorem 2** (Head Factorization)**.**

- Head Factorization: $\rightarrow_\beta{}^* \subseteq \underset{h}{\rightarrow}_\beta{}^* \cdot \underset{\neg h}{\rightarrow}_\beta{}^*$.
- Head Normalization: *M has* hnf *if and only if $M \underset{h}{\rightarrow}_\beta{}^* S$ (for some $S \in \mathcal{H}$) .*

Weak reductions in CbV

The result of interest are **values** (*i.e.* functions).

In languages, in general the reduction is *weak*, that is, it does not reduce in the body of a function.

There are three main weak schemes: left, right and in arbitrary order.

Left contexts **L**, *right* contexts **R**, and (arbitrary order) *weak* contexts **W** are defined by

$$\mathbf{L} ::= (\!|\ |\!) \mid \mathbf{L} M \mid V \mathbf{L}$$
$$\mathbf{R} ::= (\!|\ |\!) \mid M \mathbf{R} \mid \mathbf{R} V$$
$$\mathbf{W} ::= (\!|\ |\!) \mid \mathbf{W} M \mid M \mathbf{W}$$

Given a rule $\mapsto$ on Λ, *weak reduction* $\underset{w}{\rightarrow}$ is the closure of $\mapsto$ under context **W**.

A step $T \rightarrow S$ is non-weak, written $T \underset{\neg w}{\rightarrow} S$ if it is not weak. Similarly for left ($\underset{l}{\rightarrow}$ and $\underset{\neg l}{\rightarrow}$), and right ($\underset{r}{\rightarrow}$ and $\underset{\neg r}{\rightarrow}$).

▶ **Fact 3** (Weak normal forms)**.** *Given M a closed term, M is $\underset{s}{\rightarrow}$-normal* iff *$M$ is a value.*

Weak Factorization.

Let $\mathsf{s} \in \{\mathsf{w},\mathsf{l},\mathsf{r}\}$

- *weak factorization of* $\to_{\beta_v}$*:* $\to_{\beta_v}{}^* \subseteq \underset{\mathsf{s}}{\to}_{\beta_v}{}^* \cdot \underset{\neg\mathsf{s}}{\to}_{\beta_v}{}^*$.
- *Convergence:* $T \to_{\beta_v} W (W \in \mathcal{V})$ if and only if $T \underset{\mathsf{s}}{\to}_{\beta_v}{}^* V$ $(V \in \mathcal{V})$

▶ **Corollary 4.** *Given M a closed term, M has a β_v-reduction to a value, if and only if the $\underset{\mathsf{s}}{\to}_{\beta_v}$-reduction from M terminates.*

BASIC PROPERTIES OF THE CONTEXTUAL CLOSURE

If a step $T \to_\gamma T'$ is obtained by closure under *non-empty context* of a rule $\mapsto_\gamma$, then T and T' have *the same shape*, i.e. *both terms are an application (resp. an abstraction, a variable).*

▶ **Fact 5** (Shape preservation)**.**

- *Assume $T = \mathbf{C}(\!|R|\!) \to \mathbf{C}(\!|R'|\!) = T'$ and that the context $\mathbf{C}$ is* non-empty*. Then T and T' have the same shape.*
- *Hence, for any internal step $M \underset{\neg\mathsf{s}}{\to} M'$ ($\mathsf{s} \in \{\mathsf{h}, \mathsf{w},\mathsf{l},\mathsf{r},...\}$) M and M have the same shape.*

The following is an easy to verify consequence.

▶ **Lemma 6** (Redexes preservation)**.**

1. *CbN: Assume $T \underset{\neg\mathsf{h}}{\to}_\beta S$. T is a β-redex* iff *so is S.*

2. *CbV. Assume $T \underset{\neg\mathsf{w}}{\to}_{\beta_v} S$. T is a β_v-redex* iff *so is S.*

Fixed a set of redexes $\mathcal{R}$, M is w-normal (resp. h-normal) if there is no redex $R \in \mathcal{R}$ such that $M = \mathbf{W}(\!|R|\!)$ (resp. $M = \mathbf{H}(\!|R|\!)$)

▶ **Lemma 7** (Surface normal forms)**. 1.** *CbN. Let $\mathcal{R}$ be the set of β_v-redexes.*
Assume $M \underset{\neg\mathsf{h}}{\to}_\beta M'$. M is h-normal $\Leftrightarrow$ M' is h-normal.

2. *CbV. Let $\mathcal{R}$ be the set of β_v-redexes.*
Assume $M \underset{\neg\mathsf{w}}{\to}_{\beta_v} M'$. M is w-normal $\Leftrightarrow$ M' is w-normal.