Notes and Homework 1 (Abstract Rewriting)

2023-2024

Homework 1

Ex 1. Let $(A, \to)$ be an ARS.

1. Prove that if (A.) holds then (B.) holds
   A. $\forall t \in A$: $(t_1 \leftarrow t \to t_2)$ implies $(t_1 = t_2$ or $\exists u.\ t_1 \to u \leftarrow t_2)$
   B. $\forall t \in A$: if t *has a normal form* u (ie, $t \to^k u$, for some k), then all maximal reduction sequences from t *have the same length*, and all end in the same normal form u.
2. If (A.) holds, do *always all* maximal reduction sequences from t have the same length?
3. Show that the following property (C.) does not imply (B.)
   C. $\forall t \in A$: $(t_1 \leftarrow t \to t_2)$ implies $(\exists u.\ t_1 \to^= u \leftarrow^= t_2)$

Ex 2. Prove Lemma 1.

NOTES

1 Abstract rewriting system (ARS): the basics

We recall basic definitions.

Basics. An *abstract rewriting system (ARS)* is a pair $\mathcal{A} = (\mathcal{A}, \to)$ consisting of a set $\mathcal{A}$ and a binary relation $\to$ on $\mathcal{A}$ whose pairs are written $t \to s$ and called *steps*.

- We denote $\to^*$ (resp. $\to^=$) the transitive-reflexive (resp. reflexive) closure of $\to$. We write $t \leftarrow u$ if $u \to t$ (the reverse relation).
- If $\to_1, \to_2$ are binary relations on $\mathcal{A}$ then $\to_1 \cdot \to_2$ denotes their composition, *i.e.* $t \to_1 \cdot \to_2 s$ if there exists $u \in \mathcal{A}$ such that $t \to_1 u \to_2 s$.
- We write $(\mathcal{A}, \{\to_1, \to_2\})$ to denote the ARS $(\mathcal{A}, \to)$ where $\to = \to_1 \cup \to_2$.
- An element $u \in \mathcal{A}$ is $\to$-**normal**, or a $\to$-*normal form* if there is no t such that $u \to t$ (we also write $u \not\to$).
- A $\to$-*sequence* (or **reduction sequence**) from t is a (possibly infinite) sequence $t, t_1, t_2, \ldots$ such that $t_i \to t_{i+1}$. $t \to^* s$ indicates that there is a finite sequence from t to s.
  A $\to$-sequence from t is *maximal* if it is either infinite or ends in a $\to$-normal form.

We freely use the fact that the transitive-reflexive closure of a relation is a closure operator, *i.e.* satisfies

$$\to \subseteq \to^*, \qquad (\to^*)^* = \to^*, \qquad \to_1 \subseteq \to_2 \text{ implies } \to_1^* \subseteq \to_2^*. \tag{\textbf{Closure}}$$

The following property is an immediate consequence:

$$(\to_1 \cup \to_2)^* = (\to_1^* \cup \to_2^*)^*. \tag{\textbf{TR}}$$

Local vs Global Properties. An important distinction in rewriting theory is between local and global properties. A property of term t is *local* if it is quantified over only *one-step reductions* from t; it is *global* if it is quantified over all *rewrite sequences* from t. Local properties are easier to test, because the analysis (usually) involves a finite number of cases.

Commutation and Confluence Two relations $\to_1$ and $\to_2$ on $\mathcal{A}$ *commute* if $\leftarrow_1{}^* \cdot \to_2{}^* \subseteq \to_2{}^* \cdot \leftarrow_1{}^*$.

A relation $\to$ on $\mathcal{A}$ is confluent if it commutes with itself.

A classic tool to modularize the proof of confluence is Hindley-Rosen lemma.

Confluence of two relations $\to_1$ and $\to_2$ does not imply confluence of $\to_1 \cup \to_2$, however it does if they commute.

▶ **Lemma** (Hindley-Rosen). *Let $\to_1$ and $\to_2$ be relations on the set A. If $\to_1$ and $\to_2$ are* ***confluent*** *and* ***commute with each other****, then*

$$\to_1 \cup \to_2 \text{ is confluent.}$$

Local conditions. Commutation is a global condition, which is difficult to test. There are however *easy-to-check* sufficient conditions. One of the most useful such conditions is Hindley's strong commutation :

$$\leftarrow_1 \cdot \to_2 \subseteq \ \to_2{}^* \cdot \leftarrow_1{}^= \qquad \textbf{(Strong Commutation)}$$

▶ **Lemma** (Local test). *Strong commutation implies commutation.*

2 Factorization.

Both *confluence* and *factorization* are forms of commutation.

Let $\mathcal{A} = (A, \{\underset{e}{\to}, \underset{i}{\to}\})$ be an ARS.

- The relation $\to = \underset{e}{\to} \cup \underset{i}{\to}$ satisfies **e-factorization**, written $\mathtt{Fact}(\underset{e}{\to}, \underset{i}{\to})$, if

$$\mathtt{Fact}(\underset{e}{\to}, \underset{i}{\to}) : \quad \to^* \subseteq \underset{e}{\to}^* \cdot \underset{i}{\to}^* \qquad \textbf{(Factorization)}$$

- The relation $\underset{i}{\to}$ **postpones** after $\underset{e}{\to}$, written $\mathtt{PP}(\underset{e}{\to}, \underset{i}{\to})$, if

$$\mathtt{PP}(\underset{e}{\to}, \underset{i}{\to}) : \quad \underset{i}{\to}^* \cdot \underset{e}{\to}^* \ \subseteq \ \underset{e}{\to}^* \cdot \underset{i}{\to}^*. \qquad \textbf{(Postponement)}$$

Postponement can be formulated in terms of commutation, and viceversa, since clearly ($\underset{i}{\to}$ postpones after $\underset{e}{\to}$) if and only if ($\underset{i}{\leftarrow}$ commutes with $\underset{e}{\to}$). Note that reversing $\underset{i}{\to}$ introduce an asymmetry between the two relations. It is an easy result that e-factorization is equivalent to postponement, which is a more convenient way to express it.

▶ **Lemma 1.** *For any two relations $\underset{e}{\to}, \underset{i}{\to}$ the following are equivalent:*

1. $\underset{i}{\to}^* \cdot \underset{e}{\to} \subseteq \underset{e}{\to}^* \cdot \underset{i}{\to}^*$
2. $\underset{i}{\to} \cdot \underset{e}{\to}^* \subseteq \underset{e}{\to}^* \cdot \underset{i}{\to}^*$)
3. Postponement: $\underset{i}{\to}^* \cdot \underset{e}{\to}^* \ \subseteq \ \underset{e}{\to}^* \cdot \underset{i}{\to}^*$
4. Factorization: $(\underset{e}{\to} \cup \underset{i}{\to})^* \ \subseteq \ \underset{e}{\to}^* \cdot \underset{i}{\to}^*$

A local test. Hindley first noted that a local property implies postponement, hence factorization

We say that $\underset{i}{\to}$ **strongly postpones** after $\underset{e}{\to}$, if

$$\mathtt{SP}(\underset{e}{\to}, \underset{i}{\to}) : \quad \underset{i}{\to} \cdot \underset{e}{\to} \subseteq \underset{e}{\to}^* \cdot \underset{i}{\to}^= \qquad \textbf{(Strong Postponement)}$$

▶ **Lemma 2** (Local test for postponement). *Strong postponement implies postponement:*

$$\mathtt{SP}(\underset{e}{\rightarrow}, \underset{i}{\rightarrow}) \textit{ implies } \mathtt{PP}(\underset{e}{\rightarrow}, \underset{i}{\rightarrow}), \textit{ and so } \mathtt{Fact}(\underset{e}{\rightarrow}, \underset{i}{\rightarrow}).$$

It is immediate to recognize that the property is exactly the postponement analog of strong commutation; indeed it is the same expression, with $\underset{i}{\rightarrow} := \leftarrow_1$ and $\underset{e}{\rightarrow} := \rightarrow_2$.

A characterization. Another property that we shall use freely is the following, which is immediate by the definition of postponement and property **TR**

▶ **Property.** *Given a relation* $\underset{i}{\looparrowright}$ *such that* $\underset{i}{\looparrowright}^* = \underset{i}{\rightarrow}^*$,

$$\mathtt{PP}(\underset{e}{\rightarrow}, \underset{i}{\rightarrow}) \textit{ if and only if } \mathtt{PP}(\underset{e}{\rightarrow}, \underset{i}{\looparrowright}).$$

A well-known use of the above is to instantiate $\underset{i}{\looparrowright}$ with a notion of parallel reduction (as in [Takahashi])